Sharp constants in the Sobolev embedding theorem and a derivation of the Brezis-Gallouet interpolation inequality

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Sharp estimates are obtained for the constants appearing in the Sobolev embedding theorem on the two-dimensional torus. Furthermore, a version of the Brezis-Gallouet interpolation inequality is obtained with an explicit but not necessarily optimal constant in the leading term, namely, the logarithmic term. The constants are expressed in terms of the Riemann zeta-function and the Dirichlet beta-series. © *2011 American Institute of Physics.* [doi:10.1063/1.3638056]

I. INTRODUCTION AND NOTATION

Differential inequalities are a basic tool in the study of the solutions of nonlinear partial differential equations (PDEs). The most celebrated are the Poincaré and Sobolev embedding inequalities,[1] the Gagliardo-Nirenberg,[2] and Brezis-Gallouet inequalities.[3] With some exceptions, the constants on which they depend are not commonly computed explicitly. This does not present a problem in most contexts ; for example, in attempts to prove regularity for the three-dimensional Navier-Stokes equations. Nevertheless, in other situations the estimation of constants is an important exercise as they often contain geometric and number theoretic information which shed light on the overall significance and power of the inequalities themselves.[4–13] In addition, knowledge of these can be crucial for a sharp and detailed analysis of solutions of PDEs.[12–18] In this paper, we wish to provide a sharp explicit value for the constants appearing in the L^∞ norm in the Sobolev embedding theorem (SET); furthermore, we derive the Brezis-Gallouet inequality (BGI) (Ref. 3) with all the constants explicitly computed. The SET provides fundamental results on the function spaces in which the solutions of PDEs live, while the BGI gives an estimate in two-spatial dimensions for the supremum norm of a scalar or vectorial function in terms of the first spatial derivative, with the second derivative appearing in the form of a logarithmic correction. In what follows the constants are obtained in the case when the functions involved are scalar functions; the vectorial function case and applications to the global attractor of relevant dissipative PDEs will be investigated in a forthcoming paper.[19]

Let us first give some standard preliminary functional setting and notation.[20–22] Denote by $\Omega = [0, L]^d$ the d-dimensional torus; for any scalar and mean-zero function $\phi(x)$ with $x \in \Omega$ let $\|\phi\|_p^p = \int_\Omega |\phi(x)|^p \, dx$ be the Banach space of Ω-periodic functions; we also define the L^∞ norm as

$$\|\phi(x)\|_\infty = \sup_{x \in \Omega} |\phi(x)| . \tag{1}$$

For $p = 2$ we denote by $L^2(\Omega)$ the Hilbert space of Ω-periodic functions; given $n = n_1 + n_2 + \cdots + n_d$ with all the n_i non-negative integers, let

$$D^{n_1, n_2, \ldots, n_d} = \frac{\partial^{n_1 + n_2 + \cdots + n_d}}{\partial x_1^{n_1} \partial x_2^{n_2} + \cdots + \partial x_d^{n_d}} , \tag{2}$$

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and let

$$\dot{H}^n := \{\phi \mid \int_\Omega \phi dx = 0, \int_\Omega (D^{n_1,n_2,...,n_d}\phi)^2 dx < +\infty \quad \text{for} \quad n_1 + n_2 + \cdots + n_d = n\} \quad (3)$$

together with

$$\|\phi\|_{\dot{H}^n}^2 := \sum_{n=n_1+\cdots+n_d} \frac{n!}{n_1! \cdots n_d!} \|D^{n_1,n_2,...,n_d}\phi\|_{L^2}^2, \quad (4)$$

be the Sobolev space of mean zero Ω-periodic functions with up to n-derivatives in $L^2(\Omega)$. It then follows from Parseval's identity that

$$\sum_{n=n_1+\cdots+n_d} \frac{n!}{n_1! \cdots n_d!} \|D^{n_1,n_2,...,n_d}\phi\|_{L^2}^2 = L^d \left(\frac{2\pi}{L}\right)^{2n} \sum_{n=n_1+\cdots+n_d} \sum_{\vec{k}\in\mathbb{Z}^d\setminus\{\vec{0}\}} |\vec{k}|^{2n} |\phi_{\vec{k}}|^2. \quad (5)$$

In (5) the Fourier series expansion has been used for the mean zero function

$$\phi = \sum_{\vec{k}\in\mathbb{Z}^d\setminus\{\vec{0}\}} \phi_{\vec{k}} e^{2\pi i \vec{k}\cdot\vec{x}/L}, \quad (6)$$

and

$$\left(\frac{2\pi}{L}\right)^2 \vec{k}\cdot\vec{k} = \left(\frac{2\pi}{L}\right)^2 (k_1^2 + k_2^2 + \cdots + k_d^2). \quad (7)$$

By the same token the corresponding Sobolev space of mean zero periodic functions can be defined as $\dot{H}^s$ for every *real* number s; this is the same as

$$\dot{H}^s = \left\{ \phi : \phi = \sum_{\vec{k}\in\mathbb{Z}^d\setminus\{\vec{0}\}} \phi_{\vec{k}} e^{2\pi i \vec{k}\cdot\vec{x}/L}, \quad \overline{\phi}_{\vec{k}} = \phi_{-\vec{k}}, \left(\frac{2\pi}{L}\right)^{2s} \sum_{\vec{k}\in\mathbb{Z}^d\setminus\{\vec{0}\}} |\vec{k}|^{2s} |\phi_{\vec{k}}|^2 < +\infty \right\}. \quad (8)$$

Hence by extending (4) to non-integer positive values we have

$$\dot{H}^s = \{\phi : \|\phi\|_{\dot{H}^s}^2 < +\infty\}. \quad (9)$$

These Sobolev spaces, defined on the d-dimensional torus, are used below as we need to deal with the negative Laplacian $A = -\Delta$ (as a self-adjoint unbounded operator) and its fractional powers. More precisely we have the eigenvalues of the negative Laplacian $A = -\Delta$ are given by the numbers $\left(\frac{2\pi}{L}\right)^2 |\vec{k}|^2$, so the domain of its powers A^s is the set of functions such that

$$L^d \left(\frac{2\pi}{L}\right)^{4s} \sum_{\vec{k}\in\mathbb{Z}^d\setminus\{\vec{0}\}} |\vec{k}|^{4s} |\phi_{\vec{k}}|^2 = \|A^s\phi(x)\|_2^2 < +\infty. \quad (10)$$

In particular, for $s = \frac{1}{2}$ (on the torus) we have

$$\|A^{\frac{1}{2}}\phi(x)\|_2^2 = \|\nabla\phi(x)\|_2^2 = L^d \left(\frac{2\pi}{L}\right)^2 \sum_{\vec{k}\in\mathbb{Z}^d\setminus\{\vec{0}\}} |\vec{k}|^2 |\phi_{\vec{k}}|^2, \quad (11)$$

while for $s = 1$ we have (on the torus)

$$\|A\phi(x)\|_2^2 = \|(-\Delta)\phi(x)\|_2^2 = L^d \left(\frac{2\pi}{L}\right)^4 \sum_{\vec{k}\in\mathbb{Z}^d\setminus\{\vec{0}\}} |\vec{k}|^4 |\phi_{\vec{k}}|^2. \quad (12)$$

In the rest of the paper (with a minor abuse of notation), for any $s > 0$, we make the *formal* identification

$$\|A^{\frac{s}{2}}\phi(x)\|_2^2 = \|(-\Delta)^{\frac{s}{2}}\phi(x)\|_2^2 = L^d \left(\frac{2\pi}{L}\right)^{2s} \sum_{\vec{k}\in\mathbb{Z}^d\setminus\{\vec{0}\}} |\vec{k}|^{2s} |\phi_{\vec{k}}|^2, \quad (13)$$

provided it is understood that these operators are being used as differential operators "acting" on functions in $\dot{H}^s$, according to ([8]) and ([10]). For more details see Refs. [20–24].

II. SHARP CONSTANT ON THE TWO-DIMENSIONAL TORUS FOR THE SOBOLEV EMBEDDING THEOREM FOR THE L^∞-NORM

By using the notation above the two-dimensional case $d = 2$ is considered in this section where we wish to estimate explicitly the constant on the two-dimensional torus in the Sobolev embedding theorem, for any mean-zero function $\phi \in \dot{H}^s$. First let us recall that on the d-dimensional torus Ω, if $s > d/2$, for any mean zero function $\phi \in \dot{H}^s$ it is true that[21]

$$\|\phi\|_\infty \le c_s \|\phi\|_{\dot{H}^s},$$

where c_s is a positive constant depending upon s only. Our aim in this section is to obtain a sharp estimate of the constant appearing in the above inequality on the two-dimensional torus $\Omega = [0, L]^2$. We begin by proving the following:

Theorem 1: *On the two-dimensional torus $\Omega = [0, L]^2$, for every positive real number $s = 1 + \epsilon$ with $\epsilon > 0$, the L^∞ norm of a scalar function $\phi(x) \in \dot{H}^{1+\epsilon}$ satisfies the estimate*

$$\|\phi(x)\|_\infty \le [4\zeta(1+\epsilon)\beta(1+\epsilon)]^{\frac{1}{2}} L^{-1} \left(\frac{L}{2\pi}\right)^{(1+\epsilon)} \|(-\Delta)^{\frac{1+\epsilon}{2}}\phi(x)\|_2, \tag{14}$$

where $C(\epsilon) = 4\zeta(1 + \epsilon)\beta(1 + \epsilon)$ is sharp, and where $\zeta(1 + \epsilon)$ and $\beta(1 + \epsilon)$

$$\zeta(1+\epsilon) = \sum_{n\ge1} \frac{1}{n^{1+\epsilon}}, \qquad \beta(1+\epsilon) = \sum_{n\ge0} \frac{(-1)^n}{(2n+1)^{1+\epsilon}}, \tag{15}$$

are the Riemann zeta-function and Dirichlet series, respectively.

Proof: We first expand our function in Fourier series

$$\phi(x) = \sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} \phi_{\vec{k}} e^{2\pi i \vec{k}\cdot\vec{x}/L}; \tag{16}$$

to give

$$\|\phi(x)\|_\infty \le \sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} |\phi_{\vec{k}}| = \sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} \frac{(\vec{k}\cdot\vec{k})^{(1+\epsilon)/2}}{(\vec{k}\cdot\vec{k})^{(1+\epsilon)/2}} |\phi_{\vec{k}}|$$

$$\le \left(\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} \frac{1}{(\vec{k}\cdot\vec{k})^{1+\epsilon}}\right)^{1/2} \left(\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} (\vec{k}\cdot\vec{k})^{1+\epsilon} |\phi_{\vec{k}}|^2\right)^{1/2}$$

$$= \left(\frac{L}{2\pi}\right)^{(1+\epsilon)} \sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} \left(\frac{1}{(k_1^2 + k_2^2)^{1+\epsilon}}\right)^{1/2} L^{-1} \|(-\Delta)^{\frac{1+\epsilon}{2}}\phi\|_2. \tag{17}$$

The following remarkable result on series is now used:[25–27]

$$\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} \frac{1}{(k_1^2 + k_2^2)^{1+\epsilon}} = 4\zeta(1+\epsilon)\beta(1+\epsilon). \tag{18}$$

Putting together ([17]) and ([18]) gives ([14]). In order to see that $C(\epsilon) = 4\zeta(1 + \epsilon)\beta(1 + \epsilon)$ is sharp we use the extremal functions[11]

$$\phi = \sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} |\vec{k}|^{-2(1+\epsilon)} e^{2\pi i \vec{k}\cdot\vec{x}/L}. \tag{19}$$

Now first note that from the definition of ϕ it follows that

$$\|\phi(x)\|_\infty \leq \sum_{\vec{k} \in \mathbb{Z}^2 \setminus \{\vec{0}\}} |\vec{k}|^{-2(1+\epsilon)}. \tag{20}$$

Second,

$$\|\phi(x)\|_\infty \geq |\phi(0)| = \phi(0) = \sum_{\vec{k} \in \mathbb{Z}^2 \setminus \{\vec{0}\}} |\vec{k}|^{-2(1+\epsilon)}, \tag{21}$$

so we obtain

$$\|\phi(x)\|_\infty = \phi(0) = \sum_{\vec{k} \in \mathbb{Z}^2 \setminus \{\vec{0}\}} |\vec{k}|^{-2(1+\epsilon)}. \tag{22}$$

It follows that all the above inequalities become equalities and hence

$$C(\epsilon) = 4\zeta(1+\epsilon)\beta(1+\epsilon) \tag{23}$$

which cannot be improved. The proof is now complete. □

The expression (14) is general and gives the explicit value of the constant in front of *all* the Sobolev spaces $\dot{H}^s$ with $s = 1 + \epsilon$ for *every* $\epsilon > 0$. We provide here only a couple of examples by considering for simplicity the $d = 2$ torus of length 2π :

(a)   By choosing the value $\epsilon = 1$ then

$$\zeta(2) = \sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \beta(2) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^2} = K = 0.915965594177\ldots, \tag{24}$$

where K is the Catalan's constant; hence in this case we have

$$\|\phi(x)\|_\infty^2 \leq \frac{K}{6} \|\Delta\phi(x)\|_2^2. \tag{25}$$

(b)   If we take the value $\epsilon = 2$ we have

$$\zeta(3) = \sum_{n \geq 1} \frac{1}{n^3} = B = 1.20205690032\ldots, \quad \text{and} \quad \beta(3) = \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}. \tag{26}$$

Thus for the $\|\phi(x)\|_\infty^2$ we have

$$\|\phi(x)\|_\infty^2 \leq= \frac{\pi B}{32} \|(-\Delta)^{\frac{3}{2}}\phi(x)\|_2^2. \tag{27}$$

Another interesting case in our analysis is to approximate the series appearing in Theorem 1, as is well known, the sum of which is not explicitly known for all values of $\epsilon > 0$. First, an expansion of the Riemann zeta-function for small ϵ gives the representation

$$\zeta(1+\epsilon) = \frac{1}{\epsilon} + \gamma + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \gamma_n \epsilon^n, \tag{28}$$

where $\gamma \simeq 0.58$ is the Euler-Mascheroni constant and γ_n are the Stieltjes constants. Moreover, $\beta(1+\epsilon) \simeq \frac{\pi}{4} + \epsilon\beta'(1)$ with $\beta'(1) \simeq 0.19$. Bearing in mind that the exact expression is (18), we can nonetheless approximate the product of the two series as

$$4\zeta(1+\epsilon)\beta(1+\epsilon) = \pi(\gamma + \epsilon^{-1}) + 4\beta'(1) + O(\epsilon), \tag{29}$$

provided ϵ is small enough. Hence Corollary 1 below provides the explicit computation of the constant in the Sobolev embedding theorem on the $d = 2$ torus of length 2π:

Corollary 1: *On the two-dimensional torus $\Omega = [0, 2\pi]^2$, in the limit $\epsilon \to 0^+$, the L^∞ norm of a scalar function $\phi(x) \in \dot{H}^{1+\epsilon}$ satisfies the estimate*

$$\|\phi(x)\|_\infty^2 \le c_\epsilon^2 \|(-\Delta)^{\frac{1+\epsilon}{2}} \phi\|_2^2 , \tag{30}$$

where

$$c_\epsilon^2 = \frac{1}{4\pi} \left(\frac{1}{\epsilon} + \hat{\gamma} \right) + O(\epsilon) \quad \text{for} \quad \epsilon \to 0^+ , \tag{31}$$

and where $\hat{\gamma} := \gamma + \frac{4\beta'(1)}{\pi}$.

III. THE BREZIS-GALLOUET INTERPOLATION INEQUALITY

In this section, using similar arguments of Sec. II, we obtain from first principles the well known Brezis-Gallouet interpolation inequality and its explicit constant by using essentially the estimate (30), which is the sharpest result obtainable for the $\|\phi(x)\|_\infty$-norm by using a Hölder inequality. Indeed the use of a Hölder inequality is the only step where some sharpness may be lost because the lattice sum (18) is *exact* in terms of the Riemann zeta-function and the Dirichlet series. Hence if we perform any other interpolation inequality we are somehow bound to obtain results which cannot be intrinsically sharp. That said, we wish now to derive the Brezis-Gallouet inequality in its classical form, namely,

$$\|\phi\|_\infty \le c_1 \|\nabla\phi\|_2 \left[c_2 + \ln \left(\frac{L^2}{4\pi^2} \frac{\|\Delta\phi\|_2^2}{\|\nabla\phi\|_2^2} \right) \right]^{\frac{1}{2}} \tag{32}$$

with the constants explicitly computed. In fact we have the following result:

Theorem 2: *The $\|\phi(x)\|_\infty$ norm of a two spatial-dimensions scalar function $\phi(x) \in \dot{H}^2$ on the two-dimensional torus and in the limit*

$$\frac{1}{\ln \left(\frac{L^2}{4\pi^2} \frac{\|\Delta\phi\|_2^2}{\|\nabla\phi\|_2^2} \right)} = \epsilon \to 0^+ , \tag{33}$$

satisfies the explicit estimate

$$\|\phi(x)\|_\infty^2 \le \frac{e}{4\pi} \|\nabla\phi\|_2^2 \left[\ln \left(\frac{L^2}{4\pi^2} \frac{\|\Delta\phi\|_2^2}{\|\nabla\phi\|_2^2} \right) + \hat{\gamma} + O(\epsilon) \right] , \tag{34}$$

where $\hat{\gamma} := \gamma + \frac{4\beta'(1)}{\pi}$.

Proof: Let $a = \|\nabla\phi\|_2$ and $b = \|\Delta\phi\|_2$; then by using the theory of fractional powers of linear self-adjoint unbounded operators[21–24] we can interpolate $\|(-\Delta)^{\frac{1+\epsilon}{2}}\phi\|_2$ as follows:

$$\|(-\Delta)^{\frac{1+\epsilon}{2}}\phi\|_2 \le \|\Delta\phi\|_2^\epsilon \|\nabla\phi\|_2^{1-\epsilon} \equiv \left(\frac{b}{a} \right)^\epsilon a; \tag{35}$$

note that the inequality (35) holds for $0 \le \epsilon \le 1$ and for *all* functions $\phi(x) \in \dot{H}^2$. We now use (31) (adapted to the torus of length L) thereby obtaining (in the limit $\epsilon \to 0^+$)

$$\|\phi(x)\|_\infty \le L^{-1} \left(\frac{L}{2\pi} \right)^{(1+\epsilon)} \sqrt{\pi \left(\hat{\gamma} + \frac{1}{\epsilon} \right)} \left(\frac{b}{a} \right)^\epsilon a. \tag{36}$$

We now wish to minimize the above formula with respect to ϵ; define

$$f(\epsilon) = \sqrt{\hat{\gamma} + \frac{1}{\epsilon}} \left(\frac{b}{a} \frac{L}{2\pi} \right)^\epsilon . \tag{37}$$

Differentiating with respect to ϵ we find

$$\frac{df}{d\epsilon} = \left(\frac{b}{a}\frac{L}{2\pi}\right)^{\epsilon}\left[-\frac{1}{2\epsilon^2}\left(\hat{\gamma}+\frac{1}{\epsilon}\right)^{-\frac{1}{2}}+\left(\hat{\gamma}+\frac{1}{\epsilon}\right)^{\frac{1}{2}}\ln\left(\frac{b}{a}\frac{L}{2\pi}\right)\right].\tag{38}$$

Thus, we have to find the value of ϵ which minimizes $f(\epsilon)$. Within the same framework of the approximation done above for the series (small ϵ) we can neglect the $\hat{\gamma}$ constant, thereby obtaining

$$\frac{df}{d\epsilon}\simeq 0\quad\text{for}\quad\frac{1}{\epsilon}=\ln\left(\frac{L}{2\pi}\frac{b}{a}\right)^2,\qquad\left(\frac{L}{2\pi}\frac{b}{a}\right)^2\geq e.\tag{39}$$

Inserting this value of ϵ into (36) and simplifying we obtain

$$\|\phi(x)\|_{\infty}\leq\frac{a}{2\sqrt{\pi}}\left(\frac{L}{2\pi}\frac{b}{a}\right)^{\frac{1}{2\ln\left(\frac{L}{2\pi}\frac{b}{a}\right)}}\left[\hat{\gamma}+2\ln\left(\frac{L}{2\pi}\frac{b}{a}\right)\right]^{\frac{1}{2}}.\tag{40}$$

By further simplifying the logarithm in the exponent and by re-inserting the values for $a=\|\nabla\phi\|_2$ and $b=\|\Delta\phi\|_2$ we finally obtain

$$\|\phi\|_{\infty}^2\leq\frac{e}{4\pi}\|\nabla\phi\|_2^2\left[\hat{\gamma}+\ln\left(\frac{L^2}{4\pi^2}\frac{\|\Delta\phi\|_2^2}{\|\nabla\phi\|_2^2}\right)+O(\epsilon)\right],\tag{41}$$

which is (34) as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 2 derives the Brezis-Gallouet interpolation inequality from first principles in a compact and expressive way. However, it is probably not optimal; in fact there are some recent papers (see Refs. 17, 18, and 28) where sharp estimates on various domains have been derived by using the C^{α} norm and Dirichlet boundary conditions. In particular, in Ref. 28 the sharp constant $(2\pi\alpha)^{-1}$ with $0<\alpha<1$ is obtained on the unit disk in $\mathbb{R}^n$ thereby generalizing the result of Refs. 17 and 18; the same sharp constant appears in the work[11] on the whole space $\mathbb{R}^2$. It would be interesting to see if their methods could be extended to the torus by using the H^2-norm used in this paper; in fact, if we translate to our notation of taking the square of $\frac{\|\phi(x)\|_{C^{\alpha}}}{\|\nabla\phi\|_2}$, the corresponding sharp constant reads $\lambda=(4\pi)^{-1}$. Thus, it may be possible to improve our estimates on the two-dimensional torus to lower the Euler constant from the value e down to 1. However, with the methods used here, namely, the Hölder inequality and interpolation inequalities, it appears to be hard to make such an improvement; the best that we can hope for is to obtain an upper bound which has the same constant in the leading term, namely, the logarithmic term with the squares brackets in (41). This result can be achieved as follows : from Theorem 1 it can be seen that for any function $\phi(x)$, up to a constant factor, such that

$$\phi=\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}}\phi_{\vec{k}}e^{2\pi i\vec{k}\cdot\vec{x}/L}=\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}}|\vec{k}|^{-2s}e^{2\pi i\vec{k}\cdot\vec{x}/L},\tag{42}$$

which satisfies

$$\|\phi(x)\|_{\infty}^2\leq 4\zeta(s)\beta(s)\|(-\Delta)^{\frac{s}{2}}\phi\|_2^2,\tag{43}$$

the extremal functions (again, up to a constant factor) are given by

$$\phi=\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}}|\vec{k}|^{-2s}e^{2\pi i\vec{k}\cdot\vec{x}/L}.\tag{44}$$

Note that we consider periodic functions on the torus with the appropriate decay rate which ensures convergence in the corresponding appropriate spaces. So, in the light of this we introduce the space E_s of extremal functions as the set

$$\phi(x)\in E_s=\left\{\phi=\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}}|\vec{k}|^{-2s}e^{2\pi i\vec{k}\cdot\vec{x}/L}:\|\phi(x)\|_{\infty}=\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}}|\vec{k}|^{-2s}<+\infty\right\}.$$

Of course on the two-dimensional torus, the minimum value of s for a function $\|\phi(x)\|_{\infty}\in E_s$ is $s=1+\alpha$ with $\alpha>0$. For the Brezis-Gallouet inequality we need functions which are bounded in

$\dot{H}^2$, namely, functions with bounded Laplacian in L^2 on the two-dimensional torus. So we take the space of functions (and for simplicity the torus of length 2π)

$$E_{\frac{3+\alpha}{2}} = \left\{ \phi : \phi = \sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} |\vec{k}|^{-(3+\alpha)} e^{i\vec{k}\cdot\vec{x}}, 4\pi^2 \sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} |\vec{k}|^4 |\phi_{\vec{k}}|^2 = \|\Delta\phi(x)\|_2^2 < +\infty \right\},$$

with $\alpha > 0$. Now take $\phi(x) \in E_{\frac{3+\alpha}{2}}$ with $\alpha > 0$ and small enough and use (30)

$$\|\phi(x)\|_\infty^2 \leq \frac{1}{4\pi}\left(\frac{1}{\epsilon} + \hat{\gamma}\right) \|(-\Delta)^{\frac{1+\epsilon}{2}}\phi\|_2^2 ; \tag{45}$$

then multiply and divide the right hand side by

$$\|\nabla\phi\|_2^2 = (4\pi^2) \sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} (\vec{k}\cdot\vec{k}) \mid \phi_{\vec{k}} \mid^2,$$

thus obtaining

$$\|\phi(x)\|_\infty^2 \leq (4\pi)^{-1} \|\nabla\phi\|_2^2 \left(\frac{1}{\epsilon} + \hat{\gamma}\right) \frac{\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} (\vec{k}\cdot\vec{k})^{-(2+(\alpha-\epsilon))}}{\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} (\vec{k}\cdot\vec{k})^{-(2+\alpha)}} . \tag{46}$$

Observe that the term

$$D(\alpha,\epsilon) = \frac{\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} (\vec{k}\cdot\vec{k})^{-(2+(\alpha-\epsilon))}}{\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} (\vec{k}\cdot\vec{k})^{-(2+\alpha)}} = 1 + \epsilon + O(\epsilon^2),$$

provided α and ϵ are small enough and $\alpha \geq \epsilon$; in fact, the above formula is nothing else that the ratio of two lattice sums, expressed by formula (18), namely,

$$D(\alpha,\epsilon) = \frac{\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} (\vec{k}\cdot\vec{k})^{-(2+(\alpha-\epsilon))}}{\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} (\vec{k}\cdot\vec{k})^{-(2+\alpha)}} = \frac{\zeta(2+\alpha-\epsilon)\beta(2+\alpha-\epsilon)}{\zeta(2+\alpha)\beta(2+\alpha)}.$$

The estimate above can also be seen in other ways, for example, by evaluating the series by the corresponding integrals, which for these class of functions are particularly suitable; for instance if we take the particular case $\alpha = \epsilon$ we obtain

$$D(\epsilon) = \frac{\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} (\vec{k}\cdot\vec{k})^{-2}}{\sum_{\vec{k}\in\mathbb{Z}^2\setminus\{\vec{0}\}} (\vec{k}\cdot\vec{k})^{-(2+\epsilon)}} = \frac{\zeta(2)\beta(2)}{\zeta(2+\epsilon)\beta(2+\epsilon)} = 1 + \epsilon .$$

Furthermore, we note that for ϵ small enough, Theorem 2 suggests the choice

$$\epsilon = \frac{1}{\ln\left(\frac{\|\Delta\phi\|_2^2}{\|\nabla\phi\|_2^2}\right)}$$

in the function

$$4\zeta(1+\epsilon)\beta(1+\epsilon)D(\alpha,\epsilon)$$

with $D(\alpha,\epsilon)$ defined above. Hence for small $\epsilon > 0$, with $\alpha = \epsilon$, we obtain

$$\|\phi(x)\|_\infty^2 \leq \frac{\|\nabla\phi\|_2^2}{4\pi}\left[\ln\left(\frac{\|\Delta\phi\|_2^2}{\|\nabla\phi\|_2^2}\right) + \hat{\gamma}\right][1+\epsilon]. \tag{47}$$

By multiplying out the two square brackets we finally obtain

$$\|\phi(x)\|_\infty^2 \leq \frac{\|\nabla\phi\|_2^2}{4\pi}\left[\ln\left(\frac{\|\Delta\phi\|_2^2}{\|\nabla\phi\|_2^2}\right) + \eta\right], \tag{48}$$

where $\ln\left(\frac{\|\Delta\phi\|_2^2}{\|\nabla\phi\|_2^2}\right)$ is large enough and $\eta = \hat{\gamma} + 1 + O(\epsilon)$.

The above result shows that on the two-dimensional torus $\Omega = [0, 2\pi]^2$, in the limits $\epsilon \to 0^+$ and $\alpha \to 0^+$, with $\alpha - \epsilon \geq 0$, for any scalar function $\phi(x) \in E_{\frac{3+\alpha}{2}}$, our inequality holds as an *upper*

bound. We stress that this is only a partial result towards proving optimality of the leading constant, due to the specified limits for ϵ and α and also due to the "special" class of functions we use, namely, the class $E_{\frac{3+\alpha}{2}}$. To actually show that the leading constant is sharp, we would need a *lower* bound of the form

$$\|\phi\|_\infty^2 \geq \frac{\|\nabla\phi\|_2^2}{4\pi}\left[c + \ln\left(\frac{\|\Delta\phi\|_2^2}{\|\nabla\phi\|_2^2}\right)\right],$$

or a sequence where the leading constant approaches $(4\pi)^{-1}$ from below. As a comparative summary, the results of Theorem 1 are explicit and sharp while the results of Theorem 2 are explicit but not necessarily optimal. In the informal discussion made in this section, an explicit constant $(4\pi)^{-1}$ has been found as an *upper bound* only, and in the limits $\epsilon \to 0^+$ and $\alpha \to 0^+$; furthermore, the constant η is computed explicitly with no floating undetermined constants in the inequality. This (we believe) has some merit in the sense that the papers,[17, 18, 28] where the leading constant is shown to be sharp, have nevertheless a double logarithmic term in their estimate. Additionally, they have unspecified constants.

However, motivated by an earlier version of this current paper, the sharp constant $(4\pi)^{-1}$ on the two-dimensional torus has recently been obtained in Ref. 29 by using the method of Lagrange multipliers. For this reason, let us briefly describe the strategy in Ref. 29. Consider the following variational problem with constraints :

$$\frac{\|u\|_{L^\infty}^2}{\|\nabla u\|_{L^2}^2} \to \max, \quad u \in H^2(\mathbb{T}^2), \quad \int_{\mathbb{T}^2} u(x)\,dx = 0, \quad \frac{\|\Delta u\|_{L^2}^2}{\|\nabla u\|_{L^2}^2} = \delta. \tag{49}$$

It has then been proved in Ref. 29 that, for every $\delta(\mu) = \frac{\|\Delta u_\mu\|_{L^2}^2}{\|\nabla u_\mu\|_{L^2}^2} > 0$, this problem has a unique (up to shifts, scaling, and alternation of sign) solution $u_\mu(x)$,

$$u_\mu(x) = \sum_{k\in\mathbb{Z}^2-\{0\}} \frac{e^{ik\cdot x}}{k^2(1+\mu k^2)}, \qquad k^2 := k_1^2 + k_2^2. \tag{50}$$

Then by following a strategy similar to that used in Sec. I to prove the Sobolev embedding theorem, the above problem can then be cast into one for the extremal functions (see Ref. 30 where a more general expression on the n-dimensional torus is obtained)

$$\|u\|_\infty^2 \leq \frac{\|\nabla u\|_2^2}{4\pi^2}\sideset{}{'}\sum\left(\frac{1}{k^2(1+\mu k^2)}\right)(1+\mu\delta), \qquad \mu > 0, \qquad \delta = \frac{\|\Delta u\|_2^2}{\|\nabla u\|_2^2}. \tag{51}$$

It is interesting to note that when $\epsilon \to 1$ in (14), which clearly corresponds to $\mu \to \infty$ in (51), one can easily see that the two estimates merge to give the same result. In fact, by taking the limit $\mu \to \infty$ formula (51) gives

$$\lim_{\mu\to\infty} \frac{\|\nabla u\|_2^2}{4\pi^2}\sideset{}{'}\sum\left(\frac{1}{k^2(1+\mu k^2)}\right)(1+\mu\delta) = \frac{\|\nabla u\|_2^2}{4\pi^2}\sideset{}{'}\sum\left(\frac{1}{k^4}\right)\delta, \tag{52}$$

which exactly coincides with formula (14) for $\epsilon = 1$.

Another interesting case worth mentioning is the limit $\mu \to 0^+$. In this limit formula (51) becomes

$$\|u\|_\infty^2 \leq \frac{(1+\mu\delta)}{4\pi}\|\nabla u\|_2^2\left[\ln\delta + \ln(1+\ln\delta) + L\right]; L \simeq 2.15. \tag{53}$$

Note that in formula (53) it is tacitly assumed that the product $\mu\delta \ll 1$ (recall that the singularity near $\mu \to 0^+$ is an essential singularity and hence $0 < \mu\delta \ll 1$); this is achieved by "fixing" δ and then by choosing μ sufficiently small so that $0 < \mu\delta \ll 1$. So provided $0 < \mu\delta \ll 1$ the estimate (53) provides the most important estimate of (^{29}), namely, that

$$\|u\|_\infty^2 \leq \frac{1}{4\pi}\|\nabla u\|_2^2\left[\ln\delta + \ln(1+\ln\delta) + L\right], \tag{54}$$

where we have neglected the additive small quantity $\mu\delta$. The estimate (54) is the sharp estimate for the Brezis-Gallouet inequality; as one can see, this improves the estimate (34) by lowering the Euler constant e to 1: this result is contained in Ref. 29. It is nevertheless worth mentioning that the validity of (54) hinges on the condition that $\mu\delta \ll 1$. In fact it must be noted that the δ appearing in (53) *does not* (in general) depend upon μ. It is not difficult to construct situations where the product $\mu\delta$ is much larger than the Euler constant e: it is sufficient to take δ large enough so that the product $\mu\delta > e$, even though $\mu \ll 1$ (recall that the δ appearing in (53) does not depend upon μ) and so, no matter how small μ is, one can always choose δ sufficiently large so that $\mu\delta > e$. In these cases the estimate (34) does become better than the estimate (54), so it is clear that both formulas have their own merits and regimes of applicability.

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