The extinction of subharmonic vibration under the effects of gravity on the equilibrium of motion.

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Dedication

This thesis is dedicated to my parents, Mr. and Mrs. W.C.Tang, whose love and understanding have been resolute encouragement throughout my life.

Abstract

The extinction of second order subharmonic vibration response of a single degree of freedom system is investigated for the case of damped centrifugal excitation. The degree of the restoring force asymmetry resulting from the effect of gravity is expressed in terms of the parameter of static deflection.

The resonance under gravity effects is analysed theoretically for a wide range of physical conditions to determine the behavioural characteristics of the subharmonic components. The inherently coupled algebraic equations are obtained by the approximate energy method of Ritz-Galerkin and by the method of harmonic balance. These two methods are not bounded to any degree of non-linearity.

As there is no exact solution for this investigation and because of the dissipative forces inevitably introducing the problem of stability, the actual existence of the approximate solution over the frequency band-width is ascertained. There are no real roots in the instability region. The algebraic polynomial expressions cannot be satisfied simultaneously because of the accumulative effect in an

accompanying harmonic of the vibratory motion. The build-up oscillation occurs in the second order region, having a frequency the same as that of the main component of subharmonic motion. The stability criterion is derived from comparing the characteristic exponent of solution to the variational equation with damping coefficient of the system.

The response characteristics are then investigated where the polynomial equations are simplified through justifying the approximation of the fundamental harmonic as the effective amplitude of the disturbing force. The results are of comparable accuracy for cases in which gravity effects do not increase the effective non-linearity with resonance. The approximation, however, is applicable whatever the physical characteristic behaviour of the non-linearity in the region of the critical state of subharmonic extinction.

The subharmonic motion in the process of analysis is shown to exist in two opposite phases, differing by π radians. The resulting phase of periodic vibration depends upon initial conditions. The isocline graphical method is used to depict the transient motion.

The effective non-linearity is determined to be governed by the influence of gravity effects on the equilibrium of motion. The pronounced subharmonics are of the order one-half, and the extinction conditions for the resonance predominant over the higher orders are investigated through expressing the limiting inequalities in terms of the system parameters. In the critical state, complete suppression of the subharmonics is achieved. The limiting condition is then examined where damping is fixed at a convenient minimum value and the corresponding optimum limit of gravity effect tolerable can be evaluated for which the amplitude of excitation has no influence on the effective non-linearity as regard to exciting the resonance. An inequality is also presented from which the limiting frequencies of the subharmonic vibration can be predicted with reasonable accuracy. In these investigations the limiting inequalities are not dependent on the resulting variables in the non-linear phenomena.

An experimental test-rig is designed to demonstrate the subharmonic response. The values recorded from it compare favourably with the approximated theoretical results and with the experimental results obtained from the electronic analogue computer.

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Table of Contents

Page

Notation.

A, B	-	Amplitudes of the second order subharmonic of displacement.
C, D	-	Amplitudes of the first harmonic of displacement.
$Q_{\frac{1}{2}} = (A^2 + B^2)^{\frac{1}{2}}$	-	Total amplitude of the second order subharmonic.
$Q_1 = (C^2 + D^2)^{\frac{1}{2}}$	-	Total amplitude of the first harmonic.
N	-	Distance of the mean of the vibratory displacement from the static equilibrium position.
μ	-	Coefficient of the cubic term of the restoring force, $(L)^{-2}$.
$\bar{A} = \mu^{\frac{1}{2}}A$, $\bar{B} = \mu^{\frac{1}{2}}B$, $\bar{C} = \mu^{\frac{1}{2}}C$, $\bar{D} = \mu^{\frac{1}{2}}D$	-	Non-dimensional amplitudes.

$\bar{N} = \mu^{\frac{1}{2}} N$	- Non-dimensional mean amplitude.
m	- Total vibrating mass.
M_o	- Out-of-balance rotating mass causing the centrifugal excitation.
e	- The eccentricity of the rotating mass.
K	- Linear stiffness or the coefficient of the linear term of the restoring force.
F	- Non-linear restoring force exerted by the suspension.
$p = (\frac{K}{m})^{\frac{1}{2}}$	- Equivalent linear natural frequency, rad./s.
C_c	- Critical linear damping force per unit velocity.
C_1	- Linear damping force per unit velocity.
x	- Exact instantaneous displacement of the vibrating mass.
t	- Time.
$\tilde{x}$	- Approximate instantaneous displacement of vibrating mass.
Δ	- Static deflection.
$\bar{\Delta} = \mu^{\frac{1}{2}} \Delta$	- Non-dimensional static deflection.
y	- Total deflection of non-linear spring from the unstrained position.

ω	-	Frequency of the disturbing force, rad./s.
$\eta = \frac{\omega}{p}$	-	Non-dimensional frequency ratio.
$R = \frac{C_1}{C_c}$.	-	Non-dimensional damping ratio.
$Z = \frac{M_e e}{m}$	-	Amplitude of the centrifugal disturbing force.
$\bar{Z}$	-	Non-dimensional amplitude of the centrifugal disturbing force.
$Y = (\bar{\Delta} + \bar{N})$	-	The effective asymmetry of the restoring force.
$\phi_{\frac{1}{2}}$	-	Second order subharmonic phase angle.
ϕ_1	-	Phase angle lag of the first harmonic.

List of Figures

Figure		Page

Chapter VI

Chapter VII

Chapter I

Introduction

1.1. Non-linearity

That this exists in most mechanical systems where the restoring function to displacement is not directly proportional is well known. As the coefficient of the restoring forces is virtually always non-linear, the differential equation of motion of the form

$$\frac{d^2y}{dt^2} + \gamma \frac{dy}{dt} + ky + \mu f(y) = F(t) \qquad \ldots\ldots \quad (1.1,1)$$

is repeated encountered. The parameter μ may usually be small and the above equation is often linearised. In these instances linear analysis has been usually successful in the region of the harmonic resonance. For most practical purposes such an approximation often produces satisfactory results. However the phenomena of non-linear resonance constitute not only the harmonic but also a separate new component, the subharmonics, which is an essential property of non-linear systems. The subharmonics can be readily excited irrespective of the degree of non-linearity if the relationship between the independent parameters of the describing equation lies within the domain for the existence of subharmonic motion.

1.2. Subharmonic vibrations

In essence with most mechanical vibrating systems the experimental natural frequency contains numerous harmonic components. When the external forcing frequency is the same as one of the components, the strength of other harmonic displacements diminish progressively away from the resulting harmonic resonance. Subharmonic vibration is generated through exciting and sustaining components of lower order by the fundamental harmonic. Whereas with a linearised system during steady state the body can only vibrate at the same frequency as the external force.

The vibration resonance occurs at frequencies higher than the fundamental and is at a submultiple of the forcing function values. As the resonance being a direct consequence of the existence of the harmonics, the respective intensity depends upon its proximity to the forcing frequency. The pronounced amplitudes exist within certain frequency ranges and are commonly experienced when energy dissipation of a system is only adequately considered for harmonic resonance. Where in previous applications subharmonic vibrations are to be avoided and not merely classified as spurious resonances when they are encountered, a more complete analysis becomes necessary. Linearised theory would fail to explain such phenomena.

1.3. Second order subharmonics

The restoring force alone as an entity is often an odd function of displacement and can be expressed as a power series. From equation (1.1,1) the non-linear restoring function frequently takes the form

$$ky + \mu f(y) = ky + \mu_1 y^3 + \mu_2 y^5 + \ldots \qquad \ldots . \quad (1.3,1)$$

With most cases terms higher than the cubic power are negligibly small and the function is adequately described by the first two terms. However, in many practical systems the curve of the non-linear restoring force characteristic is asymmetrical. The asymmetry in cases of suspension results from the effect of gravity on the equilibrium of motion and the actual restoring force is in the form

$$ky + \mu f(y) = k(x + \Delta) + \mu_1 (x + \Delta)^3 \qquad \ldots . . \quad (1.3,2)$$

where x is the vibratory displacement and Δ is the static deflection due to the gravitational force. Subsequently the non-linear function, when satisfactorily approximated in the truncated form of a series expansion, contains a second order term. When the solution to the periodic displacement which can be represented consisting of harmonic components is expressed as a Fourier series, it is evident that the prominent subharmonic vibration would be the second order. The order sequence of subharmonics is defined by the reciprocal of its frequency.

Effects from the weights of the vibrating masses are generally neglected during the theoretical analysis. The non-linearity is often regarded as having symmetrical characteristic and as it is not possible to eliminate the static deflection that is produced, the restoring force characteristic of a hardening non-linear suspension is not satisfactorily represented by the sum of a linear and cubic term of the vibratory displacement. The static equilibrium position does not coincide with the point of symmetry, and additional terms result in the describing equation. For such cases the actual deviation from linearity in the system is larger than it is considered. The error often only becomes obvious when the vibration encountered is a dominant second order subharmonic resonance. The vibration is usually tolerated if possible over the frequency range or else it is eliminated by a costly and time consuming trial procedure.

1.4. An analysis of the investigation

The influence of gravitational force on the equilibrium of motion complicates the problem of overcoming the pronounced vibration amplitudes without undue costs and inefficiency. The resonance consists of a number of coupled dependent variables at any particular speed of the vibrating mass and often the analysis becomes cumbersome and complex to yield solutions for suppression. It would avoid much waste in terms of energy and costs if the limiting relationships for the extinction of second order subharmonics were not functions of these parameters. The main objects are then to achieve the limiting expressions containing independent variables where the values are either constant or can be controlled or determined readily.

As the position of static equilibrium does not coincide with the point of symmetry in the restoring force characteristic, the resulting effect will have considerable influence on the subharmonic response owing to the additional terms in the equation of motion. The degree of non-linearity, adequately expressed by means of the static deflection parameter, is observed to have increased. A mechanical test-rig designed to demonstrate the effect of this independent parameter will help to explain with greater clarity the influence of gravitational force on the equilibrium of motion.

The presence of dissipative forces in vibrating systems is generally inevitable for reasons obvious. With non-linear vibrations this introduces the problem of stability as the distinctive characterisation of the deviation from linearity is the possibility that various types of periodic solutions can exist for a describing equation. The amplitudes of the components other than the fundamental and the pronounced subharmonics, however, generally are small and often cannot be measured accurately in a harmonic analysis. Nevertheless, if a small variation in the displacement caused by a slight perturbation from a corresponding periodic state ~~attenuates~~ grows with a lapse of time, the polynomial algebraic expressions of the differential equation will evidently not be satisfied. The stability analysis to be carried out will verify the actual existence of the approximate solution assumed.

There are several approximate methods of analysis already developed and in usage for solving non-linear differential equations of motion. The methods considered appropriate for this investigation yield solutions from which overall results can be examined and where associated accuracy is unaffected by the extent of the non-linearity and by the magnitude of the independent parameters. The accuracy of the methods employed is estimated on the basis of comparison with experimental results obtained from an analogue computer.

It is observed in practice through traces of angular displacement by the system that resonance can occur in either one of two phases, differing by π radians for every two cycles of the same disturbing force. A clearer understanding of the vibration characteristic is achieved if the transient motion of the approximate solution is depicted by an isocline graphical method. The phase in which the periodic vibration occurs can be shown by this analysis. Also as the motion is non-linear there are various possible forms of periodic coefficients in the overall vibration shape and the relationship between initial conditions and the resulting behaviour is more readily seen if the transient state of the phenomena is examined.

Chapter II

Survey and scope of investigation

Linear concepts of practical vibratory systems will merely provide superficial understanding of their response. A more comprehensive insight into the motion is achieved if the restoring force characteristic is not idealised as directly proportional to displacement. With most realizable mechanical systems the function often has non-linear coefficients and dissipative forces are present. The resulting describing equations usually cannot be solved exactly. The vibratory motion is not pure but contains higher and other harmonics and when subjected to external disturbing force subharmonic vibrations are readily generated over certain frequencies.

Subharmonic resonances also occur in certain physical systems described by linear differential equations. This is a separate field of study where parameters of the equations vary periodically with time. A system governed by Mathieu's equation is an example. The equation of motion is linear but the periodic parameters are appropriately chosen. The term 'parametric excitation' is applied to such motion (18). The equations however may need to contain a non-linear term if the physical systems are to be protected from cumulative effects where the input energy owing to the periodic variation of the parameters is greater than the dissipated energy.

2.1. The phenomena associated with non-linearity

Non-linear restoring forces are commonly encountered in practice and in such systems the pronounced deviation from linear response is well known (8). The natural frequency is often not pure but contains other harmonics. As it varies with the amplitudes of vibration, the restoring function is commonly referred as of 'hardening' type when the natural frequency increases with the vibration amplitude. The magnitude of the restoring stiffness becomes increasingly larger with displacement. Conversely the function is said to have 'softening' characteristic when the frequency is decreased with increasing amplitude. The curves of the natural frequency are known as the backbone curves about which harmonic resonance curves for various magnitudes of excitation are generated.

The order of subharmonic resonances is usually defined by the reciprocal of its frequency and the corresponding backbone curves are obtained through multiplying the natural frequency curves by the respective order numbers. The vibrations occur at frequencies beyond the harmonic resonance and exist within certain frequencies at which the fundamental is non-resonant. It is generated through exciting one of the lower components of the natural frequency by disturbing force (13).

As the strength of the harmonics diminish progressively from the fundamental, the intensity of vibration sequence will depend upon its proximity to the forcing function (8). With symmetrical restoring force characteristics the motion contains only odd components in the displacement and as the non-linear phenomena is a direct consequence of them the predominant subharmonic resonance would be of third order. Ludeke (5) obtained results to demonstrate this trend.

2.2. Various non-linearity inducements

The causes of these non-linearity effects are numerous. They can be encountered through imperfect elasticity or overstraining of the materials. The modulus of elasticity for materials such as cast iron or concrete decreases when deformation gets larger and in consequence the restoring force characteristics are of a softening nature. Another similar case exhibiting a softening type non-linear function is found in the vibration of a pendulum in which large oscillations are executed. The restoring function is well known to be satisfactorily represented by the difference of a linear and cubic term. The non-linear stiffness yielding to large load characteristic is often employed for protection against severe shocks. It would limit relative peak displacements and the absolute acceleration, when the datums are suddenly displaced, to specified values. Young (26) formulated expressions to assist design of such non-linearity effects for protection against strong shocks.

The hardening response is commonly found in the isolation of delicate equipment. The low stiffness of the system for small displacement increasing rapidly when the displacement becomes larger offers effective protection from shock loads and forced vibrations where the limit of troublesome vibration amplitude does not exceed the range of low stiffness. Molyneux (34) utilised from a combination of systems of toggle, vertical tension and compression springs the resulting non-linear restoring forces for vibration isolation of a mass in all six degrees of freedom. Detailed analytical expressions for the stiffness of the spring configuration are determined to obtain the range of zero stiffness.

Other causes of the non-linear function are by the configuration of linear components such as sine springs, cantilevers in clamp shaped, and barrel shaped springs. With the combination of linear springs for systems where there are more than two discontinuities in the spring constant (25), non-linear hardening restoring force characteristics are produced when they successively come into contact with the vibrating mass. Heavy machinery mountings often have such non-linearity which are also encountered in journal bearings having larger clearances. Billet (28) showed for hardening characteristic of a symmetrical function the effects of a bearing on shaft whirl.

The presence of rubber bushes in flexible couplings has many advantages amongst which a greater load fluctuation is possible. The system, however, has a hardening non-linear stiffness. Practical results for pin-type couplings (42) obtained verify the existence of non-linear response. The deviation from a linear torque-deflection function is due to the appreciable increase of stiffness when the rubber mass is under compression at high load whilst the value is practically constant for moderate loading.

Non-linear vibrations are known also to exist in four bar linkage mechanism having an elastic compressible-and-extensible coupler. Crossley (16) ascertained the behaviour depended upon displacement. In large amplitudes of vibration the restoring function has symmetrical hardening characteristics and in smaller displacements it consists of the difference between a linear and cubic term. The non-linear motion inevitably results, as was demonstrated on an electronic system, in pronounced subharmonic vibrations being generated.

A representative bibliography in the field of non-linear vibrations are given in papers (8), (21), (40), and (41). The phenomena associated with non-linear response is demonstrated in a number of experiments by Ludeke. The general survey by Klotter offers a good guide to articles on non-linear oscillations and the methods available for finding approximate solutions. Clauser gives a contrast account of the various possible forms of non-linear systems. Rosenburg summarises the advances made in the study of non-linear motion and the results that have been achieved.

2.3. Non-linear function of displacement

It is common to find the non-linear displacement function expressed as a series expansion. For systems where there are two discontinuities in the spring constant Jacobsen and Jespersen (24) found experimental results were in good agreement with the approximation. Where the point of zero deflection often is the point of symmetry on the characteristic curve, the stiffness represented as a series would consist only of odd terms and with most applications adequately described by the first two terms since those higher than the cubic power are negligibly small. The resulting differential equation is usually referred to as Duffing's equation.

The differential equation of motion containing the sum of a linear and cubic term in the approximated restoring function has attracted much attention. With only odd harmonics existing in the displacements and because of a cube non-linearity the strongest possible subharmonic resonance is shown (14) to be the third order. Non-linear oscillations of Duffing's equation have been studied extensively by Burgess (14), Levenson (9), Ludeke (5), Ludeke and Pong (4), Caughey (11) and Hayashi (22).

In many mechanical applications, the effect of gravitational force influences the equilibrium of motion for it is not completely balanced by the static deflection force. Consequently, the characteristic curve of the restoring function is often asymmetrical. The static equilibrium position does not coincide with the point of symmetry and the magnitude of non-linearity is increased. With the static deflection force not balanced by the effect of gravity and as it is not possible to eliminate the static deflection parameter, the resulting differential equation of motion contains additional terms in the approximated restoring force. This consists of the sum of the first three terms in the series expansion of a non-odd type. Although the analysis will become cumbersome it is in more complete agreement with practical results for the subharmonic vibrations induced. The pronounced vibration amplitude is experienced before the third order subharmonic resonance. It is of the ~~lower~~ second order ~~one-half~~.

2.4. Investigations undertaken

The existing knowledge of second order resonances so far investigated mainly is confined to electron-tube circuits (18) and to oscillatory circuits consisting of the combined unidirectional and alternating voltages (22). The approach and procedures developed are difficult at times to apply in mechanical engineering. In the analysis by Mandelstam and Papalexi the non-linear function was restricted and was considered in systems where the possibility of self-excitation existed. The investigation of stability was restricted likewise by the perturbation method employed.

In the study of oscillatory circuits, (7,22), subharmonic resonances of the equations were not bound by the degree of non-linearity. The even order phenomena is found in systems having an odd type symmetrical function. The effective quadratic term is produced from the equivalence of a non-sinusoidal external force introduced into the system. The fundamental component was approximated following Mandelstam and Papalexi over the complete region of subharmonic response. However, the main purpose of the present investigation which is ~~is not consider~~ed in establishing the limiting relationships consisting of independent parameters influenced by gravitational force for the extinction of the subharmonic vibrations, has not been considered by the previous investigators.

The shift in the dynamic equilibrium position away from static has received relatively no attention other than by Reif, (1,2). The static deflection from dead weight of the system transmitted to supports introduces a quadratic in the restoring force and it is shown that this coefficient is the effective term of the non-linearity. The frequency position of the harmonic resonance is increased from the location calculated when the effect of gravity is neglected. The amplitudes of vibration beyond the instability region are also shown to be much larger with an additional region of unstable amplitudes below the natural frequency when disturbing forces are small. In essence the severity of second order subharmonic vibrations generated would also be heavily influenced by the displacement (3).

The limiting relationships between the extent of asymmetry represented by static deflection in the restoring force and the other independent variables of the vibratory system in the extinction of second order subharmonic vibration constitutes the main basis of this investigation. The vibrating system considered is excited by a 'centrifugal' type sinusoidal varying force, as it is common in practice to encounter external excitation through out-of-balance mass. The damping of the system is viscous and the coefficient is positive and linear. Dissipative forces generally for intended purposes are satisfactorily approximated when expressed as equivalent linear damping. The criterion of equivalence is that the amount of energy per cycle dissipated is the same.

Non-linear damping is usually associated with self-excited systems (18,20). If the damping component is negative it is readily seen that energy is introduced into the system and the vibration displacement will grow boundlessly with time. In practice for the vibrations of this nature, not destructive, it would occur in limit cycles. The solutions of the motion depend on the parameter of the first-order differential and the vibrations have a frequency very close to the natural frequency of the system. At times due to weather condition self-excited vibrations also occur in electrical transmission line wires.

The presence of viscous damping inevitably leads to a problem of stability for there is more than a single equilibrium condition since various types of periodic solution may exist in a non-linear system. With linear vibration this difficulty would not arise. The forced vibration is unaffected by the initial conditions at the start of the motion. The second order phenomena exists only so long as all the periodic points are in equilibrium and the different possibilities of periodic solution to the describing equation makes it necessary to confirm the actual existence of an approximate solution in the instability region. The problem is determined through considering the variational equation that characterises small displacement from the equilibrium state and which is subsequently reduced to a Hill's type equation where the coefficients are periodic functions and are expressed into a Fourier series. This will allow the stability conditions for the appropriate number of component regions chosen to be examined and if the solution for the investigation is to be valid they must be satisfied simultaneously. The stability criterion, stating whether the slight variation attenuates boundlessly with time, is derived by comparing the characteristic exponent of the solution with the damping coefficient (20,22).

In many applications where it is important to avoid working the mechanical systems in the pronounced vibration frequency range, an expression is derived to estimate the boundary values over which the resonance might be generated. The equation enables the frequencies to be determined with reasonable accuracy without being dependent on the values of coupled-variables of the phenomena. It is expressed as functions of the independent parameters whose influences on the response are also demonstrated on an experimental test-rig.

2.5. Theoretical methods

There are several approximate methods in existence for solving non-linear differential equations. Among the more satisfactory methods for analysis of subharmonic resonance in terms of convergence and obtaining quantitative results are the Ritz-Galerkin averaging method, the principle of harmonic balance, the Kryloff-Bogoliubov first approximation method and the perturbation method.

Graphical methods are well known to produce singular point results and are more often applied in the study of autonomous systems. The isocline graphical method, however, in conjunction with digital computers is very useful and the procedure is given in (17,22). In application to non-autonomous systems the transient state of the resonance under investigation can be examined through the integral curves drawn for the equations obtained from an approximate solution in which the coefficients are functions of a time variable. When the equations ultimately become zero with lapse of time singular points that satisfy the approximate periodic solution are produced. The phase in which the subharmonic vibration would occur for every two cycles of disturbing force depends on the conditions from which it is generated and is shown by the positions of singularities. The method is also used to determine the phase quadrant in which the subharmonics can exist.

The perturbation method is regarded as unsatisfactory for this investigation. It is only applicable to small non-linearity for convergence of the solution (18), through solving the sequence of second order linear equations produced from equating the coefficients of like powers of the non-linear parameter expressed in a series. Although the problem of secular terms, where the displacement terms contain time variable outside the trigonometric function, does not arise in the modified form of the method, it is difficult and cumbersome to apply to non-symmetrical restoring force systems. For small non-linearity Levenson (9) used this procedure to examine the types of harmonic and subharmonic components possible in Duffing's equation.

The Kryloff-Bogoliubov method (19) is also restricted to quasi-linear differential equation. The approximate solution as in the previous method is developed in the neighbourhood of linear vibration and cannot be applied to systems not containing a linear term in the restoring stiffness function. For small amplitudes of non-linear vibration the method in its first approximation is satisfactory and easy to apply, but likewise it encounters difficulty when applied to non-odd type functions.

The principle of the harmonic balance method is applicable to both linear and non-linear differential equations of motion and the method is not confined to quasi-linear functions for convergence of the solution. As a first approximation the procedure (22) is mathematically simple whilst for closer approximations large trigonometric functions are encountered, requiring much tedious calculations. It is essentially, in the analysis of non-linear systems, the solving of the unknown coefficients of the approximate solution that is expressed into a Fourier series. The polynomial algebraic expressions containing the unknown terms are obtained through equating the respective harmonic components to zero separately when having substituted into the describing equation.

Ludeke (6) and Hayashi (7) in their respective investigations employed the harmonic balance method to problems in which the non-linear function has no restriction. The difficulty from the absence of linear component or from the extent of non-linearity is not encountered and they have a negligible effect on convergence of the solution. The accuracy of the method is very satisfactory. Analysis of the motion by this method will also produce coupled-algebraic expressions and hence it allows an overall assessment on the influences of the various parameters of the system.

The Ritz-Galerkin averaging method is also not limited in its application to non-linear differential equations and the method appears just as suitable for this investigation since in application only the governing differential equation needs to be known with an assumed approximate solution that satisfies the physical boundary conditions. The degree of asymmetry in the restoring force characteristics does not reduce the effectiveness of the method. The resulting error function that varies from instant to instant when integrated with each function of the solution over an arbitary period is zero respectively, giving a set of simultaneous polynomial algebraic expressions. The general procedure is explained in (6,23), and it is based on the variational method. It is of interest to note that under certain conditions of the particular problem, Ritz's minimising and the averaging methods are identical. With the former the variational function containing energy expressions has to be derived and then subsequently minimised by differentiating with respect to each coefficient of the series expansion present in the quadratic functions. However when the generalised procedure is applied to non-linear problems the two methods become identical for Lagranges equations are considered. The describing equations for the non-linear behaviour of pendulums as vibration absorbers (35) are so obtained. The proof of a generalised problem is given by Newland (29).

Klotter (6) developed the Ritz-Galerkin averaging method in application to non-linear dynamical problems. For various systems considered the algebraic equations deduced were in a concise form, allowing the influence of various parameters involved to be easily examined. Where possible, as in cases for undamped vibrations in which the restoring force depended upon a power of the displacement, the results were compared with exact solutions and shown to be very satisfactory.

Burgess (14) applied the Ritz-Galerkin method to an autonomous system having symmetrical restoring forces. The accuracy of the method when compared with existing exact solutions improves as more odd harmonic components were considered. This method is shown to be preferable to the perturbation method in application to his investigation of third order subharmonic vibrations.

Rief (1,2) also found the averaging method to produce very satisfactory results without loss in accuracy when applied to an asymmetrical restoring stiffness vibrating body. The choice of the number of periodic terms in the approximate solution depended upon the desired accuracy and it would contain the even and odd harmonics existing in the non-linear motion. In satisfying all the physical requirements a constant term would also be included with the periodicity. The superiority of the method was demonstrated (27), within the accuracy of the analogue computer, over the other approximate methods capable of producing the algebraic equations in an analytical form.

Clearly then, the Ritz-Galerkin method and the principle of harmonic balance would appear appropriate for this investigation into the extinction of second order subharmonic vibration. The influences of the independent variables in the closed form of the polynomial algebraic expressions produced can easily be examined in relation to overall results, and the inherently coupled coefficients will present no difficulty in computing. With the accuracy of the two methods hardly affected by the magnitude of the degree of non-linearity and of the disturbing force, there would appear no undue limitation imposed on the analysis.

The stability analysis to ascertain the actual existence of the approximate solution over the limited frequency band-width is by determining the behaviour of a small displacement variation from the equilibrium states. The comparison of the characteristic exponent of the solution to the variational equation with the damping coefficient of the system will illustrate whether the slight variation grows unbounded with the lapse of time. This procedure enables the accumulative effect from the build-up of an unstable oscillation in an accompanying harmonic of the vibratory motion to be readily determined.

2.6. Empirical verification

With the number of parameters involved in the range of values considered, the theoretical results are predominantly verified by means of an electronic analogue computer. The motion is simulated to the describing equation on the computer circuits and the resulting trace recorded will contain all the physical characteristics of actual motion.

A mechanical model is also designed to demonstrate the work undertaken. Ludeke (5,32,33) is the only outstanding contributor in this field of mechanical vibration. From rational design consideration the non-linear characteristic is achieved through the combination of linear stiffnesses, similar to that by Ludeke. This arrangement allows a wide flexibility in varying the describing conditions of the motion. Mathematical expressions are formulated in the design of the non-linear suspension, and since the percentage of non-linearity itself is heterogeneous, a criterion to the permissible limits of the gravitational effects on the subharmonic resonance is also derived to prevent the vibrational phenomena to be generated reaching a dangerous amplitude. A detailed procedure is developed to determine the maximum vibration amplitudes whilst confining the model to a manageable physical size for sufficient accuracy such that comparison with theoretical results is possible. This involves formulating a concept to relate the design requirements of the non-linearity to the allowable displacement.

Chapter III

Theoretical Analysis

3.1. Equation under investigation

The single degree of freedom system, shown in Fig.(3.1-1), is restricted to vibrate under the excitation of an out-of-balance centrifugal force, $M_o e\omega^2 \cos\omega t$, in the vertical direction only. The resolved force at right angle that would introduce horizontal motion is prevented.

The vibrating mass m is supported on a non-linear suspension of negligible mass, having an odd function characteristic. The non-linear restoring force exerted by the suspension on the mass when approximated to the first two terms in the series expansion is

$$P = (ky + \alpha y^3) \qquad \ldots\ldots \quad (3.1,1)$$

where k and α are constants of the suspension and y is the total deflection. Terms of powers higher than the cubic are negligibly small and are omitted. Rewriting the expression,

$$P = K(y + \mu y^3) \qquad \ldots\ldots \quad (3.1,2)$$

where $\mu = \frac{\alpha}{k}$ and has dimensions $(L)^{-2}$

The corresponding non-linear stiffness is then,

$$\frac{dP}{dy} = K(1 + 3\mu y^2) \qquad \ldots\ldots \quad (3.1,3)$$

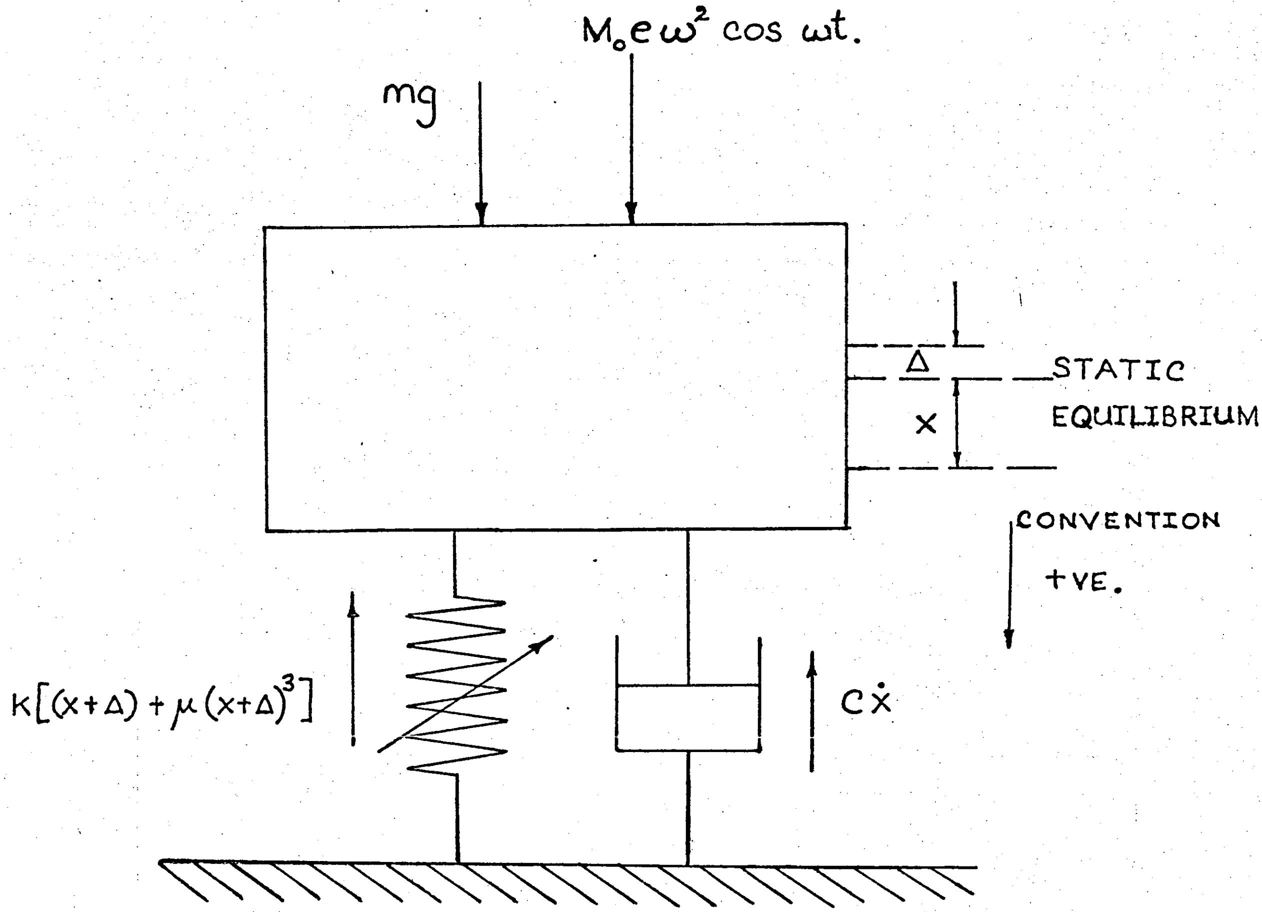


FIG.(3.1-1). THE VIBRATING SYSTEM.

The positive sign before the non-linear term implies a hardening suspension.

Under the effect of gravity the spring is compressed vertically, giving a static deflection Δ. During the vibration and for a particular instant when the total deflection of the spring is $(x + \Delta)$, where x is the dynamic displacement measured from static equilibrium position and the positive sign taken according to convention, the restoring force exerted by the suspension is

$$K\{(x + \Delta) + \mu(x + \Delta)^3\} \qquad \ldots\ldots\ldots \quad (3.1,4)$$

For this arbitary instant, the total forces acting on the mass are shown in Fig. (3.1-1). Since the restoring function of the system is non-linear, the weight of the vibrating mass would not be equal at every instant to the static force in the suspension, and hence it is included in the equation of motion.

The equation describing the actual motion of the system is then given by

$$m\ddot{x} = M_o e\omega^2 \cos\omega t + mg - C_1\dot{x} - K\{(x + \Delta) + \mu(x + \Delta)^3\}$$

$$\ldots\ldots\ldots \quad (3.1,5)$$

The equilibrium of forces for the static position gives

$$mg = K(\Delta + \mu\Delta^3) \qquad \ldots\ldots\ldots \quad (3.1,6)$$

Substituting equation (3.1,6) into (3.1,5) and dividing throughout by m would result in

$$\ddot{x} + 2Rp\dot{x} + p^2\{x + \mu(x^3 + 3\Delta x^2 + 3\Delta^2 x)\} - Z\omega^2 \cos\omega t = 0$$

$$\ldots\ldots\ldots \quad (3.1,7)$$

where $R = \frac{C_1}{C_c}$ and $Z = \frac{M_o e}{m}$, the amplitude of the disturbing force.

Introducing a new independent variable, $\theta = \omega t$, to avoid working with functions of unknown period, equation (3.1,7) becomes

$$\eta^2 x'' + 2\xi\eta x' + \{(1 + \mu 3A^2)\, x + \mu\, (x^3 + 3Ax^2)\} - Z\eta^2 \cos\theta = 0 \qquad \ldots\ldots\ (3.1,8)$$

where $\eta = \frac{\omega}{p}$ is the dimensionless frequency.

The approximate solution to equation (3.1,8) that depicts the actual motion of the body depends upon the physical state of the system and also on the accuracy required. The degree of the restoring force asymmetry, resulting from the effect of gravity, is expressed in the equation by means of the static deflection parameter.

3.2. Analysis by Ritz-Galerkin averaging method

The influence of gravitational force on the vibratory system has increased the magnitude of non-linearity as shown in equation (3.1,8). The difficulty in providing correction terms having a constant to improve on the initial solution and to account for the effect of gravity is one of the reasons for methods as the Kryloff-Bogoliubov and the perturbation method not being considered. Similar to the successive iteration procedure, the first approximation methods proceed to build the sufficiently accurate solutions from the fundamental harmonic component. This invariably means the deviation of a restoring function from linearity is assumed to be comparitively small. Hence the coefficients of non-linearity for convergence of the methods is restricted in magnitude.

In applying the Ritz-Galerkin method only the differential equation needs to be known with a properly chosen approximate solution. The solution is expressed in the form of a series and is consistent with the physical restraints as well as satisfying the boundary requirements. The development of this method is shown in Appendix I for reference, and in essence as a variational problem the energy expression of the vibrating system is minimised over the cycle, for a conservative system according to Hamiltons principle, with respect to each coeifficient in the series.

3.2.(i) The approximate solution

The choice of solution to the describing equation is determined by the boundary conditions in respect of the periodicity requirements and the physical restraints of the system. From equation (3.1,8) it is evident that the characteristic of the restoring force is not symmetrical about the static equilibrium position. The approximate solution would thus need to consist of odd and even harmonic terms. Since there is dissipative forces acting on the mass, the even and odd harmonics of displacement exist as both sine and cosine functions. During vibratory motion, a shift in the mean dynamic displacement from static equilibrium also occurs, for the point of symmetry does not coincide with the latter position and to allow for this a constant term N must be present in the approximation.

The accuracy of the approximate solution will depend upon the number of terms used in the series. With a proper choice satisfying the requirements, the approximation consisting a minimum of terms is shown (6) to yield sufficiently good results.

Thus the simplest form of the solution for this investigation is

$$\tilde{x} = N + A \cos \frac{\theta}{2} + B \sin \frac{\theta}{2} + C \cos \theta + D \sin \theta \qquad \text{..........} \quad (3.2,1)$$

As a result of the presence in positive damping the amplitudes of higher harmonics generally are much smaller than those of the fundamental, and cannot be measured with any accuracy in a harmonic analysis. Hence, they are omitted from equation (3.2,1). The slight improvement in accuracy of the approximation to equation (3.1,8) with their inclusion does not justify the considerable increase of labourious calculation. The third order subharmonic component is generally also smaller because of the magnitude of the quadratic term in relation to the cubic in the equation (3.1,7)

Since $\tilde{x}$ is not an exact solution, the equation (3.1,8) cannot be satisfied and the subsitution of equation (3.2,1) will not equal to zero but to some error function $E(\tilde{x})$.

i.e. $E(\tilde{x}) = \eta^2\tilde{x}'' + 2R\eta\tilde{x}' + \{(1 + 3\Delta^2\mu)\,\tilde{x} + (\tilde{x}^3 + 3\Delta\tilde{x}^2)\mu\} - Z\eta^2\cos\theta$
$\qquad\qquad$ .... (3.2,2)

With $E(\tilde{x})$ denoting the right-hand side of equation (3.2,2) to simplify the notation the following Ritz-Galerkin conditions then result from the application of the method to equation (3.2,2) and the solution (3.2,1), that is

$$\int_0^{4\pi} E(\tilde{x})\, d(\theta) = 0 \quad \text{...........} \quad (3.2,3)$$

$$\int_0^{4\pi} E(\tilde{x}) \cos\frac{\theta}{2}\, d(\theta) = 0 \quad \text{..........} \quad (3.2,4)$$

$$\int_0^{4\pi} E(\tilde{x}) \sin\frac{\theta}{2}\, d(\theta) = 0 \quad \text{..........} \quad (3.2,5)$$

$$\int_0^{4\pi} E(\tilde{x}) \cos\theta\, d(\theta) = 0 \quad \text{..........} \quad (3.2,6)$$

$$\int_0^{4\pi} E(\tilde{x}) \sin\theta\, d(\theta) = 0 \quad \text{..........} \quad (3.2,7)$$

The frequency of subharmonic vibrations being a submultiple of the excitation, the upper limit of integration in this case is 4π for the cycle of each period in θ.

3.2.(ii) The non-linear algebraic solutions

From equations (3.2,3) to 3.2,7) on integration yields, after simplification, the following non-linear the algebraic equations. The bar above a symbol denotes the corresponding non-dimensional quantity resulting from the multiplication of its dimensional form by $\mu^{\frac{1}{2}}$.

$$2\bar{N}\{1 + \bar{N}^2 + 3(\bar{A} + \bar{N})\} + 3(\bar{A} + \bar{N})(\bar{A}^2 + \bar{B}^2 + \bar{C}^2 + \bar{D}^2) + \frac{3}{2}\bar{C}$$

$$(\bar{A}^2 - \bar{B}^2) + 3\bar{A}\bar{B}\bar{D} = 0 \quad \ldots\ldots\ldots\ldots \quad (3.2,8)$$

$$2\bar{A}(1 - \frac{\eta^2}{4}) + 2R\eta\bar{B} + \frac{3}{2}\bar{A}\{(\bar{A}^2 + \bar{B}^2) + 2(\bar{C}^2 + \bar{D}^2)\} + 6\bar{A}$$

$$(\bar{A} + \bar{N})^2 + 6(\bar{A} + \bar{N})(\bar{A}\bar{C} + \bar{B}\bar{D}) = 0 \ldots \quad (3.2,9)$$

$$2\bar{B}(1 - \frac{\eta^2}{4}) - 2R\eta\bar{A} + \frac{3}{2}\bar{B}\{(\bar{A}^2 + \bar{B}^2) + 2(\bar{C}^2 + \bar{D}^2)\} + 6\bar{B}(\bar{A} + \bar{N})^2$$

$$+ 6(\bar{A} + \bar{N})(\bar{A}\bar{D} - \bar{B}\bar{C}) = 0 \quad \ldots\ldots\ldots\ldots \quad (3.2,10)$$

$$2\bar{C} - 2\eta^2(\bar{C} + \bar{Z}) + 4R\eta\bar{D} + 6\bar{C}(\bar{A} + \bar{N})^2 + 3(\bar{A} + \bar{N})(\bar{A}^2 - \bar{B}^2)$$

$$+ \frac{3}{2}\bar{C}\{2(\bar{A}^2 + \bar{B}^2) + (\bar{C}^2 + \bar{D}^2)\} = 0 \ldots \quad (3.2,11)$$

$$2\bar{D}(1 - \eta^2) - 4R\eta\bar{C} + 6\bar{D}(\bar{A} + \bar{N})^2 + 6(\bar{A} + \bar{N})\bar{A}\bar{B} + \frac{3}{2}\bar{D}\{2(\bar{A}^2 + \bar{B}^2)$$

$$+ (\bar{C}^2 + \bar{D}^2)\} = 0 \quad \ldots\ldots\ldots\ldots \quad (3.2,12)$$

On further simplification with the variables ϕ_1 and ϕ_2 as defined below

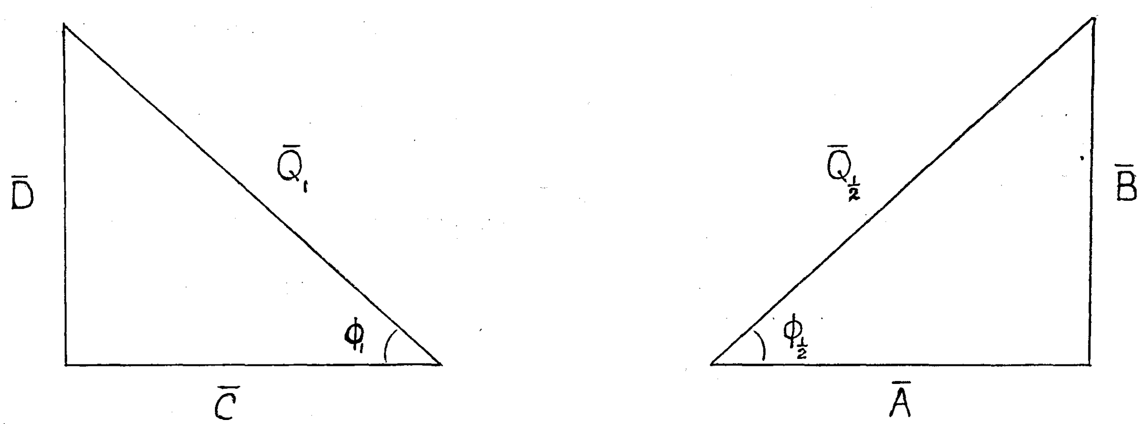


FIG. (3.2-2). THE PHASES OF THE SUBHARMONIC COMPONENTS.

gives a set of algebraic expressions in the form that has more obvious connotation. Besides the equations (3.2,8) to (3.2,12) which were too unwieldly to manipulate become more tractable.

$$2N(1 + N^2 + 3\Delta Y) + 3Y(Q_{\frac{1}{3}}^2 + Q_1^2) + \frac{3}{2}Q_1 Q_{\frac{1}{3}}^2 \cos(\phi_1 - 2\phi_{\frac{1}{3}}) = 0 \qquad \text{.............} \quad (3.2,13)$$

$$3Y\, Q_1 Q_{\frac{1}{3}} \sin(\phi_1 - 2\phi_{\frac{1}{3}}) - Q_{\frac{1}{3}} N\eta = 0 \qquad \text{..............} \quad (3.2,14)$$

$$3Y\, Q_1 Q_{\frac{1}{3}} \cos(\phi_1 - 2\phi_{\frac{1}{3}}) + (1 - \frac{\eta^2}{4})\, Q_{\frac{1}{3}} + \frac{3}{4}(Q_{\frac{1}{3}}^2 + 2Q_1^2)\, Q_{\frac{1}{3}} + 3Y^2\, Q_{\frac{1}{3}} = 0 \qquad \text{..............} \quad (3.2,15)$$

$$N(4Q_1^2 + Q_{\frac{1}{3}}^2) - 2\eta Z Q_1 \sin\phi_1 = 0 \text{ ....} \quad (3.2,16)$$

$$2Q_1^2\{(1 - \eta^2) + 3Y^2 + \frac{3}{4}(2Q_{\frac{1}{3}}^2 + Q_1^2)\} - Q_{\frac{1}{3}}^2\{(1 - \frac{\eta}{4}) + 3Y^2 + \frac{3}{4}(Q_{\frac{1}{3}}^2 + 2Q_1^2)\} - 2\eta^2 Z Q_1 \cos\phi_1 = 0 \text{ ......} \quad (3.2,17)$$

where $Y = (\Delta + N)$.

The bar above the respective coefficients are omitted for convenience. The analytical process for attaining equations (3.2,13) to (3.2,17) are shown in Appendix II for reference.

3.2.(iii) The phase relationships

As the characteristics of the predominant subharmonic component of the vibratory motion represented by equation (3.1,8) are defined by equations (3.2,14) and (3.2,15), then for the existence of the subharmonic vibration the simultaneous algebraic expressions must be satisfied for which $Q_{\frac{1}{3}}$ is not equal to zero. The solution of $Q_{\frac{1}{3}} = 0$ is ~~meaningless~~ trivial, and the equations (3.2,14) and (3.2,15) can be reduced to

$$\sin(\phi_1 - 2\phi_{\frac{1}{2}}) = \frac{R\eta}{3YQ_1} \qquad \ldots\ldots\ldots\ldots \quad (3.2,18)$$

and

$$\cos(\phi_1 - 2\phi_{\frac{1}{2}}) = \frac{-\{(1 - \frac{\eta^2}{4}) + \frac{3}{4}(Q^2_{\frac{1}{2}} + 2Q^2_1) + 3Y^2\}}{3YQ_1}$$

$$\ldots\ldots\ldots\ldots \quad (3.2,19)$$

The above two equations define the phase relationship between the subharmonic component and the harmonic, and it can be expressed by Fig. (3.2-1) (a). They can be reduced to a single algebraic expression that define the region of second order subharmonic vibration.

The phase of the harmonic component during the subharmonic resonance is determined from equations (3.2,16) and (3.2,17) for if $Q_{\frac{1}{2}} = 0$ the equations are reduced to the following expressions

$$\sin \phi_1 = \frac{2RQ_1}{\eta Z} \qquad \ldots\ldots\ldots\ldots \quad (3.2,20)$$

and

$$\cos \phi_1 = \frac{Q_1(1 - \eta^2 + 3Y^2 + \frac{3}{4}Q_1^2)}{\eta^2 Z} \qquad \ldots\ldots\ldots\ldots \quad (3.2,21)$$

from which a single equation can be obtained that is identical to the equation of the harmonic resonance whose solution consists of the first three terms. Equations (3.2,16) and (3.2,17) can be represented by Fig. (3.2-1) (b) where,

$$G = \{1 - \eta^2 + 3Y^2 + \frac{3}{4}(2Q^2_{\frac{1}{2}} + Q_1^2)\} \qquad \ldots\ldots\ldots\ldots \quad (3.2,22)$$

and

$$H = \{1 - \frac{\eta^2}{4} + 3Y^2 + \frac{3}{4}(Q^2_{\frac{1}{2}} + 2Q_1^2)\} \qquad \ldots\ldots\ldots\ldots \quad (3.2,23)$$

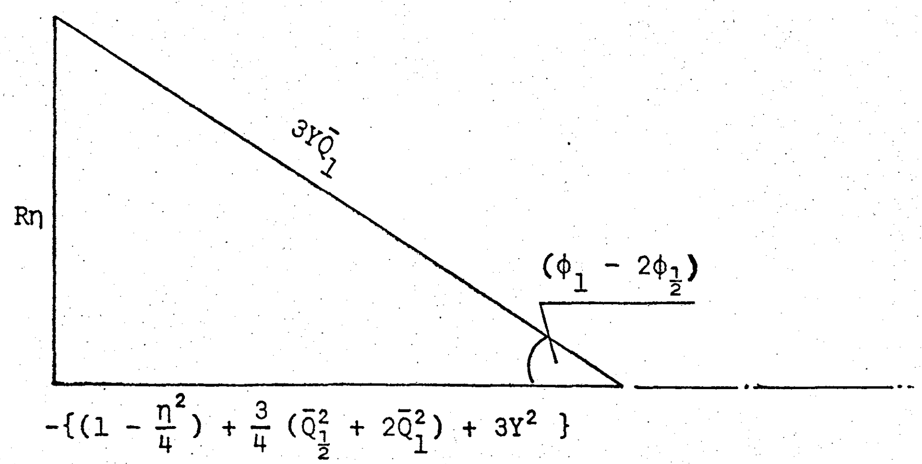


FIG.(3.2-1)(a). THE RELATIVE PHASE RELATIONSHIP.

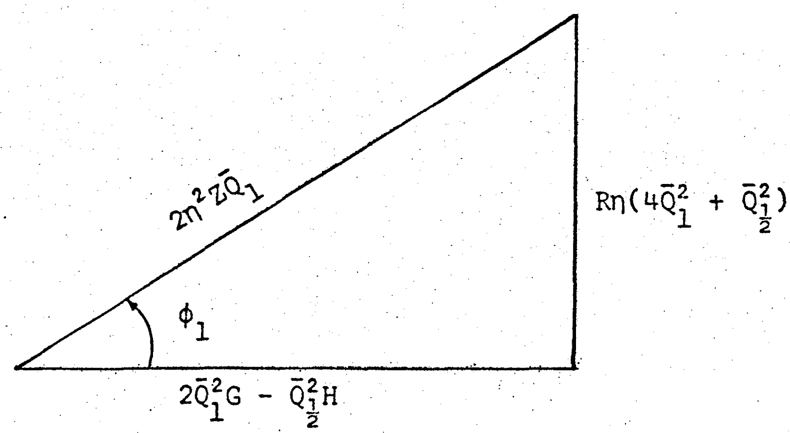


FIG.(3.2-1)(b). DIAGRAM FOR EQUATIONS (3.3,10) AND (3.3,11).

3.3. Analysis by the method of the principle of harmonic balance

This method as in the previous does not require the magnitude of the coefficient of non-linearity to be small for convergence of solution. The periodicity in vibratory motion enables the displacement of equation (3.1,8) to be approximated to a form of solution expressed in a Fourier series. The principle of this procedure is the requirements that equilibrium of forces is satisfied, and as with the Ritz-Galerkin method the solution is consistent with the physical restraints and satisfies the boundary requirements of the system. Unknown coefficients of the series are determined when the approximation is substituted into the describing equation from which sine and cosine terms of respective frequencies are set to zero; giving the simultaneous polynomial algebraic equations from which the unknowns are fixed.

For improving on an initial approximation the procedure is described in reference (22). Where there are already a number of terms in the series on first approximation, the resulting small increase of accuracy will not justify the considerable amount of calculations involved.

3.3.(i). The approximate solution

As a first approximation the solution to equation (3.1,8) is

$$\bar{x} = N + A \cos \frac{\theta}{2} + B \sin \frac{\theta}{2} + C \cos \theta + D \sin \theta \quad \ldots\ldots \quad (3.3,1)$$

The approximate solution is consistent with the physical requirements of the system. The constant N, its magnitude varying with frequency, accounts for the shift of dynamic equilibrium point from the point of symmetry and as in the previous method the higher harmonic terms are neglected. Because of damping in equation (3.1,8) their amplitudes are considerably smaller and would not improve to any effect the accuracy other than greatly increase the laboriousness of solution and the difficulty in computational procedure.

Substituting equation (3.3,1) into (3.1,8) gives

$$-\eta^2\{\frac{A}{4}\cos\frac{\theta}{2} + \frac{B}{4}\sin\frac{\theta}{2} + C\cos\theta + D\sin\theta\} + 2R\eta\{(\frac{B}{2}\cos\frac{\theta}{2} - \frac{A}{2}\sin\frac{\theta}{2}) + (D\cos\theta - C\sin\theta)\} + (1 + 3A^2\mu)\{N + A\cos\frac{\theta}{2} + B\sin\frac{\theta}{2} + C\cos\theta + D\sin\theta\} + 3A\mu\{N + A\cos\frac{\theta}{2} + B\sin\frac{\theta}{2} + C\cos\theta + D\sin\theta\}^2 + \mu\{N + A\cos\frac{\theta}{2} + B\sin\frac{\theta}{2} + C\cos\theta + D\sin\theta\}^3 - Z\eta^2\cos\theta = 0 \quad \ldots\ldots\ldots \quad (3.3,2)$$

<u>3.3.(ii). The non-linear simultaneous algebraic equations</u>

The following non-linear simultaneous algebraic expressions are obtained from equating the respective frequencies of sine and cosine terms separately to zero. The equations are produced in a closed form as in the previous analysis. This will enable an overall interpretation of the influence of the independent variables on the resonance characteristics. The bars above each coefficient are omitted after having converted the coefficient into corresponding non-dimensional quantities by multiplying the respective forms by $\mu^{\frac{1}{2}}$.

$$2N(1 + N^2 + 3A^2 + 3AN) + 3(A + N)(A^2 + B^2 + C^2 + D^2) + 3ABD + \frac{3}{2}C(A^2 - B^2) = 0 \quad \ldots\ldots\ldots \quad (3.3,3)$$

$$A(1 - \frac{\eta^2}{4}) + R\eta B + 3A(A + N)^2 + 3(A + N)(AC + BD) + \frac{3}{2}A\{\frac{1}{2}(A^2 + B^2) + (C^2 + D^2)\} = 0 \quad \ldots\ldots \quad (3.3,4)$$

$$B(1 - \frac{\eta^2}{4}) - R\eta A + 3B(A + N)^2 + 3(A + N)(AD - BC) + \frac{3}{2}$$

$$B\{ \tfrac{1}{2}(A^2 + B^2) + (C^2 + D^2)\} = 0 \quad \ldots\ldots\ldots \quad (3.3,5)$$

$$C - \eta^2(C + Z) + 2R\eta D + 3C(A + N)^2 + \frac{3}{2}(A + N)(A^2 - B^2)$$

$$+ \frac{3}{2} C \{(A^2 + B^2) + \tfrac{1}{2}(C^2 + D^2)\} = 0 \quad \ldots\ldots \quad (3.3,6)$$

$$D(1 - \eta^2) - 2R\eta C + 3D(A + N)^2 + 3AB(A + N) + \frac{3}{2} D\{(A^2 + B^2)$$

$$+ \tfrac{1}{2}(C^2 + D^2)\} = 0 \quad \ldots\ldots\ldots \quad (3.3,7)$$

The expansion of squared and cubed of the approximation in equation (3.3,2) is shown in Appendix III together with detail simplification of resulting trigonometric identities to attain equations (3.3,3) to 3.3,7).

On further simplification of the equations, with reference to the previous definitions for variables $\phi_{\frac{1}{2}}$ and ϕ_1, to give more obvious physical interpretation produces

$$2N(1 + N^2 + 3AY) + 3Y(Q_{\frac{1}{2}}^2 + Q_1^2) + \frac{3}{2} Q_{\frac{1}{2}}^2 Q_1^2 \cos(2\phi_{\frac{1}{2}} - \phi_1) = 0$$

$$\ldots\ldots\ldots\ldots \quad (3.3,8)$$

$$R\eta Q_{\frac{1}{2}} + 3YQ_1 \sin(2\phi_{\frac{1}{2}} - \phi_1) Q_{\frac{1}{2}} = 0 \quad \ldots\ldots\ldots \quad (3.3,9)$$

$$3YQ_1 \cos(2\phi_{\frac{1}{2}} - \phi_1) Q_{\frac{1}{2}} + (1 - \frac{\eta^2}{4}) Q_{\frac{1}{2}} + \frac{3}{4}(Q_{\frac{1}{2}}^2 + 2Q_1^2) Q_{\frac{1}{2}}$$

$$+ 3Y^2 Q_{\frac{1}{2}} = 0 \quad \ldots\ldots\ldots\ldots \quad (3.3,10)$$

$$R(4Q_1^2 + Q_{\frac{1}{2}}^2) - 2\eta Z Q_1 \sin\phi_1 = 0 \quad \ldots\ldots\ldots\ldots \quad (3.3,11)$$

$$2Q_1^2\{1 - \eta^2 + 3Y^2 + \frac{3}{4}(2Q_{\frac{1}{2}}^2 + Q_1^2)\} - Q_{\frac{1}{2}}^2\{1 - \frac{\eta^2}{4} + 3Y^2$$

$$+ \frac{3}{4}(Q_{\frac{1}{2}}^2 + 2Q_1^2)\} - 2\eta^2 Z Q_1 \cos\phi_1 = 0 \quad \ldots\ldots\ldots \quad (3.3,12)$$

The above equations are identical to the non-linear algebraic expressions (3.2,13) to (3.2,17) obtained from the Ritz-Galerkin averaging method. Thus these two methods are equivalent when their respective approximations contain the same terms of harmonics.

3.4. Isocline phase-plane analysis

As the region of second order subharmonic vibration is defined either by equations (3.2,14) and (3.2,15) or by equations (3.3,9) and (3.3,10), subharmonics having the same magnitude can exist in two phases, differing by π radians. Analysis of the approximate solution for the vibratory motion of the system in the transient state by isocline graphical method will illustrate that the vibrations can occur in either phase, depending upon initial conditions from which the phenomena is generated.

3.4.(i). The approximate transient state solution

To show the resulting second order subharmonic vibration and also the relative phase relationship between two different stable amplitudes of a frequency depend upon the initial state of the system preceding the excitation of the subharmonic, an approximate transient solution of equation (3.1,8) is assumed in the form of equation (3.2,1), that is

$$\bar{R} = N + A(\theta) \cos \frac{\theta}{2} + B(\theta) \sin \frac{\theta}{2} + C(\theta) \cos \theta + D(\theta) \sin \theta \qquad \ldots\ldots\ldots \quad (3.4,1)$$

in which coefficients N,A,B,C, and D are slowly varying functions of θ that ultimately become constant values when steady state is reached. Under this assumption and that the coefficient of damping is relatively small, second order differential terms together with those of first differential multiplied by $\bar{R}$ are neglected on substituting equation (3.4,1) into equation (3.1,8).

Hence the following equations are obtained,

$$\eta^2\{-\frac{dA}{d\theta}\sin\frac{\theta}{2} - \frac{A}{4}\cos\frac{\theta}{2} + \frac{dB}{d\theta}\cos\frac{\theta}{2} - \frac{B}{4}\sin\theta - 2\frac{dC}{d\theta}\sin\theta$$

$$- C\cos\theta + 2\frac{dD}{d\theta}\cos\theta - D\sin\theta\} + 2R\eta\{-\frac{A}{2}\sin\frac{\theta}{2}$$

$$+ \frac{B}{2}\cos\frac{\theta}{2} - C\sin\theta + D\cos\theta\} + \{(1 + 3A^2\mu)\tilde{X}$$

$$+ (\tilde{X}^3 + 3A\tilde{X}^2)\mu\} - Z\eta^2\cos\theta = 0 \quad \ldots\ldots \quad (3.4,2)$$

If the above expression is to yield the periodic solution, it must ultimately with the lapse of time be satisfied.

3.4.(ii). The polynomial expressions of the integral curve

Equating the coefficients containing the non-oscillatory term N, and those of $\cos\frac{\theta}{2}$ and $\sin\frac{\theta}{2}$ in the above equation (3.4,2) respectively to zero gives

$$N(1 + 3A^2\mu) + 3A\mu(N^2 + Q^2_{\frac{1}{2}} + Q^2_{\frac{1}{2}}) + \mu(N^3 + \frac{3}{2}NA^2 + \frac{3}{2}NB^2$$

$$+ \frac{3}{4}A^2C - \frac{3}{4}B^2C + \frac{3}{2}NC^2 + \frac{3}{2}ND^2 + \frac{3}{2}ABD) = 0 \quad .. \quad (3.4,3)$$

$$\eta^2(-\frac{A}{4} + \frac{dB}{d\theta}) + R\eta B + (A + \mu(\frac{3}{4}A^3 + 3N^2A + \frac{3}{4}B^2A + \frac{3}{2}AQ_1^2$$

$$+ 3NAC + 3NBD) + 3A\mu(2NA + AC + BD) + 3A^2A\mu) = 0$$

$$\ldots\ldots \quad (3.4,4)$$

$$-\eta^2(\frac{dA}{d\theta} + \frac{B}{4}) - R\eta A + \{B(1 + 3A^2\mu) + \mu(\frac{3}{4}B^3 + 3N^2B$$

$$+ \frac{3}{4}A^2B + \frac{3}{2}BQ_1^2 + 3NAD - 3NBC) + 3A\mu(2NB + AD - BC)\} = 0$$

$$\ldots\ldots \quad (3.4,5)$$

The expansion of $\tilde{X}^2$ and $\tilde{X}^3$ and the resulting simplification of trigonometric identities are shown in Appendix III for reference. The coefficients of cos and sine terms are not considered as the analysis is intended on the transient state of second order subharmonic vibration. Besides as it will be seen in the results of the previous analysis that there is no actual solution as to the existence of the vibratory motion during the resonance of the fundamental harmonic. It is also part of the investigation to ascertain at a later stage that when the predominant subharmonics occur the fundamental harmonic component is of non-resonant type and that it can be reasonably approximated independent of the periodicity for a certain physical characteristic of non-linearity.

On further simplification and after converting equations (3.4,3) to (3.4,5) into non-dimensional quantities gives

$$N^3 + 3AN^2 + N\{1 + 3A^2 + \frac{3}{2}(A^2 + B^2) + \frac{3}{2}(C^2 + D^2)\} + \{\frac{3}{2} A$$
$$(A^2 + B^2 + C^2 + D^2) + \frac{3}{4} C\,(A^2 - B^2) + \frac{3}{2} ABD\} = 0 \qquad \text{.....} \quad (3.4,6)$$

$$\frac{dB}{d\theta} = \frac{1}{\eta^2}\,\{\eta^2 \frac{A}{4} - R\eta B - A(1 + 3A^2) - \frac{3}{4} A(A^2 + B^2) - 3N^2 A$$
$$- \frac{3}{2} AQ_1^2 - 3NAC - 3NBD - 3A(2NA + AC + BD)\} \ \text{...} \ (3.4,7)$$

$$\frac{dA}{d\theta} = \frac{1}{\eta^2}\,\{ - \eta^2 \frac{B}{4} - R\eta A + B(1 + 3A^2) + \frac{3}{4} B(A^2 + B^2) + 3N^2 B$$
$$+ \frac{3}{2} BQ_1^2 + 3NAD - 3NBC + 3A(2NB + AD - BC)\} \ \text{...} \ (3.4,8)$$

Hence, together with equation (3.4,6), the integral curves of

$$\frac{dA}{dB} = \frac{\{- \eta^2 \frac{B}{4} - R\eta A + B(1 + 3A^2) + \frac{3}{4} B(A^2 + B^2) + 3N^2 B + 3NAD + \frac{3}{2} BQ_1^2 - 3NBC + 3A(2NB + AD - BC)\}}{\{\eta^2 \frac{A}{4} - R\eta B - A(1 + 3A^2) - \frac{3}{4} A\,(A^2 + B^2) - 3N^2 A - 3NAC - \frac{3}{2} AQ_1^2 - 3NBD - 3A(2NA + AC + BD)\}} \qquad \text{.....} \ (3.4,9)$$

can be plotted in A, B plane. When steady state is reached N(6), A(θ), C(θ) and D(θ) are constants and the periodic solutions are satisfied by the conditions

$$\frac{dH}{d\theta} = \frac{dA}{d\theta} = \frac{dB}{d\theta} = 0 \qquad \text{.........} \quad (3.4,10)$$

Thus for the periodic solution of equation (3.1,6), equation (3.4,9) is equal to zero and the coefficients satisfying simultaneously the conditions of equation (3.4,10) give singular points on the AB plane. The periodic solutions are determined by equations (3.4,7) and (3.4,8) both equal to zero. The resulting equations would be identical to equations (3.3,4) and (3.3,5).

Chapter IV

Stability Analysis of periodic solution

The behaviour of linear systems when subjected to periodic external force will have motion that consists in addition to the forced vibration of the same frequency as the disturbing force the transient state component of the natural frequency. If the function of displacement is non-linear this principle of superposition will no longer be applicable. For in such systems there are a number of separate periodic vibrations as well as those having the same period as the external force. With realizable physical systems evidently always include energy dissipation, it is readily seen that for the existence of the various types of vibration the problem of stability analysis is unavoidable.

The instability of a vibration is not necessarily due to its own component of the corresponding harmonic in the vibratory motion. The build-up of an oscillation can occur in a region in which the component is of order a multiple of the vibration frequency. This means that the accumulative effect can be of the same frequency as the vibration frequency and it will no longer permit the continuation of the original vibration. Thus stability condition for only the first harmonic region of the subharmonic frequency is not sufficient. The regions for adequate consideration must then be determined from the order of displacement in the variational equation and from the number of terms necessary in the approximate solution.

4.1. The variational equation

The analysis of stability is investigated through considering a small variation δx from the periodic state of equilibrium, and $x_o(\tau)$ as the periodic solution of equation (3.1,7) for which its stability and regions of existence are to be investigated. τ is a non-dimensional time, that is

$$\tau = t \cdot p$$

Equation (3.1,7) then becomes

$$\frac{d^2x}{d\tau^2} + 2R\frac{dx}{d\tau} + \{x + \mu(x^3 + 3\Delta x^2 + 3\Delta^2 x)\} - Z\eta^2 \cos \eta\tau = 0 \qquad \text{......... (4.1,1)}$$

Substituting $(x_o + \delta x)$ in place of x and neglecting higher orders of δx than the first, a linear variational equation for δx is obtained (20,43).

$$\frac{d^2\delta x}{d\tau^2} + 2R\frac{d\delta x}{d\tau} + \{1 + 3\Delta^2 + 3x_o^2 + 6\Delta x_o\}\,\delta x = 0 \qquad \text{.... (4.1,2)}$$

Equation (4.1,2) is in non-dimensional form through multiplying the respective component by $\mu^{\frac{1}{2}}$. The equation characterizes a small variation δx from the periodic solution.

The behaviour of δx with time determines the stability of the solution to equation (3.1,8). The solution is defined as stable if all solutions of δx are bounded as τ tends to infinity. The solution $x_o(\tau)$ is unstable if δx grows unbounded as $\tau \to \infty$.

Using the transformation $\delta x = e^{-R\tau}\,\xi$ .... (4.1,3)

to eliminate the first derivative term gives

$$\frac{d^2\xi}{d\tau^2} + \{(1 - R^2 + 3A^2) + 3x_o^2 + 6Ax_o\}\,\xi = 0 \quad \ldots\ldots \quad (4.1,4)$$

This is a linear equation in ξ where the coefficient is a periodic function of τ and can be developed into a Fourier series. Thus substituting $x_o\,(\tau)$ in the form of equation (3.2,1) the variational equation (4.1,4) is then transformed to a Hill's type equation, that is

$$\frac{d^2\xi}{d\tau^2} + \{C_o + 2\sum_{u=1}^{4} C_u \cos\left(\frac{u\eta\tau}{2} - \sigma_u\right)\} = 0 \quad \ldots\ldots \quad (4.1,5)$$

where $C_o = \{1 - R^2 + 3(A + N)^2 + \frac{3}{2}\,(Q_{\frac{1}{2}}^2 + Q_1^2)\}$; $\quad \ldots\ldots \quad (4.1,6)$

$$C_u^2 = C_{us}^2 + C_{uc}^2\ , \text{ and } \sigma_u = \tan^{-1}\frac{C_{us}}{C_{uc}}\ ,$$

$C_{1c} = \frac{3}{2}\,(2NA_{\frac{1}{2}} + A_{\frac{1}{2}}A_1 + B_{\frac{1}{2}}B_1 + 2AA_{\frac{1}{2}})$, $\;C_{1s} = \frac{3}{2}\,(2NB_{\frac{1}{2}} + A_{\frac{1}{2}}B_1$

$- B_{\frac{1}{2}}A_1 + 2AB_{\frac{1}{2}})$, $\;C_{2c} = \frac{3}{2}\,(2NA_1 + \frac{1}{2}\,(A_{\frac{1}{2}}^2 - B_{\frac{1}{2}}^2) + 2AA_1)$,

$C_{2s} = \frac{3}{2}\,(2NB_1 + A_{\frac{1}{2}}B_{\frac{1}{2}} + 2AB_1)$, $\;C_{3c} = \frac{3}{2}\,(A_{\frac{1}{2}}A_1 - B_{\frac{1}{2}}B_1)$,

$C_{3s} = \frac{3}{2}\,(A_{\frac{1}{2}}B_1 + B_{\frac{1}{2}}A_1)$, $\;C_{4c} = \frac{3}{4}\,(A_1^2 - B_1^2)$, and $C_{4s} = \frac{3}{2}\,A_1B_1$

4.2. The conditions of stability

If the simultaneous algebraic expressions of equation (3.1,8) are not satisfied, the actual existence of the approximation, equation (3.2,1), for only so long as the coefficients are stable is examined for the conditions where the appropriate number of regions of the vibratory motion are to be satisfied simultaneously.

4.2.(i). The characteristic determinant of the stability criteria

By Floquet's theory (15) a particular solution of equation (4.1,5) is given in the form

$$\xi = e^{\gamma\tau} . \phi(\tau) \qquad \text{.........} \quad (4.2,1)$$

where γ is the characteristic exponent, real or imaginary determined by the coefficients of the Fourier series in equation (3.2,1), and $\phi(\tau)$ is the periodic function of time (τ) in which the period is the same or twice the fundamental harmonic period of the series. The number of terms for $\phi(\tau)$ in the Fourier series development depends on the describing equation (3.2,1) and the accuracy required.

Thus from previous consideration,

$$\phi(\tau) = a_o + a_1 \cos\left(\frac{\eta\tau}{2} - \delta_1\right) + a_2 \cos(\eta\tau - \delta_2) \ \ldots \ (4.2,2)$$

where a_o, a_1, a_2, δ_1 and δ_2 are new parameters to be determined.

From equations (4.1,3) and (4.2,1) the conditions defining the stability of $x_o(\tau)$ that is the approximate periodic solution of equation (3.1,7), is readily seen to be given by

$$\text{the real part of } (\gamma - R) < 0 \qquad \text{.......} \quad (4.2,3)$$

and at the boundary of stability by

$$\text{the real part of } (\gamma - R) = 0 \qquad \text{.......} \quad (4.2,4)$$

Since γ can either be real or imaginary, the equation (4.2,3) is equivalent to

$$R > 0 \ \text{ and } R^2 \geqslant \gamma^2 \qquad \text{.......} \quad (4.2,5)$$

If equation (4.2,3) is positive, the small variation δx diverges boundlessly with increase of τ and consequently the approximate solution (3.2,1) does not actually exist.

A particular solution of equation (4.1,5) is then ,

$$\xi = e^{\tau\gamma}\{a_o + a_1 \cos\left(\frac{\eta\tau}{2} - \delta_1\right) + a_2 \cos(\eta\tau - \delta_2)\} \quad \ldots \; (4.2,6)$$

Substituting into equation (4.1,5) gives

$$\frac{d^2\phi}{d\tau^2} + 2\gamma \frac{d\phi}{d\tau} + \{(\mu^2 + C_o) + 2 \sum_{u=1}^{4} C_u \cos\left(\frac{u\eta\tau}{2} - \sigma_u\right)\}\phi = 0$$

$$\ldots\ldots\ldots\ldots \; (4.2,7)$$

With $\phi(\tau)$ as defined by equation (4.2,2), where a_o the constant term is to allow for a zero frequency term contained in the non-odd type displacement, the substitution into equation (4.2,7) and equating the coefficients of the constant and of those of the respective same frequencies through applying the principle of harmonic balance gives a set of linear homogenous equations from which the parameters of equation (4.2,2) can be determined.

$$(\gamma^2 + C_o)\, a_o + C_{1c}\, a_1 \cos\delta_1 + C_{1s}\, a_1 \sin\delta_1$$
$$+ C_{2c}\, a_2 \cos\delta_2 + C_{2s}\, a_2 \sin\delta_2 \;=\; 0$$

$$2C_{1c}\, a_o + (C_{2c} - \frac{\eta^2}{4} + \gamma^2 + C_o)\, a_1 \cos\delta_1$$
$$+ (\gamma\eta + C_{2s})\, a_1 \sin\delta_1 + (C_{1c} + C_{3c})$$
$$a_2 \cos\delta_2 + (C_{1s} + C_{3s})\, a_2 \sin\delta_2 \;=\; 0$$

$$2C_{1s}\, a_o + (C_{2s} - \gamma\eta)\, a_1 \cos\delta_1 + (\gamma^2 + C_o$$

$$- \frac{\eta^2}{4} - C_{2c}) a_1 \sin \delta_1 + (C_{3s} - C_{1s})$$

$$a_2 \cos \delta_2 + (C_{1c} - C_{3c}) a_2 \sin \delta_2 = 0$$

...... (4.2,8)

$$2C_{2c} a_o + (C_{1c} + C_{3c}) a_1 \cos \delta_1 + (C_{3s} - C_{1s})$$

$$a_1 \sin \delta_1 + (\gamma^2 + C_o - \eta^2 + C_{4c}) a_2 \cos \delta_2$$

$$+ (2\gamma\eta + C_{4s}) a_2 \sin \delta_2 = 0$$

$$2C_{2s} a_o + (C_{1s} + C_{3s}) a_1 \cos \delta_1 + (C_{1c} - C_{3c})$$

$$a_1 \sin \delta_1 + (\gamma^2 + C_o - \eta^2 - C_{4c}) a_2 \sin \delta_2$$

$$+ (C_{4s} - 2\gamma\eta) a_2 \cos \delta_2 = 0$$

The equations of (4.2,8) must satisfy for all values of a_o, a_1, a_2, δ_1 and δ_2 which do not vanish. Thus the characteristic determinant of the coefficients must be equal to zero. Also since the determinant is dependent on γ, hence

$$\Delta_{det.}(\gamma) = 0 \qquad \text{.......} \quad (4.2,9)$$

From the condition of (4.2,4), at the boundary between stable and unstable region, equation (4.2,9) becomes

$$\Delta_{det.}(R) = 0 \qquad \text{.......} \quad (4.2,10)$$

and from the conditions of equations (4.2,3) and (4.2,5) lead to

$$\Delta_{det.}(R) > 0 \qquad \text{.......} \quad (4.2,11)$$

for the stability of the periodic solution.

Hence, the characteristic determinant leading to a criteria of the stability limit takes the form

$$\Delta_{det.}(R) = \begin{vmatrix} R^2+C_o & C_{1c} & C_{1s} & C_{2c} & C_{2s} \\ 2C_{1c} & R^2+C_o-\frac{\eta^2}{4}+C_{2c} & R\eta+C_{2s} & C_{1c}+C_{3c} & C_{1s}+C_{3s} \\ 2C_{1s} & C_{2s}-R\eta & R^2+C_o-\frac{\eta^2}{4}-C_{2c} & C_{3s}-C_{1s} & C_{1c}-C_{3c} \\ 2C_{2c} & C_{1c}+C_{3c} & C_{3s}-C_{1s} & R^2+C_o-\eta^2+C_{4c} & 2R\eta+C_{4s} \\ 2C_{2s} & C_{1s}+C_{3s} & C_{1c}-C_{3c} & C_{4s}-2R\eta & R^2+C_o-\eta^2-C_{4c} \end{vmatrix} = 0 \quad .. \ (4.2,12)$$

where C_{ic} and C_{is} are as defined by the equations (4.1,6),

and $i = 1, 2, 3, 4$.

The expansion of the above determinant and from substituting the equations of (4.1,6) into it, a relation giving the boundary of any unstable regions on the resonance curves can be obtained. The stability criteria derived will locate the region of the accumulative effect of an unstable component in the non-linear motion.

The direct differentiation of the expression describing the subharmonic resonance to obtain the vertical gradients at the stability limits is not the most appropriate way to determine the instability region. The prodecure, besides, is complicated and very laborious. To use $\frac{d\eta^2}{dQ_1} = 0$ as a stability criterion will not readily illustrate that the approximate solution does not exist in the instability region or the cause of the instability that is due to a build-up effect in one of the harmonic oscillation of the vibratory motion.

4.2.(ii). The vertical tangency

The use of an equivalence between the criteria of equation (4.2,12) and the Routh-Hurwitz criterion of stability will ascertain that the characteristic curve of displacement amplitude against frequency has a vertical tangency at the stability limit.

From the non-linear algebraic expressions, that may be satisfied for more than one set of values of the dependent variables, that if

$$S \equiv \text{equation } (3.2,8) = 0$$

$$Y_1 \equiv \text{equation } (3.2,9) = 0$$

$$X_1 \equiv \text{equation } (3.2,10) = 0 \qquad \ldots\ldots \quad (4.2,13)$$

$$Y_2 \equiv \text{equation } (3.2,11) = 0$$

$$X_2 \equiv \text{equation } (3.2,12) = 0$$

the stability condition, derived from the Routh-Hurwitz criterion (22,45) for the describing equation (3.1,6), at the boundary is given by

$$\nabla = \begin{vmatrix} \frac{\partial S}{\partial N} ; & \frac{\partial S}{\partial A} ; & \frac{\partial S}{\partial B} ; & \frac{\partial S}{\partial C} ; & \frac{\partial S}{\partial D} ; \\ \frac{\partial X_1}{\partial N} ; & \frac{\partial X_1}{\partial A} ; & \frac{\partial X_1}{\partial B} ; & \frac{\partial X_1}{\partial C} ; & \frac{\partial X_1}{\partial D} ; \\ \frac{\partial Y_1}{\partial N} ; & \frac{\partial Y_1}{\partial A} ; & \frac{\partial Y_1}{\partial B} ; & \frac{\partial Y_1}{\partial C} ; & \frac{\partial Y_1}{\partial D} ; \\ \frac{\partial X_2}{\partial N} ; & \frac{\partial X_2}{\partial A} ; & \frac{\partial X_2}{\partial B} ; & \frac{\partial X_2}{\partial C} ; & \frac{\partial X_2}{\partial D} ; \\ \frac{\partial Y_2}{\partial N} ; & \frac{\partial Y_2}{\partial A} ; & \frac{\partial Y_2}{\partial B} ; & \frac{\partial Y_2}{\partial C} ; & \frac{\partial Y_2}{\partial D} ; \end{vmatrix} = 0 \quad \ldots\ldots (4.2,14)$$

On differentiating equations (3.2,8) to (3.2,12) with respect to the non-dimensional frequency η, a set of simultaneous equations is obtained.

Solving these resulting expressions will yield

$$\frac{dN}{d\eta} = \frac{\nabla_1}{\nabla} ; \quad \frac{dA}{d\eta} = \frac{\nabla_2}{\nabla} ; \quad \frac{dB}{d\eta} = \frac{\nabla_3}{\nabla} ; \quad \frac{dC}{d\eta} = \frac{\nabla_4}{\nabla} ; \quad \frac{dD}{d\eta} = \frac{\nabla_5}{\nabla} ; \quad \ldots\ldots\ldots (4.2,15)$$

where ∇_i (i = 1, 2, 3, 4, 5) is the characteristic determinant of equation (4.2,14) in which the ith column is replaced by the following column of terms.

$$-\frac{dS}{d\eta} ; \quad -\frac{dX_1}{d\eta} ; \quad -\frac{dY_1}{d\eta} ; \quad -\frac{dX_2}{d\eta} ; \quad -\frac{dY_2}{d\eta} ; \quad \ldots\ldots (4.2,16)$$

Since $Q_2^2 = A^2 + B^2$, and $Q_1^2 = C^2 + D^2$, the gradient of the characteristic curve displaying the vibration amplitude against frequency is given by

$$\frac{dQ_2}{d\eta} = \frac{\partial Q_2}{\partial A} \cdot \frac{dA}{d\eta} + \frac{\partial Q_2}{\partial B} \cdot \frac{dB}{d\eta}$$

$$\text{............ } (4.2,17)$$

and $$\frac{dQ_1}{d\eta} = \frac{\partial Q_1}{\partial C} \cdot \frac{dC}{d\eta} + \frac{\partial Q_1}{\partial D} \cdot \frac{dD}{d\eta}$$

The substitution of equation (4.2,15) into equation (4.2,17) will give

$$\frac{dQ_2}{d\eta} = \frac{A}{Q_2} \frac{V_2}{V} + \frac{B}{Q_2} \frac{V_3}{V} = \frac{1}{Q_2 V} (AV_2 + BV_3)$$

and .... (4.2,18)

$$\frac{dQ_1}{d\eta} = \frac{1}{Q_1 V} (CV_4 + DV_5)$$

Thus with the stability limits at the boundary occuring when $V = 0$ it is readily seen from the equations of (4.2,18) that

$$\frac{dH}{d\eta} = \frac{dQ_2}{d\eta} = \frac{dQ_1}{d\eta} = \infty \qquad \text{............ } (4.2,19)$$

Equation (4.2,19) shows that a vertical tangency of the resonance curves occurs at the stability limit which is determined from the equations of (4.2,8) for the first and second component regions.

Chapter V

Analogue Computing

5.1. The instruments.

The analogue computer has been regarded not only to help bridge the media between intuitions and exact analysis, but it is also used as a 'working' model (1, 4, 10, 12, 14) if the equation of motion, to be simulated, adequately describes the actual motion. In this investigation such difficulty does not arise.

An S.C.40 System Computer analogue, Fig.(5.1-1), is used to obtain the experimental results to equation (3.1,7) for the wide range of independent parameters. The machine comprises in addition to the computing elements housed in a temperature controlled oven a digital voltmeter and relay switching circuits from which individual parts of the program can be monitored at any instant. The necessity of calibrations, voltage measurements and potentiometer settings are carried out through this address system to the respective outputs.

The non-linear function of displacement is generated by means of transistorised mark space multipliers. These are slightly less accurate than the servo-type but they have a much wider bandwidth, whilst the servo multipliers are restricted in useage to lower frequencies. Computing outputs are observed by monitoring the corresponding traces on an Airmec type 279 display oscilloscope, item B in Fig.(5.1-1). The results are recorded

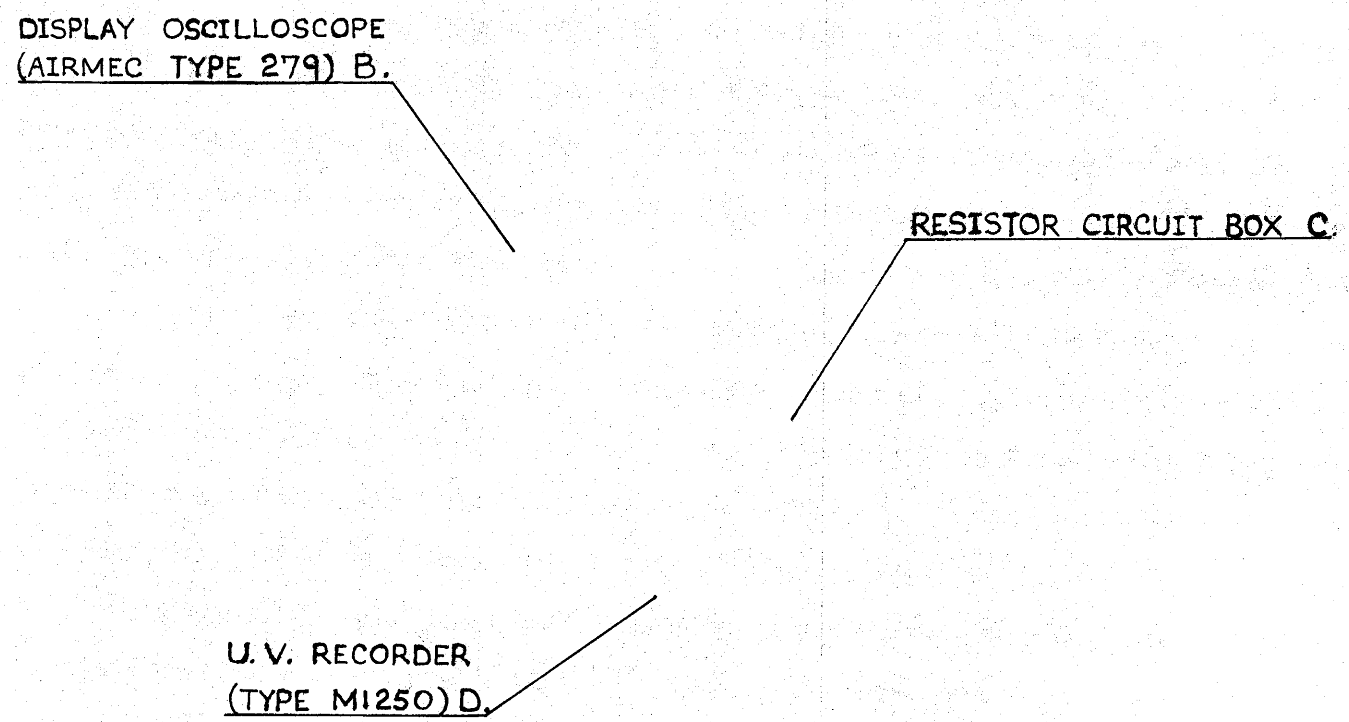
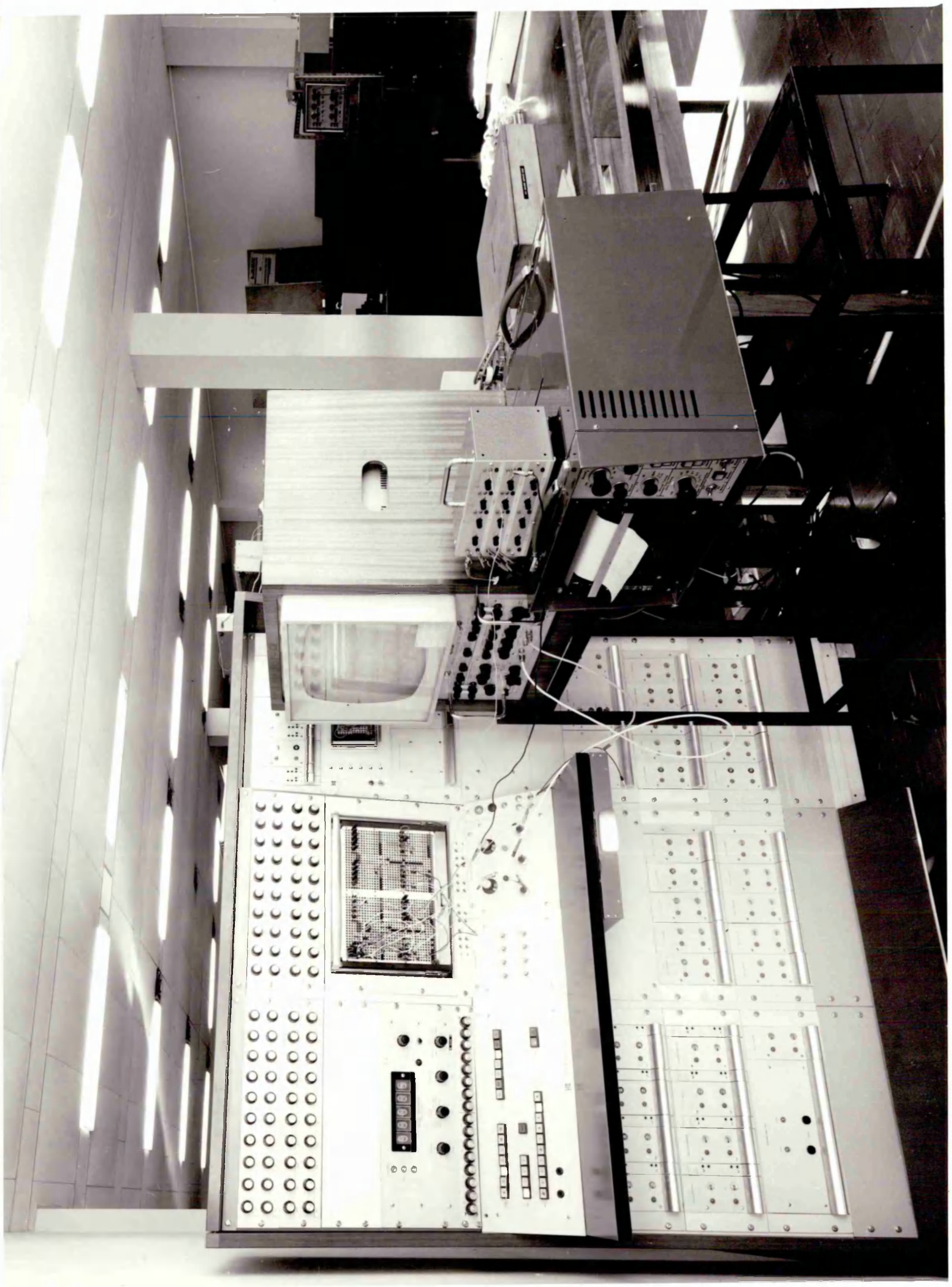


FIG.(5.1-1). THE S.C. 40 SYSTEM COMPUTER ANALOGUE.



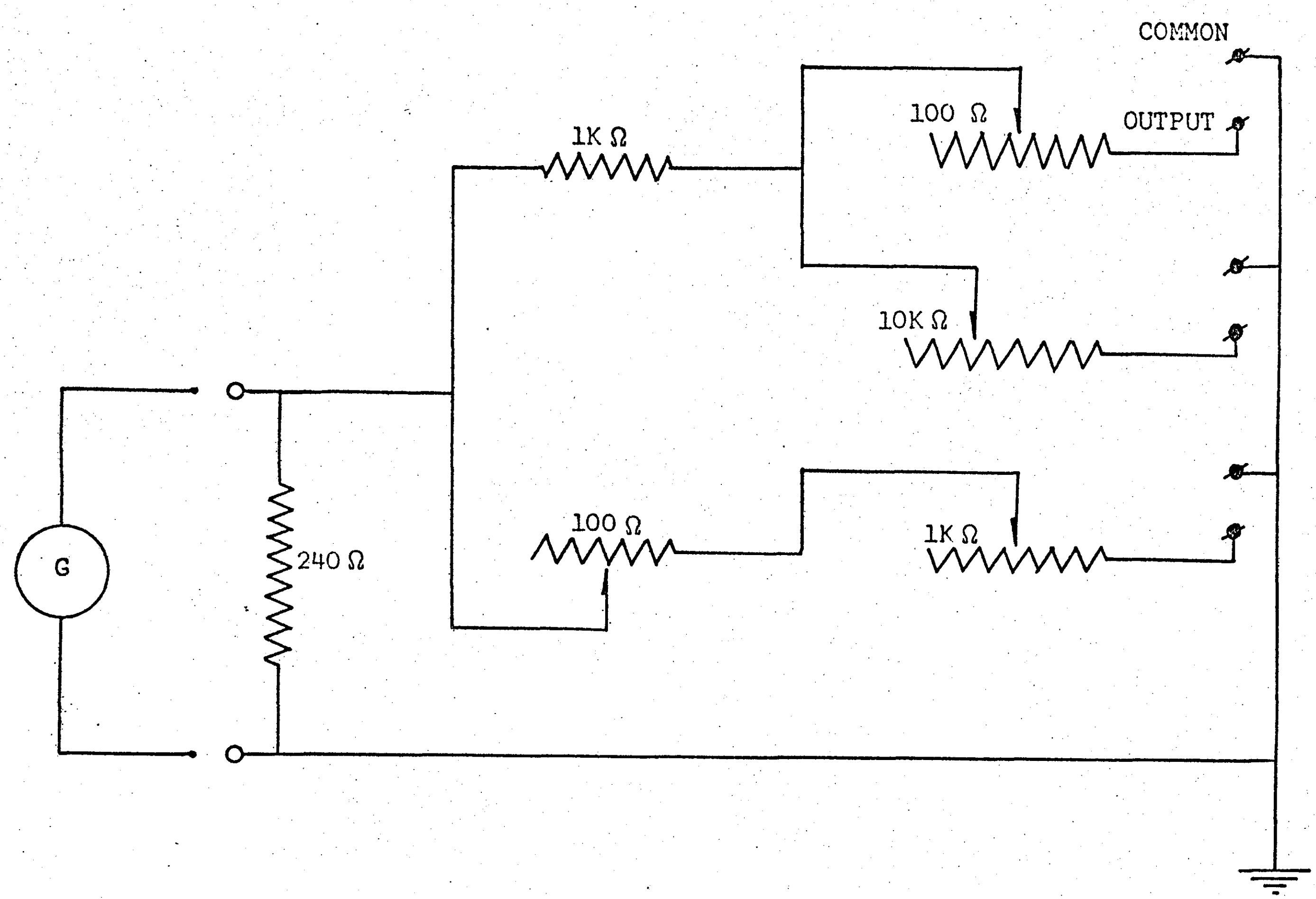


FIG.(5.1-2). RESISTOR CIRCUIT FOR CONTROL BOX.

via a resistor circuit, item C, by a Southern Instrument ultra-violet recorder type M1250, item D. The resistor circuit shown in Fig.(5.1-2) allows sufficient deflection magnitude off the galvanometer when needed to maintain the least error from subsequent measurement of the traces.

5.2. Programming.

5.2.(i). Scaling

The fundamental circuit programs for the second order subharmonic solutions to equation (3.1.7) are shown in Fig. (5.2-1). The difficulties which may be encountered when setting up the program are from the problems of scaling as it is necessary to relate the computer variables directly to those of the investigation. The independant parameters of the describing equation and the coefficients of the solution are expressed in voltages and time of the computer. The necessity for scaling is not only to relate the behaviour of the physical system to the two variables, but as with all working models there are real limitations on possible magnitudes and rates of change with respect to time. Often, amplitude and time scaling are necessary if the equations are to be simulated efficiently for a wide selection of the independent parameters.

The problem of amplitude scaling becomes less simple when computing for solutions to non-linear motion. In a computer study of linear differential equations all amplitude scale factors can be uniformly changed by merely altering the magnitude of the forcing function input and of the initial conditions. It is not possible to apply such a procedure to equation (3.1.7) simply because the principle of superposition does not hold. Amplitude scaling to each output may be necessary either to avoid exceeding the operating specification or to prevent large errors being introduced to the results on a percentage basis.

Because of design limitations the built-in error of the computer is lowest at maximum loading of the outputs. Thus to ensure good accuracy of the solution the amplitude scaling is calculated to attain high loading of the amplifiers and multipliers. Generally, this involves several scaling sequences for each set of independent parameters. The machine loses its linearity if the limiting voltages were exceeded. However the outputs of the various computing elements are appropriately determined to achieve not lower than ninety per cent of the stablised voltage. The error that is introduced during the computation owing to drift of the reference level of operational amplifiers, multipliers and reference supplies is then less than 5 per-cent.

The solution of equation (3.1,7) requires the use of the multipliers to raise the power of the displacement. Often it becomes necessary to time scale the differential equation especially when the servo-multipliers are used to generate the non-linear motion. The scaling of actual time to one suitable on the machine is by making the substitution

$$\tau = S \times t \qquad \text{.........} \quad (5.2,1)$$

where τ is the computer time and S is the time scale.

Since the machine computes for only a period of three to four minutes without loss in accuracy the use of servo-multipliers will not necessarily increase the accuracy of the solution for a longer rate is required to attain the steady state from transient motion.

As the equation (3.1,7) describes a centrifugally excited system, real time can be used by making $p = \frac{1}{1}$. Because of the scaling requirements it will give a higher frequency response. Otherwise, at low operating speed, the drift in the computing elements will cause deterioration of the accuracy. This means the use of transistorised mark space multipliers is more suitable and there is also little difficulty then with the recording facilities for the respective outputs. Hence the problem of frequency limitation does not arise for the wide range of values necessary for a thorough investigation.

If the amplitude scaling is carried out as previously discussed there is no concern as to whether the outputs of the multipliers exceeds the design maximum. The actual operation of the multipliers produces an output equal to the product of the two inputs divided by the computer reference with a complete reversal in phase to the resulting variable. Thus two inputs not overloaded in their respective circuits will give an output within limits.

5.2.(ii). The computing equation

By making μ equal to unity the dependent variables of the vibration have non-dimensional values. As the frequency of the output by previous consideration is converted to actual time of equation (3.1.7), the non-linear response of the system to various permutation of independent parameters will be as depicted by the output traces. Thus the behaviour resulting from gravitational influence on the magnitude of non-linearity is monitored directly on the oscilloscope.

The describing equation is reduced to the form

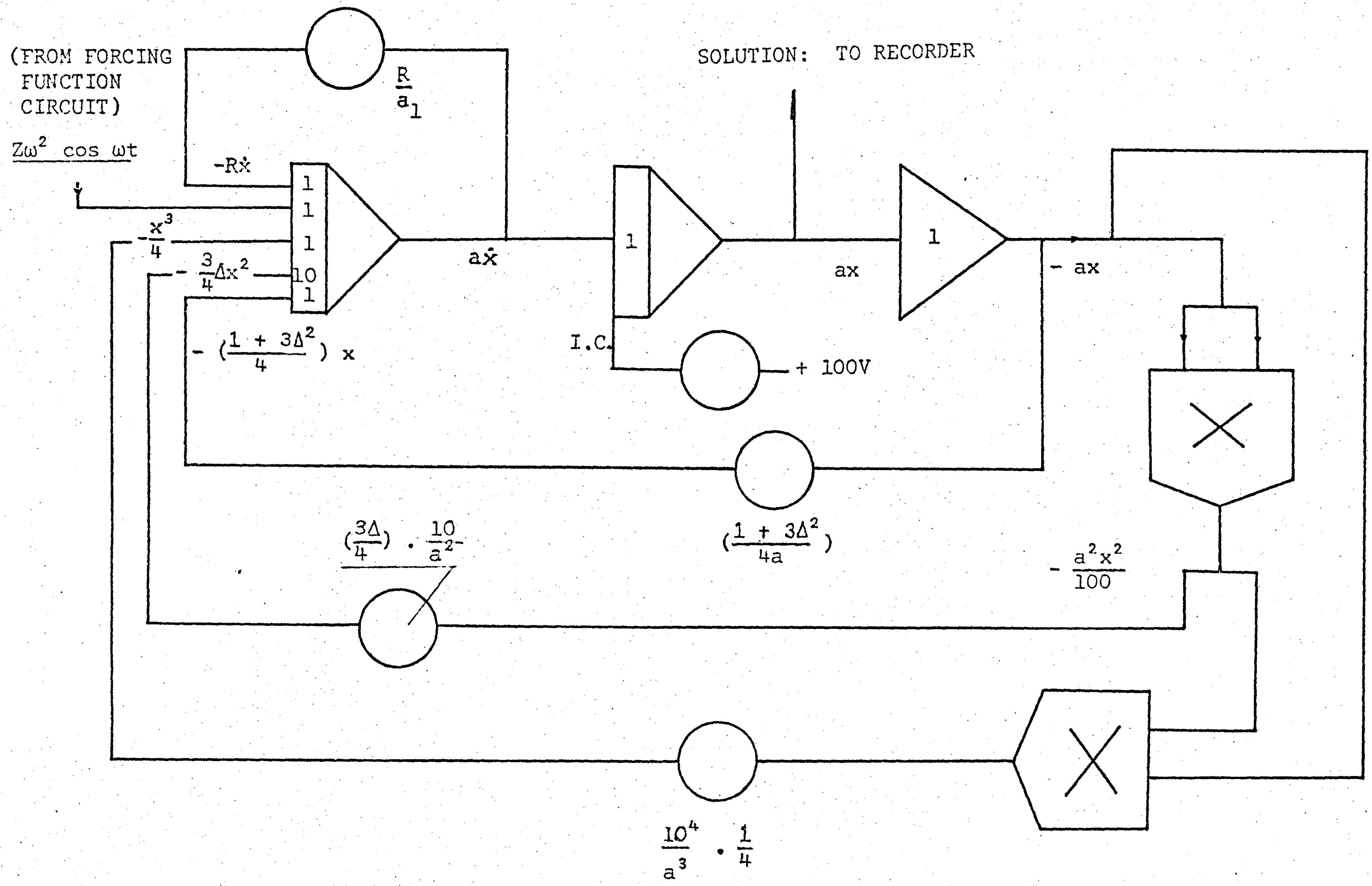
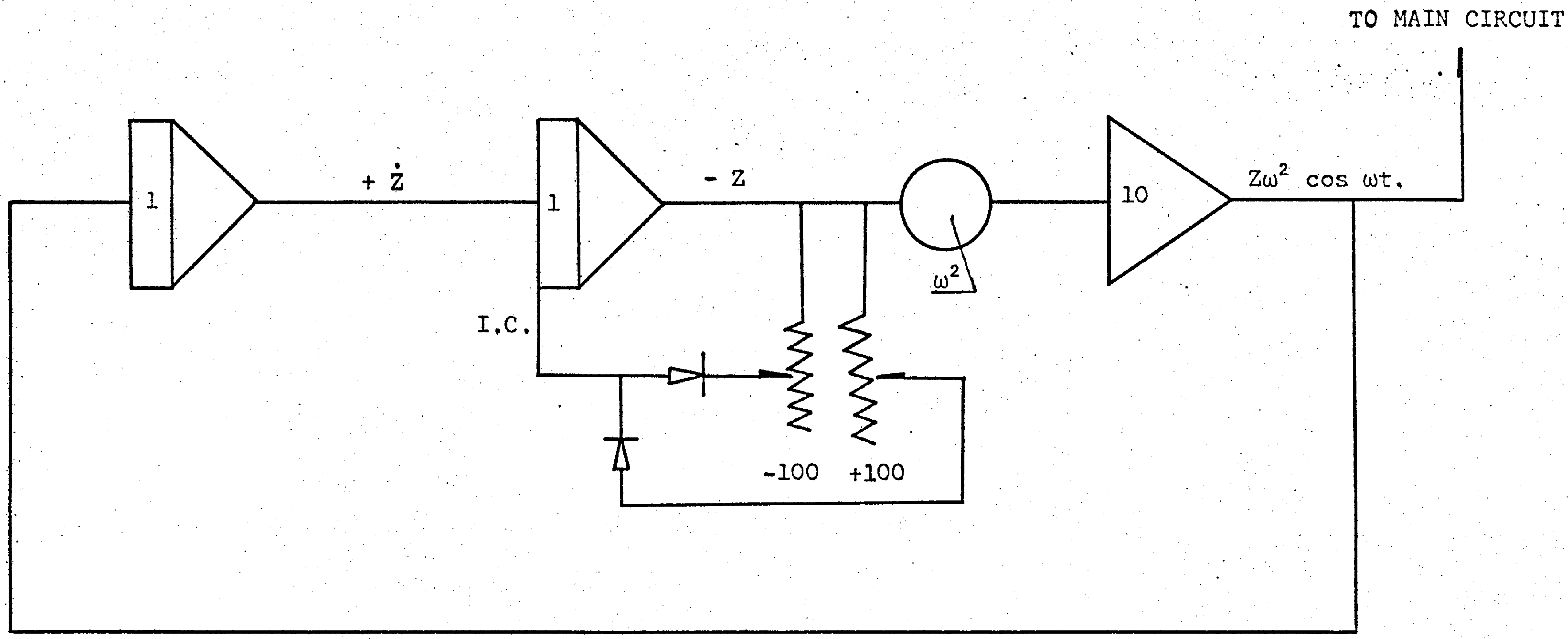


FIG.(5.2-1).(a) ANALOG COMPUTER PROGRAM FOR EQUATION (5.1.1.)



$$\frac{d^2Z}{dt^2} + \omega^2 Z = 0$$

FIG.(5.2-1)(b) COS. FUNCTION GENERATOR

$$\ddot{x} = - R\dot{x} - \frac{1}{4} \{ x(1 + 3A^2) + 3Ax^2 + x^3 \} + \bar{X}\omega^2 \cos \omega t \quad \ldots (5.2.2)$$

The computing circuits programmed for the system are shown in Fig. (5.2-1). To start circuit Fig. (5.2-1)(b) it is necessary to have an initial condition. Since the disturbing force is a cosine function the initial condition is accordingly fed into the second integrator.

5.3. Computation

5.3.(i). The procedure.

The coefficients of the potentiometers are set only after the complete interconnections of the circuit components have been made, and the patch panel have been inserted into the machine. This will then take into account the loading effects of the potentiometer settings on the various integrators, amplifiers and multipliers. The accuracy of the settings by means of control circuitry in the console to the degital voltmeter is at least 0.05 per cent full scale.

Because of inherent damping the duration of computing preferably is not longer than three to four minutes. Such precaution will prevent errors of some magnitude otherwise being introduced into the results. However the problem is more difficult when computing for solutions to non-linear systems. In cases where the coefficients of damping are relatively small, severe amplitudes are excited during the transient motion. Since they require sufficient time to be suppressed, the effects from the need to attain a steady state within a period are felt in both circuits.

Although theoretically a periodic force can only do work on a motion having the same frequency, with a non-linear system the predominant component of subharmonic vibration is maintained by the periodic force at a submultiple of its frequency. Energy is supplied to sustain this non-linear phenomena via the fundamental harmonic. Thus, if the computation exceeds the reasonable period, the magnitude of the excitation is reduced on both counts and the accuracy of the results deteriorates rapidly.

The potentiometer coefficients are checked frequently to avoid loss in accuracy through small variations in the supply reference voltage. This problem arises particularly when the analogue computer is operated before it is allowed a sufficient time period to attain the working temperature condition of maximum efficiency. If the relative potentiometer settings limit one of the coefficients to small values, the loss in accuracy can be of a significant degree.

5.3.(ii). The minimization of error.

To minimise such error in both the main and forcing function circuits when generating the pronounced vibrations, initial conditions are introduced to the input of the first differential in circuit Fig.(5.2-1)(a) during the trial runs.

The values are set according to the theoretical results. This will produce at the outset of computing the displacement voltage which varies from an amplitude and frequency near the region of the experimental results. When steady-state conditions are achieved the small decrease in magnitudes of the vibration amplitude and of the disturbing force are recorded. On the basis that the computing circuits describing the subharmonic motion of equation (3.1,7) are from equation (5.2,2), the corresponding outputs are monitored efficiently. Hence when these values are used as initial conditions in the actual run, steady-state subharmonic vibration is produced half-way through the period of computation.

The experimental results obtained by this procedure are in general in good agreement with the theoretical analysis. The experimental value of a particular frequency is located without difficulty for either the lower or the upper branch of the Q_2 frequency response through the voltage set to the input of the first differential. As expected from the consideration of chapter IV, section 4.2, the location of the experimental points in the instability region is not possible due to the build-up of an oscillation which has the same frequency as the subharmonic frequency. Thus there is no actual solution in that region if the determinant of equation (4.2,12) is smaller than zero. The 'jump' phenomena where it exists can be observed on either increasing or reducing the potentiometer value in the forcing function circuit.

5.4. Analysis of experimental wave-forms.

Experimental solutions for each selected frequency value to equation (3.1,7) are obtained for various permutations of independent parameters and are recorded in time wave-forms. A trace which display the steady state subharmonic resonance of second order is shown in Fig.(5.4-1)(a).

5.4.(i). Time wave-forms represented as Fourier series.

The periodic curve representing the vibration displacement is analysed into harmonic components by dividing for each cycle of vibration, which is 4π for every period in θ, into twelve equally spaced ordinates. The y-ordinates of the wave-form are measured by means of a Pharol chart reader equipped with vernier scales, and the values subsequently are reduced to a dimensionless state through multiplying and dividing by the appropriate factors.

Thus, with the time wave-form expressed as a Fourier series, that is

$$f(\theta) = \frac{a_o}{2} + \sum_{\kappa=1}^{\kappa=\alpha} (a_\kappa \cos \kappa\theta + b_\kappa \sin \kappa\theta) \qquad \ldots\ldots \quad (5.4.1)$$

the components are determined by

$$a_o = \frac{1}{6} + \sum_{j=1}^{12} y_j \qquad \ldots\ldots \quad (5.4.2)$$

$$a_\kappa = \frac{1}{\pi} \sum_{j=1}^{12} y_j (\cos \kappa\theta_j) \,.\, \delta(\theta) \qquad \ldots\ldots \quad (5.4.3)$$

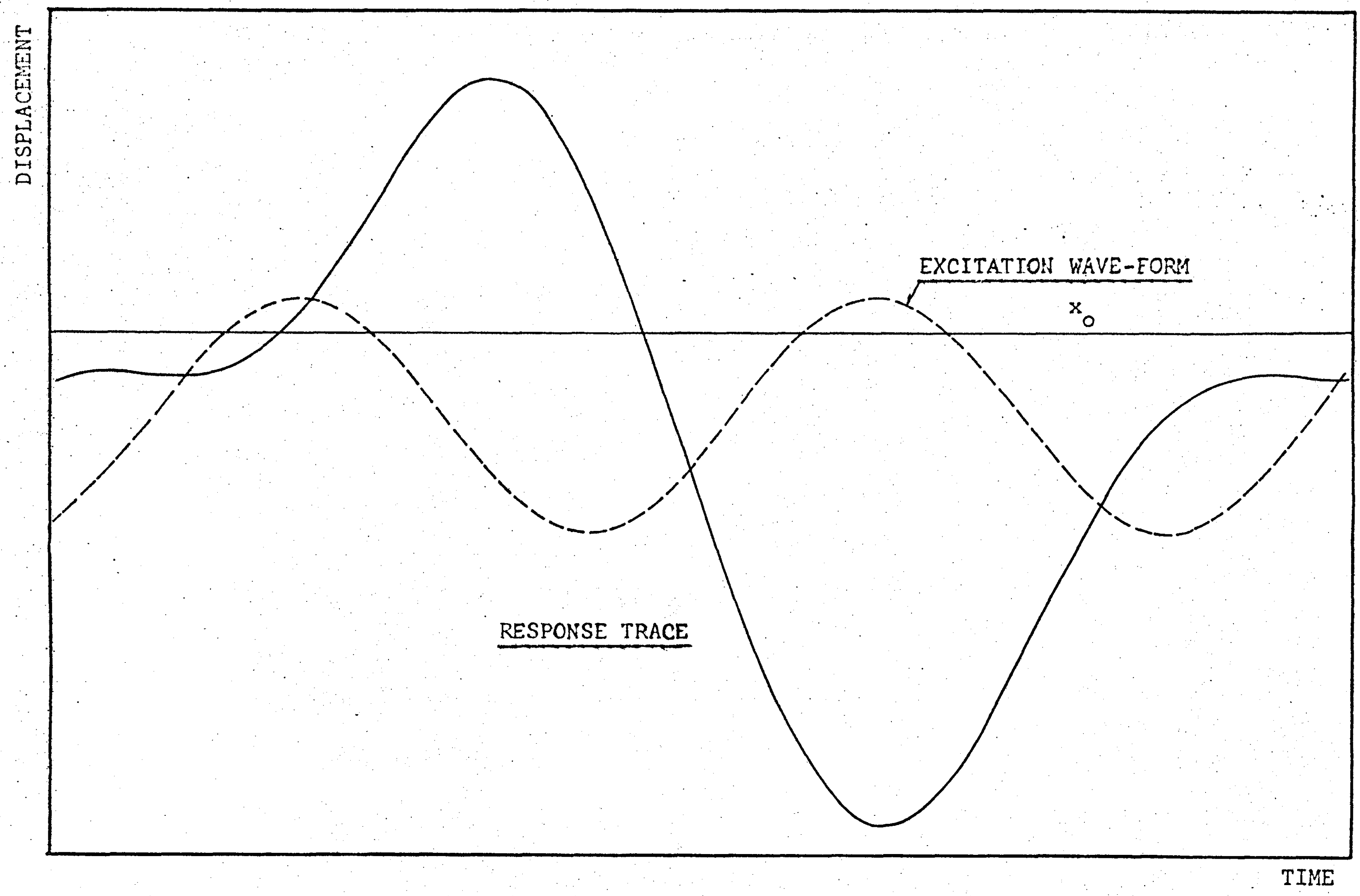


FIG.(5.4-1)(a) ANALOGUE COMPUTER TIME WAVEFORM

$\bar{\Delta} = 1.0 \quad \bar{Z} = 0.4 \quad R = 0.25 \quad \eta = 3.6$

$\bar{Q}_{\frac{1}{2}} = 1.07 \quad \bar{Q}_1 = 0.52 \quad \bar{N} = -0.422 \quad \bar{Q}_{\frac{3}{2}} = 0.03$

where

$$\theta_j = \{ \frac{\pi}{6} + (j - 1) . \frac{\pi}{6} \}$$

Since $\delta(\theta)$ here is a constant, that is $\delta(\theta) = \frac{\pi}{6}$, then

$$a_\kappa = \frac{1}{6} . \sum_{j=1}^{12} y_j \cos \kappa\theta_j \qquad \ldots\ldots \quad (5.4,4)$$

Similarly,

$$b_\kappa = \frac{1}{6} \sum_{j=1}^{12} y_j \sin \kappa\theta_j \qquad \ldots\ldots \quad (5.4,5)$$

As the coefficients of the series can be expressed in the form

$$f(\theta) = M + Q_1 \cos (\theta - \phi_1) + Q_2 \cos (2\theta - \phi_2) + \ldots \qquad \ldots. \quad (5.4,6)$$

where $Q_\kappa = (a_\kappa^2 + b_\kappa^2)^{\frac{1}{2}}$,

and $\phi_\kappa = \tan^{-1} \dfrac{b_\kappa}{a_\kappa}$,

then with the aid of degital computing, equations (5.4,2), (5.4,4) and (5.4,5) will give for the wave-forms of equation (5.2,2) the mean dynamic displacement and the respective harmonic components of the non-linear motion.

The non-dimensional frequency of the recorded trace is determined from knowing the time scale of the wave-form, that is

$$\eta = \frac{4\pi}{\ell} \cdot \frac{S}{p} \qquad \text{.... (5.4.7)}$$

where

S is the time gradient of the wave-form in cm. per sec.,
ℓ the length of each cycle of vibration in cm., and
p is the linear natural frequency in radians per sec.

The sensitivity of the amplitude-frequency traces can be maintained when needed by merely increasing the paper speed and adjusting the resistor circuit. This will prevent wide fluctuation in the accuracy of the coefficients and of the frequencies determined.

The control of the paper speed of the U.V. recorder regulates the x-axis of the wave-forms. Hence, it allows the suitable number of equally spaced divisions in the length of each cycle of vibration to be made accurately, and thus reducing the percentage error that is introduced in the approximation of the actual wave-form in the harmonic analysis. The appropriate adjustment of the resistor circuit enables a greater deflection off the galvanometer of the U.V. recorder to be obtained. The accuracy of the results is then readily maintained, for the measurements of the large amplitudes of the wave-forms are taken by means of the Pharol chart reader which gives readings up to the second decimal place.

5.4.(ii). The measureable subharmonic components.

The results for a particular set of parameters are presented in Table (5.4-2). Although the above expressions are approximates, generally the twelve ordinates give sufficiently good results. If higher accuracy is required the number of ordinates can be increased.

As observed from the table, the third order of even subharmonic components is very small. In fact the third order components cannot be effectively measured with the Tharol chart reading up to the second decimal place. The error introduced from the analysis of the trace is as high as 200 per cent. With the computation error estimated to be less than 3 per cent it is reasonable to say that only the first three amplitudes of the subharmonic components are measureable. Thus the inclusion of the third and higher order components to the approximate solution will have negligible improvement upon the accuracy of the theoretical results. Energy supplied by the disturbing force principally is absorbed to sustain the resonance condition.

A graph of the harmonic components that are measureable in the experiment is shown in Fig.(5.4-1)(b). The vectorial addition of each succeeding component will give a trace identical to the waveform recorded if a sufficiently close number of ordinates are considered. The effect of the shift in phase for the respective components is readily seen in the results illustrated.

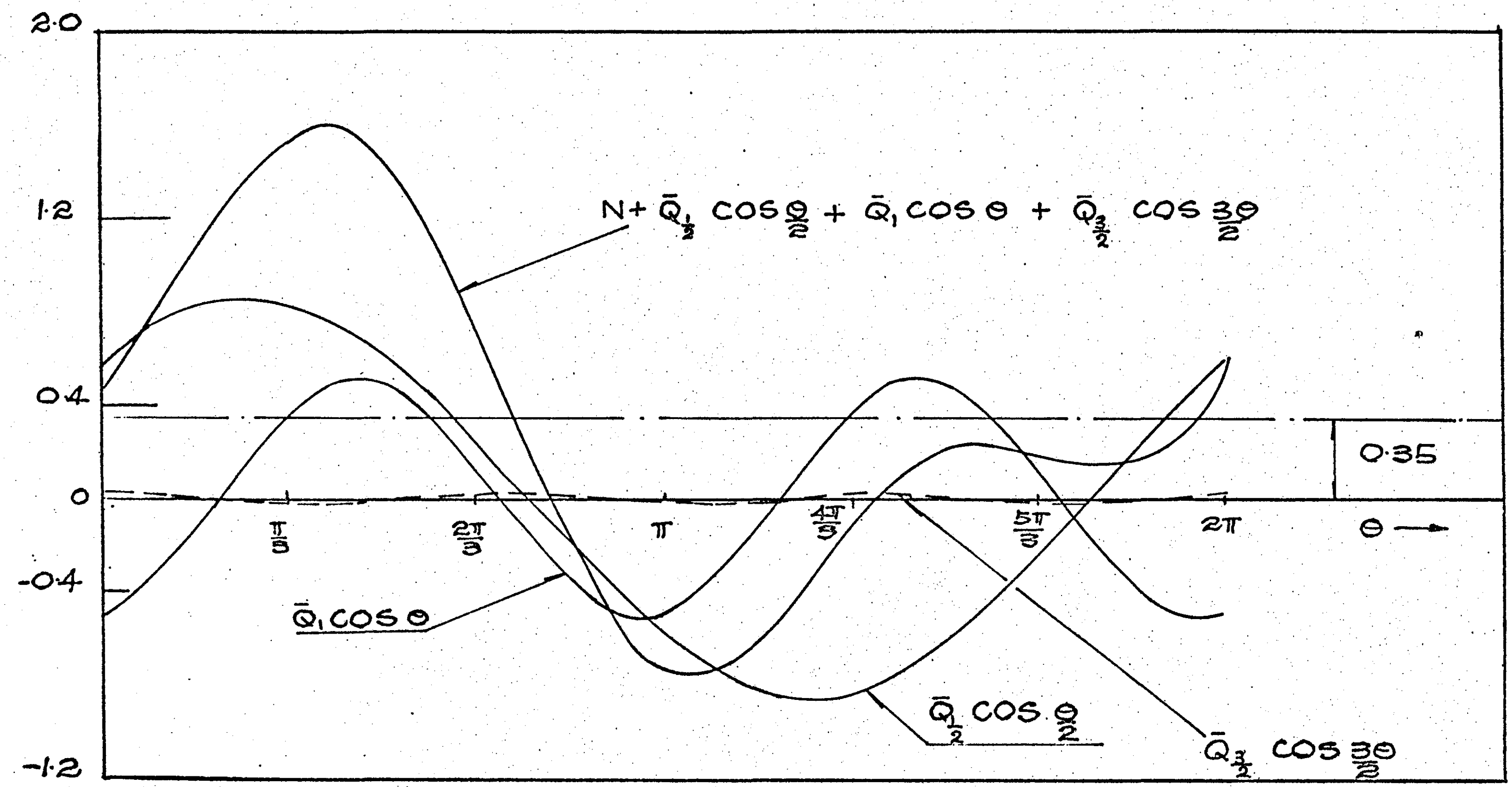


FIG. (5.4-1)(b). ANALOGUE COMPUTER TIME WAVEFORM AND THE FIRST THREE MEASURABLE SUBHARMONICS.

$\bar{\Delta} = 1.0 \quad \bar{Z} = 0.4 \quad R = 0.25 \quad \eta = 3.78$

η	$\bar{Q}_{\frac{1}{2}}$	$\bar{Q}_1$	$\bar{Q}_{\frac{3}{2}}$	$\bar{N}$
3.15	0.073	0.588	0.001	- 0.12
3.2	0.196	0.569	0.005	- 0.15
3.22	0.41	0.558	0.014	- 0.17
3.28	0.68	0.51	0.02	- 0.21
3.34	0.88	0.496	0.03	- 0.28
3.41	0.94	0.49	0.03	- 0.31
3.45	1.04	0.495	0.03	- 0.36
3.51	1.1	0.508	0.03	- 0.395
3.6	1.07	0.52	0.03	- 0.422
3.71	0.93	0.53	0.03	- 0.39
3.78	0.86	0.516	0.03	- 0.35
3.85	0.76	0.505	0.03	- 0.31
3.98	0.61	0.49	0.02	- 0.23
4.09	0.48	0.49	0.02	- 0.18
4.24	0.23	0.47	0.01	- 0.11

TABLE.(5.4-2). EXPERIMENTAL RESULTS OF EQUATION (3.1,7)

FOR $\bar{\Delta} = 1.0$, $\bar{Z} = 0.4$, $R = 0.25$,

$p^2 = 1.0$ and $\mu = 1.0$.

Chapter VI

Second order subharmonic vibration.

6.1. Numerical evaluation.

From the consideration of chapter III, section (3.1) and (3.2), it is evident from the respective corresponding equations which define the domain of subharmonic resonance that one solution is $Q_{\frac{1}{2}} = 0$. If this component is not equal to zero then the resulting simultaneous equations must be satisfied for the periodic solution of equation (3.2,1). This gives

$$3YQ_1 \sin(\phi_1 - 2\phi_{\frac{1}{2}}) - R\eta = 0 \qquad \text{..........} \quad (6.1,1)$$

and

$$3YQ_1 \cos(\phi_1 - 2\phi_{\frac{1}{2}}) + (1 - \frac{n^2}{4}) + 3Y^2 + \frac{3}{4}(Q_{\frac{1}{2}}^2 + 2Q_1^2) = 0 \qquad \text{..........} \quad (6.1,2)$$

6.1.(i). The simultaneous polynomial expressions

Since equations (6.1,1) and (6.1,2) can be represented by Fig.(3.2-1)(a), a single equation for the subharmonic response in terms of the variables and parameters is obtained. The equation which is a polynomial expression of $Q_{\frac{1}{2}}$ to the fourth power is presented in the form that has greater physical meaning to the equation in describing the vibration resonance. Thus, after laborious simplification gives

$$\eta^2 = \{ 4 + 3X + 12Y^2 - 8R^2 \} \pm 4 \{ 9Q_1^2 Y^2 - R^2(4 + 3X + 12Y^2 - 4R^2) \}^{\frac{1}{2}} \quad \ldots\ldots\ldots (6.1,3)$$

where $X = (Q_{\frac{1}{3}}^2 + 2Q_1^2)$.

It is readily seen the important component $Q_{\frac{1}{3}}$ of the motion exists only in a limited frequency range. Equation (6.1,3) will be further discussed in a following section.

The elimination of variables $\phi_{\frac{1}{3}}$ and ϕ_1 from the remaining simultaneous equations will reduce the original five equations to three. The resulting polynomial expressions are still very complex but they are rearranged into more suitable forms whereby the remaining variables are determined with less difficulty.

Using the trigonometric definition of Fig. (3.2-1)(a), equation (3.2,18) after simplification gives,

$$N = \frac{- \{ 9\Delta^2(Q_{\frac{1}{3}}^2 + Q_1^2) - 1.5Q_{\frac{1}{3}}^2 (1 - \frac{\eta^2}{4} + 0.75X + 3\Delta^2) \}}{6N^3 + a_1N^2 + b_1N + c_1} \quad \ldots (6.1,4)$$

where $a_1 = 24\Delta$,

$b_1 = (6 + 36\Delta^2 + 4.5\, Q_{\frac{1}{3}}^2 + 9Q_1^2$,

and $c_1 = (6 + 18\Delta^3 + 9\Delta X)$.

The roots of N can now be more conveniently determined by applying an iteration method. It gives for a selected value of $Q_{\frac{1}{3}}$ and its corresponding frequency a single real root of N.

Finally, with equations (3.2,16) and (3.2,17) represented by Fig. (3.2-1)(b), a single equation describing the behaviour of the accompanying fundamental harmonic component during the subharmonic vibration can also be obtained. Since the expression is a polynomial to the eighth power and is inherently coupled to equations (6.1,3) and (6.1,4), the component Q_1 is more suitably determined by the iteration method. Hence after simplification the complicated algebraic equation is presented in the form

$$Q_1 = - \frac{\left\{ Q^4_{\frac{1}{2}} \left\{ R^2\eta^2 + (1 - \frac{\eta^2}{4})(1 - \frac{\eta^2}{4} + 6Y^2 + 1.5Q^2_{\frac{1}{2}}) \right\} + 9Q^4_{\frac{1}{2}} Y^2 (Y^2 + \frac{Q^2_{\frac{1}{2}}}{2}) + \frac{9}{16} Q^8_{\frac{1}{2}} \right\}}{2.25\, Q_1^7 + a_2 Q_1^5 + b_2 Q_1^3 - c_2 Q_1} \qquad \text{......} \quad (6.1,5)$$

where

$$a_2 = \{ 6(1 - \eta^2) + 18Y^2 + 4.5\, Q^2_{\frac{1}{2}} \} ,$$

$$b_2 = \{ 16R^2\eta^2 + 4(1 - \eta^2)(1 - \eta^2 + 6Y^2) + 3Q^2_{\frac{1}{2}} (1 + 3Y^2) - \frac{21}{4} Q^2_{\frac{1}{2}}\ \eta^2 + 36Y^4 \}$$

$$c_2 = \{ 4\eta^2 (\eta^2 Z^2 - 2R^2 Q^2_{\frac{1}{2}}) + Q^2_{\frac{1}{2}} (4 - \eta^2)(1 - \eta^2) + Q^2_{\frac{1}{2}} Y^2 (24 - 15\eta^2 + 36Y^2) + 6Q^4_{\frac{1}{2}} (3Y^2 + 1 - \frac{\eta^2}{4}) + 2.25\, Q^6_{\frac{1}{2}} \}$$

As with the roots of N, equation (6.1,5) is presented in the form such that quick convergence of its roots are possible.Although the three simultaneous expressions are still very complex, the coefficients, $Q_{\frac{1}{2}}$, Q_1, and N, can be readily determined by the following procedure.

6.1.(ii). Iteration procedure for solving the three polynomial algebraic expressions.

It is readily seen from the three simultaneous equations (6.1,3) to (6.1,5) that there are four resulting unknowns, namely, η, $Q_{\frac{1}{2}}$, N and Q_1. With the equations as presented in the above forms, a suitable method for solving the variables is developed and is shown in Figs. (6.1-1) and (6.1-2).

Because of the three expressions for Q_3, N and Q_1 being polynomial algebraic equations, the procedure of calculating all the roots of one equation at a time will introduce considerable difficulty at a later stage. For an example, in determining Q_3 from the expansion of equation (6.1,3), the prescribing of a frequency value together with the initial approximates of N and Q_1 will give in certain regions two real roots as well as two complex values of Q_3. These roots when subsequently used to evaluate closer approximates of the other two variables will produce complication on solving the equation of Q_1. This expression is to the eighth power and the roots obtained will not necessarily be in the corresponding order of two real values and three pairs of complex results as the coefficients of equation (6.1,5) initially are determined from poor approximations of Q_3 and N.

It is much simpler to evaluate the results of solution (3.2,1) from the equations presented in the above forms. By fixing the value of Q_3 and designating initial approximates for N and Q_1, it will then always give for η one set of roots for which corresponding closer values of N and Q_1 are subsequently determined.

In the initial stage of the calculation when the amplitude of the Q_3 component is small, the magnitudes for N and Q_1 are easily estimated from equations (3.2,13) and (6.1,5) with $Q_3 = 0$. Then for a fixed value of Q_3, the frequencies are calculated and the results are used to determine closer approximates of the corresponding variables by means of the iteration method described in (46). With the aid of a high speed degital computer, the whole process is repeated until the roots of η, N and Q_1 satisfy the equations of (6.1,3), (6.1,4) and (6.1,5) simultaneously. For the next value of Q_3 set, the magnitudes of N and Q_1 from the preceding step are taken as initial values and the whole process is repeated.

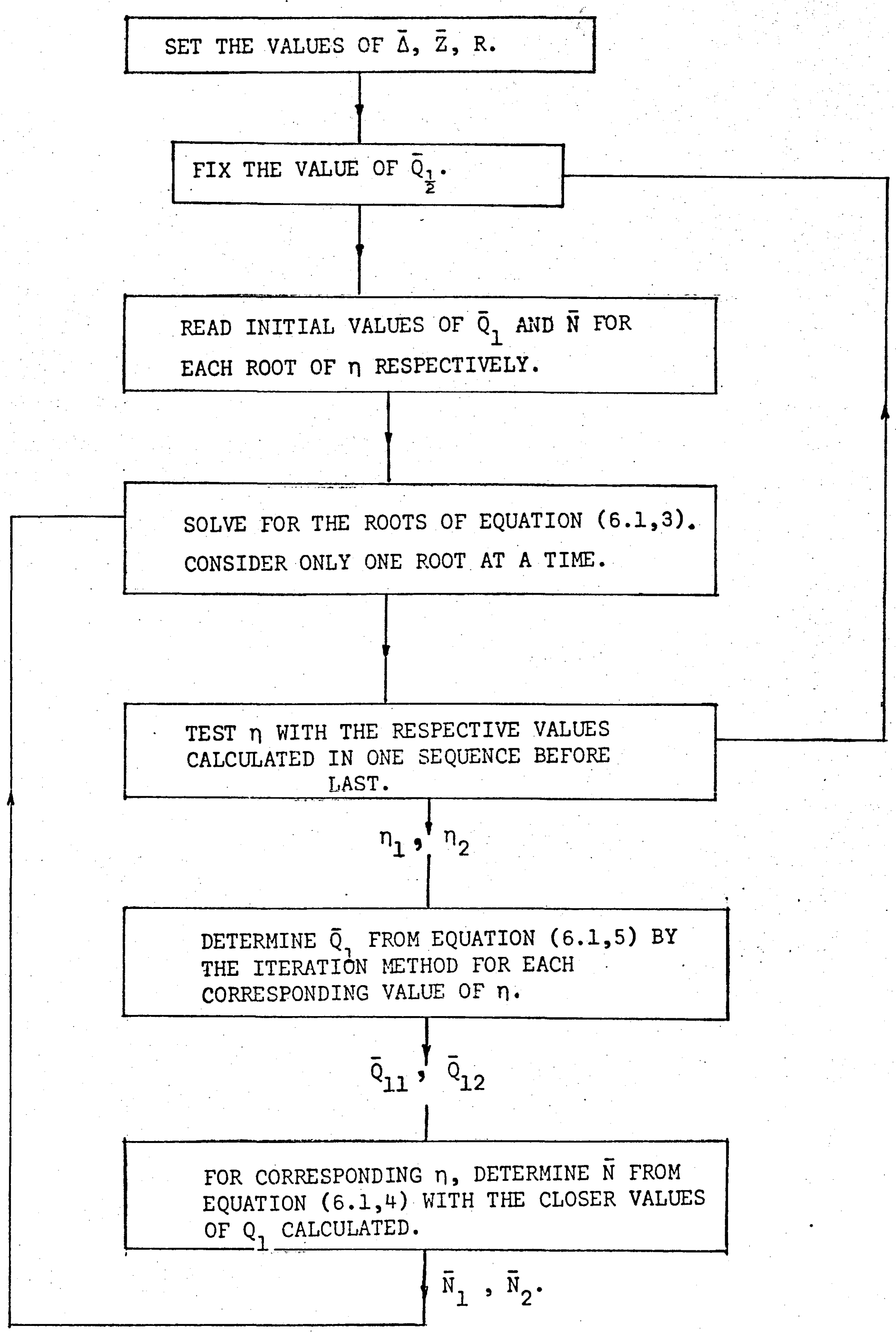


FIG.(6.1-1). GENERAL FLOW-DIAGRAM FOR DETERMINING THE NON-LINEAR ALGEBRAIC EQUATIONS.

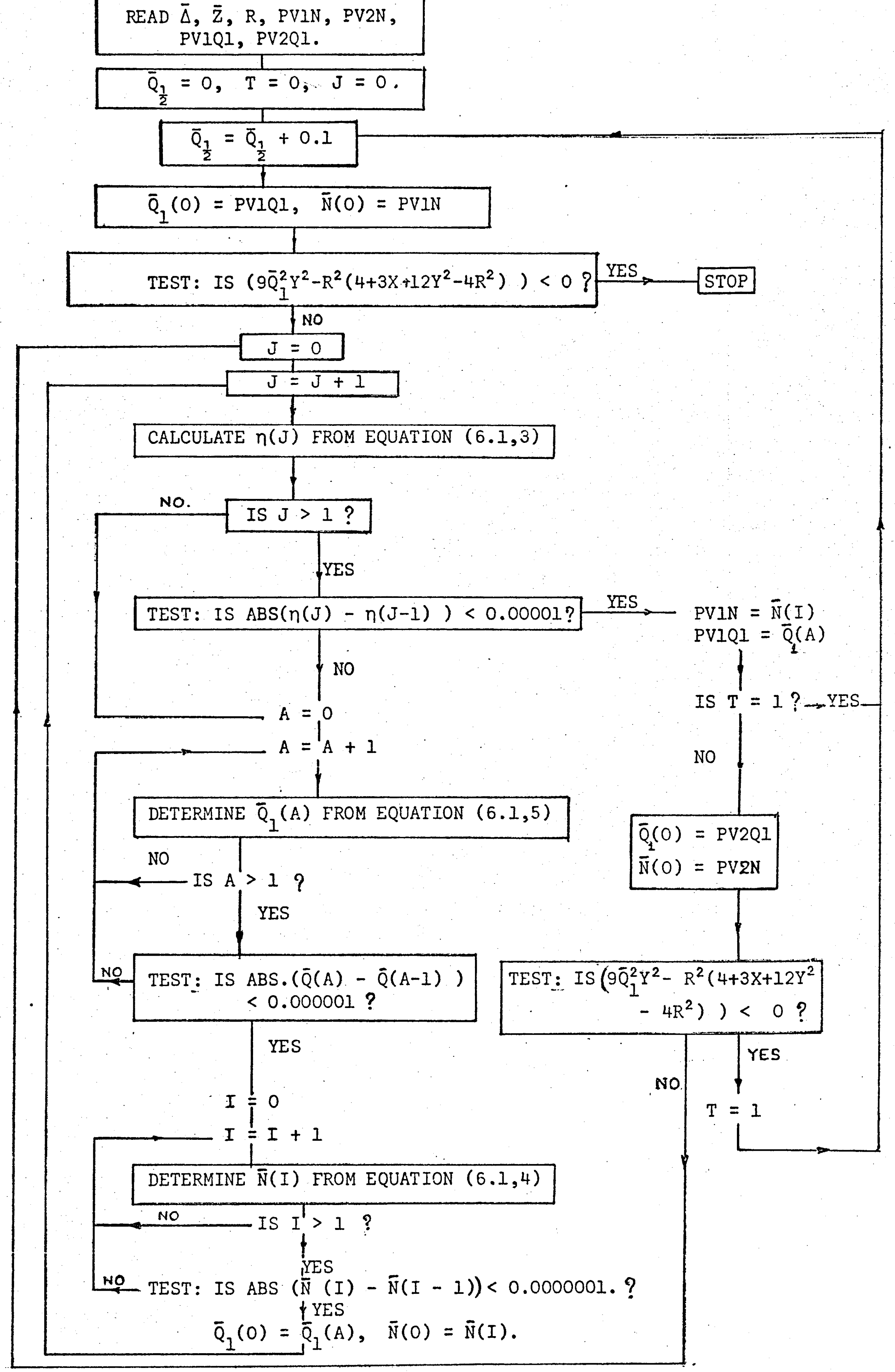


FIG.(6.1-2). FLOW DIAGRAM FOR DETERMINING THE SUBHARMONICS.

A more complete procedure is described in the flow diagram, Fig. (6.1-2) that follows.

6.2. The effects of damping

The influence of gravitational force on the system defined by equation (3.1,7) is readily seen from Figs.(6.3-1), (6.3-4) and (6.3-5). Its effects are significant and cannot be disregarded especially when there is damping, which would be the case for most realizeable physical systems. It is evident the frequency becomes a function of static deflection whatever the magnitudes of the vibrating displacement. Otherwise during large amplitudes the forcing frequency is practically independent of gravity effect as illustrated in Fig. (6.3-4) where all the curves virtually coalesce. It will be more apparent in section (6.3) that with the presence of damping the effective non-linearity is governed principally by the amplitude of $\bar{A}$ and the magnitude of the disturbing force. Because of the gravity effects on the equilibrium of motion the response frequency and the subharmonic amplitudes are determined by the relationship between these two parameters, however small the value of $\bar{A}$ is in relation to the other parameter.

From previous consideration of chapter V it is shown within the accuracy of the analogue computer solutions that only the first three components are measureable. This indicates the smallness of the other subharmonics in the presence of the positive coefficient of the first differential. Since the effective term of non-linearity is $3A\mu x^2$, the higher order of subharmonic vibration cannot be generated if the damping rate of the dashpot is fixed at the critical value in which real root of the predominant subharmonic component cannot exist. Hence, the accuracy of the approximate five term solution of the equation (3.2,1) will be sufficient under the existing condition for the investigation.

6.2.(i). The non-validity of the periodic solution

It is readily seen during the consideration of chapter IV that there is more than one possible periodic solution to equation (3.1,7). The presence of damping in the forced response introduces a real possibility that equation (3.2,1) over certain domains of the subharmonics is not valid. The solution will give complex roots through the phase variation of the $Q_{1/3}$ amplitudes and it is readily shown by the form of equation (6.1,3). The actual cause of the non-existence of the solution in the region is due to an unstable build-up in one of the harmonic oscillation of the motion. Thus, if the expression for the $Q_{1/3}$ component gives imaginary values, it is obvious the other two simultaneous equations (6.1,4) and (6.1,5) will immediately cease to be viable.

As the condition of stability for the system is defined by the equation (4.2,11), an approximate expansion of the determinant, $\Delta_{det.}(R)$, will provide an indication to the unstable region of the harmonic of the subharmonic vibration. Then, if the unstable build-up oscillation does not occur in the $Q_{1/3}$ harmonic, it will imply that for the particular frequency, equations (6.1,3) and (6.1,4) which define the $Q_{1/3}$ intrinsic behaviour can be satisfied in the unstable frequency although equation (3.2,1) is not valid.

In determining the set of roots, $Q_{1/3}$, N and Q_1, that will satisfy equations (6.1,3) and (6.1,4) a simple numerical procedure is used. It merely consists of calculating the values of the expressions to each instant of altering in a sequence the three variables, whereby Q_1 is repeated for all values of $Q_{1/3}$ as when the latter is fixed, Q_1 is varied for various magnitudes of N. The remainder values for a particular frequency are then plotted with respect to one of the coefficients of the solution. At the instant where the two curves intersect at zero datum, the coefficients, $Q_{1/3}$, N and Q_1, satisfy two of the three equations.

The sets of roots shown in Fig. (6.2-1) are from a random selection of independent parameters. Their substitution into the three simultaneous algebraic expressions show explicitly the non-existence of the solution (3.2,1) in this domain of second order subharmonics. Although Q_5, N and Q_1 satisfy the equations (6.1,3) and (6.1,4), these values when used in equation (6.1,5) exhibit the build-up of an unstable harmonic. Since the component is in the region of the fundamental harmonic of the subharmonic vibration, the region of instability is of the second order. This is evident from equation (4.1,5). It has the same frequency as the subharmonic vibration and consequently the second order phenomena is excited with negative damping. Thus it is readily seen that over certain region, the subharmonic resonance cannot be sustained in the steady-state is because of the accumulative effect of an oscillation in which the principal frequency is half that of the disturbing force.

6.2.(ii). Numerical analysis of instability

From having determined the coefficients of the solution in equations (3.2,13) to (3.2,17), the parameters of C_{nc} and C_{ns} can readily be calculated. The stability limits for the various independent parameters of the system can be obtained through the consideration of section (4.2) in which equation (4.2,12) is derived. The fact that the physical system has a periodic solution other than equation (3.2,1) over the instability is shown by the expansion of this determinant.

Since it is apparent that the build up of the harmonic is in the second order region where its oscillation has the same frequency as the subharmonics, it will be sufficeit to approximate the equation (4.2,2) to the first three variables. The expansion of the resulting determinant then gives after simplification for the criterion of stability,

	$\bar{Q}_{\frac{1}{2}}$	η	$\bar{N}$	$\bar{Q}_1$	EQUATION (6.1,3)	EQUATION (6.1,4)	EQUATION (6.1,5)
(a)	0.4	3.028	- 0.1699	0.6138	0.0018	0.0001	- 8.875
	1.0	3.259	- 0.3054	0.4956	0.0015	- 0.0002	- 8.196
	1.607	3.96	- 0.68	0.622	0.005	- 0.018	-15.06
	1.655	4.0	- 0.674	0.62	- 0.005	- 0.004	-15.28
(b)	1.136	3.4	- 0.469	0.54	- 0.003	- 0.004	- 9.34
	1.248	3.44	- 0.472	0.541	- 0.016	- 0.003	- 9.54
(c)	0.5	2.536	- 0.156	0.562	0.0002	0.0015	- 5.67

TABLE.(6.2-1). NUMERICAL VALUES IN THE UNSTABLE REGION.

(a) FOR $\bar{\Delta} = 1.0$, $\bar{Z} = 0.4$, $R = 0.15$.

(b) FOR $\bar{\Delta} = 0.8$, $\bar{Z} = 0.4$, $R = 0.15$

(c) FOR $\bar{\Delta} = 0.6$, $\bar{Z} = 0.4$, $R = 0.15$.

$$\Delta_{det.}(R) = (R^2 + C_o) \{ (R^2 + C_o - \frac{\eta^2}{4})^2 + R^2\eta^2 - C_2^2 \}$$

$$- 2C_1^2 (R^2 + C_o - \frac{\eta^2}{4}) + 2C_{2c} (C_{1c}^2 - C_{1s}^2) + 4C_{1c}C_{1s}C_{2s} > 0 \qquad \ldots\ldots \quad (6.2,1)$$

As the above criterion accounts for the stability condition in the region of the second order subharmonic vibration, then any unstable oscillation in those frequencies will not satisfy the above expression. The equation also readily provides an effective way of determining whether the coefficient values of the approximate solution for a particular subharmonic frequency are real roots. If the roots that are determined in the preceding section (6.2.(i)) are substituted into equation (6.2,1) the resulting value should be less than zero.

Using the criterion of equation (6.2,1) as to the validity of the solution (3.2,1), the following sample calculation is shown to test the stability condition for a particular set of parameters. From Fig. (6.3-1) for the parameters $\bar{A} = 1.0$, $\bar{Z} = 0.4$, $R = 0.15$, $\eta = 3.8$ and for the response of $Q_{\frac{1}{2}} = 1.38$ gives

$$(4.813)(1.203^2 + 0.325 - 3.132) - 10.28\,(1.203) + 1.574\,(4.74)$$
$$+ 1.788\,(2.22)\,1.61 < 0 \ .$$

Since the resulting value is negative, the subharmonic vibration, as expected, cannot exist. The continuation of the vibration is destroyed by the accumulative effects in the fundamental harmonic of the subharmonic motion. Hence the subharmonic solution does not actually exist.

For completeness, the criterion is applied to the calculated response of $Q_{\frac{1}{2}} = 0.8$ for the parameters $\bar{A} = 1.0$, $\bar{Z} = 0.4$, $R = 0.25$ and $\eta = 3.31$.

This gives

$$(4.02)\,(1.28^2 + 0.687 - 0.789) - 11.5(1.28) + 1.34(2.79) + 4.876(2.066)\,(0.584) > 0.$$

A steady-state periodic solution is shown to exist for the corresponding results of the above subharmonic response. Admittedly, the equation is not sufficiently precise to pin-point the limits of instability because of the laboriousness of expanding a five by five determinant. But for most practical purposes the above criterion allows the region of instability and the existence of solution to a centrifugally excited system to be satisfactorily ascertained without much difficulty.

6.3. Discussion of results

In the theoretical analysis of chapter III, it is readily seen that provided the respective approximate solutions of the two analytical methods satisfy the boundary requirements and are consistent with the physical restraints of the system, the two methods are equivalent. The resulting corresponding polynomial expressions are identical when the solutions contain the same number of harmonic terms. As observed from Figs. (6.3-1) and (6.3-2) the methods are neither restricted to small magnitudes of non-linearity nor to small values of the other independent parameters of the system for convergence of the solution. From the several approximate methods in existence the Ritz-Galerkin and the harmonic balance methods are thus the most appropriate in application to this investigation. On the basis of comparison with the experimental results of the analogue, the theoretical results are satisfactory even for cases of severe gravitational influence on the restoring force.

The time-wave traces produced from the analogue computations depict the actual motion of the non-linear system that is described by equation (3.1.8). Although, as already discussed, the analogue computing technique is not a straight forward procedure the possible errors are likely prevented or minimised. The computation error on a percentage basis as a whole is estimated to be lower then three per cent.

Since the dependent variables and the frequency are calculated for selected subharmonic amplitudes of $Q_{\frac{1}{3}}$, it is then appropriate to define the accuracy of the results from either one of the terms. However, as the variation in the behaviour of the coefficients of N and of Q_1 is more perplex, it is preferable that the comparison of the results is made with the corresponding frequency. In general, the worst frequency error between the theoretical and experimental results of $Q_{\frac{1}{3}}$ is estimated to be not greater than five per cent. The analogue computer solutions practically coincide with the theoretical curves. The error might probably be higher in the early stages of the subharmonic response but for most applications the accuracy should be satisfactory.

It is inevitable that with the amplitudes of the fundamental harmonic smaller, the error would be increased. It is introduced largely through the measurements of the traces. The discrepancies between the experimental and the theoretical results are tolerable for the Fharol chart only measures up to the second decimal place.

Since the magnitude of the third order component is very small it cannot be measured effectively. The error can be as high as 200 per cent, although in the resonance state the component might become sufficiently pronounced to be measured. The frequency error would still be as large as fifty per cent; giving an indication of the smallness of the subharmonic components of the third and higher orders.

It is readily seen that the response characteristics within the frequency ranges are functions of the parameters governing the physical system. Effects from the variation in the physical mass, or from the degree of asymmetry, on the subharmonic vibration are ably illustrated in the behaviour of the component $Q_{\frac{1}{2}}$. This component predominates over higher odd and even subharmonics that exist in the vibratory motion, and that its frequency range is increased with the magnitude of the non-linearity terms. The vibration response is also much heavier over the greater band of the working speed. Consequently, although the third order subharmonics would exist it is obvious that $Q_{\frac{1}{2}}$ is far the more important component of the subharmonic motion under the influence of gravitational forces.

The vibration has general similarity with the harmonic resonance but in other aspects the subharmonic resonance is much stronger. It is readily evident that by making $R = 0$ in equations (6.1,3) and (6.1,5) the harmonic resonance occurs over a smaller band-width.

The significant influence of the effects of static deflection on the subharmonic vibration is readily seen from Figs. (6.3-1) and (6.3-4). With the other two governing parameters of the system fixed and the remaining variables in the neighbourhood, from which the subharmonic component would start, being practically indifferent to the changes as shown by Figs.(6.3-6), (6.3-9), (6.3-11) and (6.3-14), the variation in the magnitudes of $\bar{A}$ must hence be the principal influence as to the position of the frequency. The effects on the resonance amplitudes are more clearly described by the $Q_{\frac{1}{2}}$ response illustrated in Fig.(6.3-1). The severe increase in the amplitudes of $Q_{\frac{1}{2}}$ corresponding to the change in the degree of asymmetry demonstrates its significance, and that the effective term of the non-linearity is $3A\mu x^2$. It is apparent that a limiting relationship governing the subharmonic vibration exists between $\bar{A}$ and the other describing parameters, and as the effective degree of the non-linearity is the second order then subharmonic is not likely to be excited if the state of the physical system lies outside the region favourable to generating the motion. Since it is essential to avoid introducing the phenomena or working within the frequencies, it will be useful knowledge to have in the design stages if the critical expressions in terms of these independent variables are known. This is further examined at a later stage and the limiting equations for such purposes are derived.

Fig.(6.3-3) illustrates the vibration response for increasing values of viscous damping. Because of the severness of the non-linearity it is necessary that the damping is sufficiently large to suppress the amplitudes. With such magnitudes, the harmonic resonance is practically linearised whilst the subharmonic vibration still has displacement more than twice the harmonic motion. This is readily seen by the vectorial addition of the coefficients as shown in Fig.(5.4-1). It will indeed seem uneconomical to extinct the second order subharmonic solely interms of viscous damping. Besides the magnitude of the viscous force is proportional to the linear stiffness rate that in not permitted to be low if the coefficient of non-linearity, governed from the change in the gradient of the restoring stiffness, is to be kept to a minimum. However, the effective non-linearity of the system can be reduced through $\bar{A}$, since in the early stages of $Q_{\frac{1}{2}}$ the constant N which accounts for the influence of gravitational force is mainly dependent upon the magnitude of $\bar{Z}$.

as seen in Fig.(6.3-7). Hence it is obviously less expensive as the subharmonic vibration is not significantly sensitive to damping if limiting relationships are derived to allow for any permutation of the values between the parameters.

Fig.(6.3-4) shows that, in the absence of damping, the influence of gravitational force is progressively reduced until at very large amplitudes the cubic term, μx^3, is the effective non-linearity. In this region the curves virtually coalesce for all values of $\bar{A}$ and the frequency is independent of the parameter. The weight being supported at the base is proportionately reduced as the vibration increases and the curve of the restoring force approaches asymptotically the position of symmetrical characteristics. However, such a situation is unlikely to arise in practise because of the necessity for the presence of damping.

The increasingly heavy subharmonic vibration for corresponding values of $\bar{Z}$ is characterised by the behaviour of $Q_{\frac{1}{3}}$ in Fig.(6.3-2). Theoretical and experimental results are also shown again for the case of $\bar{A}$ = 1.0 in which the effects of non-linearity are strongest. The general effects of increasing the disturbing force are the widening of the two branches of the response and of the frequency range of resonance.

The vibratory motions for the various magnitudes of $\bar{Z}$ are observed not necessarily to have the 'family' of curves characteristics. Such is the case with the state of the response in the region of harmonic resonance. Because of the magnitude of the disturbing force, the harmonic component Q_1 is practically indifferent to the frequency changes over the lower branch of $\phi_{\frac{1}{3}}$ as illustrated in Fig.(6.3-12). With the mean dynamic displacements also proportionately increased, the values of the expression on the right of the minus sign of equation (6.1,3) as a result are lower. Hence the vibration response of the lower phase of the resonance occurs at higher frequencies. The upper sign of the equation corresponds to the other branch of the Q_1 response. With smaller amplitudes of $\bar{Z}$ the resonance condition is experienced more suddenly during the early stages.

It is apparent then from Fig.(6.3-2) and from the discussion of an earlier paragraph that the strength of the displacements and the resonance frequencies are governed mainly by $\hat{A}$ and $\bar{Z}$ whilst R determines the finite limits to the vibration amplitudes. The heavier viscous damping although virtually linearise the harmonic resonance, the subharmonic vibration generated under the same condition of static deflection results in amplitudes twice as severe.

Because of the influence of gravitational force on the system there is a shift in the mean dynamic displacement from the position of static equilibrium. The position of dynamic equilibrium does not coincide with either the origin of symmetrical restoring force characteristics or the static equilibrium position. The displacement as will be seen is not negligible and to allow for it theoretically, the constant term, N, is included in the approximate solution. The significant variations in the characteristics of the mean dynamic deflection with frequency during the vibratory motion are shown in Figs.(6.3-6) to (6.3-10).

From Fig.(6.3-7) it is readily observed in the neighbourhood from which the subharmonic vibration will start, that the mean dynamic displacement is increased whenever the system is subjected to a larger disturbing force. Since the values of N are always negative, the direction of the mean of the vibratory motion is in opposite sense to the static deflection and hence it is towards the point of symmetrical restoring forces. Thus, as the disturbing forces become larger the proportion of the dead weight transmitted to the supports prior to the subharmonic vibration being excited will diminish. This means that the magnitude of the effective non-linearity is correspondingly reduced before the phenomenon is generated, and that the effective non-linearity is strongest when $\bar{A}$ is large and the system is excited by small amplitude of disturbing forces. Furthermore, in the preceding paragraph, Fig.(6.3-2) illustrates that extinction of the vibration can be achieved through lowering the magnitude of the disturbing force. Hence, for fixed values of $\bar{A}$ and R, there must be two probable limits of $\bar{Z}$ for which no subharmonic motion can exist. It would facilitate in the design stages to avoid experiencing such vibrational phenomena if an equation, defining the boundary locus in terms of the known parameters of a system, is derived.

Since in general there are two limiting values of $\bar{Z}$, the gradient of the locus defining the boundary will have at that instant of fixed values of $\bar{A}$ and R both positive and negative values. It can be readily seen from the corresponding graphs for the two physical conditions that where the gradient is positive on reducing the magnitude of the disturbing forces, the mean of the vibratory motion moves away from the position of static equilibrium as the vibration increases. This means that the effective asymmetry of the restoring forces is progressively reduced and it is a minimum in the region of maximum vibration. Consequently, although the influence of gravitational force is most prominent during the early stages of the resonance response the degree of $\bar{A}$ determines the magnitude of the response. The amount of weight resting on the supports eventually is reduced by more than fifty per cent, and the percentage is increased with the magnitude of either $\tilde{A}$ or $\tilde{Z}$.

For changes of the physical conditions in which the gradient of the locus is negative, the effective non-linearity of the system is found to have the opposite characteristics. The weight transmitted to the supports is increased with the vibration and consequently the magnitude of the restoring forces is raised proportionately over the resonance region. This is clearly illustrated in Fig.(6.3-5) and (6.3-10). The system can vibrate to considerable amplitudes and over a large frequency range. The resonance will exist until the energy supplied to sustain the motion is insufficient that is when

$$R^2(4 + 3X + 12Y^2 - 4R^2) > 9Q_1^2Y^2$$

Thus, unless the physical conditions that are defined by the governing parameters of the system lie outside the locus, the effects of gravitational force on the non-linear suspension cannot be neglected. The effective term of non-linearity is $3\tilde{A}\mu x^2$, however small $\tilde{A}$ is in relation to the amplitude of excitation.

Fig.(6.3-8) shows the variation in the positions of the mean dynamic displacement with frequency for different values of viscous damping R. Although the heavy damping suppresses the amplitudes, it is readily observed that a considerable amount of the weight is still transfered to the supports, and the effective degree of non-linearity remains practically unchanged over the frequency band-width. It is then not surprising, with such magnitude of the restoring force and with the main component $Q_{\frac{1}{2}}$ absorbing proportionately less energy by having only half the frequency of the disturbing force, that the subharmonic vibration is twice as severe as the harmonic motion. If in practice the amplitude of $\bar{\bar{Z}}$ is larger, the position of the mean dynamic displacement is towards that of static equilibrium and such characteristic effectively increases the magnitude of non-linearity with the increase of resonance. Clearly, when the system is under the influence of gravitational effects, the resonance that is generated within the subharmonic frequencies is not significantly affected by damping and it is uneconomical to extinct the vibration by this means alone.

Figs.(6.3-11) to (6.3-15) illustrate the characteristics of the harmonic component in the vibratory motion. In general, the existence of $Q_{1/2}$ will not significantly alter the response of Q_1. This can be observed from the comparison of solutions where the curves of the harmonic resonance are also plotted for the various physical conditions. The increase in the main component of motion does not necessarily mean proportionate rise in Q_1 unless the increase in vibration is maintained through greater weight transfered to the supports. The coefficient Q_1 is the component by which energy is supplied to the system. The values are correspondingly increased with $\bar{\bar{Z}}$, as shown in Fig.(6.3-12), for it is the effective amplitude of the disturbing force in the neighbourhood before the subharmonic vibration is generated. The insignificant change of Q_1 over the stable branches of the subharmonic response is due mainly to the magnitude of $\bar{\bar{Z}}$ relative to the damping present, since the component having a higher oscillation than $Q_{1/2}$ is more affected by the rate of energy dissipation. However, in the predominant region of $Q_{1/2}$ there is no significant difference in the values, whatever the degree of damping as can be readily observed from Fig.(6.3-13).

From the comparison of Figs.(6.3-6) and (6.3-10) it is evident, if the effective non-linearity is not reduced over the period of resonance, the amount of weight transmitted to the supports is through an increasing influence of the disturbing force. This is shown by the growth of the harmonic component in Fig.(6.3-15). With the restoring forces continually larger, the vibration becomes stronger with frequency, until the magnitude of $\bar{Z}$ is insufficient to supply the necessary energy for the existing subharmonic motion. The response of Q_1 under such circumstances differs from the values of the harmonic resonance.

However, since the behaviour of Q_1 is dependent upon the magnitude of the disturbing force, it will be an immense simplification of the calculation of $Q_{\frac{1}{2}}$ in the process to derive the equations for the critical relationships, if a reasonable approximation is made for the harmonic component of motion. Even when the system is subjected to the above physical conditions, the equation subsequently derived is not a poor inequality since for the limiting conditions to the critical relationship there is no practical difference in the values within the respective frequencies. Thus, with the vibration not stable during the resonance oscillation of this component, and as it is also shown that the subharmonics cannot actually exist along with the resonant condition, such an approximation appears quite legitimate. The possibility is further examined in the following section.

Figs.(6.3-16) to (6.3-19) show the phases of the main component of subharmonic motion with frequency for the two branches of the response. It is readily seen that the curves of $Q_{\frac{1}{2}}$ have values between 0 and $\frac{\pi}{2}$. The magnitude of the phase relationship is evident from Figs.(3.2-1)(a) and (b), for if damping is absent ϕ_1 will either be zero or π, and similarly with $(\phi_1 - 2\phi_{\frac{1}{2}})$. Since the possibility is that $(\phi_1 - 2\phi_{\frac{1}{2}})$ is either 0, or π, then for the condition $\phi_1 = \pi$ the corresponding phase of the two branches of $\phi_{\frac{1}{2}}$ will have values either $\frac{\pi}{2}$ or zero. This means the response of $Q_{\frac{1}{2}}$ is in the same phase quadrant as the disturbing force. However, from the consideration of section (3.4) the phase response depends upon the initial physical conditions as $Q_{\frac{1}{2}}$ can also be generated in the phase quadrant lagging by π radians.

The influence of gravitational force on the phase $\phi_{\frac{1}{2}}$ is readily explained by the curves in the above figures mentioned. For most cases the maximum amplitudes of vibration is reached for $\phi_{\frac{1}{2}}$ lower than $\frac{\pi}{6}$. If the effective non linearity relative to viscous damping is small, the gradient of the curves is always positive with increasing frequency. Otherwise, the negative gradients correspond to the regions of instability and vertical tangency characteristics exist at the positions in which the 'jump' phenomena occurs. For a fixed value of $\phi_{\frac{1}{2}}$, the frequency increases with $\bar{A}$, and with heavy damping the change in the signs of the gradient is eliminated as expected with instability. The relationship between $\phi_{\frac{1}{2}}$ and η is then basically similar to those of linear curves characteristics.

It is obvious from the harmonic resonance characteristic that in the subharmonic frequency region, the phase of the Q_1 component is in the second quadrant. As the absence of the component will eliminate the existence of $Q_{\frac{1}{2}}$, it is not surprising that the increase in the amplitudes of vibration lowers the phase lag, ϕ_1. The rate of the phase reduction with frequency is less pronounced if the effective non-linearity of the system becomes larger with the vibration. Otherwise a lower phase lag of the component through which energy is supplied to maintain the subharmonic vibration is necessary. In the region of maximum resonance, a vibratory state is reached where the out of phase between the component and excitation is lowest. This condition exists for whatever values of the parameters, hence irrespective of the physical condition of the system. This is readily evident from Figs.(6.3-20) to (6.3-23).

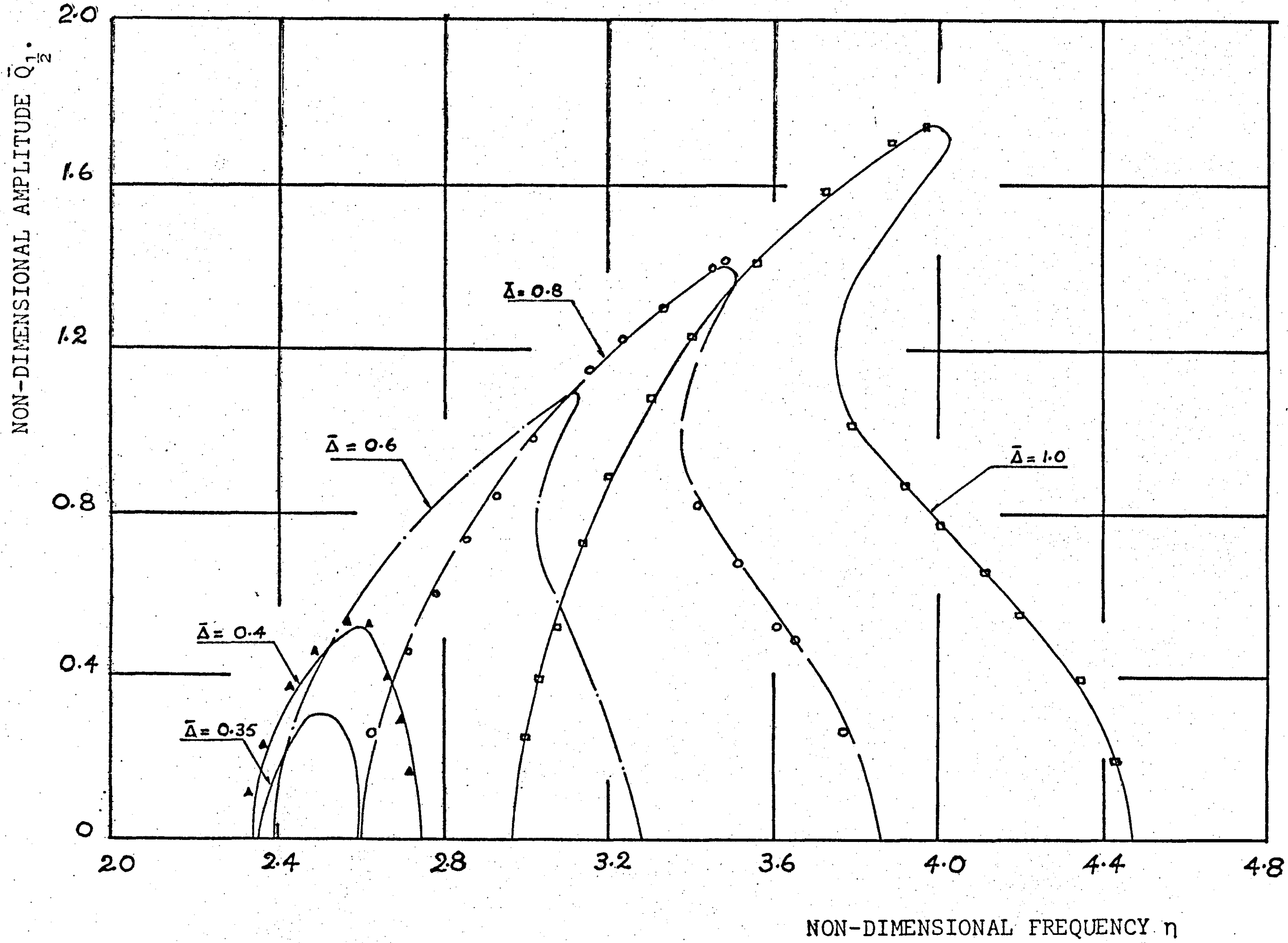


FIG.(6.3-1). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{Z} = 0.4$, $R = 0.15$.

— — — ——— THEORETICAL RESULTS

□ ○ ▲ EXPERIMENTAL RESULTS.

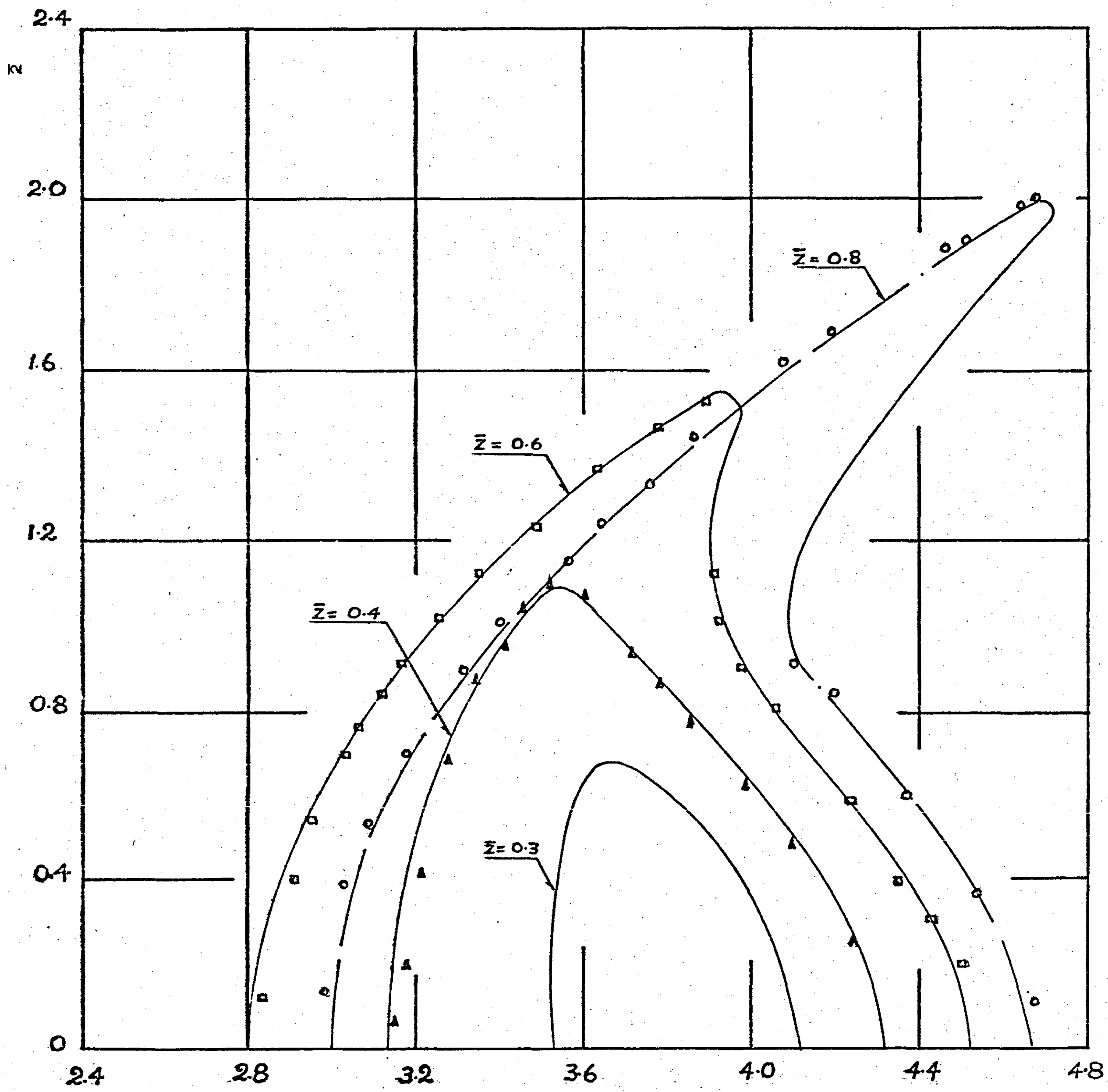


FIG.(6.3-2). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION

RESPONSE CURVES FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{\Delta} = 1.0$, $R = 0.25$

— — ——— THEORETICAL RESULTS.

□ ○ ▲ EXPERIMENTAL RESULTS.

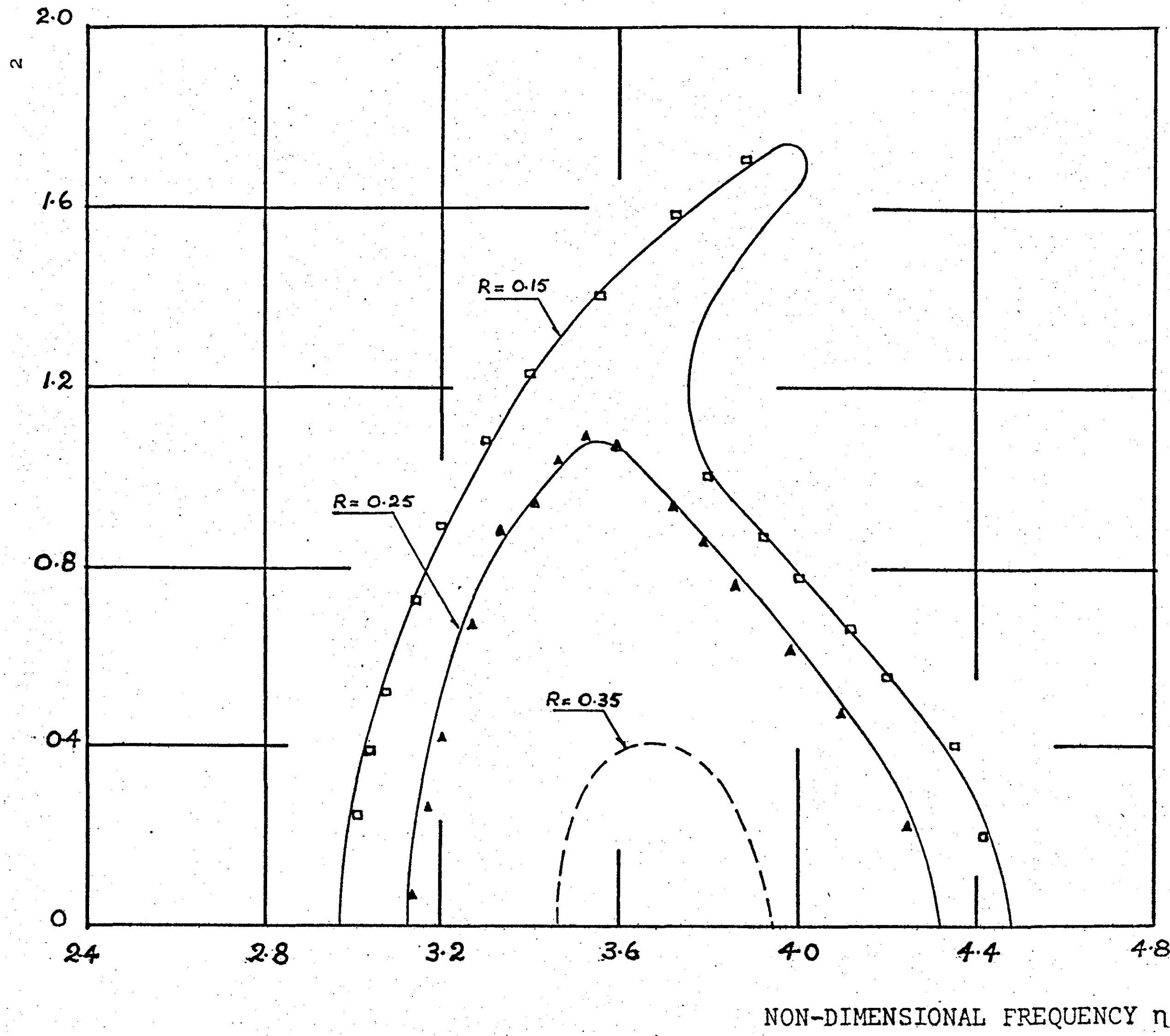


FIG.(6.3-3). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION

RESPONSE CURVES FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{\Delta} = 1.0$, $\bar{Z} = 0.4$

– – – ——— THEORETICAL RESULTS.

▫ ▲ EXPERIMENTAL RESULTS.

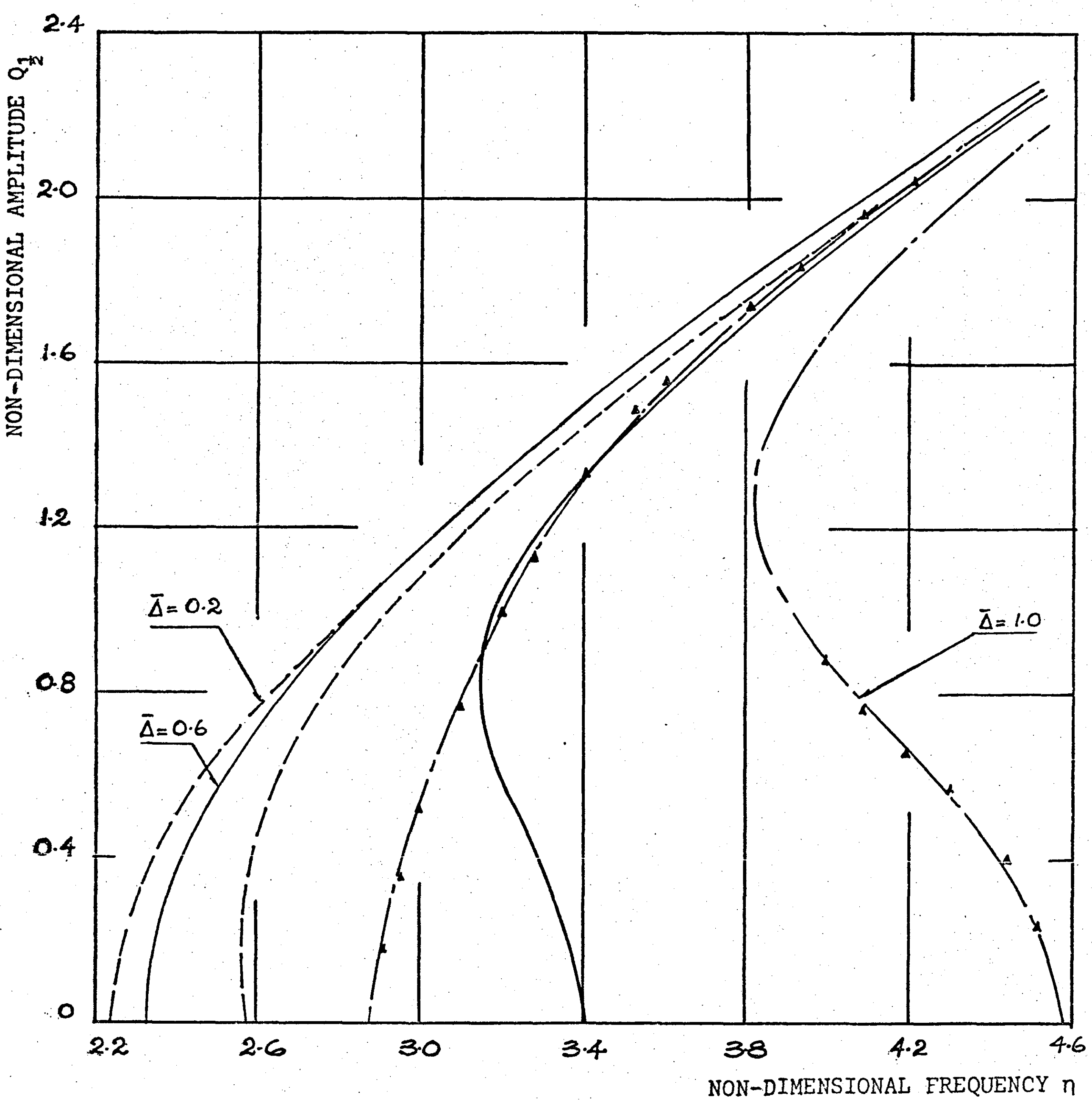


FIG.(6.3-4). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{Z} = 0.4$, $R = 0$.

--- ——— ——— THEORETICAL RESULTS.

▲ EXPERIMENTAL RESULTS.

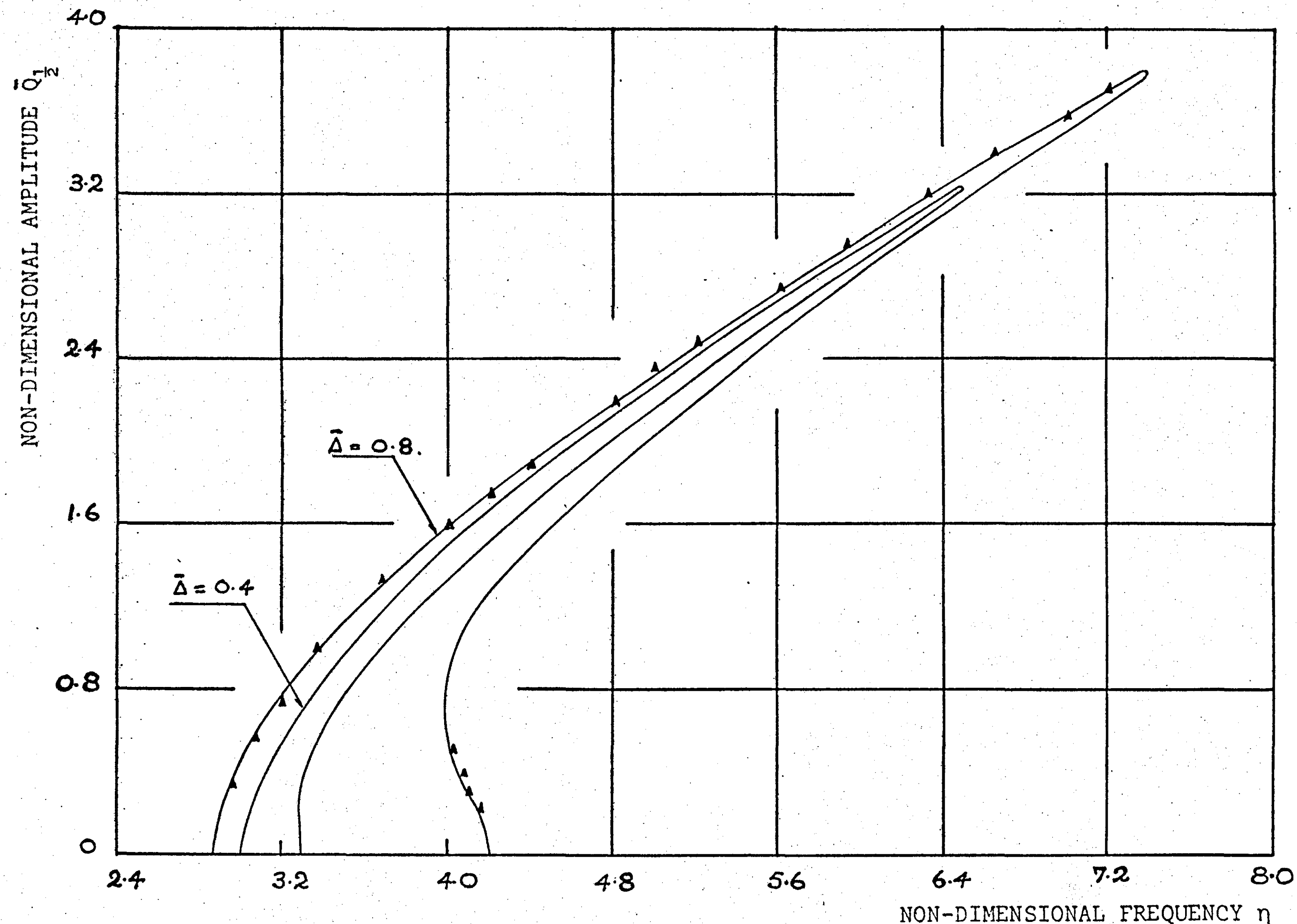


FIG.(6.3-5). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION

RESPONSE CURVES FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{Z} = 0.8$, $R = 0.15$.

——— THEORETICAL RESULTS.

▲ EXPERIMENTAL RESULTS.

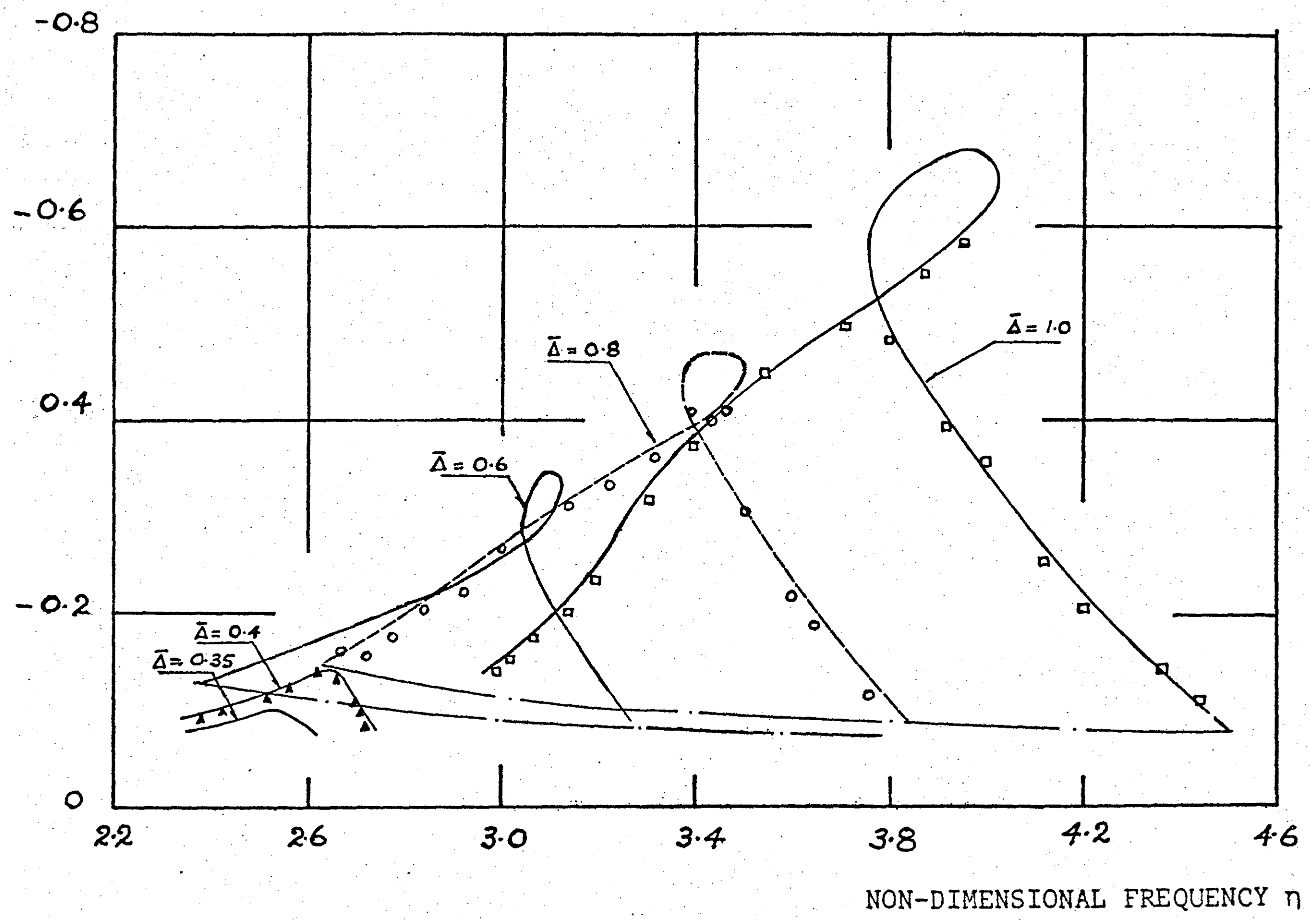


FIG.(6.3-6). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION

RESPONSE CURVES FOR $\bar{N}$. $\bar{Z}$ = 0.4, R = 0.15.

—————·————— HARMONIC RESONANCE.

- - - - ——— THEORETICAL RESULTS.

□ ○ ▲ EXPERIMENTAL RESULTS.

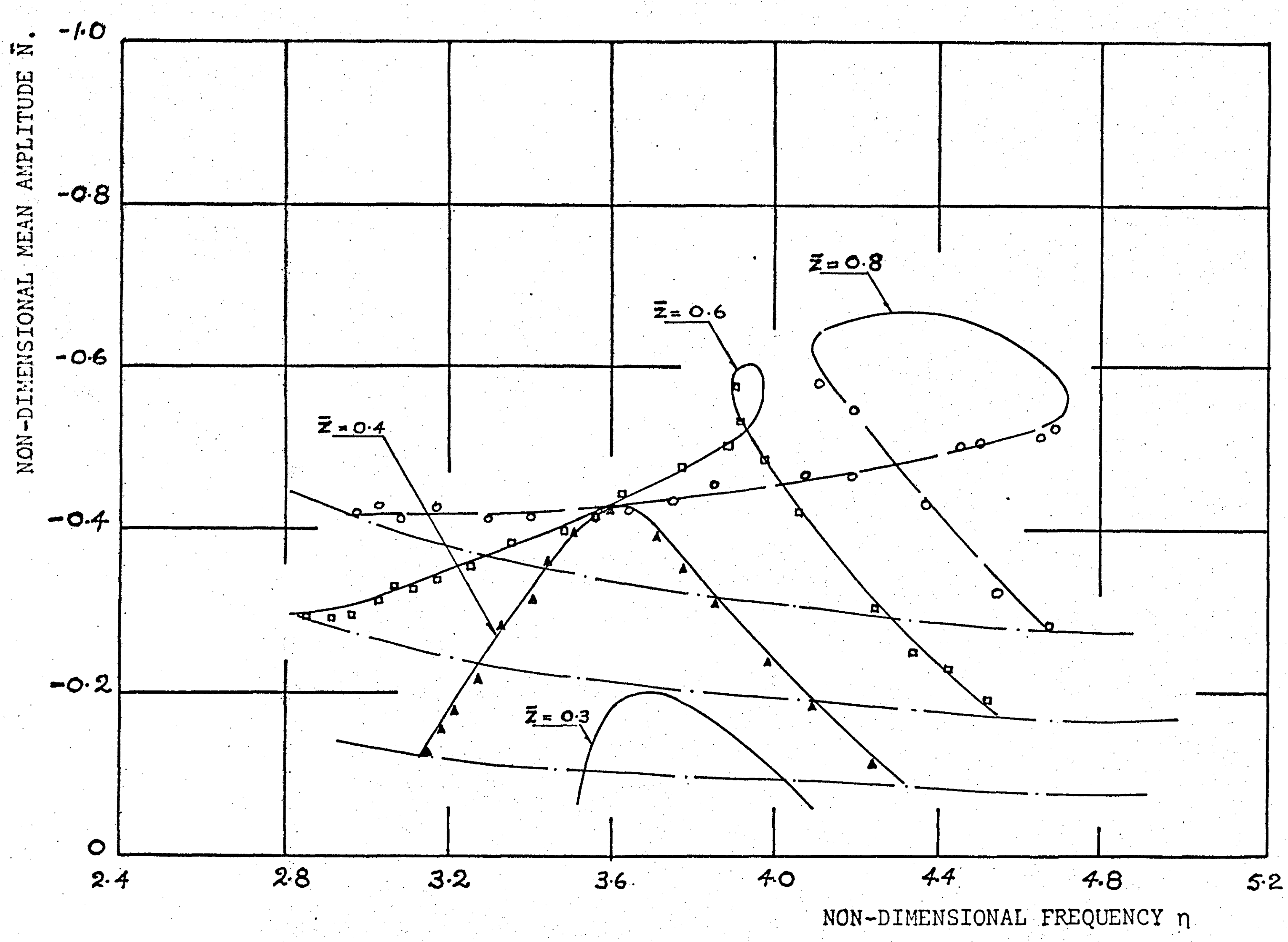


FIG.(6.3-7). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION

RESPONSE CURVES FOR $\bar{N}$. $\bar{\Delta} = 1.0$, $R = 0.25$

—·— HARMONIC RESONANCE.

— — — THEORETICAL RESULTS.

□ ○ ▲ EXPERIMENTAL RESULTS.

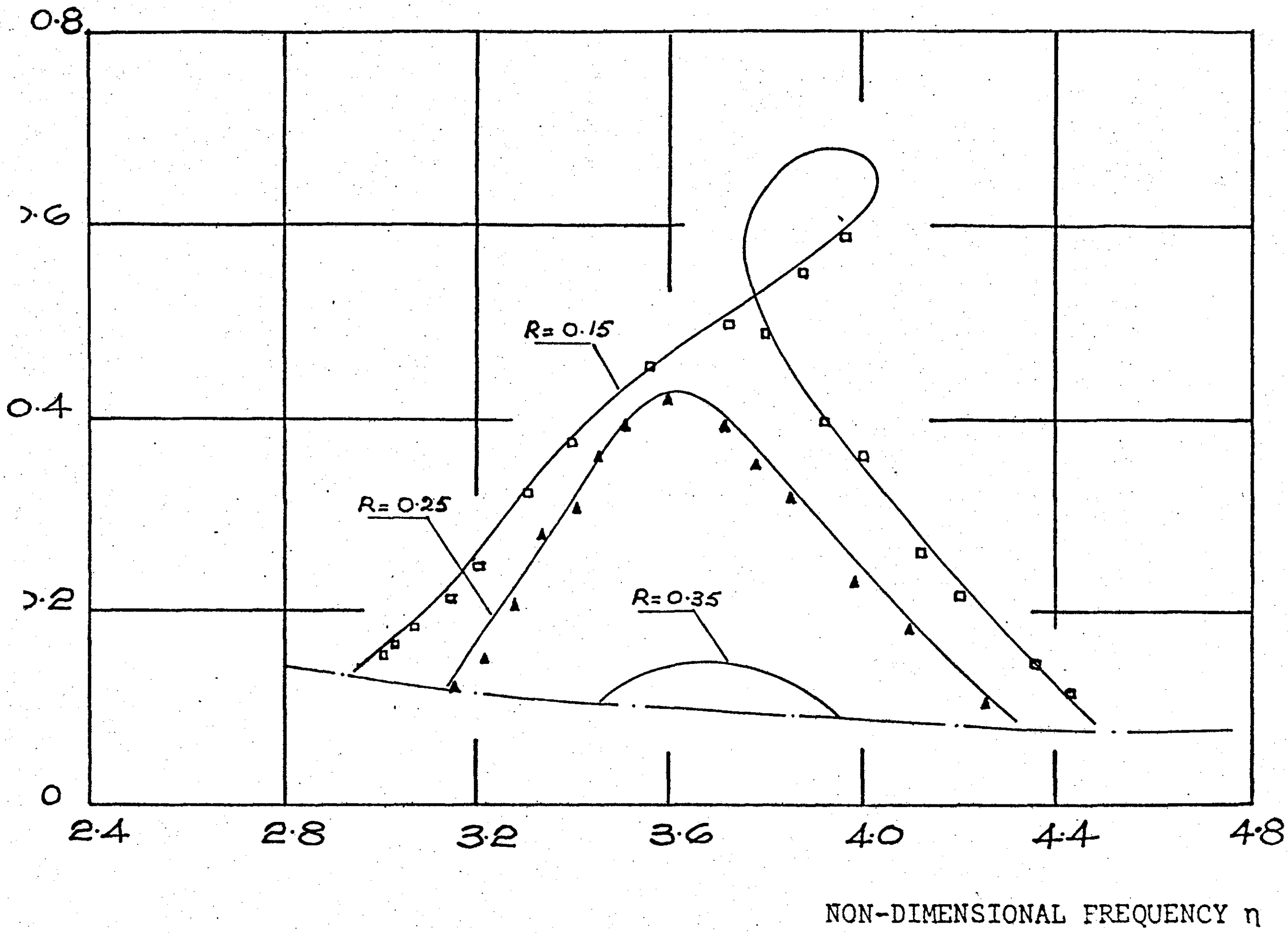


FIG.(6.3-8). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR $\bar{N}$. $\bar{\Delta} = 1.0$, $\bar{Z} = 0.4$.

—— · —— HARMONIC RESONANCE.

———— THEORETICAL RESULTS.

□ ▲ EXPERIMENTAL RESULTS.

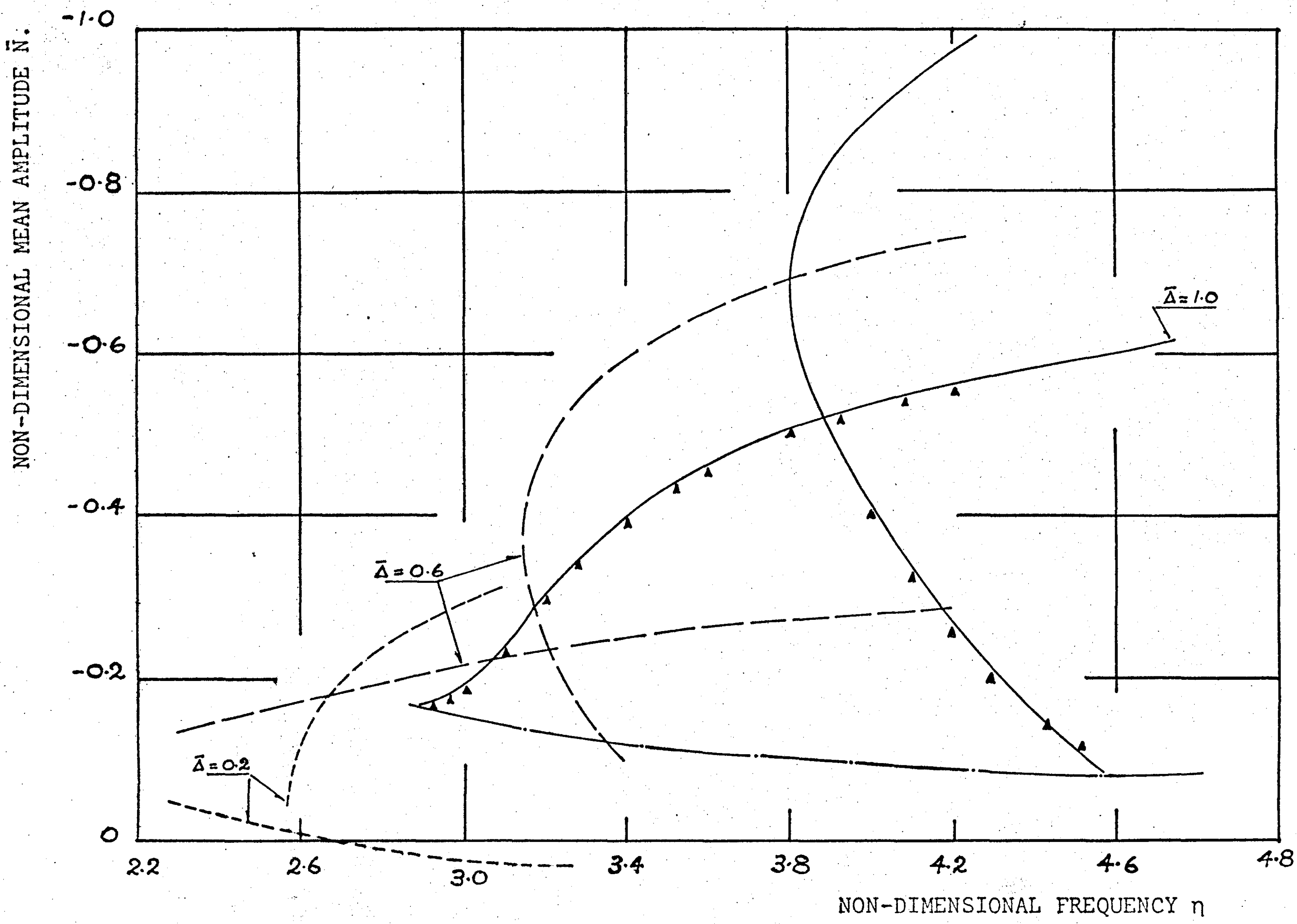


FIG.(6.3-9). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION

RESPONSE CURVES FOR $\bar{N}$. $\tilde{Z} = 0.4$, $R = 0$.

—— · —— HARMONIC RESONANCE.

- - - — — — —— THEORETICAL RESULTS.

▲ EXPERIMENTAL RESULTS.

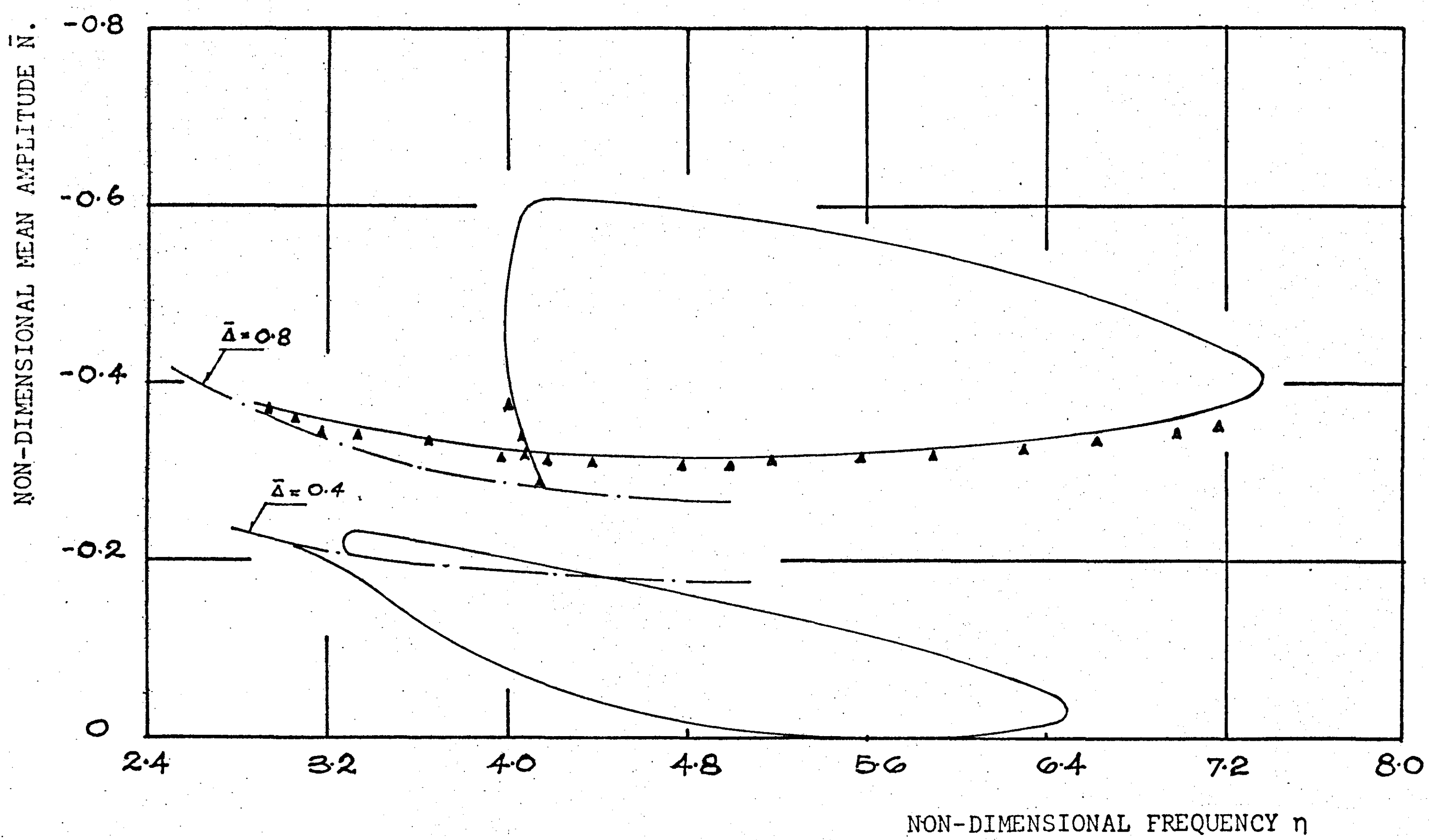


FIG.(6.3-10). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR $\bar{N}$. $\bar{Z}$ = 0.8, R = 0.15.

—·— HARMONIC RESONANCE.

—— THEORETICAL RESULTS.

▲ EXPERIMENTAL RESULTS.

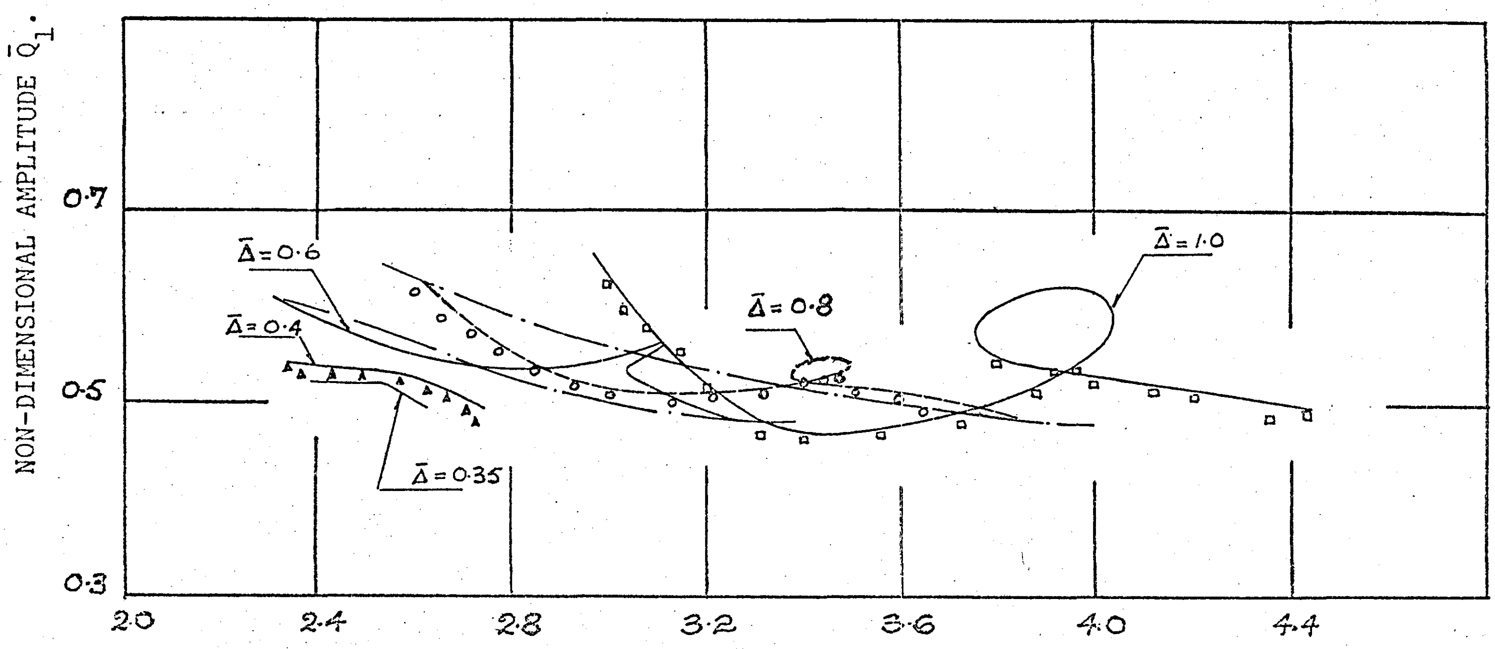


FIG.(6.3-11). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR $\bar{Q}_1$. $\bar{Z} = 0.4$, $R = 0.15$.

—·— HARMONIC RESONANCE.

– – – —— THEORETICAL RESULTS.

□ ○ △ EXPERIMENTAL RESULTS.

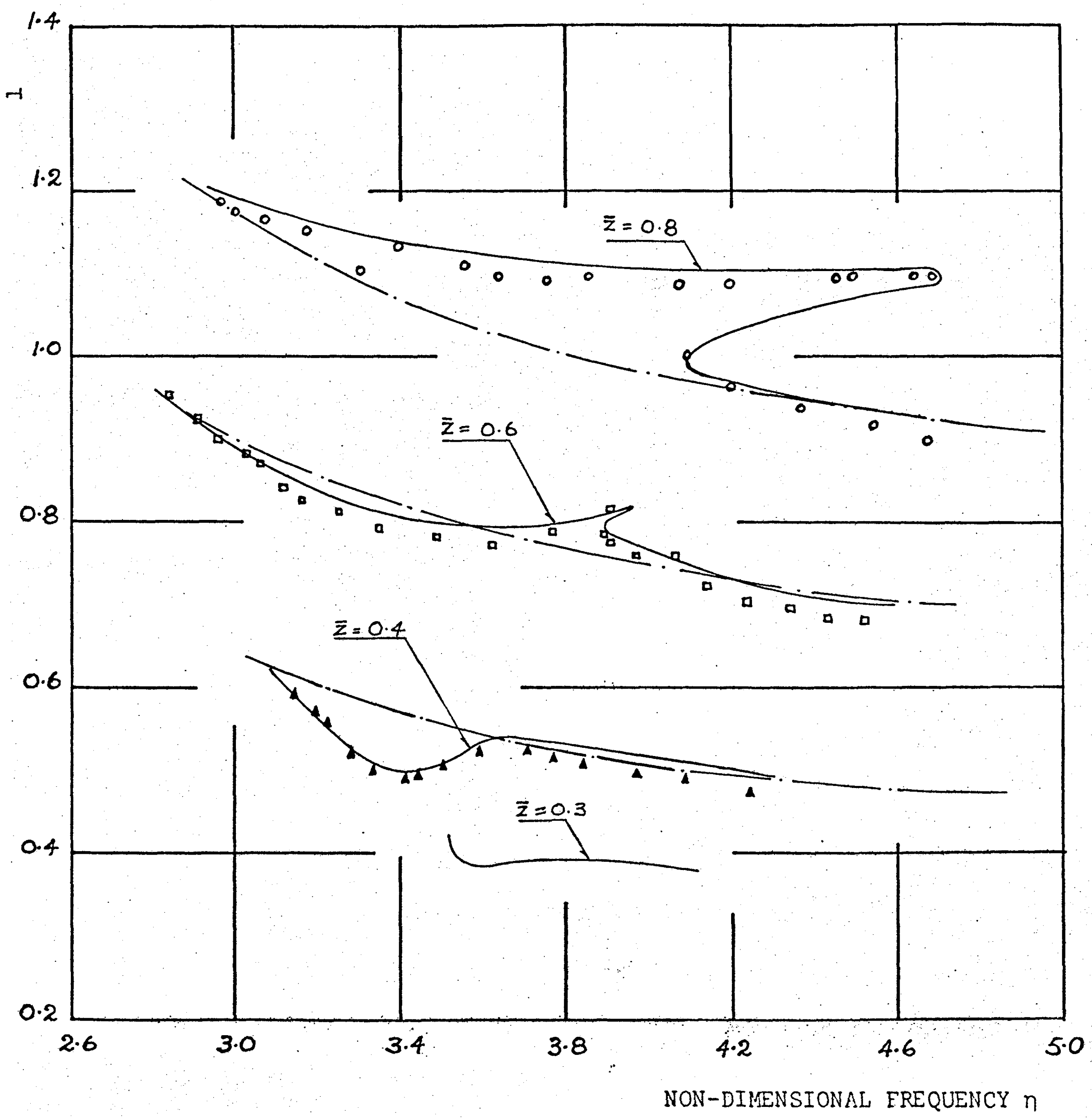


FIG.(6.3-12). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR $\bar{Q}_1$. $\bar{\Delta}$ = 1.0, R = 0.25.

——·—— HARMONIC RESONANCE.

———— THEORETICAL RESULTS.

□ ○ ▲ EXPERIMENTAL RESULTS.

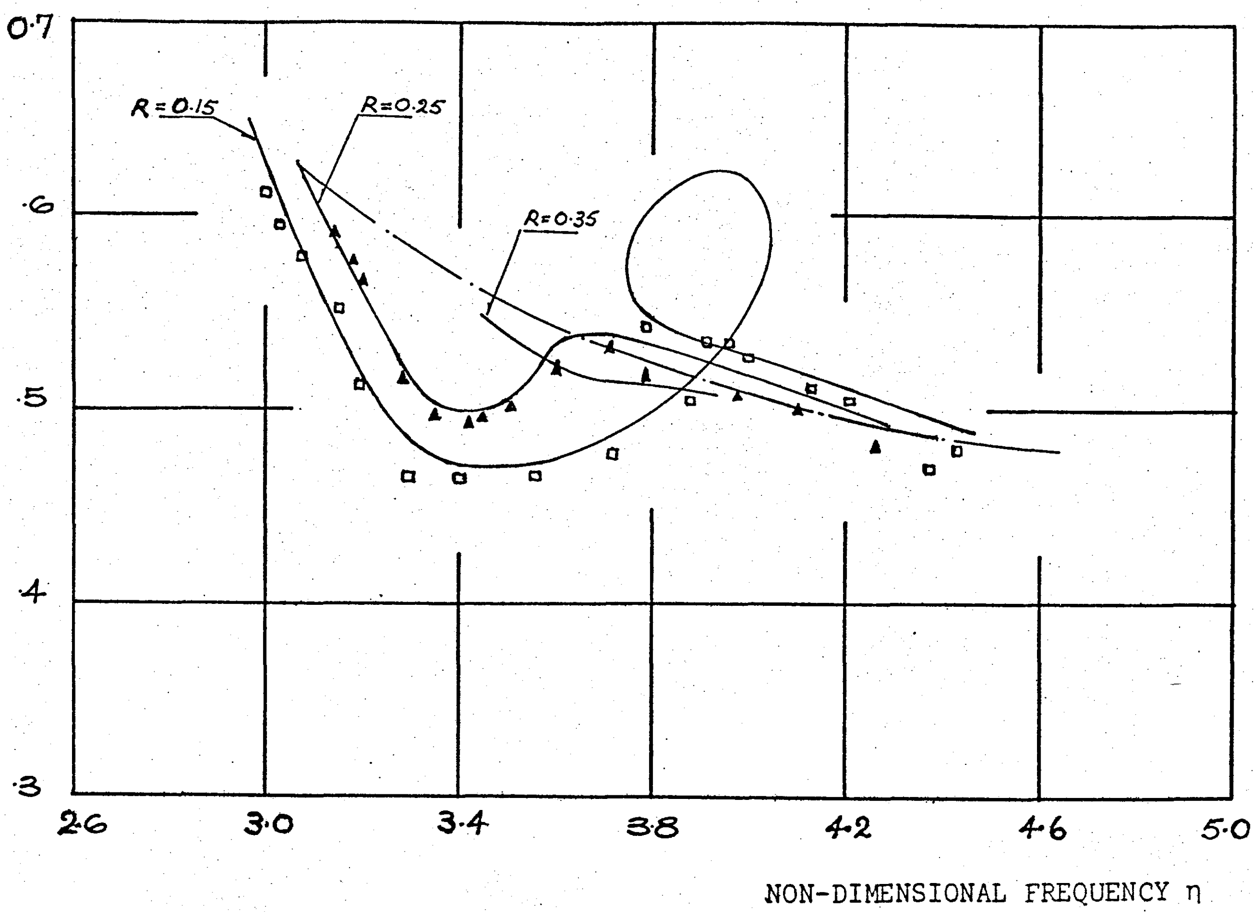


FIG.(6.3-13). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION

RESPONSE CURVES FOR $\bar{Q}_1$. $\bar{\Delta} = 1.0$, $\bar{Z} = 0.4$.

——— · ——— HARMONIC RESONANCE.

————— THEORETICAL RESULTS.

□ R=0·15. ▲ R=0·25 EXPERIMENTAL RESULTS.

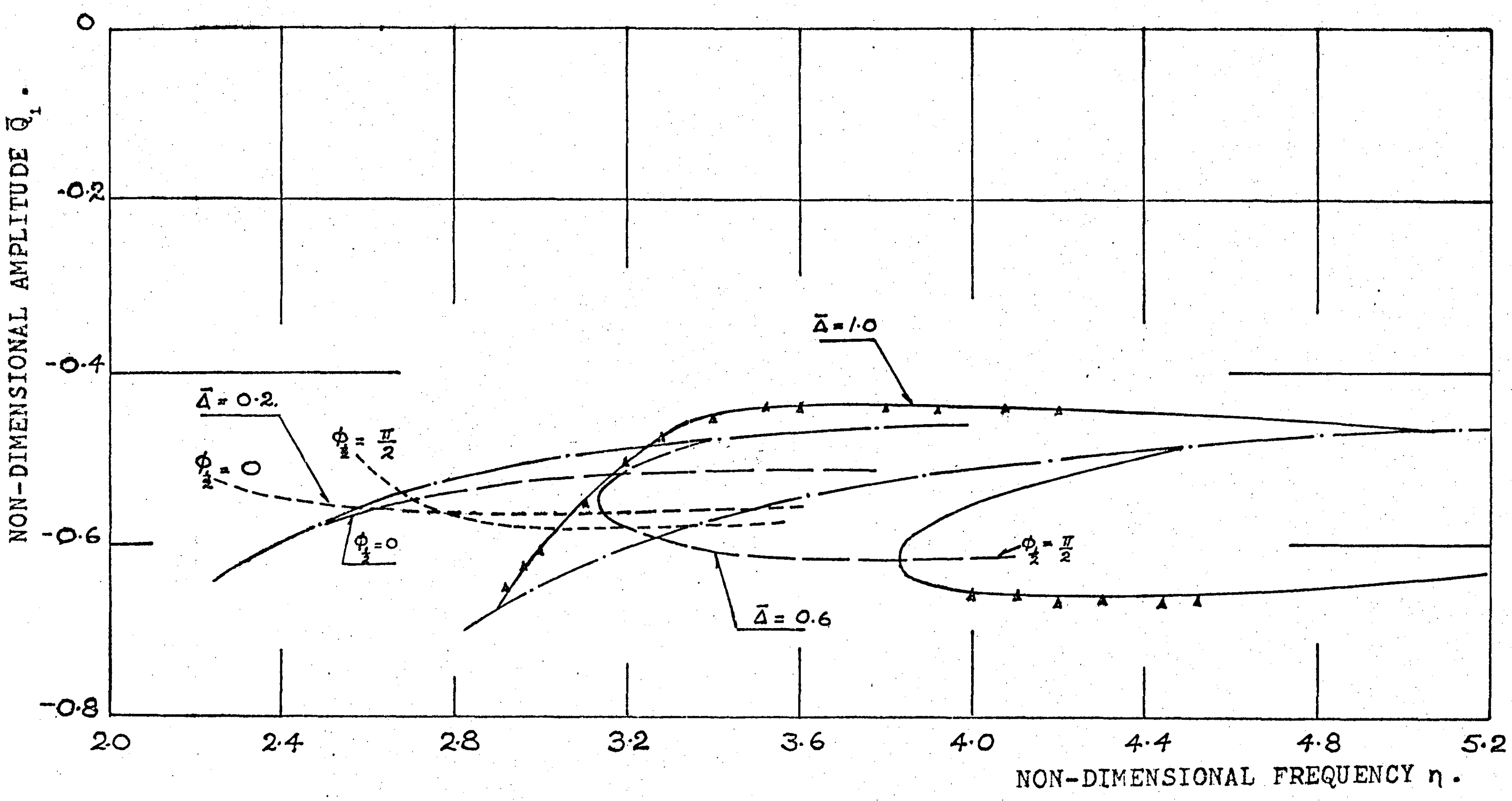


FIG.(6.3-14). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR $\bar{Q}_1$. $\bar{Z} = 0.4$, R = 0.

—— · —— HARMONIC RESONANCE.

– – – – —— —— ———— THEORETICAL RESULTS.

▲ $\bar{\Delta} = 1{\cdot}0$ EXPERIMENTAL RESULTS.

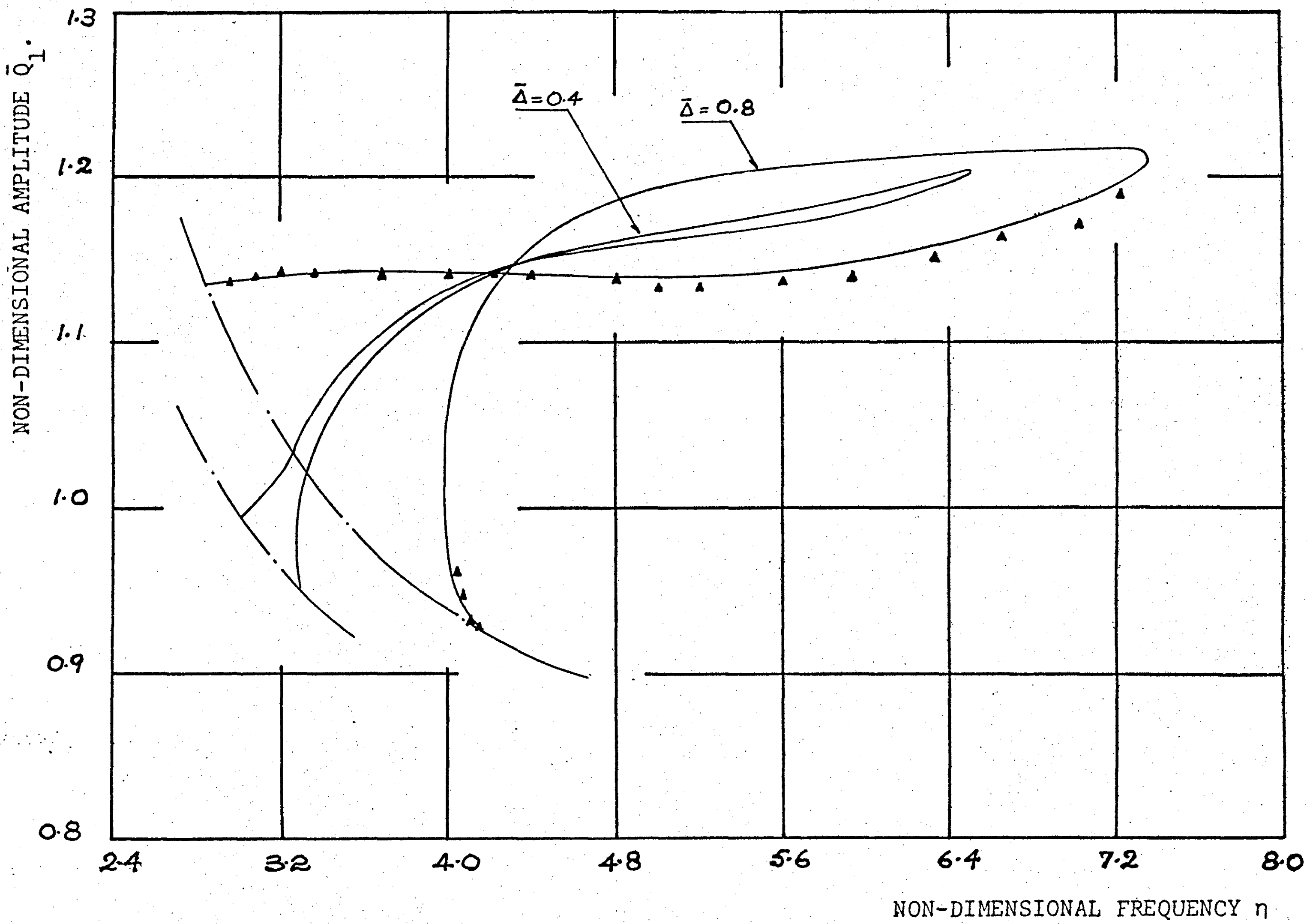


FIG.(6.3-15). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR $\bar{Q}_1$. $\bar{Z}$ = 0.8, R = 0.15.

—·— HARMONIC RESONANCE.

——— THEORETICAL RESULTS.

▲ EXPERIMENTAL RESULTS.

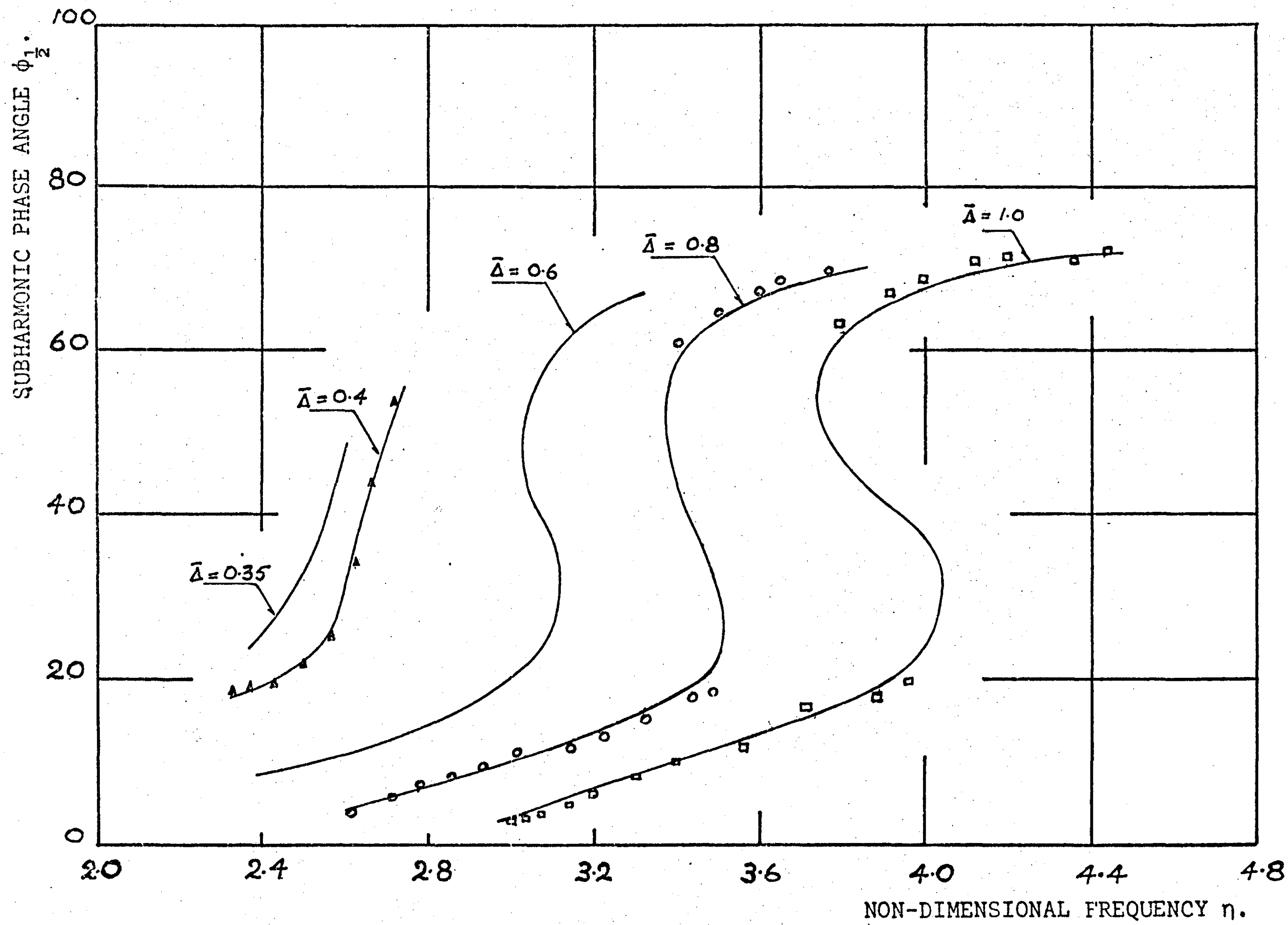
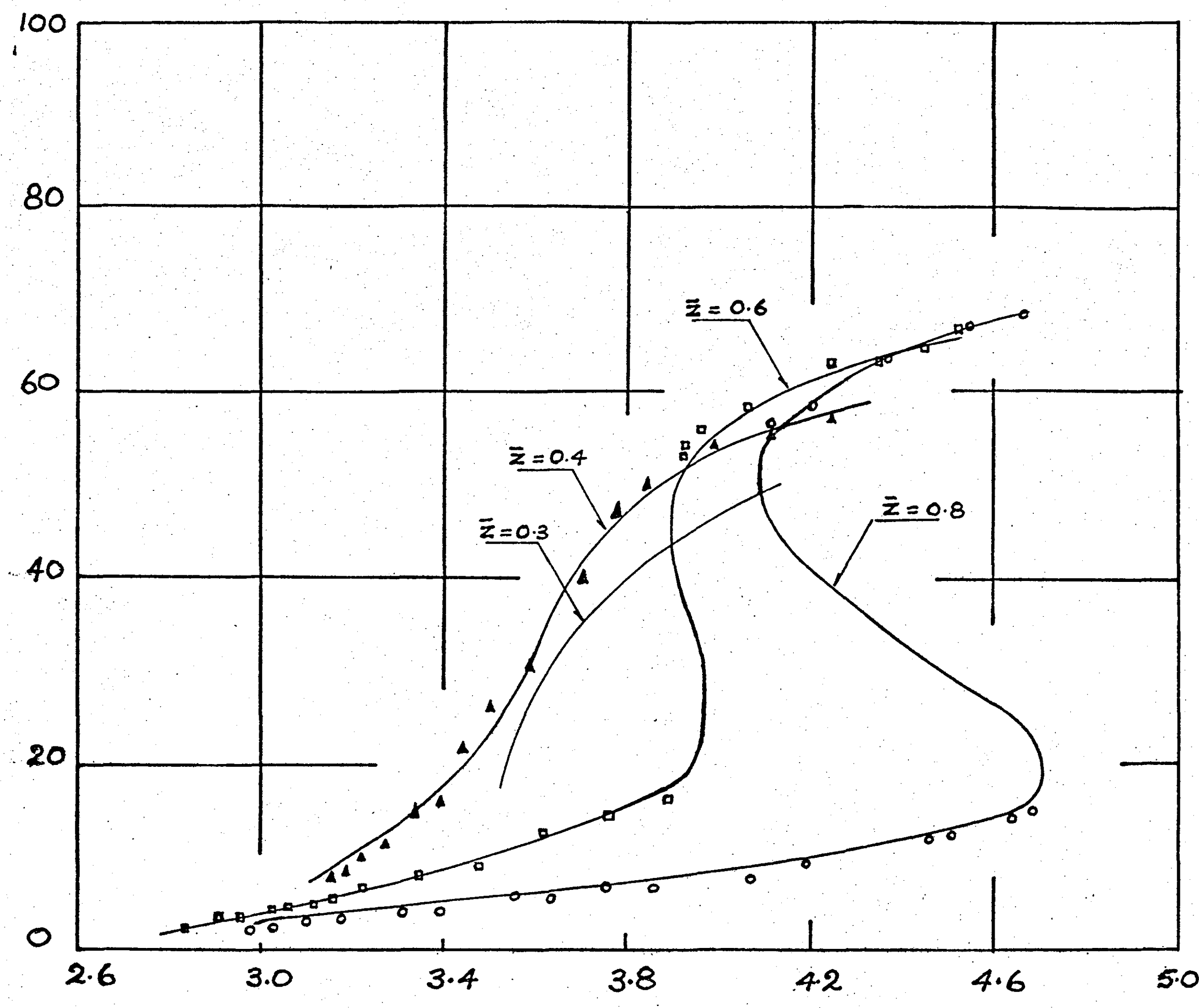


FIG.(6.3-16). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR $\phi_{\frac{1}{2}}$. $\bar{Z} = 0.4$, $R = 0.15$.

——— THEORETICAL RESULTS.

□ ○ ▲ EXPERIMENTAL RESULTS.



NON-DIMENSIONAL FREQUENCY η.

FIG.(6.3-17). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR $\phi_{\frac{1}{2}}$. $\bar{\Delta}$ = 1.0, R = 0.25.

———————— THEORETICAL RESULTS.

▲ $\bar{z}$=0.4, ○ $\bar{z}$=0.8, □ $\bar{z}$=0.6. EXPERIMENTAL RESULTS.

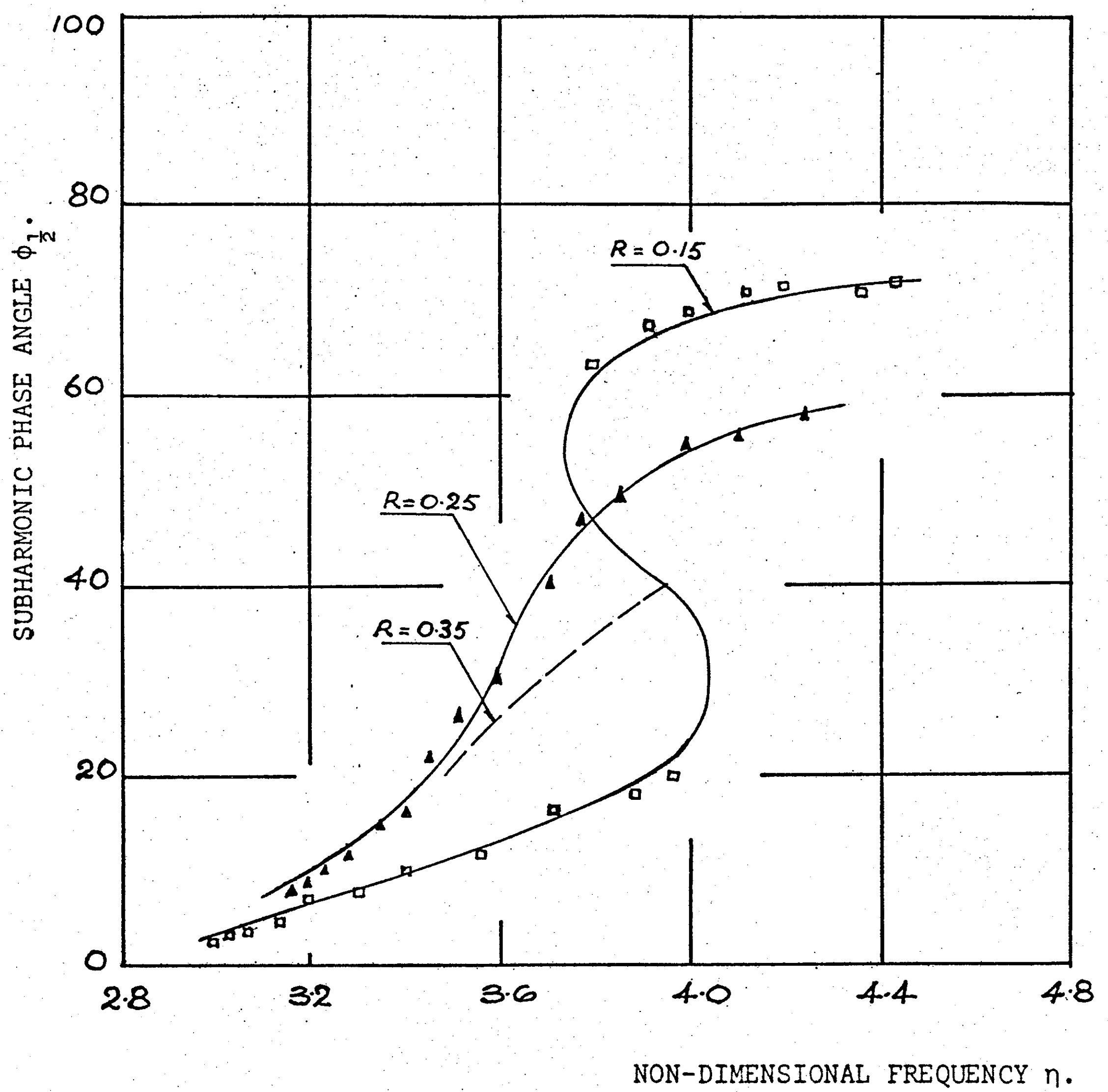


FIG.(6.3-18). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR $\phi_{\frac{1}{2}}$. $\bar{\Delta} = 1.0$, $\bar{Z} = 0.4$.

— — — ——— THEORETICAL RESULTS.

□ ▲ EXPERIMENTAL RESULTS.

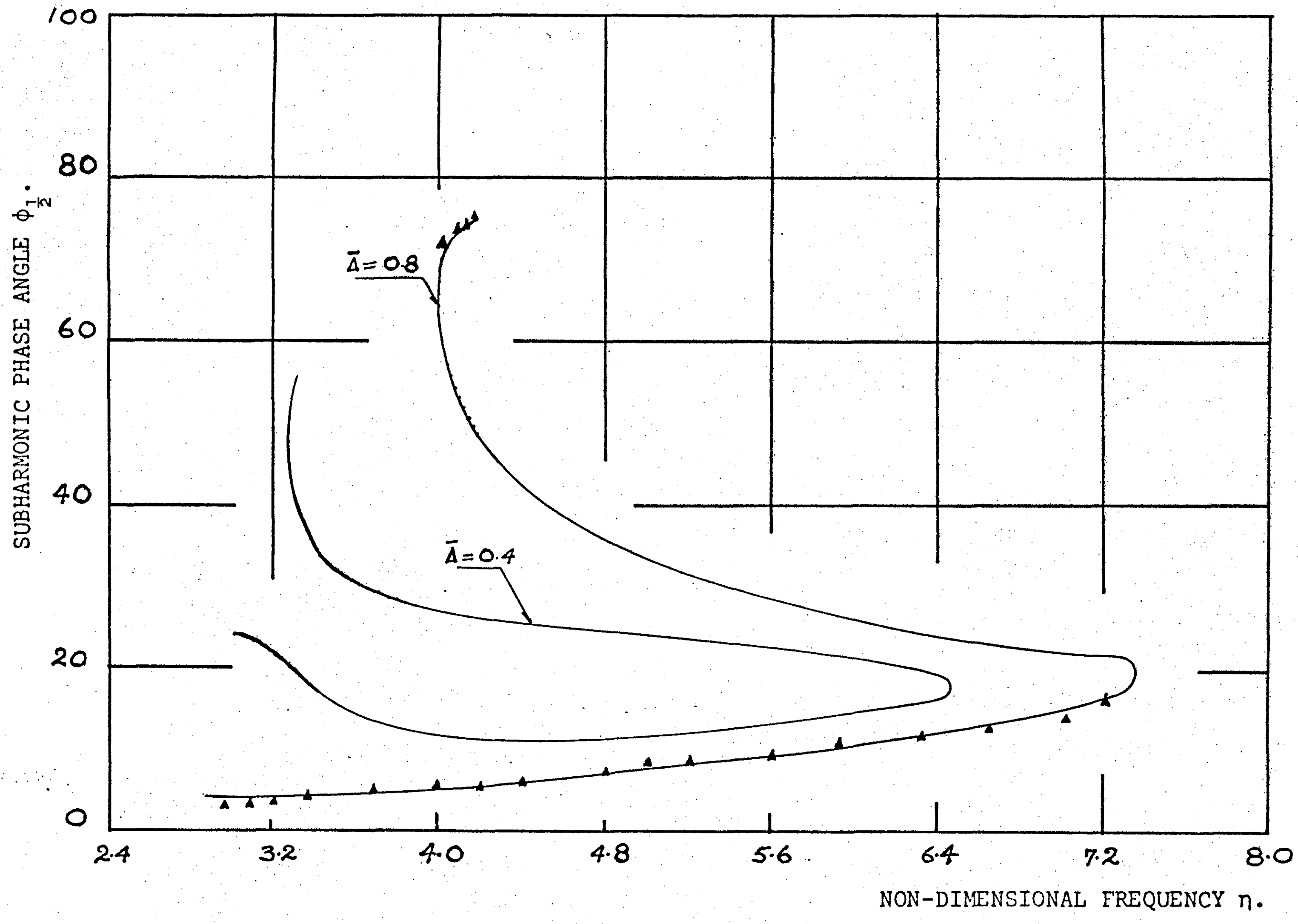


FIG.(6.3-19). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR $\phi_{\frac{1}{2}}$. $\bar{Z}$ = 0.8, R = 0.15.

——— THEORETICAL CURVES.

▲ EXPERIMENTAL CURVES.

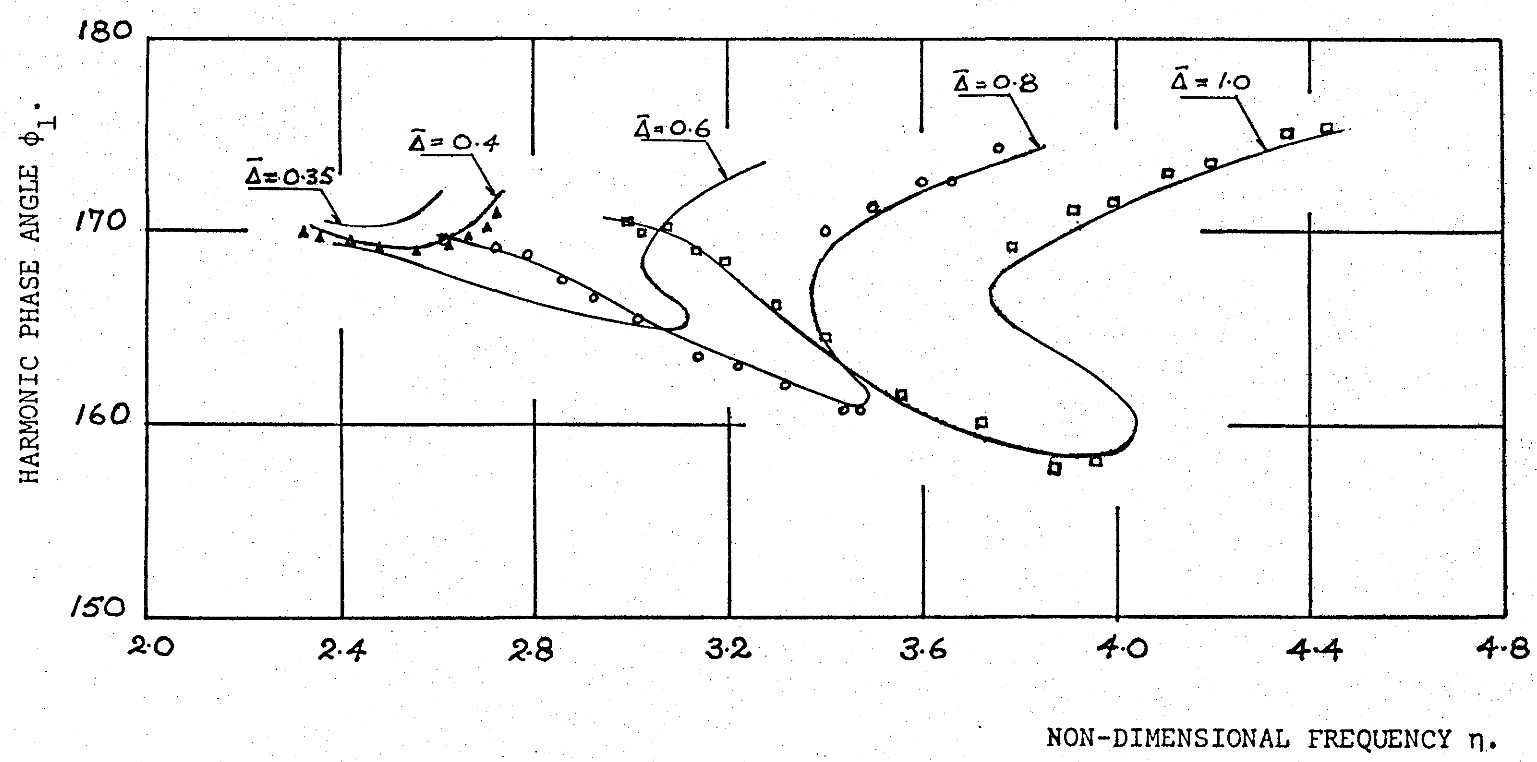


FIG.(6.3-20). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR ϕ_1. $\bar{Z}$ = 0.4, R = 0.15.

———— THEORETICAL RESULTS.

□ $\bar{\Delta}$=1.0, ▲ $\bar{\Delta}$=0.4, ○ $\bar{\Delta}$=0.8. EXPERIMENTAL RESULTS.

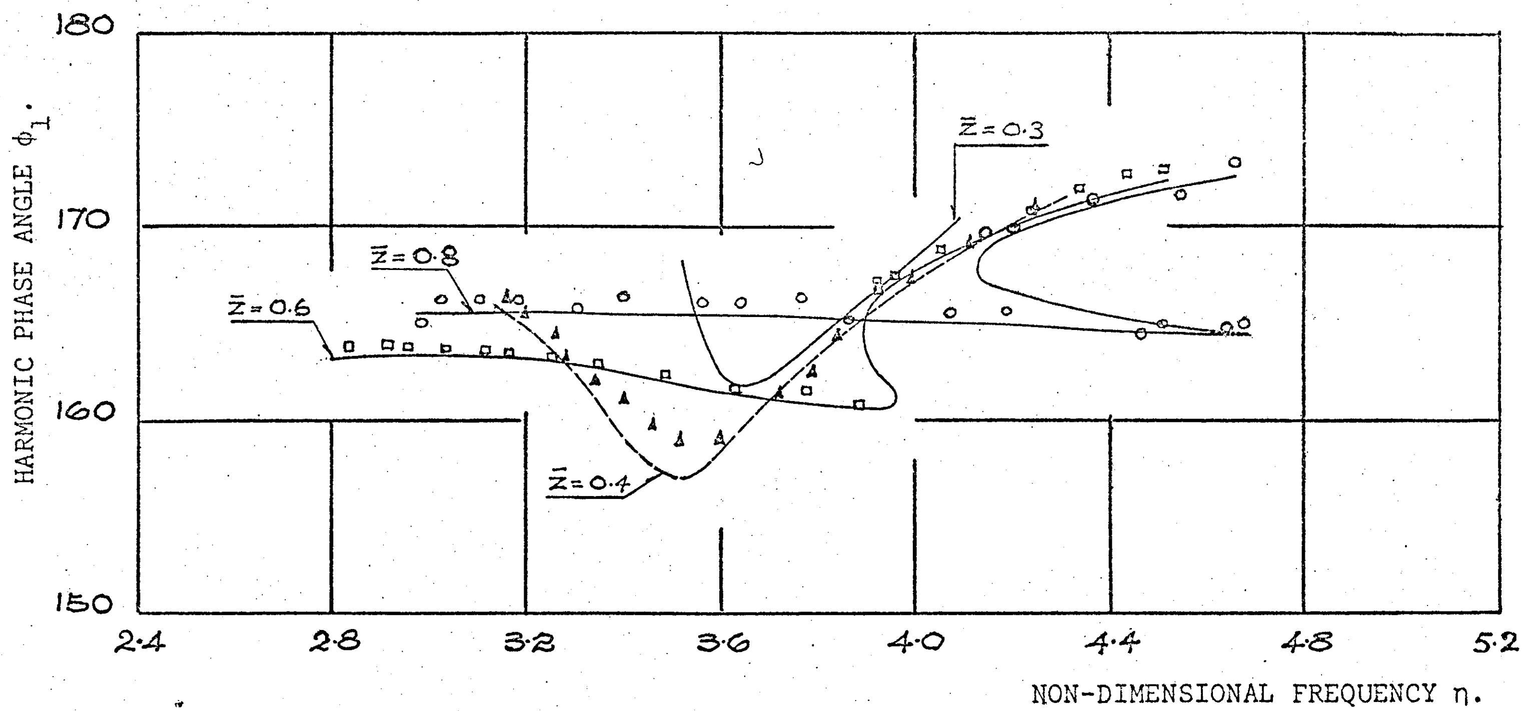
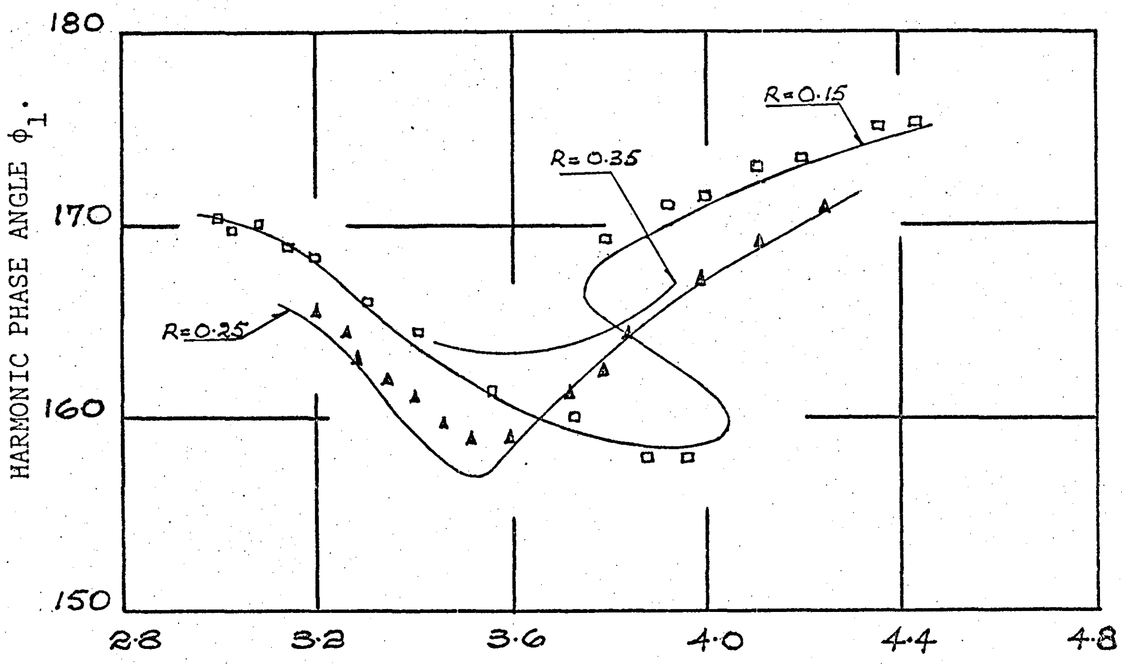


FIG.(6.3-21). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONCE CURVES FOR ϕ_1. $\bar{\Delta}$ = 1.0, R = 0.25.

---- ———— THEORETICAL RESULTS.

○ $\bar{Z}$ = 0·8, ◻ $\bar{Z}$ = 0·6, ▲ $\bar{Z}$ = 0·4. EXPERIMENTAL RESULTS.



NON-DIMENSIONAL FREQUENCY η.

FIG.(6.3-22). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR ϕ_1. $\bar{\Delta} = 1.0$, $\bar{Z} = 0.4$.

——— THEORETICAL RESULTS.

□ △ EXPERIMENTAL RESULTS.

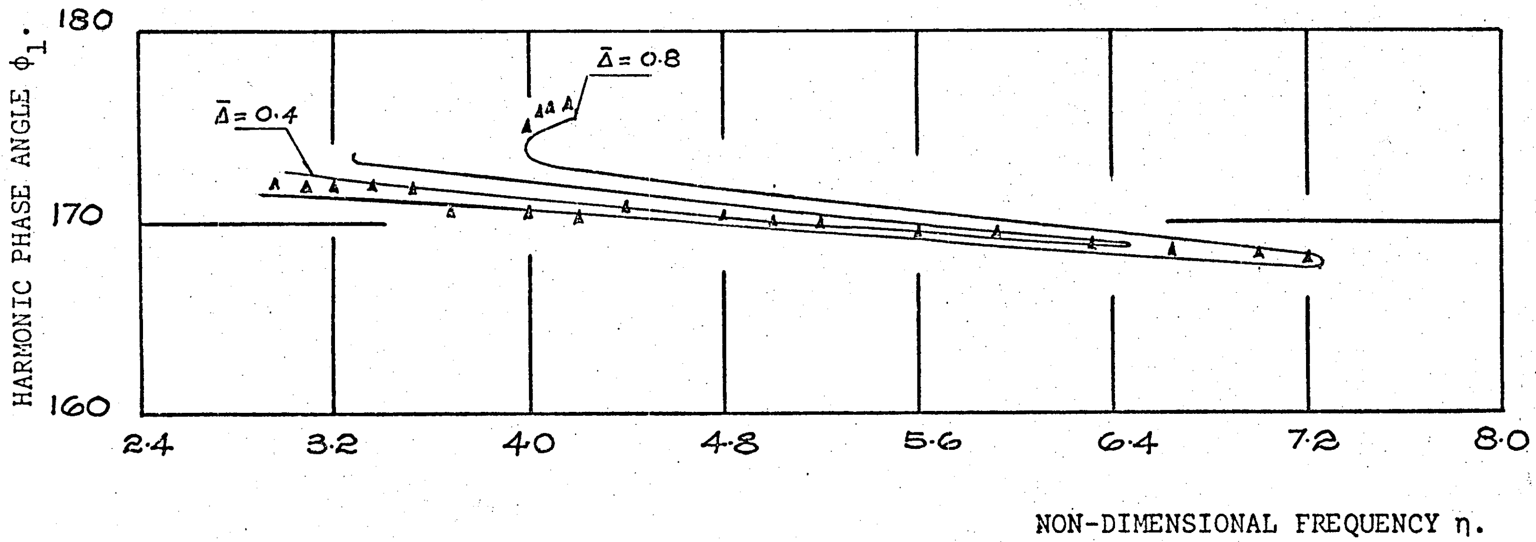


FIG.(6.3-23). SUBHARMONIC SOLUTION OF 5-TERM APPROXIMATION.

RESPONSE CURVES FOR ϕ_1. $\bar{Z} = 0.8$, $R = 0.15$.

——————— THEORETICAL RESULTS.

▲ $\bar{\Delta} = 0.8$. EXPERIMENTAL RESULTS.

6.4. The subharmonic response in which the accompanying harmonic component Q_1 is approximated independent of frequency.

6.4.(i). Approximation of the fundamental harmonic

The consideration of the minimum five-term solution to equation (3.1,8) ©is seen to produce complicated algebraic expressions that describe the behaviour characteristics of the subharmonic motion. It would simplify the calculations for the more important components of the motion if reasonable approximations are made to the fundamental harmonic, as suggested in the preceding discussion.

Since, in general, there is no significant alteration during the predominance of $Q_{\frac{1}{3}}$ to the response of the accompanying harmonic component Q_1, and as its magnitude is principally influenced by that of the disturbing force, it is not an unreasonable approximation if the harmonic component is regarded as the effective amplitude of $\bar{Z}\eta^2$. This merely implies that in the subharmonic motion the accompanying oscillation of Q_1 is non-resonant. Besides, from the analysis of the vibrational characteristics of the preceding sections such an approximation is quite legitimate. It is shown that the build-up of an unstable oscillation occurs in the same region as the harmonic component and there is no actual solution to the existence of the motion under the resonant condition of the harmonic Q_1.

Finally, in the region of the subharmonic frequencies, Fig.(6.3-13) shows that the harmonic resonance curve will practically be indifferent to the changing conditions of damping. The value of a harmonic curve for a particular frequency of damping did not differ significantly from the corresponding values of different damping coefficients.

Hence, if the approximate solution for the harmonic resonance condition is given by

$$\tilde{Z} = N + Q_1 \cos \omega t \qquad \ldots\ldots\ (6.4,1)$$

the substitution of equation (6.4,1) into (3.1,8), in which $R = 0$, subsequently yield an equation in the form

$$2Q_1 \{ 1 - \eta^2 + 3Y^2 + \frac{3}{4} Q_1^2 \} - 2\eta^2 \ddot{Z} = 0 \quad \ldots\ (6.4,2)$$

Equation (6.4,2) can also be achieved from equation (6.1,5) by making Q_2 and R both equal to zero.

Since the solution for the natural frequency of a system under the influence of gravitational force is also adequately expressed by the equation (6.4,1), the substitution of the equation into equation (3.1,8), in which both $\ddot{Z}$ and R are zero, gives after applying the principle of harmonic balance the following equation,

$$\eta_o^2 = 1 + 3(\Delta + N)^2 + \frac{3}{4} Q_1^2 \qquad \ldots\ldots\ (6.4,3)$$

Equation (6.4,3) represents the 'back-bone' curve of the system represented by equation (3.1,8). Since the approximate solution describes the motion for which Q_1 is the predominant component, equation (6.4,2) then becomes,

$$2Q_1 \{ n_o^2 - n^2 \} = 2n^2\bar{Z}$$

which leads to

$$Q_1 = \frac{n^2\bar{Z}}{\{n_o^2 - n^2\}} \qquad \ldots\ldots\ldots \quad (6.4,4)$$

Thus, with the harmonic component of the subharmonic motion approximated to equation (6.4,4), and since for predominantly second order subharmonics the vibration occurs in the frequency region where

$$n \approx 2n_o \qquad \ldots\ldots\ldots \quad (6.4,5)$$

the amplitudes of the harmonic Q_1 over the frequency band-width is given by the approximation,

$$Q_1 \approx -\frac{4}{3}\bar{Z} \qquad \ldots\ldots\ldots \quad (6.4,6)$$

The negative sign merely signifies the phase quadrant of the component during the existence of subharmonic vibration.

The values from the above relationship for the various magnitudes of $\bar{Z}$ are a reasonable approximation in the predominant region of $Q_{\frac{1}{2}}$. This is readily seen by the comparison with the curves, that are plotted for the Q_1 component, in Figs.(6.3-11) to (6.3-14).

6.4.(ii). Subharmonic equations independent of the periodicity of the harmonic component Q_1.

The harmonic Q_1 is the largest coefficient of the accompanying oscillations in the subharmonic motion and it is approximated as a direct function of $\tilde{Z}$. The equation (6.1,5) describing the behaviour of this component during the existence of $Q_{\frac{1}{3}}$ can be omitted and the five simultaneous expressions that were too unwieldy to manipulate are then reduced to two equations containing three unknowns.

Hence, the substitution of equation (6.4,6) into (6.1,3) and (6.1,4) gives on simplification,

$$\eta^2 = \frac{1}{3} \{ (12 + 36Y^2 + 9Q_{\frac{1}{3}}^2 + 32Z^2 - 24N^2) \pm 4(144Z^2Y^2 - N^2 (36 + 27Q_{\frac{1}{3}}^2 + 64Z^2 + 108Y^2 - 36N^2))^{\frac{1}{2}} \} \quad \ldots\ldots (6.4,7)$$

$$\text{and } N = \frac{- \{ 64\Delta^2 Z^2 + Q_{\frac{1}{3}}^2 (18\Delta^2 - 6 + 1.5\eta^2 - 16Z^2) - 4.5Q_{\frac{1}{3}}^4 \}}{\{ 24N^3 + a_3 N^2 + b_3 N + c_3 \}} \quad \ldots\ldots (6.4,8)$$

where $a_3 = 96\Delta$,

$b_3 = \{ 144\Delta^2 + 24 + 18Q_{\frac{1}{3}}^2 + 64Z^2 \}$

$c_3 = \{ 72\Delta^3 + 24\Delta + 36\Delta Q_{\frac{1}{3}}^2 + 128 \Delta Z^2 \}$

Although equations (6.4,7) and (6.4,8) are still complicated, the results from the two expressions provide a means of assessing the justification in the use of the approximation to achieve the limiting inequality of the second and higher orders of subharmonic motion. If the results of the two important components of the motion do not vary significantly from the previous values, the conditions for the critical relationships, obtained through the simplification of the polynomial expressions by equation (6.4,6), are derived with acceptable accuracy.

From an initial appraisal it is evident that the above equations (6.4,7) to (6.4,8) would produce similar response characteristics. For by assuming viscous damping to be small, in view that the investigation is on predominant subharmonic vibration, equation (6.4,7) is reduced to

$$\eta^2 = 4 \{ 1 + \frac{3}{4} Q^2_{\frac{1}{2}} + 3Y^2 \} + \frac{4}{3} \{ 8Z^2 \pm 12Z (\Lambda + N) \} \quad ..(6.4,9)$$

If we regard the natural frequency of the subharmonic motion, relative to the amplitude of $Q_{\frac{1}{2}}$, represented as

$$\eta^2_N = (1 + \frac{3}{4} Q^2_{\frac{1}{2}} + 3Y^2) \quad ...... \quad (6.4,10)$$

it is then evident from (6.4,9) that the vibration response and the frequency range are heavily influenced, as before, by the magnitudes of $\bar{Z}$ and $\bar{\Lambda}$. Also as the plus or minus sign in equations (6.4,7) and (6.4,9) corresponds for a fixed value of $Q_{\frac{1}{2}}$, the frequency that is either higher or lower than its natural frequency, then with the effective non-linearity of the system large the two branches of the response are generated about the curve of $2n_b$. The condition for this is that the amplitude of the disturbing force must be smaller than the magnitude of the static deflection, since N is always negative in the presence of damping and as seen from the previous results the displacement from origin of symmetrical restoring force characteristics in the early stages are influenced by $\bar{Z}$. For the case of R = 0, the condition is given by

$$(\bar{A} + N) > \frac{2}{3}\bar{Z} \qquad \ldots\ldots \quad (6.4,11)$$

Hence, if $\bar{Z}$ is large, it is obvious that both branches of $Q_{\bar{z}}$ start at frequencies greater than $2n_b$.

The theoretical results of equations (6.4,7) and (6.4,8) are obtained through a similar procedure to that which is described in section (6.1)(ii), whereby the values of $Q_{\bar{z}}$ are prescribed with corresponding initial approximates of N for selected magnitudes of the independent parameters of the system. The iteration sequence is identical to Figs.(6.1-1) and (6.1-2) and the method for convergence of the roots is explained in (46). In the instability region where it is impossible to achieve the convergence of solution, the numerical procedure which is as described in section (6.2)(i) is used.

6.4.(iii). Discussion of results

From the comparison of the important components of the motion, it appears in general that there is no reason why the approximation represented by equation (6.4.6) cannot be applied with reasonable accuracy to derive the limiting inequalities to the second and higher orders of subharmonic vibration. The justification in the use of the approximate is by examining the response characteristics that are produced through the simplified calculations.

Since the subharmonic vibration is shown to be heavily influenced by the effects of gravitational force on the equilibrium of motion, it may be appropriate to consider first the case in which the effective non-linearity of the system is the strongest. It is readily seen from the curves of Fig.(6.3-1) and (6.4-1) that, although when the harmonic Q_1 is approximated the vibration of the lower branch of ϕ_1 occurs for the small displacements at higher frequencies, the general results are practically identical. In arriving at equation (6.4.6), the approximation is considered for the region where the subharmonic component Q_1 predominates. Consequently, it is expected that a certain discrepancy exists at the start of the subharmonic vibratory motion. In this neighbourhood, often the actual value of the accompanying harmonic oscillation is higher for a particular amplitude of $\bar{E}$. The values for the expression on the right of the plus or minus sign are then much smaller and hence the higher frequency is obtained initially. However, the maximum relative error of the frequency, determined for a fixed value of Q_2, is about five per cent which for most practical purposes is generally satisfactory. If the magnitude of $\bar{\Lambda}$ is reduced, this discrepancy in the early stages is eliminated as shown by the graphs mentioned. With the prevalent regions of the subharmonic vibration, the approximation takes into consideration the gravitational influence acting on the system and the results of the important components are virtually identical to those of the preceding section

It is interesting to compare the method of approximating Q_1 with the case of the second order subharmonics investigated by Hayashi. In the relationship of the accompanying harmonic component to the disturbing force, the harmonic amplitude for the second order frequency region is approximated to the corresponding value of its linearised response. If the same reasoning is applied here, the equivalent amplitude of a centrifugally excited system is given by $Q_1 = (\frac{\tilde{Z}\eta^2}{1-\eta^2})$. The problem arises when the fundamental harmonic amplitude is approximated independent of the frequency. Although by assuming the subharmonics exists in the frequency region of order 2 will give the same value as equation (6.4.6), the statement under the influence of gravitational force on the equilibrium of motion is theoretically inaccurate. It is readily observed that the response frequencies are significantly higher and this makes the linearised harmonic component smaller for a value of $\tilde{Z}$ than equation (6.4.6). Thus, if the corresponding procedure is adopted here, the substitution of the approximate into equation (6.1.3) gives throughout a much reduced vibration response. With an approximation not giving a close indication of the vibration characteristics, an obvious difficulty is created at a later stage when deriving the limiting equations of reasonable accuracy.

The results of the various physical states of damping from the simplified calculations are not significantly different to those of Figs.(6.3-3) and (6.3-8). As mentioned earlier, the condition in which equation (6.4.6) was derived would give a smaller harmonic amplitude Q_1 in the early stages of the lower frequency branch. Whilst with the approximation the shift of the position of dynamic equilibrium from static equilibrium is unaltered by viscous damping as shown by Figs.(6.4-9) and (6.4-11), it is not surprising then higher frequency errors are obtained for the initial values of Q_3. However the percentage error is reduced when the coefficients are raised, because the difference between the actual values of Q_1 is decreased. If R is relatively large, the effect of approximating is reversed. Since Q_1 is larger than the true amplitude at the other branch of the response, higher frequencies are obtained at the other end. Nevertheless, in both instances, the simplified calculations also produce, for whatever the values of viscous damping, virtually identical results in the region of predominant Q_3. This is readily seen from Figs. (6.3-3) and (6.4-3).

With the harmonic Q_1 independent of the frequency, the same magnitude of the component influences both ends of the response. For the physical conditions of Figs.(6.4-7), (6.4-9) and (6.4-12), it has no practical affect on the effective non-linearity of the system. But if the amplitude of $\tilde{Z}$ is large, the approximation will result in the non-linearity being raised or lowered at the respective starting-point of the motion. The reason is because if the harmonic amplitude ~~is~~ approximated to be smaller it theoretically means a larger amount of the weight is transfered to the supports, as shown from the comparison of Figs.(6.3-7) and (6.4-8). Consequently for the case of $\tilde{Z}$ = 0.8, the use of equation (6.4.6) produces results to suggest the subharmonic vibration of the smaller ϕ_3 response occurs at lower frequencies than it actually is.

The discrepancy in the results disappears, when the difference in the values of Q_1 is reduced, hence with the magnitude of the disturbing force. If at the end the disturbing force is too small, a higher frequency is determined for a fixed value of $Q_{\frac{1}{3}}$ in the same branch of the response, because a smaller amplitude of Q_1 is attributed by the approximation whilst the degree of non-linearity remains unchanged. However, as with all previous results examined, there is practically little difference over the regions of the predominant subharmonic resonance between the curves illustrated by Figs.(6.4-2), (6.4-8) and (6.3-2), (6.3-7).

From the discussion of section (6.3) it is evident that, if the behaviour of the restoring forces differs from the general characteristics by the effective degree of the non-linearity increasing with the vibration, the argument for the use of equation (6.4.6) to simplify the calculations of the subharmonic response breaks down. Otherwise it is seen as a reasonable indication of the subharmonic response and the limiting values of the inequalities are given. In the above state, the accuracy rapidly deteriorates for the approximation becomes progressively smaller than the actual harmonic component of solution (3.2.1). Consequently, the substitution restricts the magnitude of the predominant subharmonic $Q_{\frac{1}{3}}$ and the resonance frequencies as illustrated by the comparison of Figs.(6.3-5) and (6.4-4).

Since the term N is correspondingly lowered with static deflection and whilst virtually unaltered with a smaller magnitude of the disturbing force, then as observed in the preceding section that for a fixed value of $\bar{A}$ there exists two critical magnitudes of the disturbing force. Thus for the physical state of large amplitude of $\bar{Z}$, the same value of critical $\bar{A}$ can be ascertained through the inequality derived by considering the situation in which the amplitude of $\bar{Z}$ changes. The use of equation (6.4.6) in the neighbourhood of the limitation then does not effect significantly the state of the response. The difference in values is practically insignificant. Hence, there is no reason why the approximation cannot be used with satisfactory accuracy for the boundary relationship of the subharmonic vibration which result under the influence of gravitational force on the equilibrium of motion.

With the restoring force characteristics where the effective non-linearity increases with resonance, it will seem that the use of a graphical method to attain the tractable representation of the results is not the most appropriate procedure to apply for an indication of the vibratory behaviour in reducing the effect of static deflection. This is because the effective non-linearity is not proportionately modified as readily seen from the curves of Fig.(6.3-10). Besides, an overall illustration of the vibration response by graphical method is very laborious. The situation of diminishing response from the reduction of gravity effect on the equilibrium of motion will not be evident in practise for such restoring force characteristics and the heavy vibration ceases abruptly at the limiting $\bar{\lambda}$.

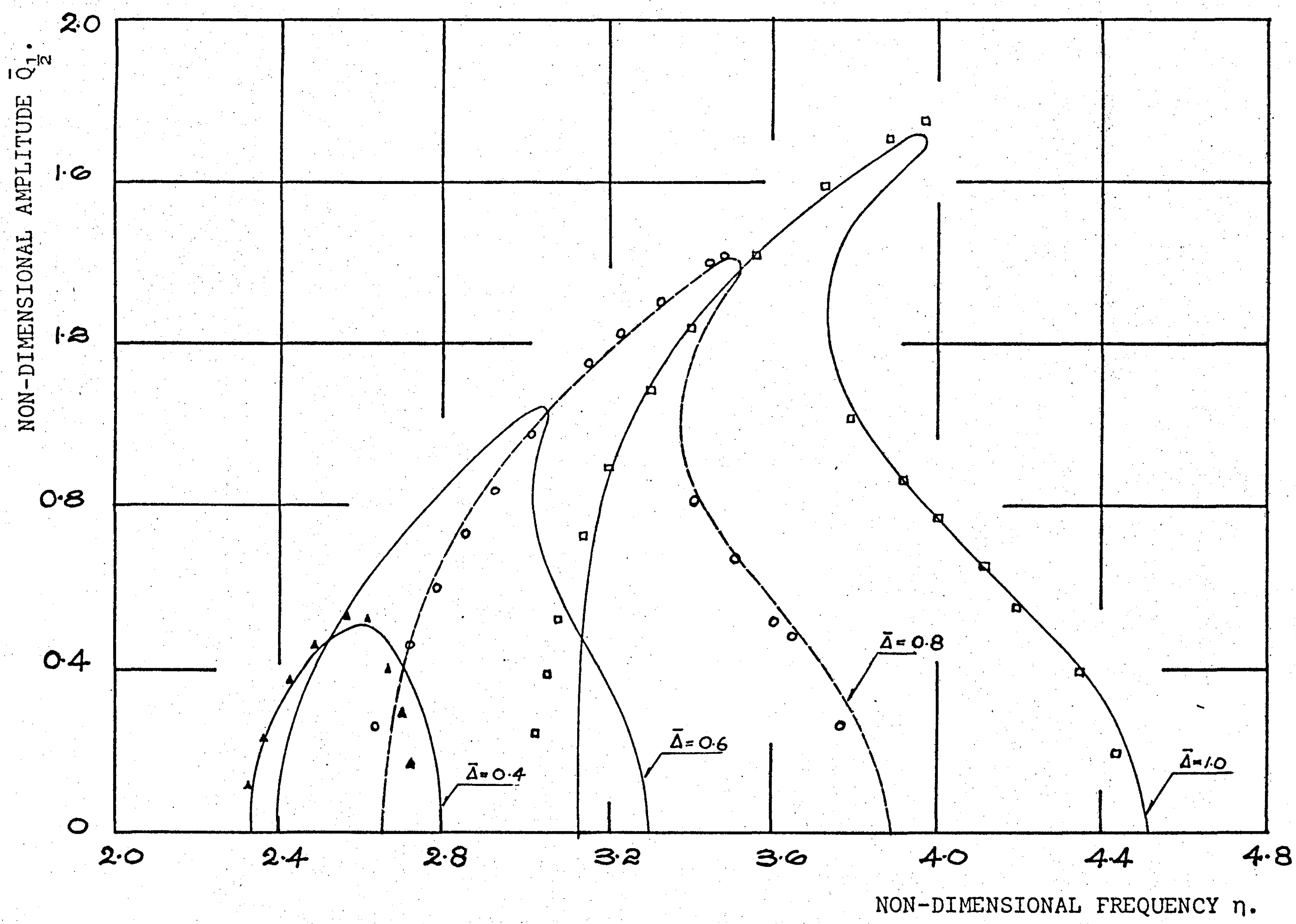


FIG.(6.4-1). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVE FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{Z}$ = 0.4, R = 0.15.

– – – – ———— THEORETICAL RESULTS.

▲ $\bar{\Delta}$=0·4, ○ $\bar{\Delta}$=0·8, □ $\bar{\Delta}$=1·0. EXPERIMENTAL RESULTS.

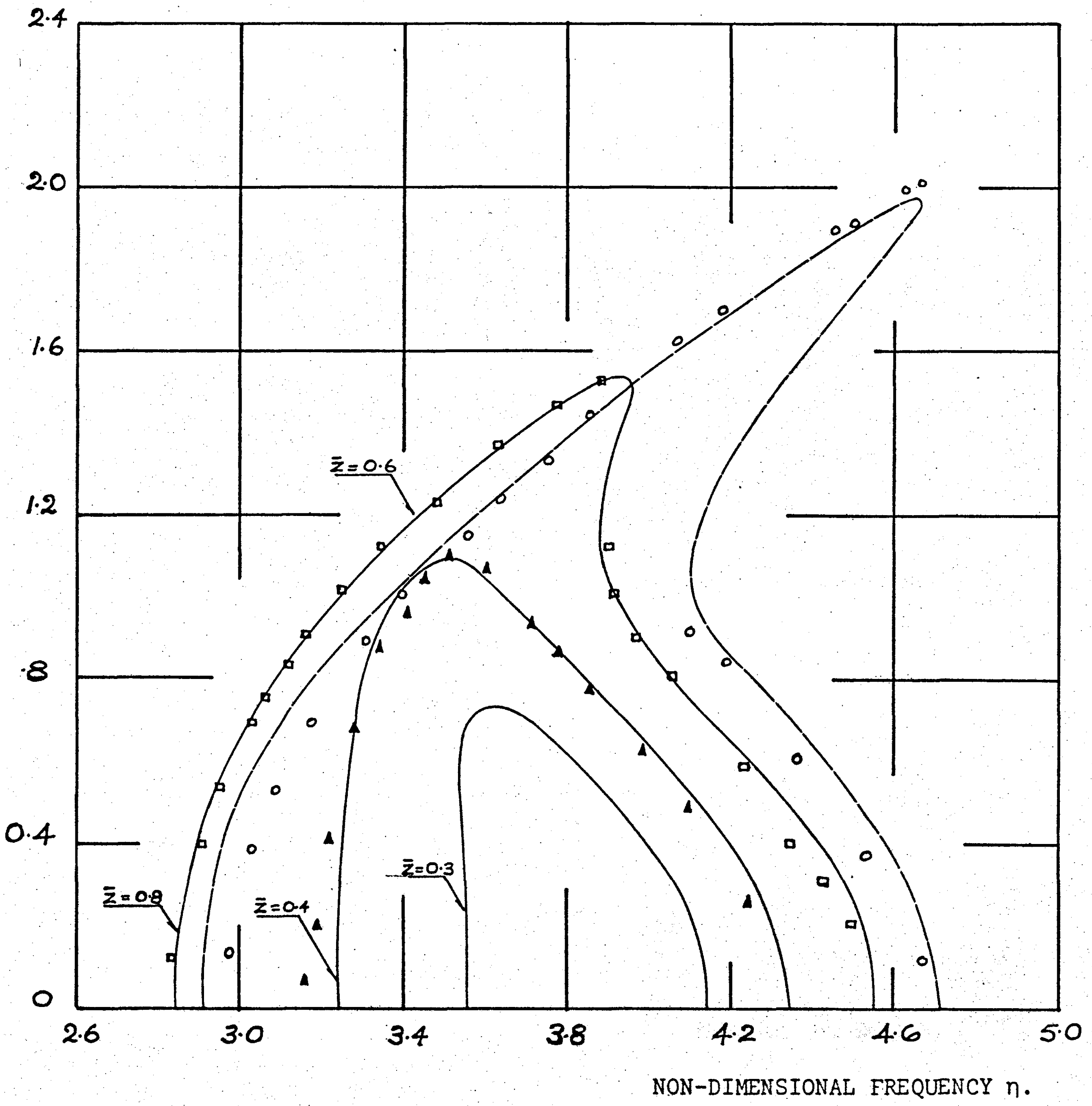


FIG.(6.4-2). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{\Delta} = 1.0$, R = 0.25.

— — — ——— THEORETICAL RESULTS.

○ □ ▲ EXPERIMENTAL RESULTS.

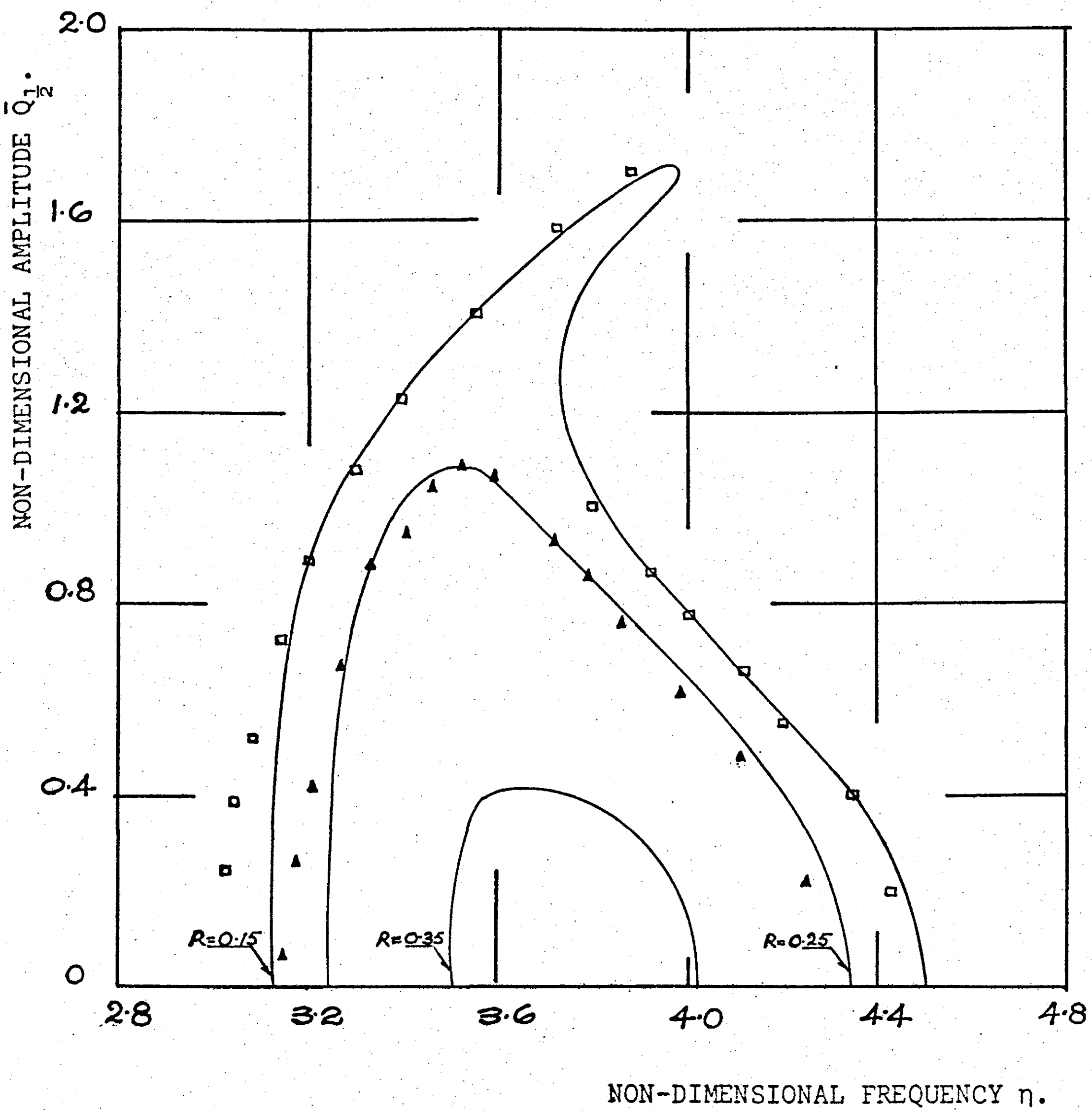


FIG.(6.4-3). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{\Delta} = 1.0$, $\bar{Z} = 0.4$.

——— THEORETICAL RESULTS.

□ ▲ EXPERIMENTAL RESULTS.

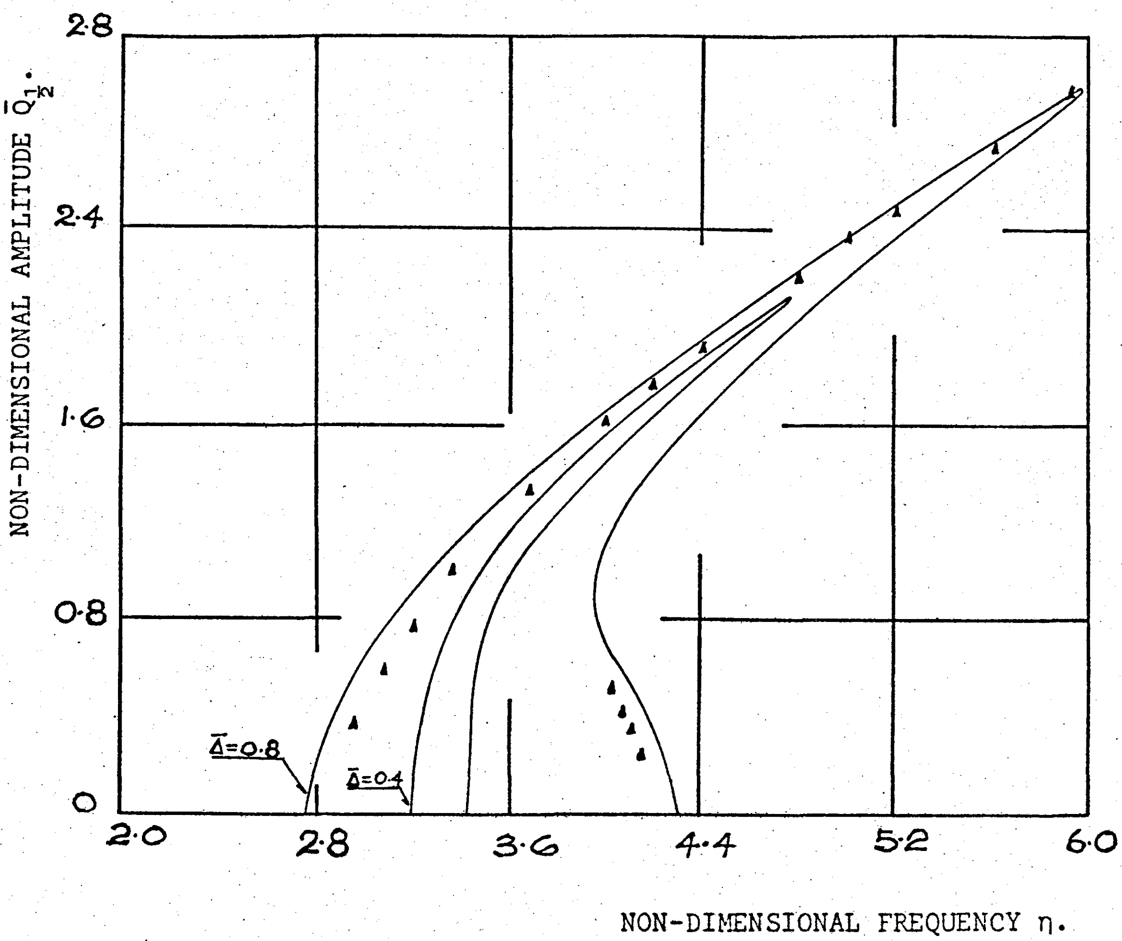
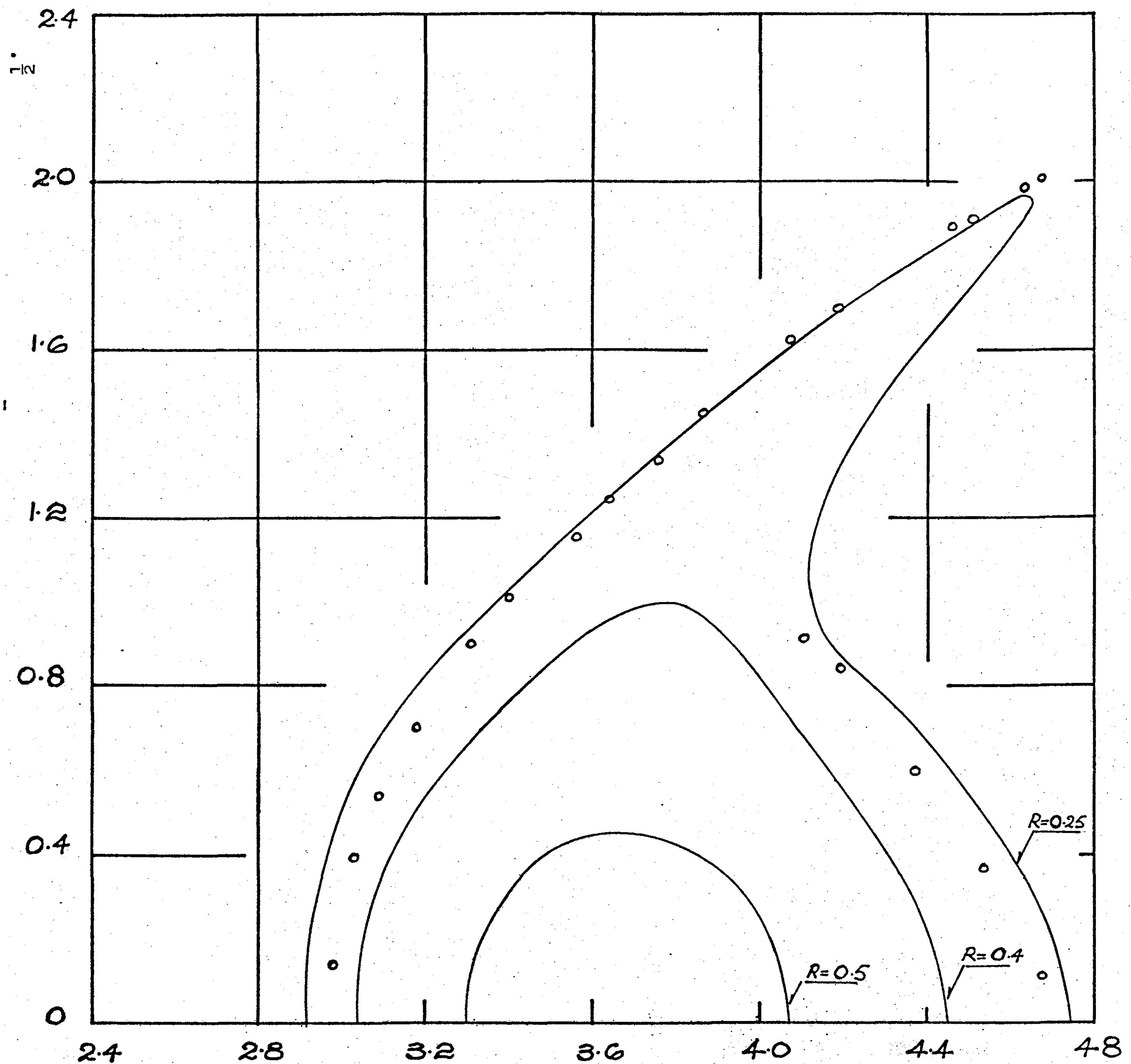


FIG.(6.4-4). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{Z}$ = 0.8, R = 0.15.

——————— THEORETICAL RESULTS.

▲ $\bar{\Delta}$=0.8 EXPERIMENTAL RESULTS.



NON-DIMENSIONAL FREQUENCY η.

FIG.(6.4-5). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{\Delta} = 1.0$, $\bar{Z} = 0.8$.

——— THEORETICAL RESULTS.

o EXPERIMENTAL RESULTS.

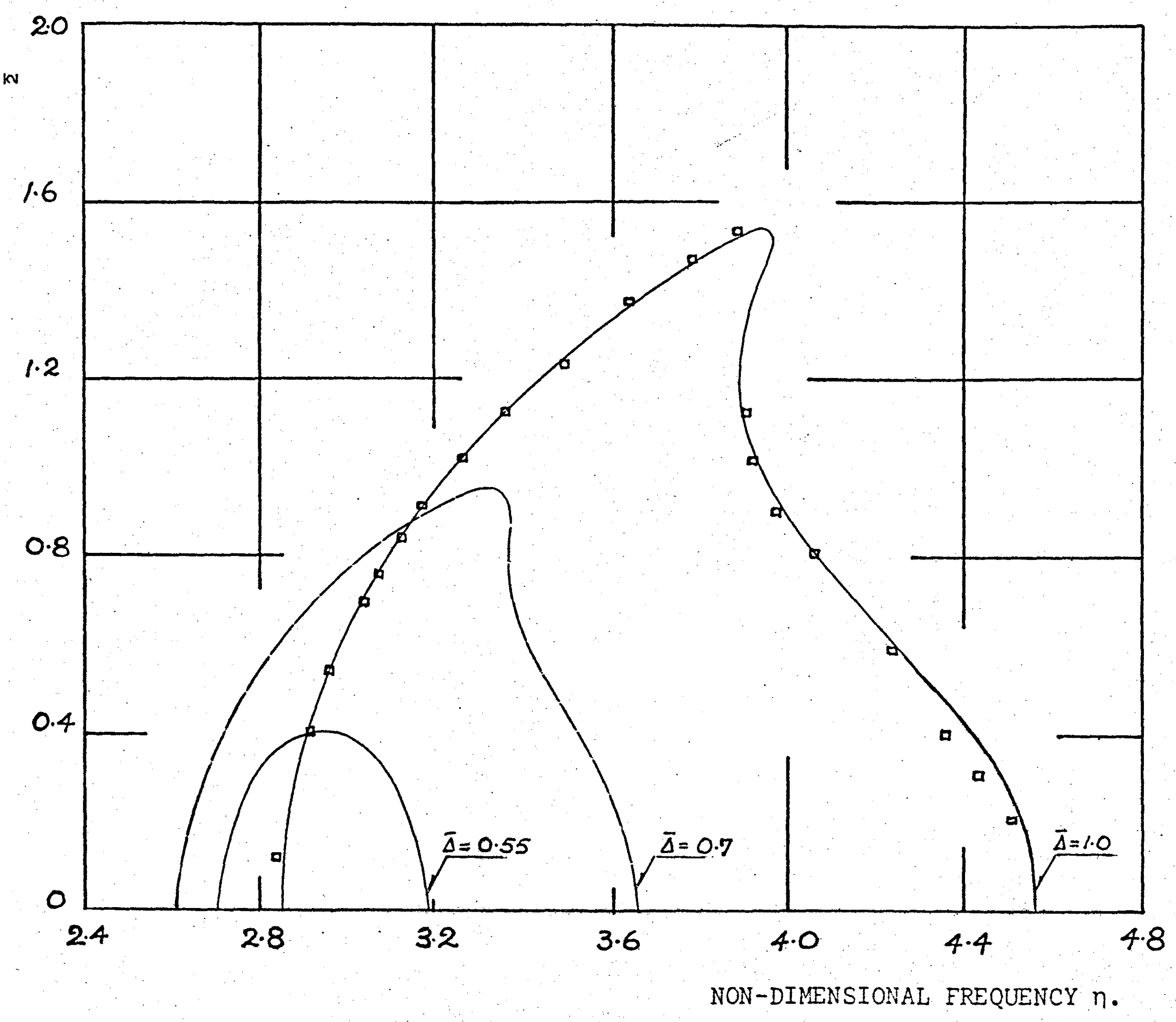


FIG.(6.4-6). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{Z} = 0.6$, $R = 0.25$.

— — — — ——— THEORETICAL RESULTS.

▫ EXPERIMENTAL RESULTS.

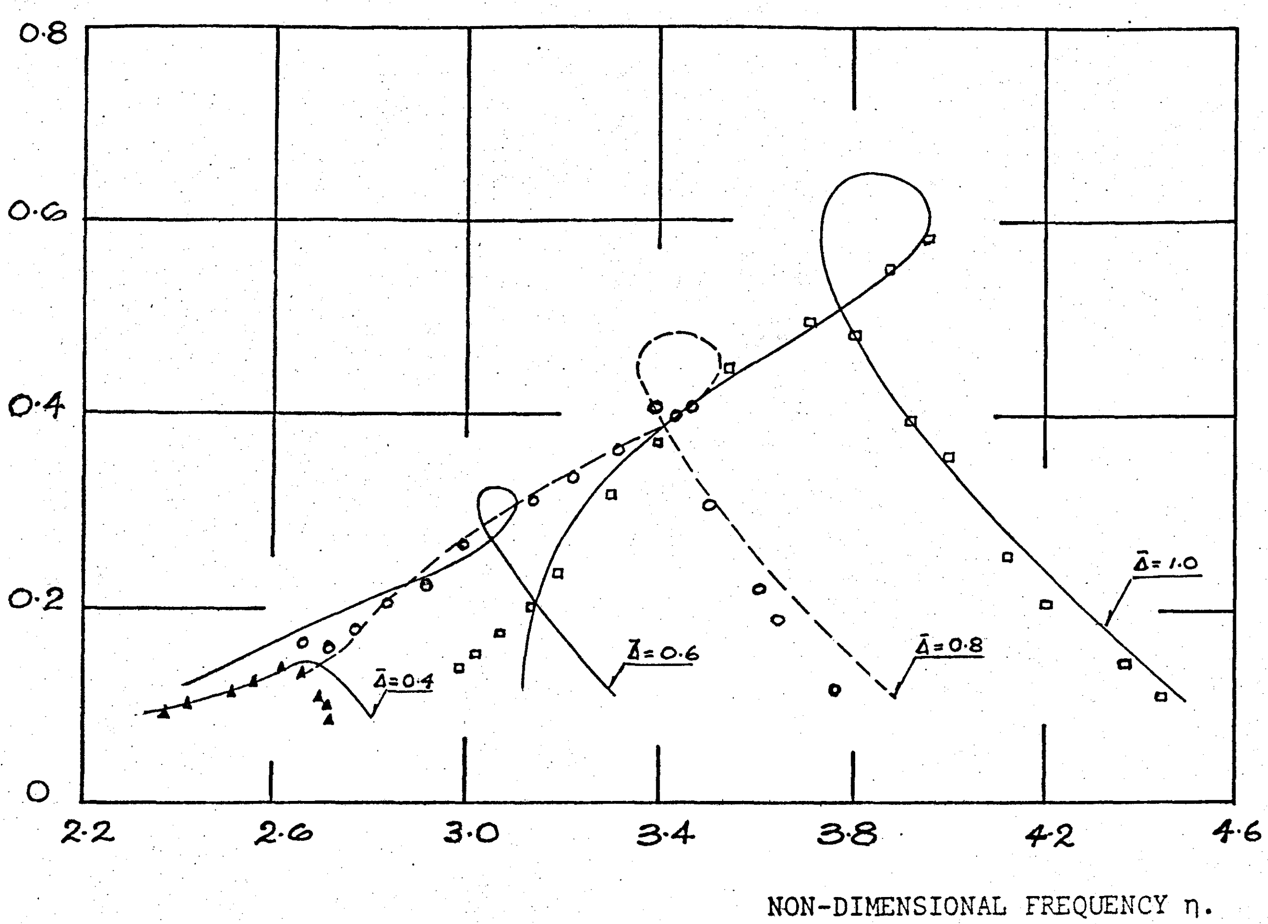


FIG. (6.4-7). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{N}$. $\bar{Z}$ = 0.4, R = 0.15.

- - - ——— THEORETICAL RESULTS.

□ $\bar{\Delta}$=1.0, ○ $\bar{\Delta}$=0.8, ▲ $\bar{\Delta}$=0.4. EXPERIMENTAL RESULTS.

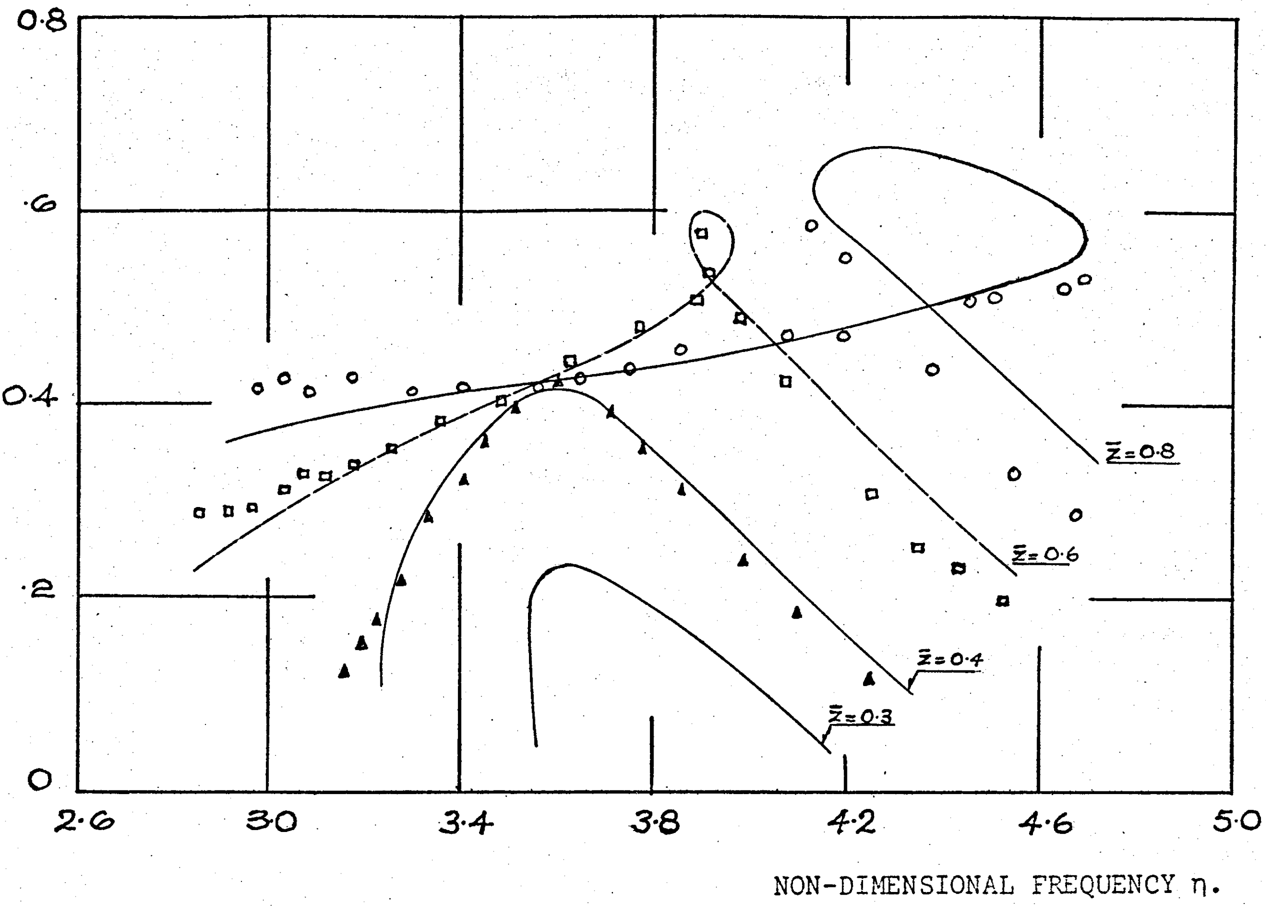


FIG.(6.4-8). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{N}$. $\bar{\Delta}$ = 1.0, R = 0.25.

— — — ———— THEORETICAL RESULTS.

▲ $\bar{Z}$=0·4, ▫ $\bar{Z}$=0·6, ○ $\bar{Z}$=0·8. EXPERIMENTAL RESULTS.

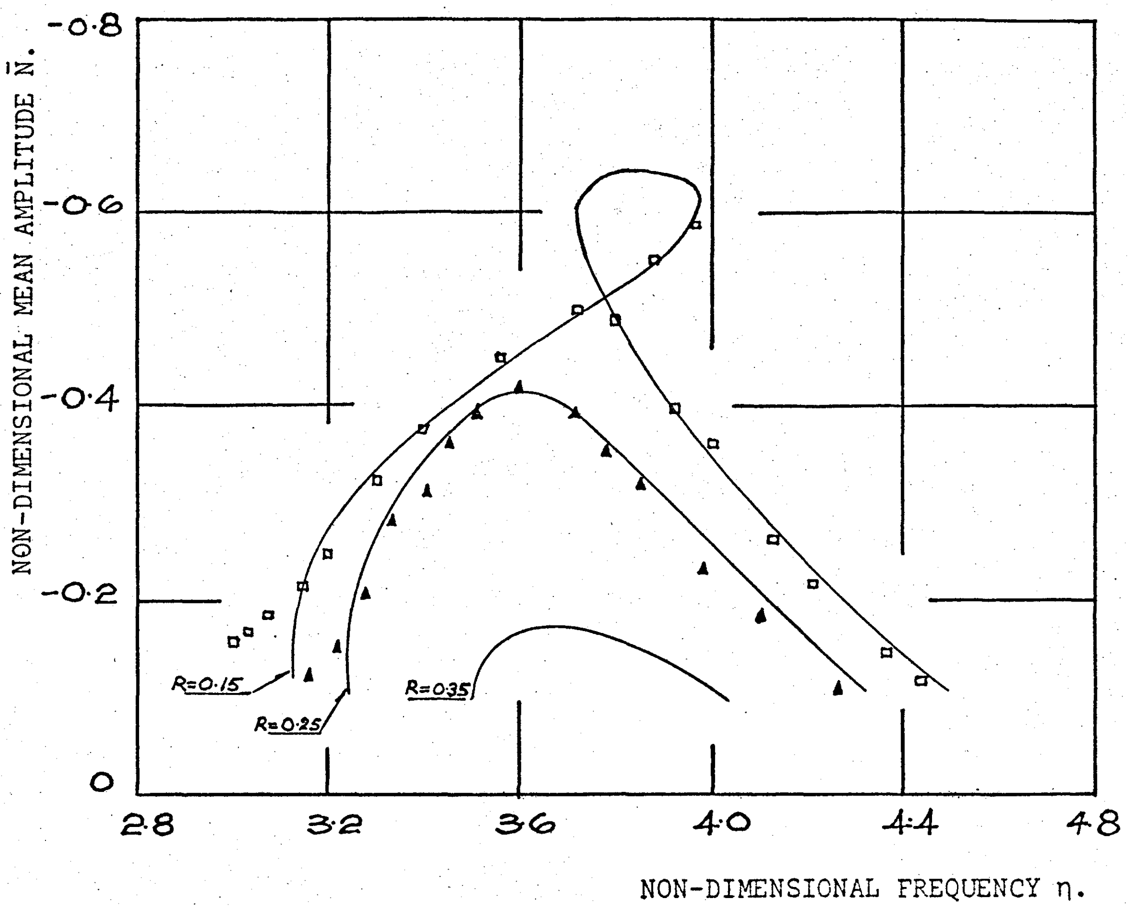


FIG.(6.4-9). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{N}$. $\bar{\Delta} = 1.0$, $\bar{Z} = 0.4$.

——— THEORETICAL RESULTS.

▫ ▲ EXPERIMENTAL RESULTS.

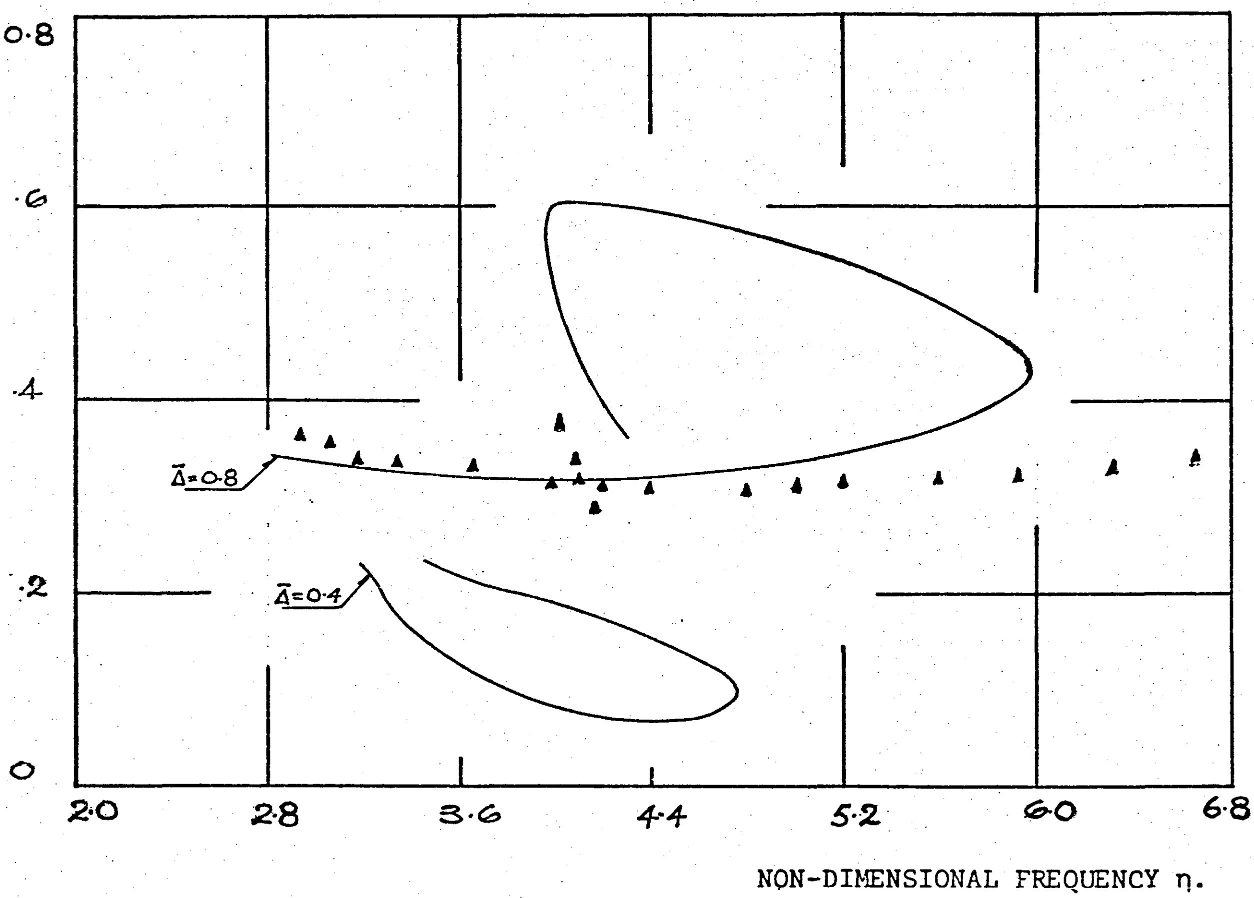


FIG.(6.4-10). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{N}$. $\bar{Z}$ = 0.8, R = 0.15.

——— THEORETICAL RESULTS.

▲ EXPERIMENTAL RESULTS.

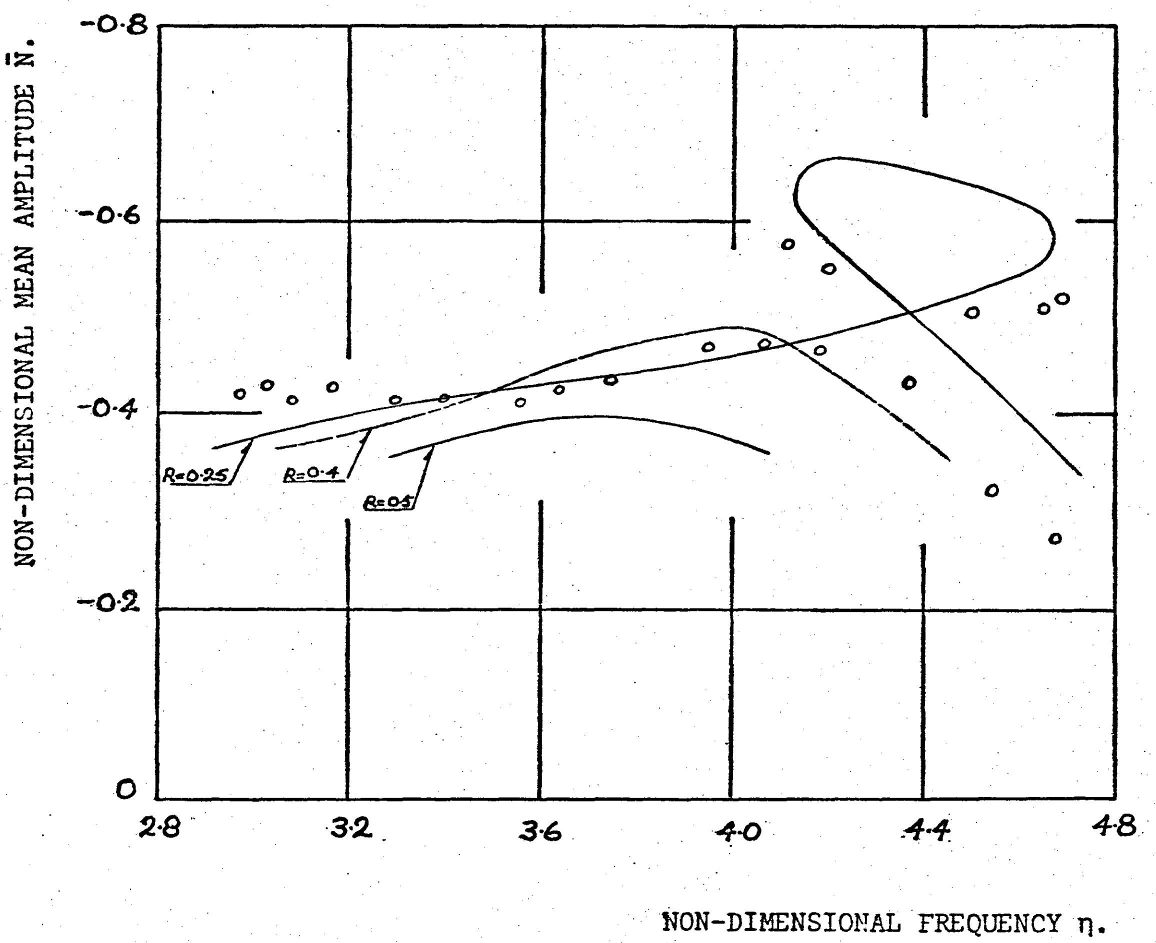
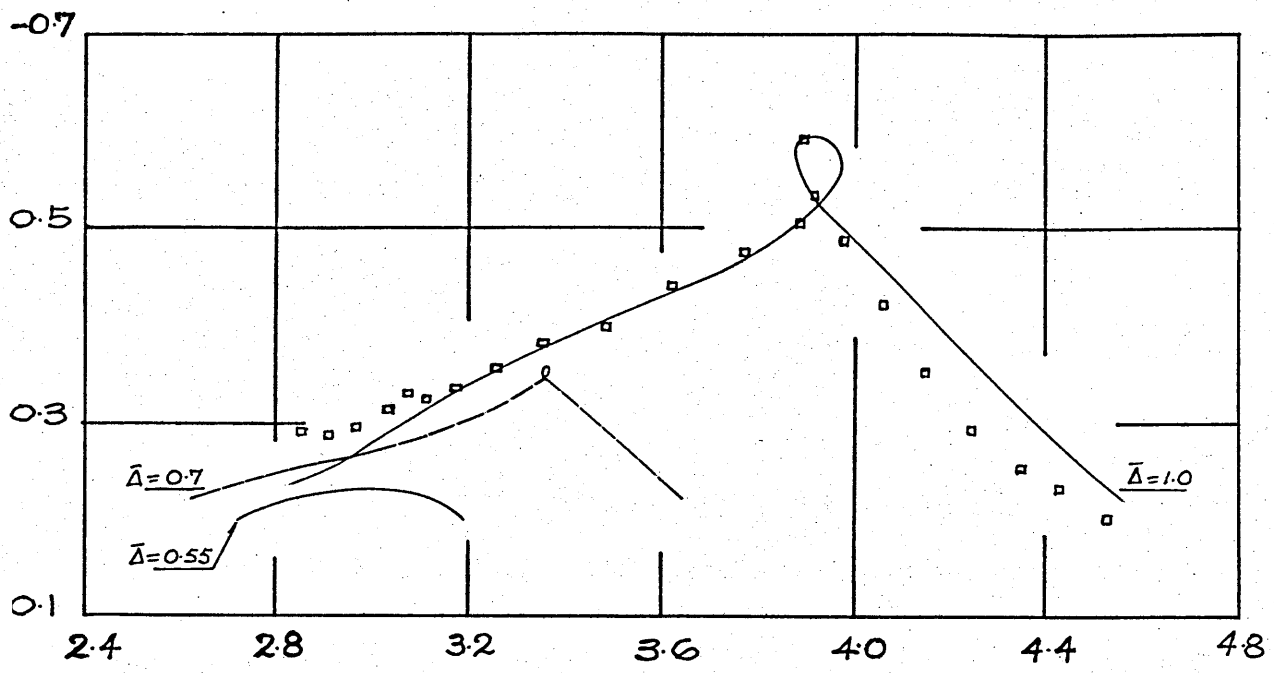


FIG. (6.4-11). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{N}$. $\bar{\Delta} = 1.0$, $\bar{Z} = 0.8$.

---- ——— THEORETICAL RESULTS.

o R=0.25 EXPERIMENTAL RESULTS.



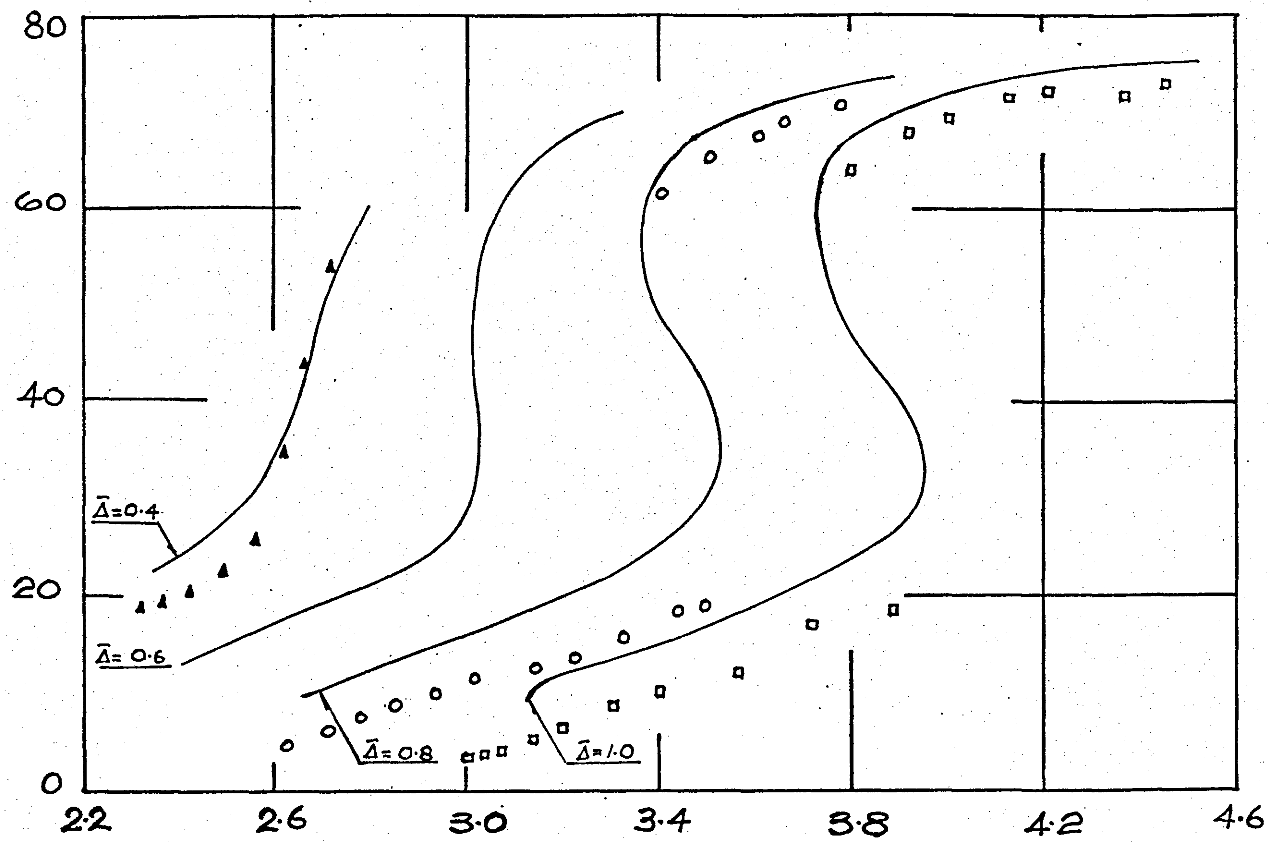
NON-DIMENSIONAL FREQUENCY η.

FIG.(6.4-12). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{N}$. $\bar{Z}$ = 0.6, R = 0.25.

---- ——— THEORETICAL RESULTS.

□ EXPERIMENTAL RESULTS.



NON-DIMENSIONAL FREQUENCY η.

FIG.(6.4-13). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\phi_{\frac{1}{2}}$. $\bar{\bar{Z}} = 0.4$, $R = 0.15$.

——————— THEORETICAL RESULTS.

▲ $\bar{\Delta}=0{\cdot}4$, ○ $\bar{\Delta}=0{\cdot}8$, ◻ $\bar{\Delta}=1{\cdot}0$. EXPERIMENTAL RESULTS.

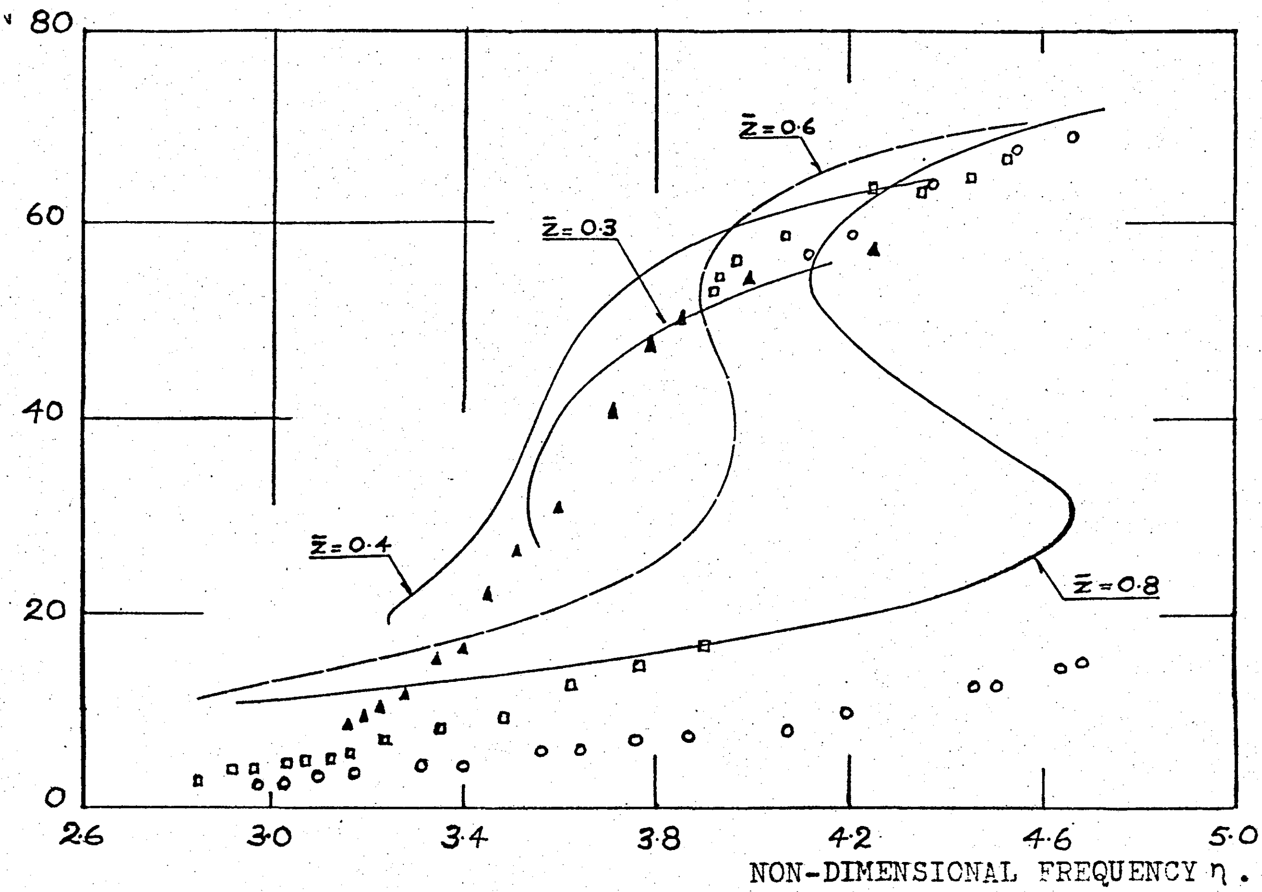


FIG.(6.4-14). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\phi_{\frac{1}{2}}$. $\bar{\Delta}$ = 1.0, R = 0.25.

— — — ———— THEORETICAL RESULTS.

▲ $\bar{z}$=0.4, ▫ $\bar{z}$=0.6, ○ $\bar{z}$=0.8. EXPERIMENTAL RESULTS.

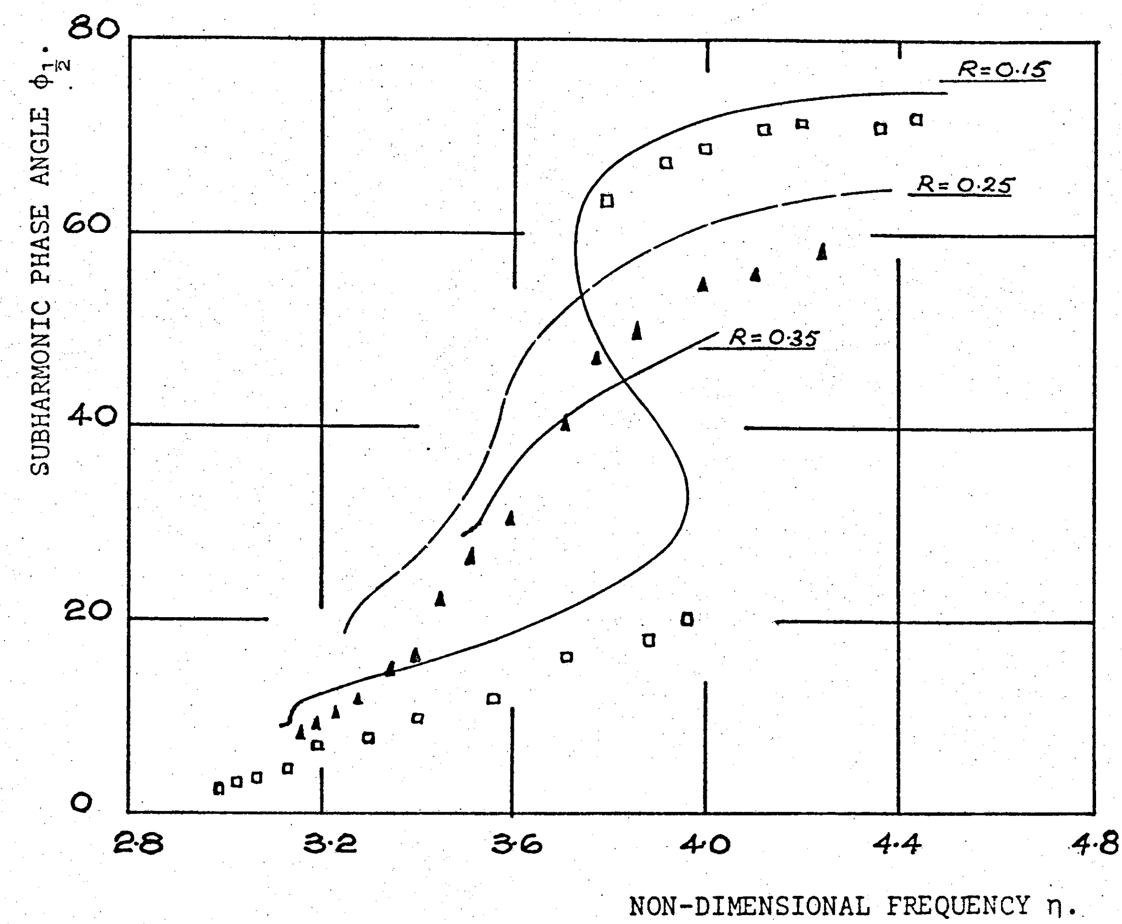


FIG.(6.4-15). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\phi_{\frac{1}{2}}$. $\bar{\Delta} = 1.0$, $\bar{Z} = 0.4$.

— — — ——— THEORETICAL RESULTS.

▲ R=0·25, □ R=0·15. EXPERIMENTAL RESULTS.

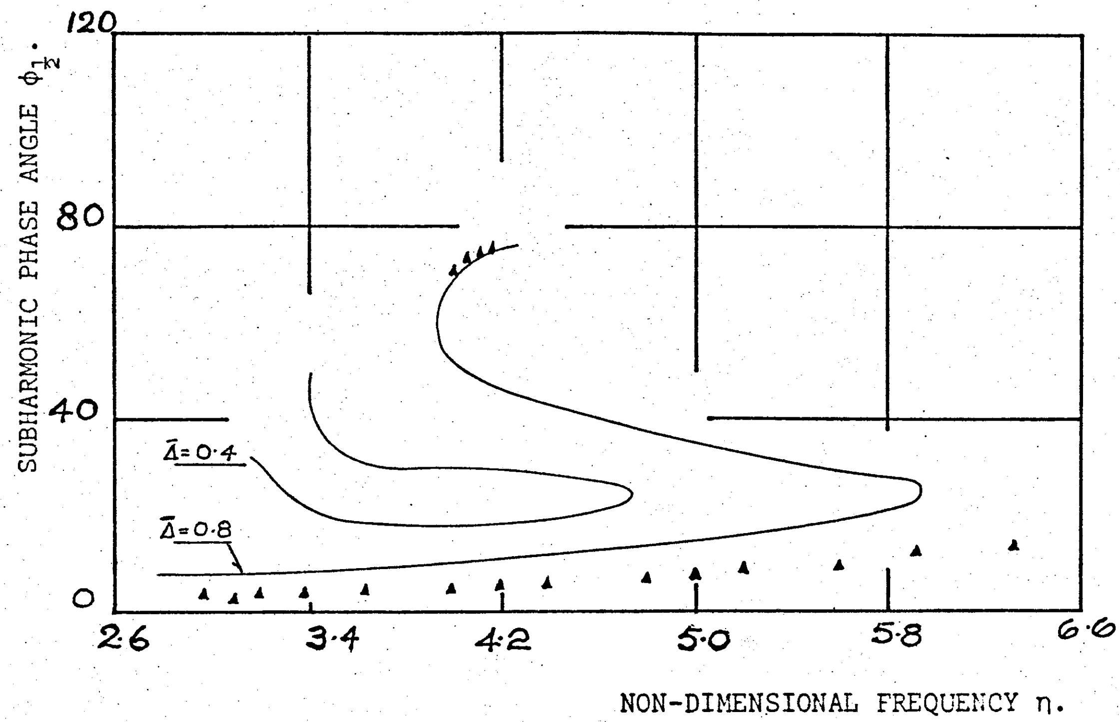


FIG. (6.4-16). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\phi_{\frac{1}{2}}$. $\bar{Z}$ = 0.8, R = 0.15.

——— THEORETICAL RESULTS.

▲ EXPERIMENTAL RESULTS.

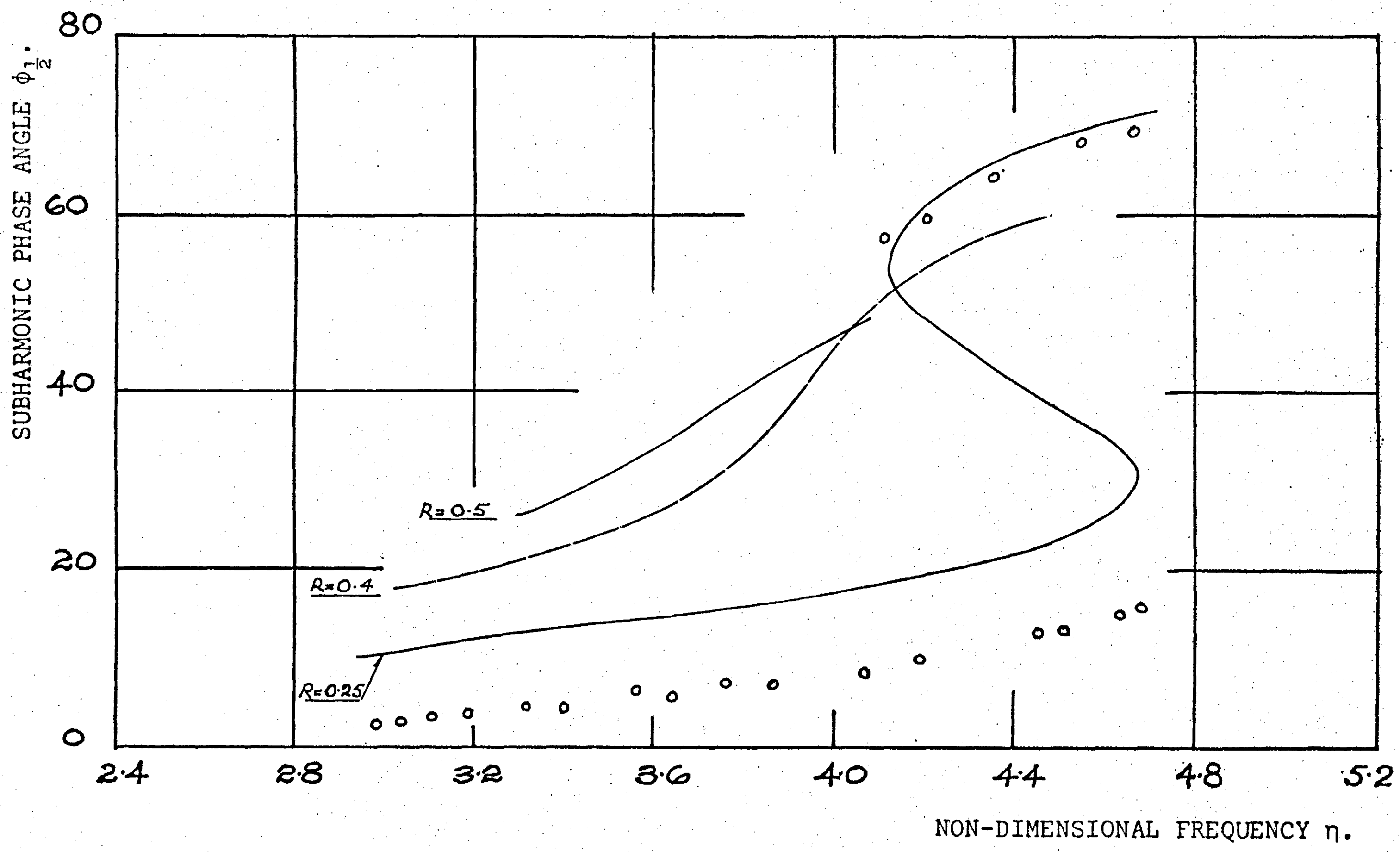
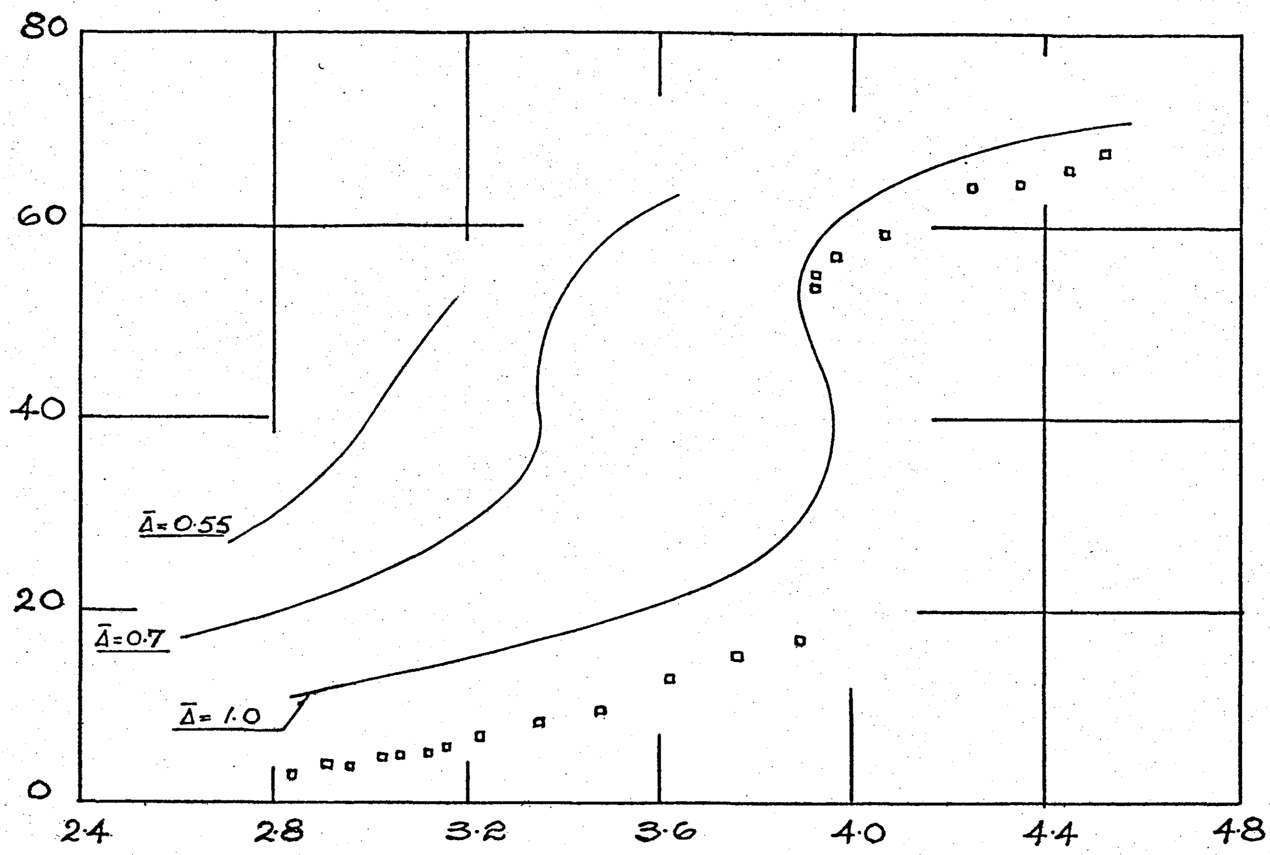


FIG.(6.4-17). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\phi_{\frac{1}{2}}$. $\bar{\Delta} = 1.0$, $\bar{Z} = 0.8$.

— — — ——— THEORETICAL RESULTS.

o EXPERIMENTAL RESULTS.



NON-DIMENSIONAL FREQUENCY η.

FIG.(6.4-18). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\phi_{\frac{1}{2}}$. $\bar{Z}$ = 0.6, R = 0.25.

——— THEORETICAL RESULTS.

□ EXPERIMENTAL RESULTS.

6.5. The transient solution

6.5.(i). Numerical analysis

From the consideration of chapter III, section (3.4), the characteristic behaviour of the subharmonic coefficients during the transient state of motion can be determined from equations (3.4,6) and (3.4,9). Since the time which is expressed in θ does not appear explicitly in the latter equation, the integral curves of the approximate transient solution are conveniently plotted on the AB plane. Ultimately, with the lapse of time, equation (3.4,9) would become zero to yield singular points which correlate with the coefficients of the steady state periodic solution and hence to give the amplitude of the subharmonic component, Q_1.

It is shown by the results of the vibrational characteristics of section (6.3) that the vibration cannot actually exist along with the resonant condition of the accompanying harmonics. An accumulative effect would result over this period from an oscillation that subsequently destroys the motion. Since the accompanying harmonic components are restricted to oscillations of a nonresonant nature, it is not out of order to make a reasonable approximation for Q_1 independent of periodicity as readily seen by the results of the preceding section. Although this approximation is restricted to systems in which the effective non-linearity does not increase with resonance, it is however quite adequate for the intended purpose of this section. Thus, with the substitution of equation (6.4,6), equations (3.4,6) to (3.4,9) yield

$$N^3 + 3AN^2 + \{ 1 + 3A^2 + \frac{3}{2}(A^2 + B^2) + \frac{8}{3}Z^2 \} N + 1.5A$$

$$\{ A^2 + B^2 + \frac{16}{9} Z^2 \} - Z \{ A^2 - B^2 \} = 0 \quad ..... \quad (6.5,1)$$

$$\frac{dB}{d\theta} = \frac{1}{\eta^2} \{ \frac{A}{4} \eta^2 - R\eta B - A(1 + 3A^2) - \frac{3}{4} A(A^2 + B^2) - 3N^2 A$$
$$- \frac{8}{3} AZ^2 + 4NAZ - 3A(2NA - \frac{4}{3} AZ) \} \quad ..... \quad (6.5,2)$$

$$\frac{dA}{d\theta} = \frac{1}{\eta^2} \{ - \frac{B}{4} \eta^2 - R\eta A + B(1 + 3A^2) + \frac{3}{4} B (A^2 + B^2)$$
$$+ 3N^2 B + \frac{8}{3} BZ^2 + 4NBZ + 3A(2NB + \frac{4}{3} BZ) \} \quad ..... \quad (6.5,3)$$

and for the singular points, the integral curves of

$$\frac{dA}{dB} = \frac{\text{Equation (6.5,3)}}{\text{Equation (6.5,4)}} \quad .... \quad (6.5,4)$$

must ultimately with the lapse of time yield singularities in which equation (6.5,4) is equated to zero.

When steady-state is reached, equation (6.5,4) is zero and the magnitude of the singular point on the AB plane corresponds to the amplitude of the subharmonic component $Q_{\frac{1}{2}}$.

It is evident from the above equations that if initial values of A and B are prescribed, the value of Z is readily determined from equation (6.5,1). Subsequently the integral curves of equation (6.5,4) that depict the transient state of the approximated solution can be drawn for any initial condition given to the subharmonic coefficients. As the calculations

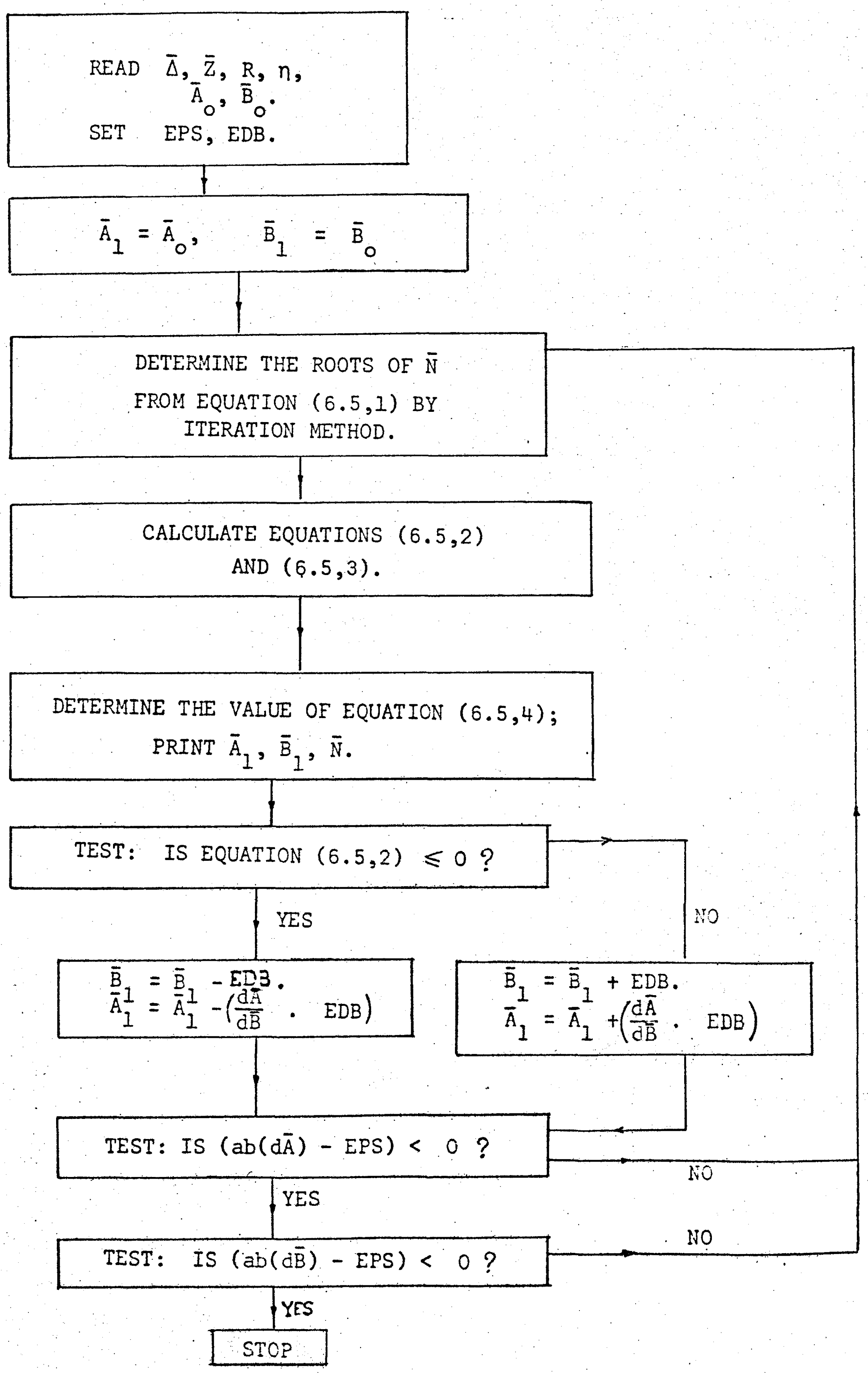


FIG.(6.5-1). FLOW DIAGRAM FOR THE INTEGRAL CURVES OF EQUATION (6.5,4).

involved are rather laborious a computing program is developed and the general flow diagram is shown in Fig.(6.5-1).

6.5.(ii). Discussion

Figs.(6.5-2) and (6.5-3) show for a set of parameter values the singular points of a selected frequency. The simultaneous algebraic equation, which describes the subharmonic vibration as a quadratic squared in $Q_{\frac{1}{2}}$, gives the four stable foci, as illustrated in Figs.(6.5-3) by the points 1, 2, 5 and 6. The distance between the singularities and the origin determines the amplitude of the $Q_{\frac{1}{2}}$ response and it is readily seen the respective magnitudes are identical to the $Q_{\frac{1}{2}}$ components determined from the equations (6.4,7) and (6.4,8).

From the results of the previous analysis, the two roots of a selected frequency are shown to exist within the first quadrant. However, it was mentioned during the analysis of the algebraic expressions (3.2,14), (3.2,15) and (3.3,9), (3.3,10) that the predominant component of subharmonic vibration can also exist in a phase quadrant differing by π radians. This is illustrated by Fig.(6.5-3). Since the singular points 1 and 2 which have the same amplitude are of the opposite phase, it implies that the motion can exist in either the first or the third quadrant. and that the resulting second order subharmonic vibration depends upon the initial conditions of the coefficients A and B. The angular distance between the singular points of the same amplitude corresponds to one cycle of the disturbing force.

Again with reference to equations (3.2.14) and (3.2.15) one possible solution is for Q_s zero. Although such a solution is meaningless it is represented by the singular point 7, at the origin. Since the singularity is unstable this means all the initial conditions of A, B lead to a response of subharmonic vibration. The phase of the resulting motion is given by the relationship that is illustrated in Fig. (6.5-3).

In either phase of the response there is a singularity of a saddle point, and its amplitude corresponds to the value in the unstable region. The singular points 4 and 5 are inherently unstable for reasons already discussed in sections (6.3) and (6.4), and the integral curves which pass through them divide the plane into stable and unstable regions. Thus the amplitude and phase of the response is determined by the region in which the initial conditions are.

Since the constant term N also varies with Q, the integral curves can be represented in a three dimensional plane ABZ. However, the curves of Figs.(6.5-2) and (6.5-3) demonstrate adequately the intended purpose of investigating the transient solution and there is little justification in applying the more laborious procedure. The values of the singularities are shown in Fig.(6.5-4).

It is evident the phase-plane method is not the most suitable to employ in terms of producing continuous results. Besides there are limitations to the method. It is not applicable if there are a number of steady state responses and the initial conditions must be prescribed in the region of steady state for the assumption that the amplitude and phase of the oscillation vary slowly to be valid. In addition to the assumption that the second differential of the predominant amplitude components are insignificant, the use of the phase-plane analysis in the investigation of the problem restricts the damping coefficient to small magnitude. In the application of this method, there is also the difficulty of examining the influences of the independent parameters in relation to the overall results.

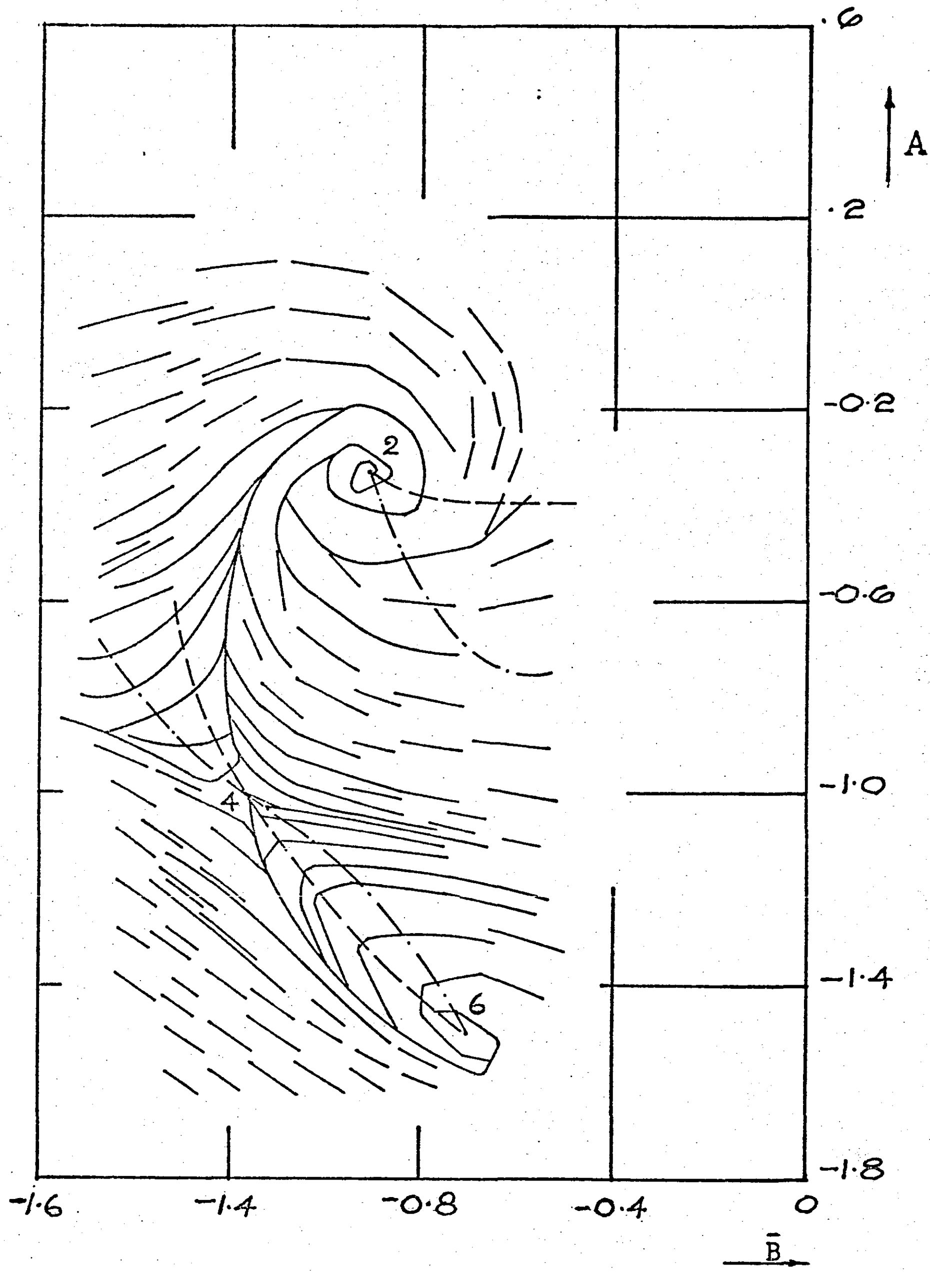


FIG.(6.5-2). INTEGRAL CURVES OF EQUATION (6.5,4) IN THE $\bar{A}\bar{B}$ PLANE.

SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED FOR SYSTEM PARAMETERS $\bar{\Delta} = 1.0$, $\bar{Z} = 0.4$, $R = 0.15$ AND $\eta = 3.84$.

SINGULAR POINT 2. $\bar{Q}_{\frac{1}{2}} = 0.97$, $\bar{N} = -0.453$.

SINGULAR POINT 4. $\bar{Q}_{\frac{1}{2}} = 1.53$, $\bar{N} = -0.63$.

SINGULAR POINT 6. $\bar{Q}_{\frac{1}{2}} = 1.65$, $\bar{N} = -0.53$.

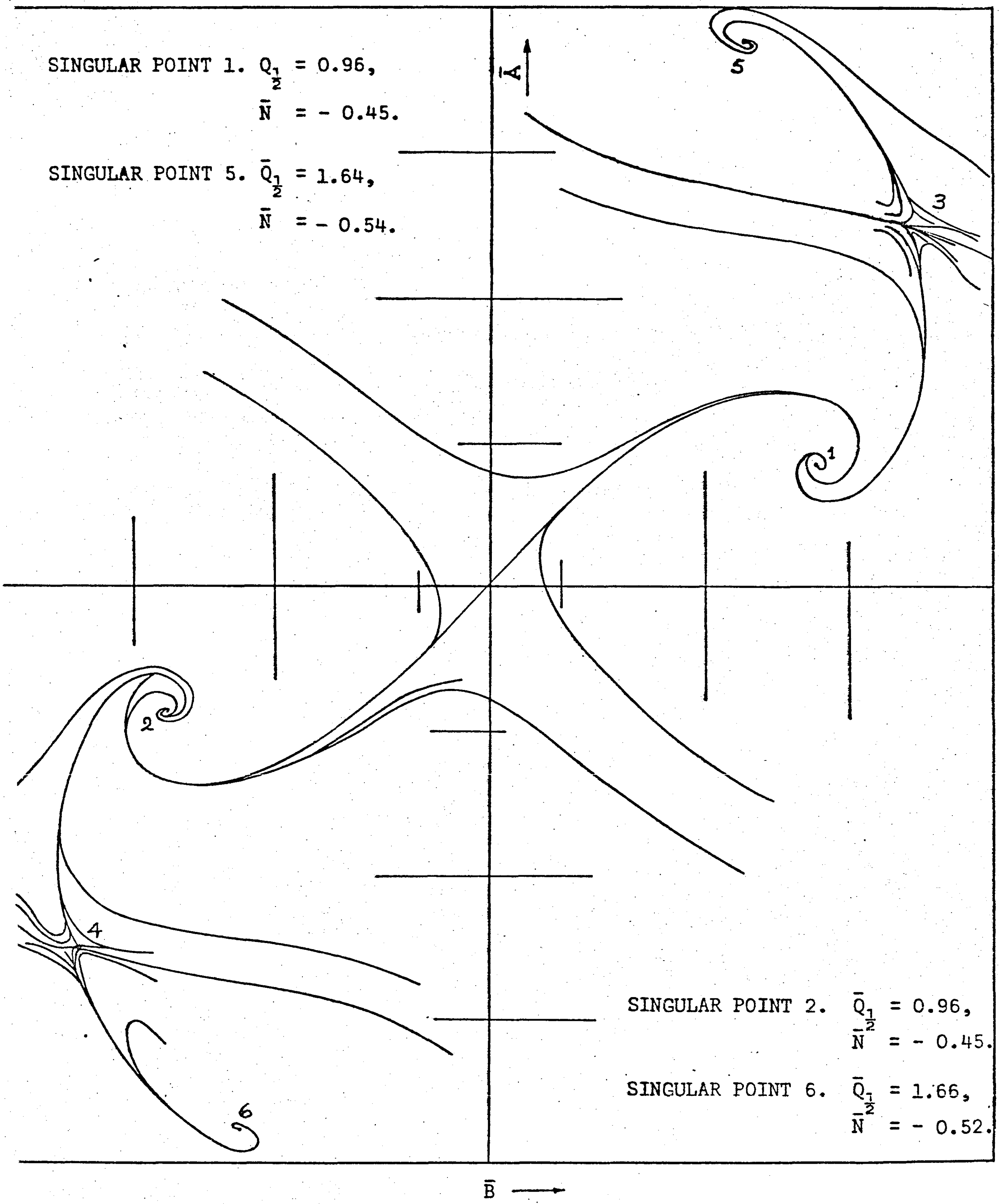


FIG.(6.5-3). INTEGRAL CURVES OF EQUATION (6.5,4) IN THE $\bar{A}\bar{B}$ PLANE.

SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED FOR SYSTEM PARAMETERS $\bar{\Delta}$ = 1.0, $\bar{Z}$ = 0.4, R = 0.15 AND η = 3.84.

SINGULAR POINTS	η	$\bar{A}$	$\bar{B}$	$\bar{Q}_{\frac{1}{2}}$	$\bar{N}$
1	3.84	0.34	0.904	0.965	- 0.449
2	3.84	- 0.344	- 0.9035	0.966	- 0.4503
3	3.84	1.018	1.13	1.52	- 0.615
4	3.84	- 1.0	- 1.13	1.51	- 0.62
5	3.84	1.48	0.685	1.64	- 0.538
6	3.84	- 1.502	- 0.704	1.66	- 0.522

TABLE.(6.5-4). THE SINGULAR POINTS OF FIG.(6.5-3),
FOR $\bar{\Delta} = 1.0$, $\bar{Z} = 0.4$, $R = 0.15$.

Chapter VII

Experimental Design

7.1. Initial consideration

In the early stages of considering the form of an actual model that represents equation (3.1,8) with sufficient accuracy, it is decided that this is attained with the least difficulty through limiting the eventual physical size. Besides, often in practise the out-of-balance force experienced is relatively small.

The initial notion is for a system of vertical displacement and in particular the design of non-linear spring supports by Molyneux (34) appears to give the requirements of the restoring forces characteristics. However, the concept appears not practical for there is the problem of maintaining only a single-degree-of-freedom system if the mass is not to be guided in some way. The inevitable restriction will likely create serious doubts at a later stage as to the condition of viscous damping and hence the tolerable allowance in the approximation of the restrictive forces.

From the need of avoiding the use of such restraint, the physical model of the required accuracy becomes a more feasible proposition by having an inertia body executing angular vibration about a pivot. The angular displacement results are readily transformed to its equivalent motion.

The general consideration of the next stage is the form of the non-linear restoring forces and the frequency range of the system. The idea of using a combination of linear springs to introduce the asymmetrical characteristics seems best. It readily allows a mathematical expression to be obtained for the non-linearity. At the same time, since the stiffness also influences the linear natural frequency, this gives a larger latitude in varying the frequency range.

There are other advantages in the use of linear springs. One of them is that the coefficient of non-linearity can be altered with ease. By merely either changing the initial stiffness rate or the pronounced hardening spring, the various magnitudes of the effects resulting from the influence of gravitational force on the system, if required, can be demonstrated. In many applications where the initial stiffness of a suspension increases rapidly when the displacement becomes larger, the relationship between the two stiffness rates principally defines the coefficient of non-linearity μ, and hence it also determines the magnitude of $\bar{A}$.

In addition, as the impressed vibration is by an out-of-balance centrifugal force, the magnitude of the excitation which is proportional to the square of the forcing frequency is more likely to be constant in the higher speed range and the combination of linear springs enables the natural frequency to be changed without difficulty to a suitable value, if it ever becomes necessary on account of small fluctuations of the disturbing force.

A fractional horse-power motor driving an out-of-balance mass seems perfectly adequate for generating the centrifugal disturbing force. If the design speed is sufficiently high, the problem of the non-uniformity in speed control will probably not arise. The driving torque in comparison to the small variable friction forces is considerably greater. However, necessary precautions will need to be considered to ensure the resolved

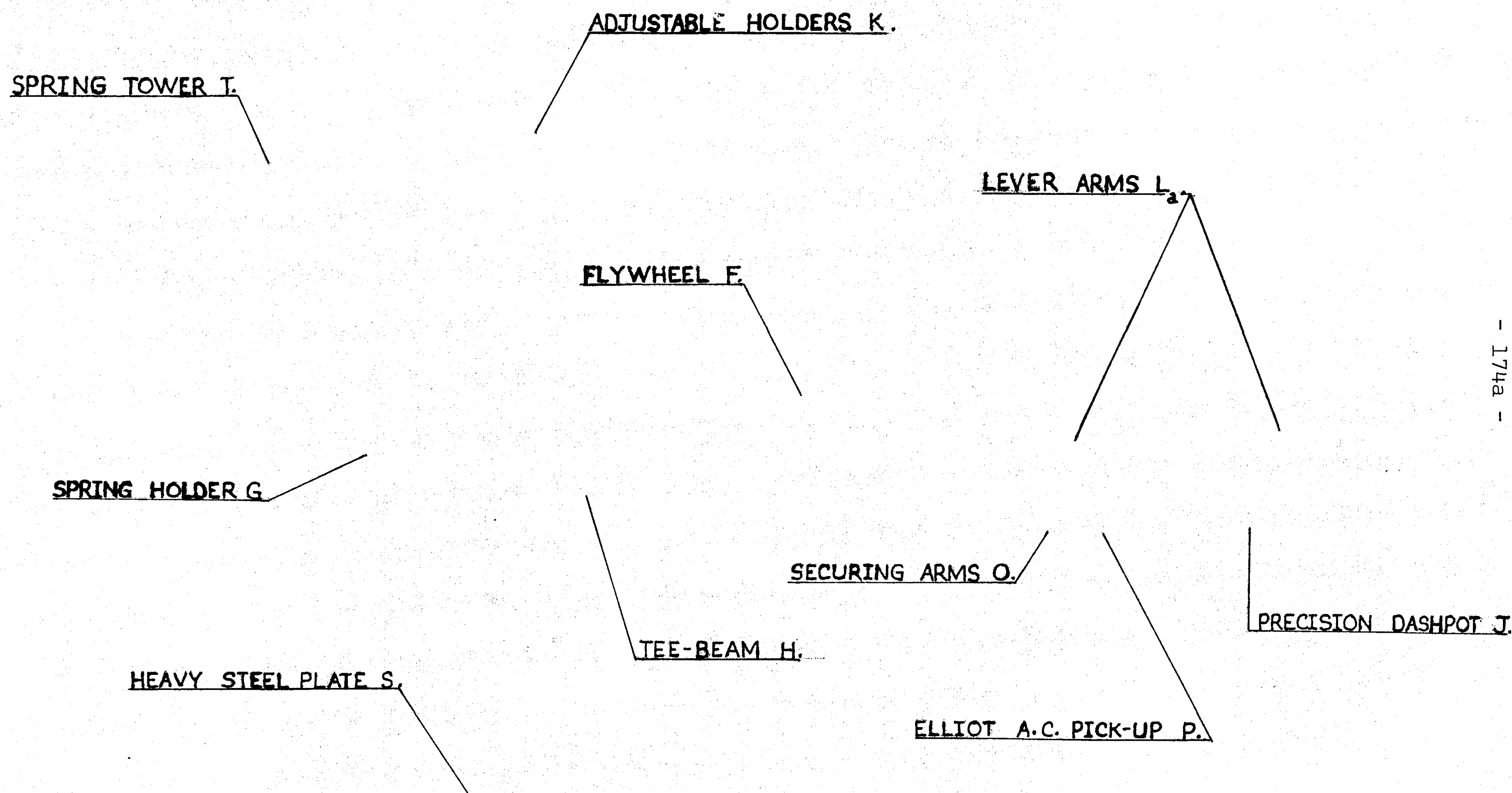
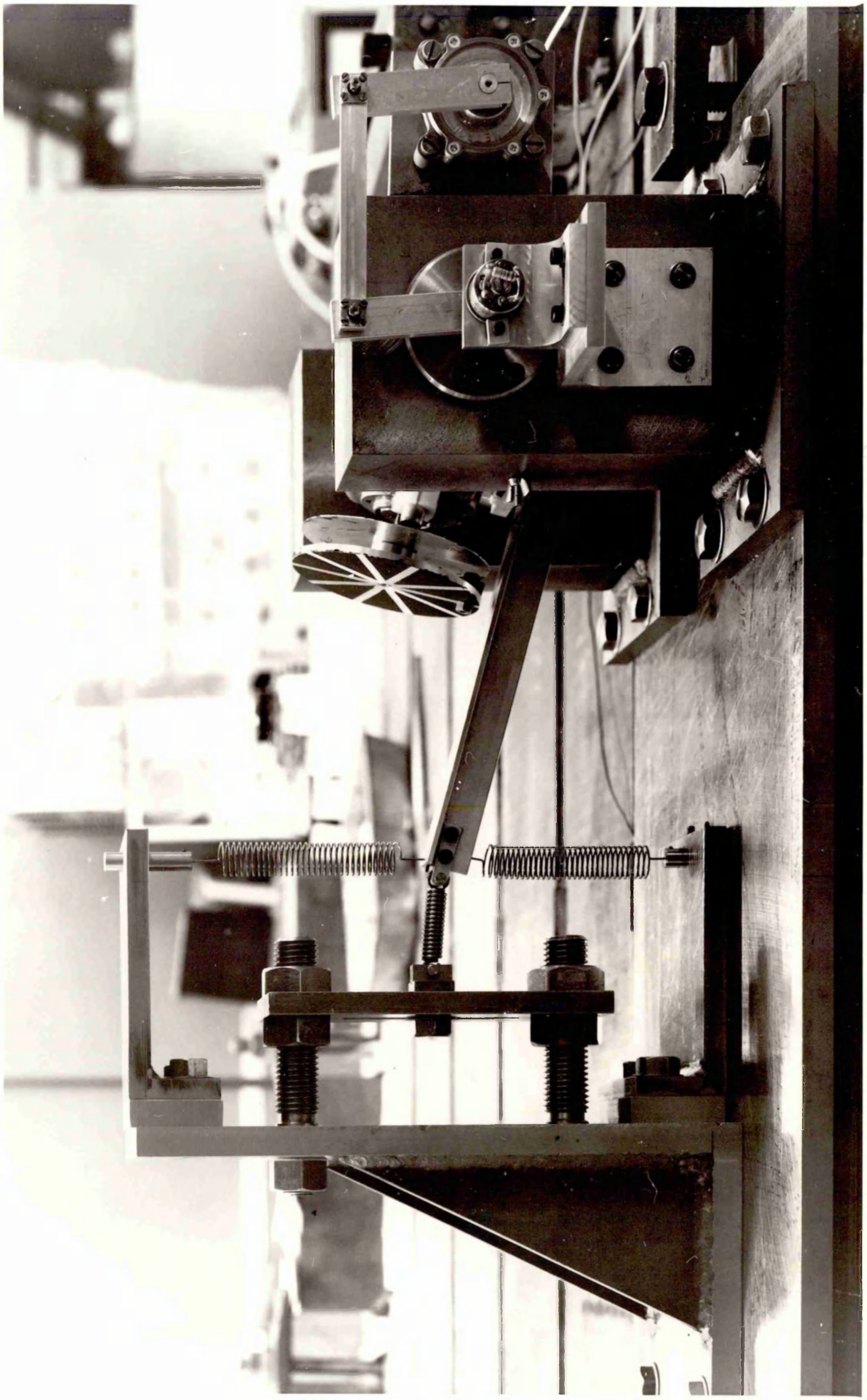


FIG. (7.1-1)(a). THE DESIGNED EXPERIMENTAL SYSTEM.



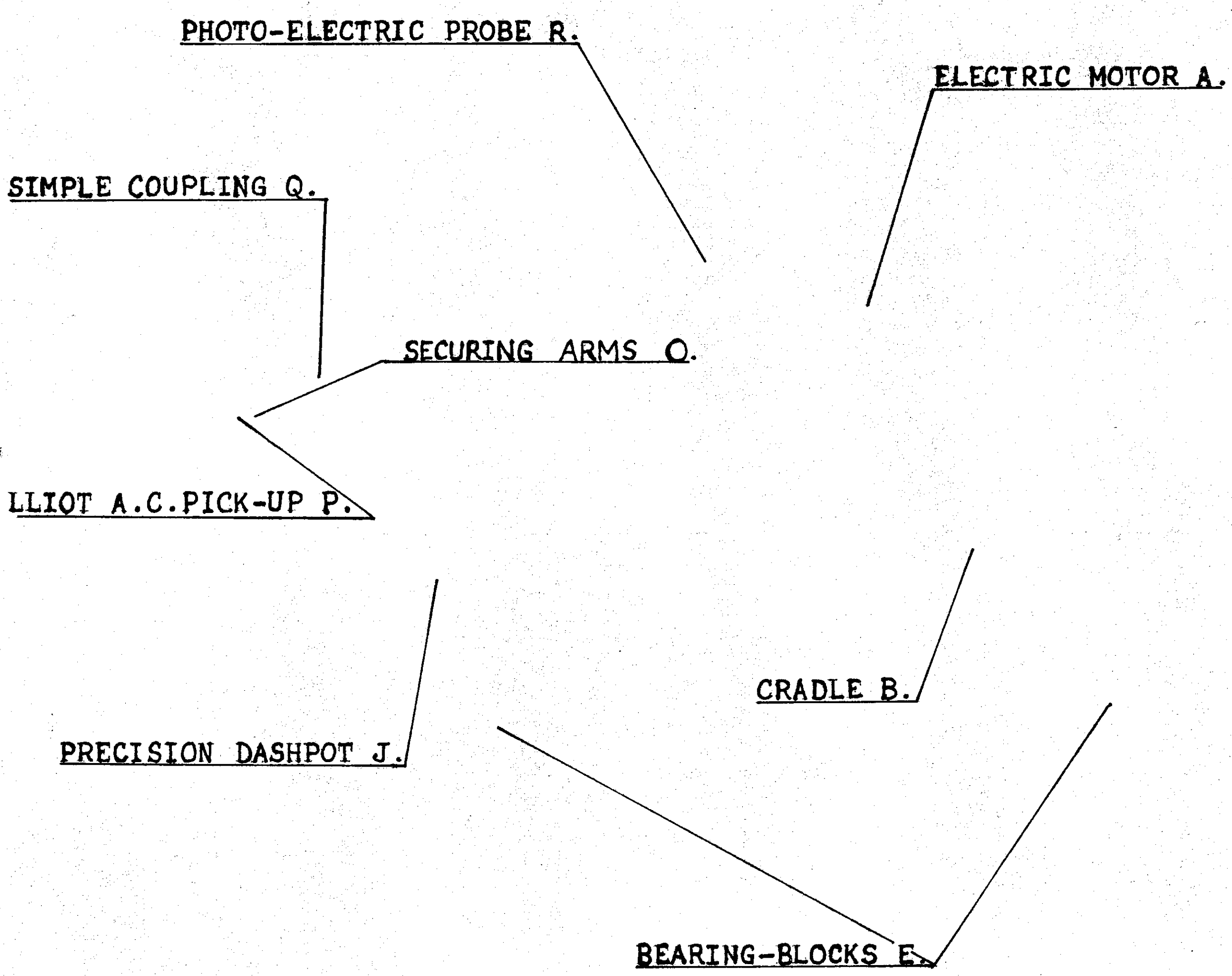
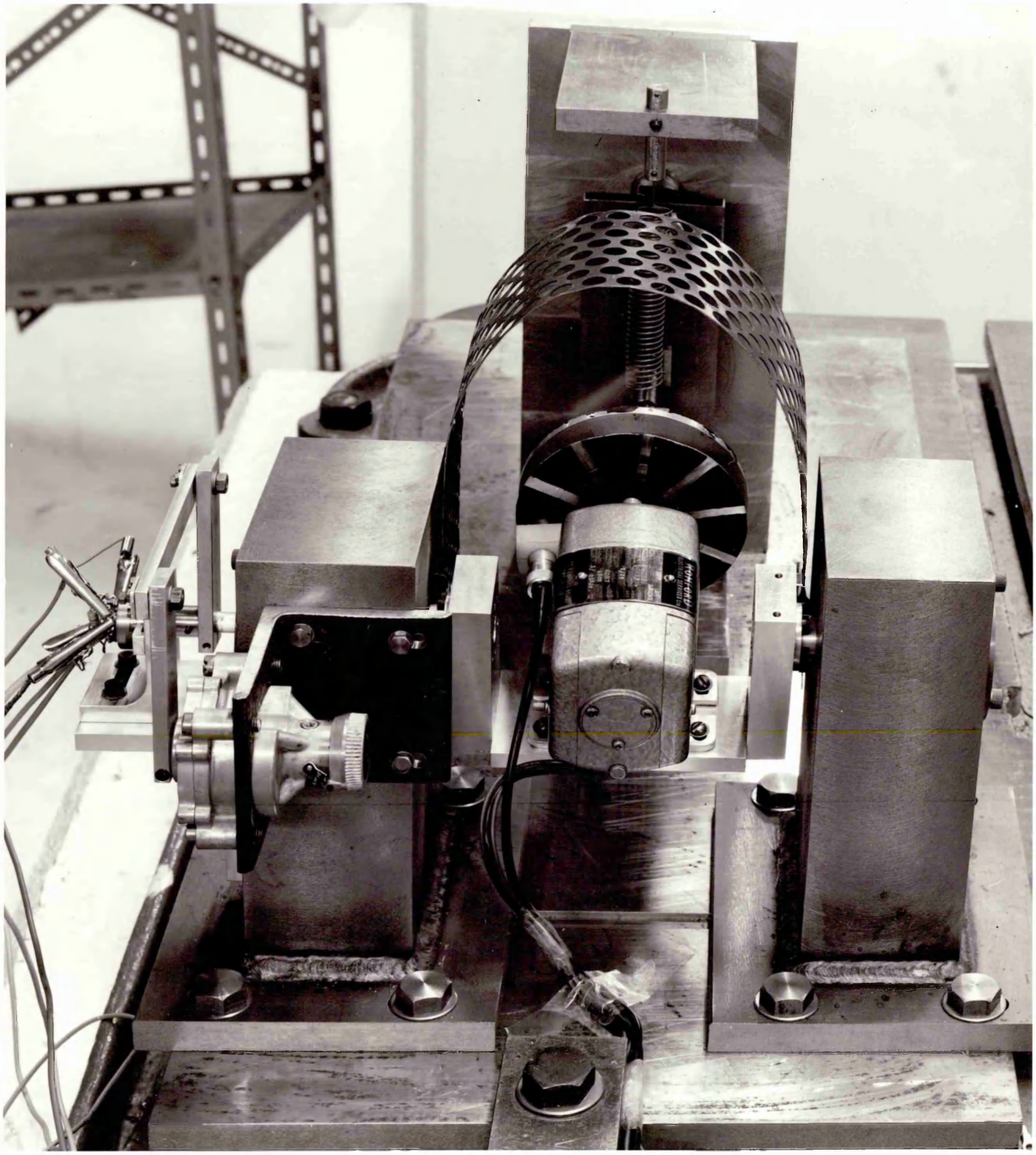


FIG.(7.1-1)(b). THE DESIGNED EXPERIMENTAL SYSTEM.



U. V. RECORDER X.

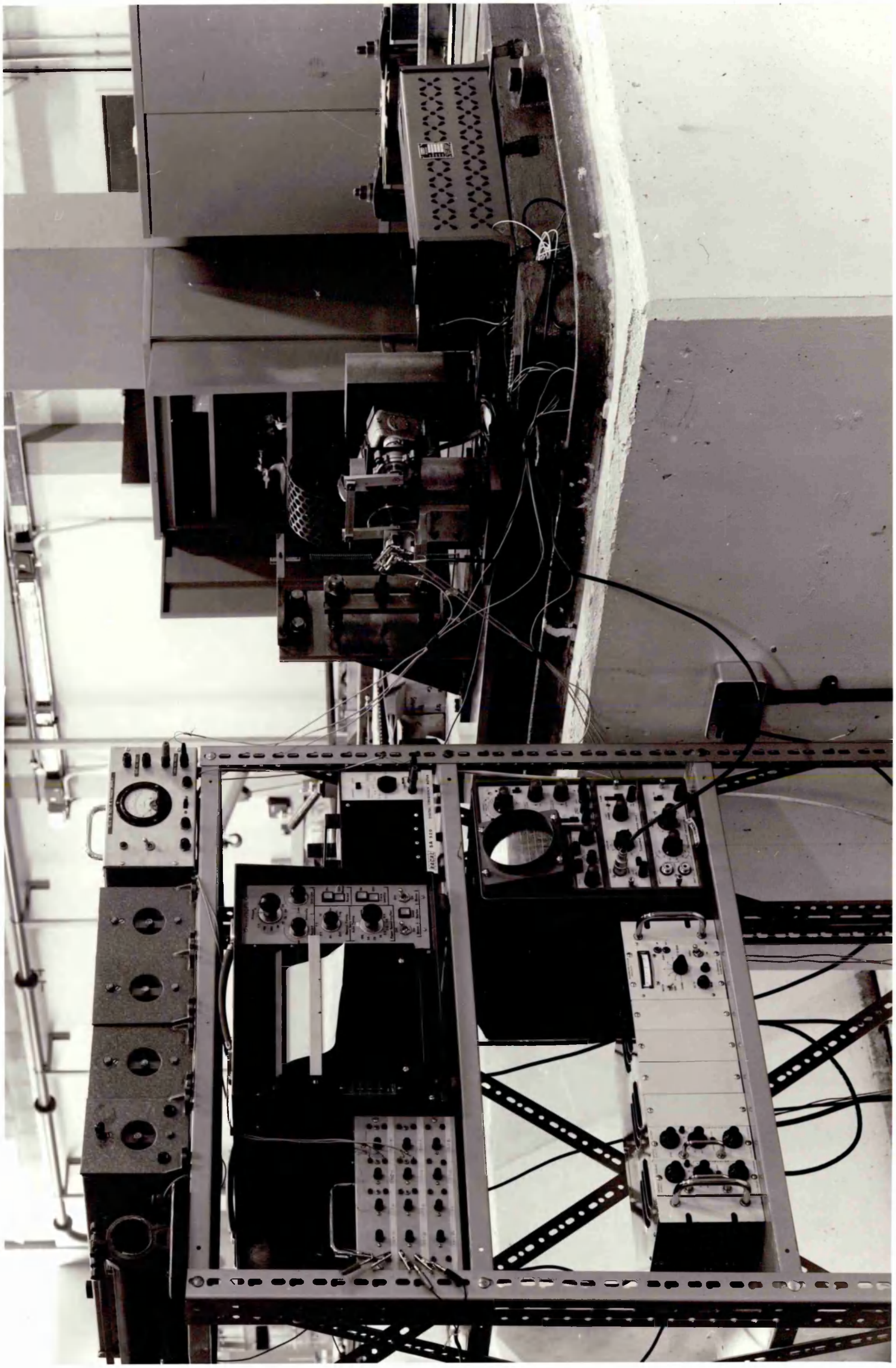
RESISTOR CIRCUIT BOX Y.

DEGITAL FREQUENCY METER W.

MOTOR SPEED CONTROL RESISTOR

RANK S.E. 429 CARRIER SYSTEM M.

FIG. (7.1-1)(C). THE DESIGNED EXPERIMENTAL SYSTEM.



force at right angle is not allowed to destroy the single-degree-of-freedom. With such measures taken into consideration a sinusoidal trace should then be obtained, depicting the uniformity of the disturbing force acting on a truely single-degree-of-freedom system.

The oscillating inertia is conveniently realised by an arrangement where the motor is suitably mounted on a beam. The whole system is to be pivoted on an axle at one end with the non-linear stiffnesses at the other.

The finalised form of the model which must have an equation of motion that is identical to the describing equation of Fig. (3.1-1) is shown in Figs. (7.1-1)(a) to (c).

7.2. Non-Linear restoring forces

7.2.(i). The mathematical expression of the non-linear stiffness

The combination of linear springs is illustrated in Fig. (7.2-1)(a). By logical design consideration, the arrangement of the springs in that order is similar to one used by Ludeke (32).

The vertical springs have equal stiffnesses of $\frac{K_1}{2}$ and each has an independent tension adjustment. This arrangement facilates the levelling of the transverse spring. The transverse spring has a stiffness K_2 and a length ℓ. Both its ends are mounted on plain light-weight bearings and an allowance for the attachment of the spring on to the beam gives a larger moment arm than the vertical springs, as seen from Fig.(7.2-1)(b).

If the whole system is deflected vertically from F to E through a distance δ then the tension in the transverse spring is given by

$$P = P_o + K_2 \{ (1 + \frac{DB^2}{\ell^2})^{\frac{1}{2}} \ell - \ell \} \quad \ldots\ldots\ldots \quad (7.2,1)$$

where P_o is the initial tension for $\delta = 0$.

Expanding the above equation gives

$$P = P_o + K_2 \{ (1 + \frac{1}{2}\frac{DB^2}{\ell^2} - \frac{1}{8}\frac{DB^4}{\ell^4} + \ldots\ldots) \ell - \ell \} \;..\; (7.2,2)$$

From the trigonometric identity of the configuration shown in Fig. (7.2-1),

$$DB = (L + c) \tan \phi. \quad \ldots\ldots\ldots \quad (7.2,3)$$

Subsequently, the design requirement is that the angle of displacement, ϕ, is restricted to

$$\tan \phi \approx (\phi + \frac{\phi^3}{3}) \quad \ldots\ldots\ldots \quad (7.2,4)$$

Hence, on substituting equations (7.2,3) and (7.2,4), the values of the ratio $\frac{\delta}{L}$ to the fourth power and higher orders, being very small, are neglected. Equation (7.2,2) then becomes

$$P = P_o + K_2 \{ (1 + \frac{1}{2\ell^2} (L + c)^2 \frac{\delta^2}{L^2}) \ell - \ell \}$$

$$= P_o + K_2 \{ \frac{1}{2\ell} (L + c)^2 \frac{\delta^2}{L^2} \} \quad \ldots \quad (7.2,5)$$

In view of the condition of equation (7.2,4), the angle θ is limited to $\sin \theta \approx \tan \theta$ that is the ratio of $\frac{1}{3} \cdot (\frac{L + c}{L})^2 \cdot \frac{\delta^2}{\ell^2}$ and higher powers are small and they can be neglected. The oscillating angle of the transverse spring at the instant of displacement is approximately given by

$$\sin \theta = \frac{1}{\ell} (L + c) \frac{\delta}{L} \quad \ldots\ldots\ldots \quad (7.2,6)$$

and, from the design of the attachment for the transverse spring, the initial tension P_o can be adjusted appropriately. Thus if P_o is reduced conveniently to zero, the vertical component of the restoring force exerted by the transverse spring in the sence BD is then given by

$$P_{BD} = \kappa_2 \{ \frac{1}{2\ell^2} (L + \varepsilon)^2 \cdot \frac{\delta^2}{L^2} \} (L + \varepsilon) \cdot \frac{\delta}{L} \quad \ldots\ldots (7.2,7)$$

The allowance, ε, for the attachment of the horizontal spring gives a bigger torque about the axis of oscillation. Consequently, the equivalent non-linear restoring force of the stiffness κ_2, acting in line with the vertical springs, is greater and it is expressed by

$$P_{EF} = \kappa_2 \{ \frac{1}{2\ell^2} \frac{(L + \varepsilon)^4}{L^4} \delta^3 \} \quad \ldots\ldots (7.2,8)$$

At that instant of the deflection, δ, the corresponding effective force of the vertical springs is $\kappa_1 \delta$. Thus, the full expression for the restoring forces exerted by the combination of springs in the vertical sence is

$$P_v = \kappa_1 \{ \delta + \frac{\kappa_2}{\kappa_1} \frac{1}{2\ell^2} \frac{(L + \varepsilon)^4}{L^4} \delta^3 \} \quad \ldots\ldots (7.2,9)$$

Hence, the non-linear stiffness for the arrangement of the springs is

$$\{ \kappa_1 + \frac{3\kappa_2}{2\ell^2} (\frac{L + \varepsilon}{L})^4 \delta^2 \} \quad \ldots\ldots (7.2,10)$$

Having determined the mathematical expression for the restoring force of the system, an equation of motion can now be obtained.

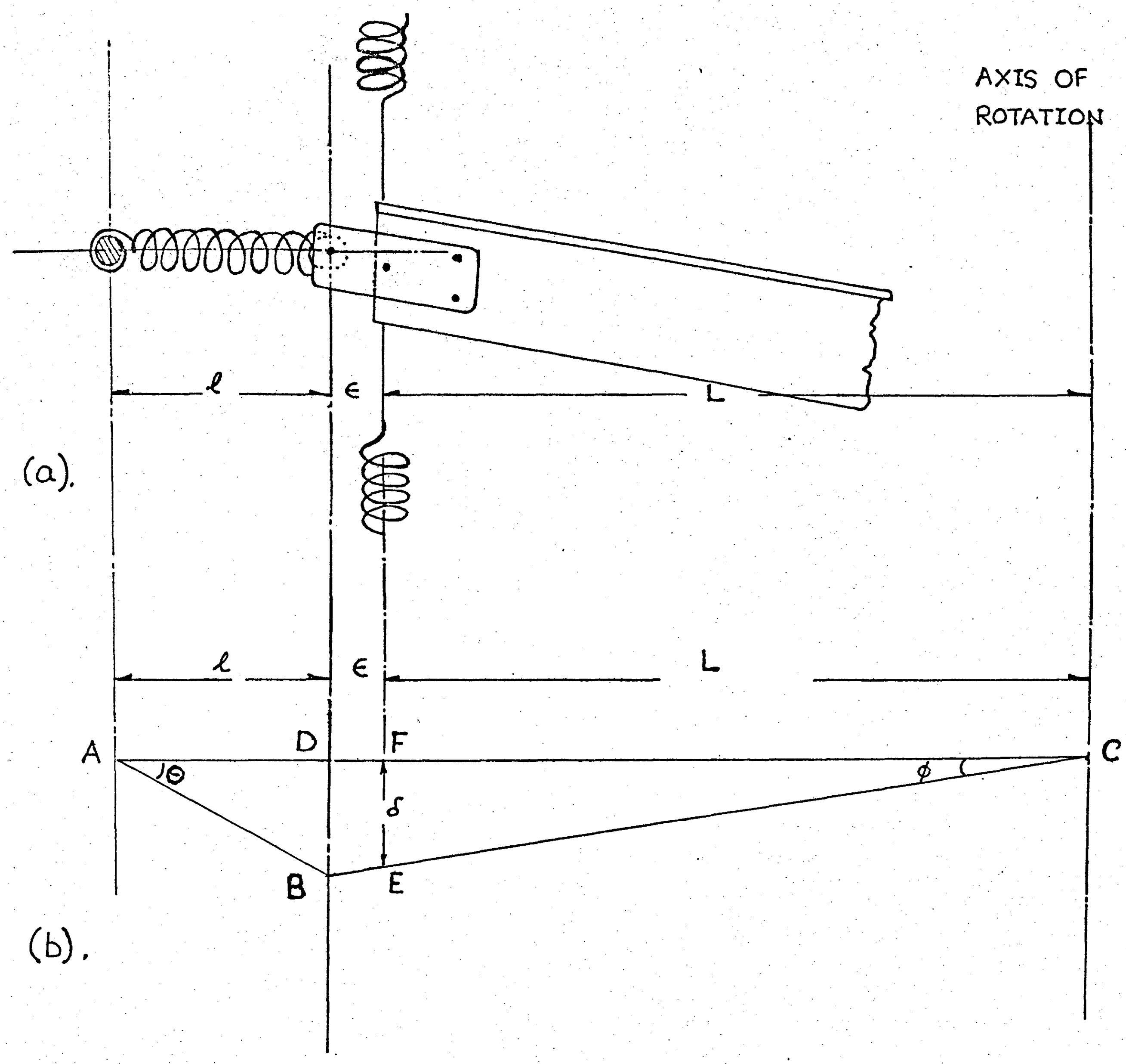


FIG. (7.2-1). THE NON-LINEAR RESTORING FORCE CONFIGURATION.

7.2.(ii). The describing equation of the designed system

If P_1 and P_2 are the initial tensions of the upper and lower vertical springs when the angle of deflection ϕ about the axis of rotation is zero, the torque acting on the system when it is statically deflected (assuming the principal axis of the motor and other masses of the cradle lie in line with the axis of rotation) is

$$(P_2 - P_1)\, L + M_b . L_b . g . a_b$$

where M_b is the mass per unit length of the beam, L_b is the length of the beam and a_b is the moment arm of the beam. This torque is then approximately equal and opposite to the torque resulting from the non-linear stiffnesses, that is

$$(P_2 - P_1)\, L + M_b . L_b . a_b . g \;=\; K_1 \left\{ \Delta + \frac{K_2}{K_1 2\ell^2} \left(\frac{L + \varepsilon}{L}\right)^4 \Delta^3 \right\} L \qquad \ldots\ldots \quad (7.2,11)$$

where Δ is the static deflection.

Hence, for a particular instant when the total deflection is $\delta = (x + \Delta)$, where x is the dynamic displacement measured from static equilibrium, the equation of motion describing the system of Fig. (7.1-1) is of the form

$$I_t\, \ddot{\phi} + C_1 \dot{\phi} + K_1 L \left\{ x + \frac{K_2}{K_1} \frac{1}{2\ell^2} \frac{(L + \varepsilon)^4}{L^6} (x^3 + 3\Delta x^2 + 3\Delta^2 x) \right\}$$

$$= m_0 \omega^2 r h \,.\, \cos \omega t \qquad \ldots\ldots \quad (7.2,12)$$

where I_t is the total inertia of the system,

C_1 is the coefficient of viscous damping,

m_o is the out-of-balance mass,

r is the distance of the mass centroid to the shaft centre

and h is the moment arm of the centrifugal force.

The relationship between the single-degree-of-freedom system executing angular oscillation and the system described by equation (3.1,7) is elementary. If m and C are the terms referring to the parameters of the equivalent system shown in Fig.(3.1-1) such that with reference to Fig.(7.2-2),

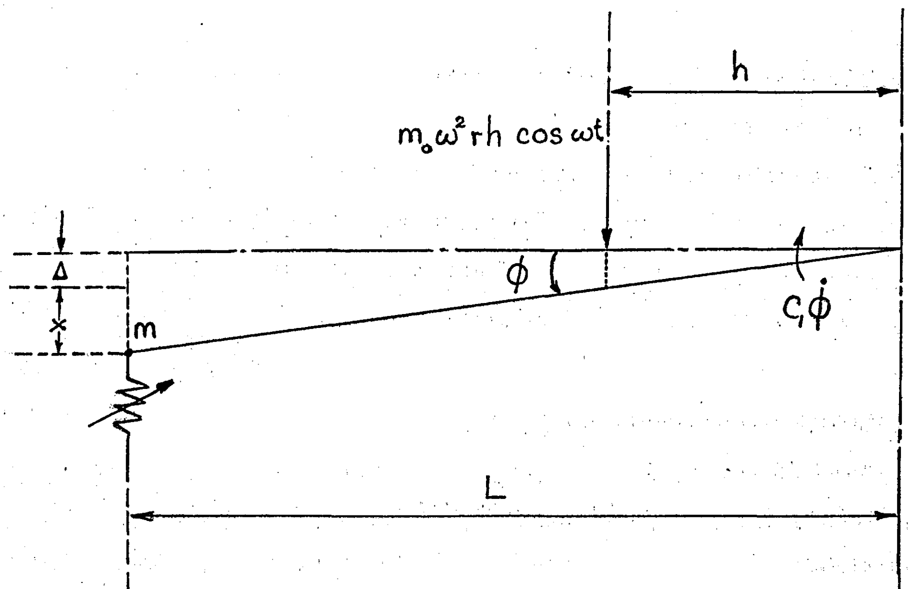


FIG.(7.2-2). THE VIBRANT CONFIGURATION OF THE EXPERIMENTAL SYSTEM.

$I_t = mL^2$ and $C_1 = CL^2$, the physical equation then becomes

$$mL\ddot{x} + CL\dot{x} + K_1 L \left\{ x + \frac{K_2}{K_1} \frac{1}{2\ell^2} \left(\frac{L + \ell}{L}\right)^4 (x^3 + 3\Delta x^2 + 3\Delta^2 x) \right\}$$

$$= m_o \omega^2 rh \cos \omega t \qquad \ldots\ldots\ldots \quad (7.2,13)$$

which on simplification reduces to

$$\ddot{x} + 2Kp\dot{x} + p^2 \{ x + \frac{K_2}{K_1} \frac{1}{2\ell^2} (\frac{L + c}{L})^4 (x^3 + 3\Delta x^2 + 3\Delta^2 x) \}$$
$$= \frac{m_o \omega^2}{m} \frac{rh}{L} \cos \omega t \qquad \text{........} \quad (7.2,14)$$

if $\bar{Z} = \frac{m_o}{m} \frac{rh}{L}$, the above equation is identical to the describing equation of Fig. (3.1-1). Comparing the expression of (7.2,14) with equation (3.1,7) it is evident that in equation (7.2,12) the designed system has identical restoring force characteristics in which the coefficient μ of the non-linear displacement is given by

$$\mu = \frac{1}{2} \frac{K_2}{K_1} \frac{1}{\ell^2} (\frac{L + c}{L})^4 \qquad \text{.......} \quad (7.2,15)$$

This has dimensions of length^{-2} which makes the displacement, if equation (7.2,14) is multiplied throughout by $\mu^{\frac{1}{2}}$, dimensionless.

It is readily seen from equation (7.2,15) that various degrees of non-linearity can be obtained by changing the length of the transverse spring. Hence, since the vibration response is a function of the spring length, it is more convenient to use the same spring rate after having determined the upper limit of stiffness K_2 whilst varying its length to achieve the various magnitudes of non-linearity then to construct a set of springs similar in length but differing in every other respect.

7.2.(iii). The percentage of non-linearity

For an indication of the design requirements, it is helpful to adopt a procedure of assessing the magnitude of non-linearity. As the definition for this is arbitrary, since the degree of non-linearity itself is heterogeneous even with the beam fixed at the optimum length for maximum deflection, it is not unreasonable to express the restoring force of the non-linear terms as a percentage of the force by linear displacement that is

$$\% \text{ of non-linearity} = \frac{1}{2}\,\frac{\kappa_2}{\kappa_1}\,\frac{1}{\ell^2}\left(\frac{L + c}{L}\right)^4 (3\Delta^2 + 3\Delta x + x^2) \qquad \ldots\ldots \quad (7.2,16)$$

In general, as shown by the above equation, the ~~degree~~ percentage of non-linearity for a fixed spring ratio increases with the displacement. However it is desireable that the amplitude is limited by design considerations to a maximum value. This will then take into consideration the approximations made in arriving at the mathematical expression for the coefficient μ within the range of non-linearity. Thus, with the moment arm L approximately set through the use of an optimum length of the beam and for a suitable transverse spring length ℓ such that the ratio of $(\frac{\delta}{\ell})^2$ and of higher powers are small, the spring ratio $\frac{\kappa_2}{\kappa_1}$ and hence the upper limit of κ_2 are determined for the designed allowable magnitude of displacement which gives about a hundred per cent non-linearity.

7.3. Design procedure

7.3.(i). An estimation of the maximum displacement

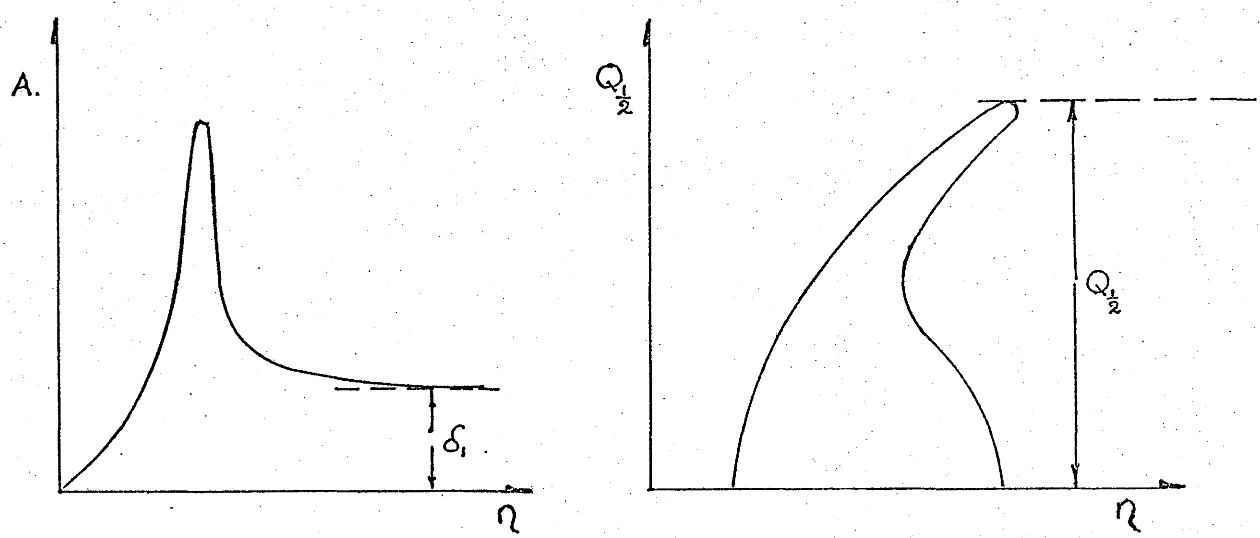
For the arrangement of the system illustrated in Fig. (7.1-1), to demonstrate the subharmonic vibration of the second order, it will appear that the displacement at the position of the springs increases in proportion to the beam length. The beam, however, constitutes a significant magnitude of the systems inertia. This means a larger displacement is not necessarily obtained through having a greater length on which the motor is mounted but that an optimum length exists for which the vibrating amplitude is a maximum.

To achieve the most suitable design condition for the vertical amplitude, the difficulty in formulating a mathematical expression to relate the tolerable allowable displacement to the inertia of the system and the beam length is overcome if linear oscillation is considered initially as to form a useful guide. Although by applying the linear theory to the system envisaged will considerably simplify the problem, it is not unreasonable with reference to Fig. (7.3-1) to say that at a one hundred per cent non-linearity

$$\delta = r\delta_1 \qquad \ldots\ldots\ldots \quad (7.3.1)$$

where r is the appropriate constant.

With the standard linear solution of $\phi = \phi \cos (\omega t + \beta)$ where



(a) LINEAR CHARACTERISTIC. (b) NON-LINEAR CHARACTERISTIC.

FIG. (7.3-1). RESPONSE CHARACTERISTICS FOR A CENTRIFUGAL EXCITATION.

$$\hat{\phi} = \frac{Z\omega^2}{I_t} \{ (p^2 - \omega^2)^2 + (\frac{C}{I_t}\omega)^2 \}^{\frac{1}{2}}$$

and $Z = m_o rh$ .... (7.3,2)

then in the region of the maximum subharmonic resonance frequency, in which ω is greater than the natural frequency such that $\frac{1}{\eta^2}$ is small, the amplitude is practically a constant with change in frequency and it can be approximated with the substitution of

$$(\frac{C}{I_t}\omega) = (\frac{C}{C_c}.2p.\omega)$$

to $\hat{\phi} \approx \frac{Z}{I_t}$ .... (7.3,3)

Hence, for the length, L, of axis-to-spring the above expression gives the linear end deflection δ_1 as

$$\delta_1 \approx (\frac{Z}{I_t}.L) \qquad \text{....... (7.3,4)}$$

Thus, with reference to equation (7.3,1), an approximate expression of the limit of subharmonic displacement for a given magnitude of the disturbing force is obtained in terms of the inertia and the length L. This relationship

$$\delta = (\frac{Z}{I_t}.L)r \qquad \text{....... (7.3,5)}$$

gives the condition of a hundred per cent non-linearity at the frequency of the allowable maximum displacement, and it is independent of any linear spring terms. The displacement can take any value within the approximation that the ratio of $(\frac{\delta}{\varepsilon})^3$ and of higher powers are small, and the equation (7.3,5) is complementary to equation (7.2,16) in determining the transverse spring rate.

It is readily seen that the increase in the length of the beam and hence in L will not give in proportion a larger displacement. Since the inertia of the system is also a function of L, it is evident that for the condition of $(\frac{Z}{\delta})$ a minimum, a length of the beam L_b can be found which makes the allowable displacement a maximum. The optimum beam length is determined at a later stage, after the evaluation of the principal oscillating inertia of the system.

7.3.(ii). The centrifugal excitation

The inertia of the excitation constitutes a major proportion of the systems total inertia. This takes into consideration the out-of-balance mass, the flywheel, a fractional horse-power motor and a cradle on which the equipments generating the disturbing forces are mounted.

As the magnitude of the disturbing force is proportional to the square of the frequency, the motor of variable speed characteristics which drive the out-of-balance mass must have an effective control system to maintain the constant excitation. It is decided that by running the whole system in the high speed range together with a good speed regulating circuit, the problem of frequency instability from driving against the variable friction forces is unlikely to be any real concern. Hence, a linear natural frequency of about 6Hz is used to determine the working band-width that is required from the motor. This approximate value is considered to be in the region of the lower limit possible.

Otherwise there is the difficulty of achieving the prescribed degree of static deflection with too low a rate of κ_1.

The motor which is selected must then have a range of between 700 to 2500 r.p.m. if the subharmonic vibration is to be demonstrated for the various degrees of asymmetry, in which the influence of gravity effects on the response is significant. A higher linear natural frequency is not considered because it is apparent from the equations (7.2,16), (7.3,5) and from the relationship

$$\kappa_1 = \frac{p^2 I_t}{L^2} \qquad \ldots\ldots \quad (7.3,6)$$

that otherwise there is difficulty to keep the system to a reasonable size that is in proportion to the amplitude of vibration generated when the spring rates are in the neighbourhood of about one hundred per cent non-linearity.

In view of equation (7.3,5) it is evident that to obtain a satisfactory non-linear deflection for a reasonable magnitude of the disturbing force the total inertia is kept to a minimum value. Since the inertia of the excitation, taking into consideration the flywheel, the attachment of the out-of-balance weight and the cradle, constitutes a major proportion of I_t, a fractional horse-power motor of the above frequency range is used.

The most suitable motor available is a twentieth horse-power d.c. series wound motor of weight three pounds. Together with a simple yet effective regulating circuit shown in Fig. (7.3-2), good control of the stability of the disturbing force is attained. The armature is excited by a 110 volt d.c. supply and the required speed of the motor can be obtained to within satisfactory accuracy through the coarse and fine regulation of the armature supply voltage. The variable resistance is also connected in parallel to the field circuit. By this means the altering of the field strength improves further the attaining of the motor speed. If the speed decreases, the back e.m.f. drops and this allows an increase of the armature current which also will influence the field strength. Consequently, from an increase in the torque produced and with the changes in the flux, the required speed is maintained more efficiently.

Since the motor is a non-uniform body, it is necessary to evaluate the position of the principal axis. This will facilate the determining of its inertia. Also it allows the motor to be suitably mounted at a later stage on a cradle such that the principal axis is as near coincident with the axis of vibration as possible. In this manner the inertia value of the motor is kept to a minimum in concurrence with the criterion of equation (7.3,5).

If the motor rests on a knife-edge, the force at a known distance from it due to its own weight distribution is easily recorded. Then from knowing the weight of the motor, the position of the required axis about one direction is calculated. Hence, if the procedure is repeated with the motor resting on its adjacent side, the principal axis is readily located.

It is apparent that since the mass distribution is non-uniform an accurate calculation of the moment of inertia for the mass of the motor is difficult. However, an experimental determination of the value is found to be satisfactory. The

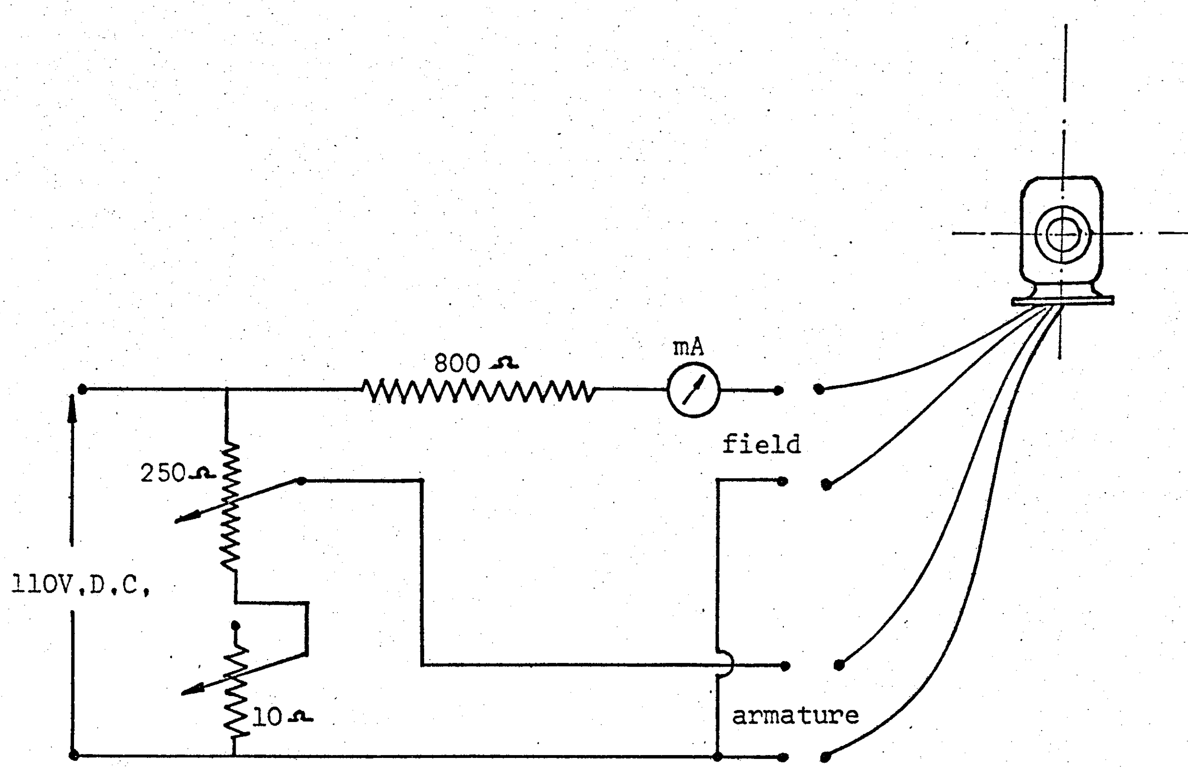


FIG.(7.3-2). MOTOR SPEED REGULATING CIRCUIT.

error is ascertained to be less than three per cent. Hence, the inertia of the motor determined experimentally from applying directly the elementary principle of the trifilar suspension has a value of 0.018 $lbf.in.s^2$.

Fig.(7.3-3) shows a schematic diagram of the apparatus illustrated in Fig.(7.1-1)(b). The motor A is suitably mounted on the cradle B which has side-extensions C. They are designed and machined to give a fine fit into the bearings D, that are housed in the respective bearing-blocks E. The bearings which are subsequently locked in position by means of the washers and nuts, having a $\frac{5}{8}$ in. B.S.F. thread, enables the whole system to oscillate freely about the axis of the bearings. Since the cradle is part of the vibrating structure, duralumin is used for the construction. The low-rate of mass distribution of the material together with a high tensile strength serves adequately the purpose of carrying the motor. The inertia value of the cradle B is then reduced to 0.0071 $lbf.in.s^2$. about the axis of vibration.

The flywheel F that is shown in Fig.(7.1-1)(b) is also made from duralumin and it is attached to the shaft of the motor at its sleeve by means of grub-screws. The forcing energy that causes the vibration of the system is generated when the flywheel is adapted later to take an out-of-balance mass m_o. Together with the cradle of the preceding paragraph, the flywheel and m_o are a part of the vibrating structure and their respective inertias will influence the magnitude of $\bar{Z}$ which is governed by the relationship

$$\bar{Z} = \frac{m_o rh}{I_t} \qquad \text{....... (7.3.7)}$$

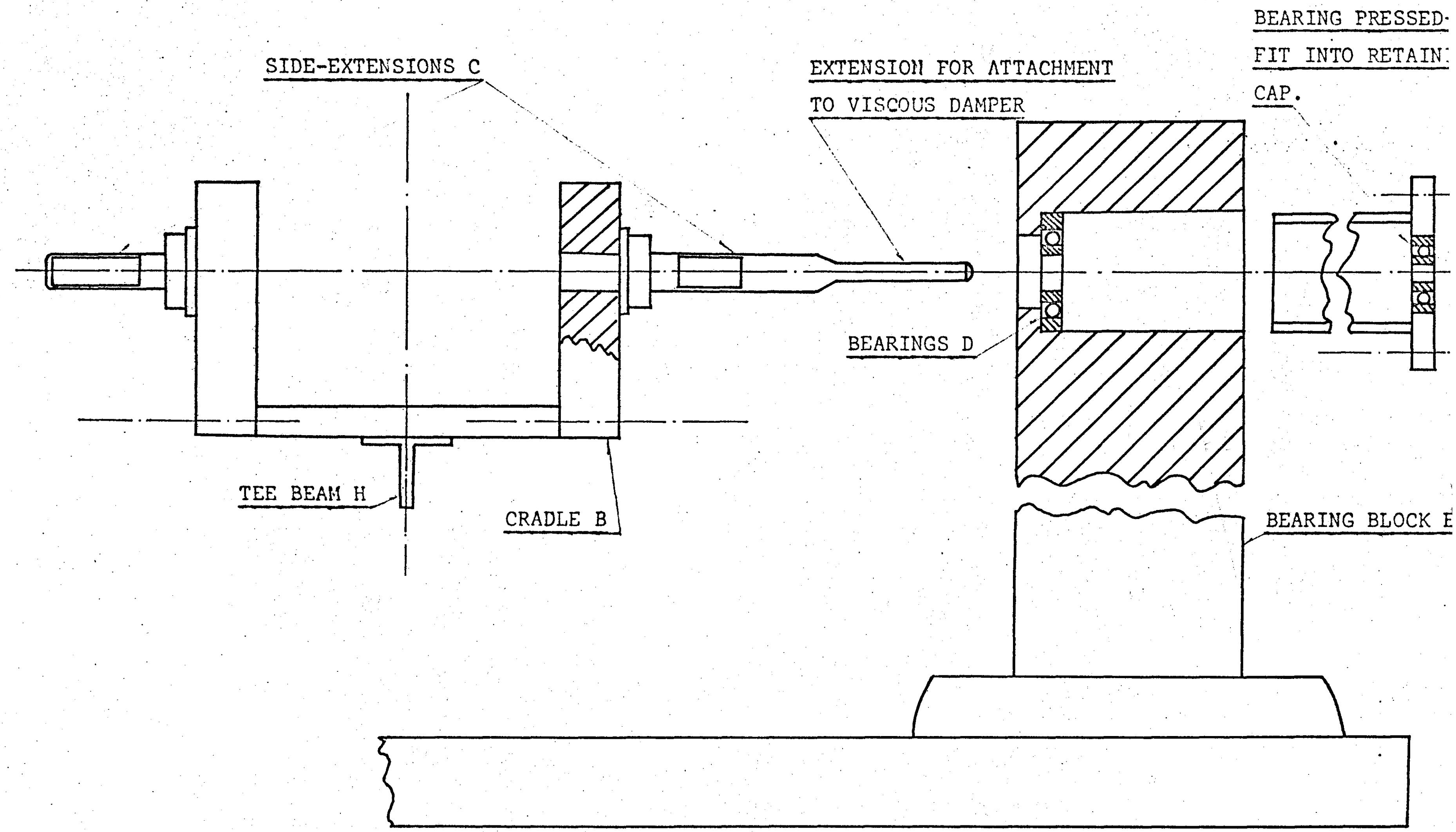


FIG.(7.3-3). THE CRADLE AND BEARING BLOCK

where r is the radius of m_o to the shaft centre,
and h is the distance of the flywheel to the axis of oscillation.

It is apparent from the above equation that the increase in value of m_o and in the diameter of the flywheel do not necessarily give a significantly larger $\bar{Z}$. Also in conformity with the earlier discussion of the need to keep the total inertia low, their moments of inertia about the axis of vibration are restricted to within the design requirements of the non-linear suspension. The dimensions of the flywheel F are thus determined in relation to the size of the whole system and to the out-of-balance mass for a reasonable magnitude of $\bar{Z}$.

From the assumption that the inertia of the beam I_b is in the region of half the total value of the excitation components, m_o is evaluated for an approximate $\bar{Z}$. It is probable that the actual value of I_b is greater but the difference will not be significant in view of equation (7.3,5). For m_o = 0.08 lb., the moment of inertia of the flywheel, allowing for the attachment of the out-of-balance mass is effectively reduced to 0.0114 lbf.in.s^2.

The motor A is run through its speed range, with and without the flywheel, when thewhole system is assembled. This is to ascertain that the motor has no inherent out-of-balance force. The flywheel is then adapted to take the mass m_o at a measured distance from its centre. The required value of the out-of-balance mass which is determined for an approximate $\bar{Z}$ is found from comparing the weights of the flywheel before and after the attachment. The difference is effectively the out-of-balance mass.

7.3.(iii). The vibrating beam

A significant proportion of the total inertia of the system is contributed by the beam H (see Fig.7.3-3). As the systems inertia value increases correspondingly with the beam length that also determines the limit allowable for the displacement, an optimum length of the beam may exist which makes the deflection a maximum. Because there is no undue loading at either of the end conditions, it is assumed flexural deflection of the beam does not arise. A mathematical expression for the moment of inertia about the vibration axis is then developed for the state in which the ratio of any given Z value to the amplitude is a function of the beam length. Hence, the design of the system, can be examined to ascertain whether there is a beam length which makes the permissible deflection a maximum.

The beam H extends beyond the axis in one direction as illustrated in Fig.(7.3-4). In this manner, with the motor A suitably positioned such that one of its diametral principle axis is in line with the axis of oscillation, the magnitude of the systems total inertia is reduced correspondingly. Then, if A_x is the sectional area of the beam and S the second moment of area of the cross-section about the neutral axis which passes through the centroid, an approximate expression for the moment of inertia about the centre of gravity of any considered beam length L_b is given by

$$I_g = ML_b \left(\frac{S}{A_x} + \frac{L_b^2}{12}\right) \qquad \ldots\ldots \quad (7.3.8)$$

where M is the mass per unit length of the beam.

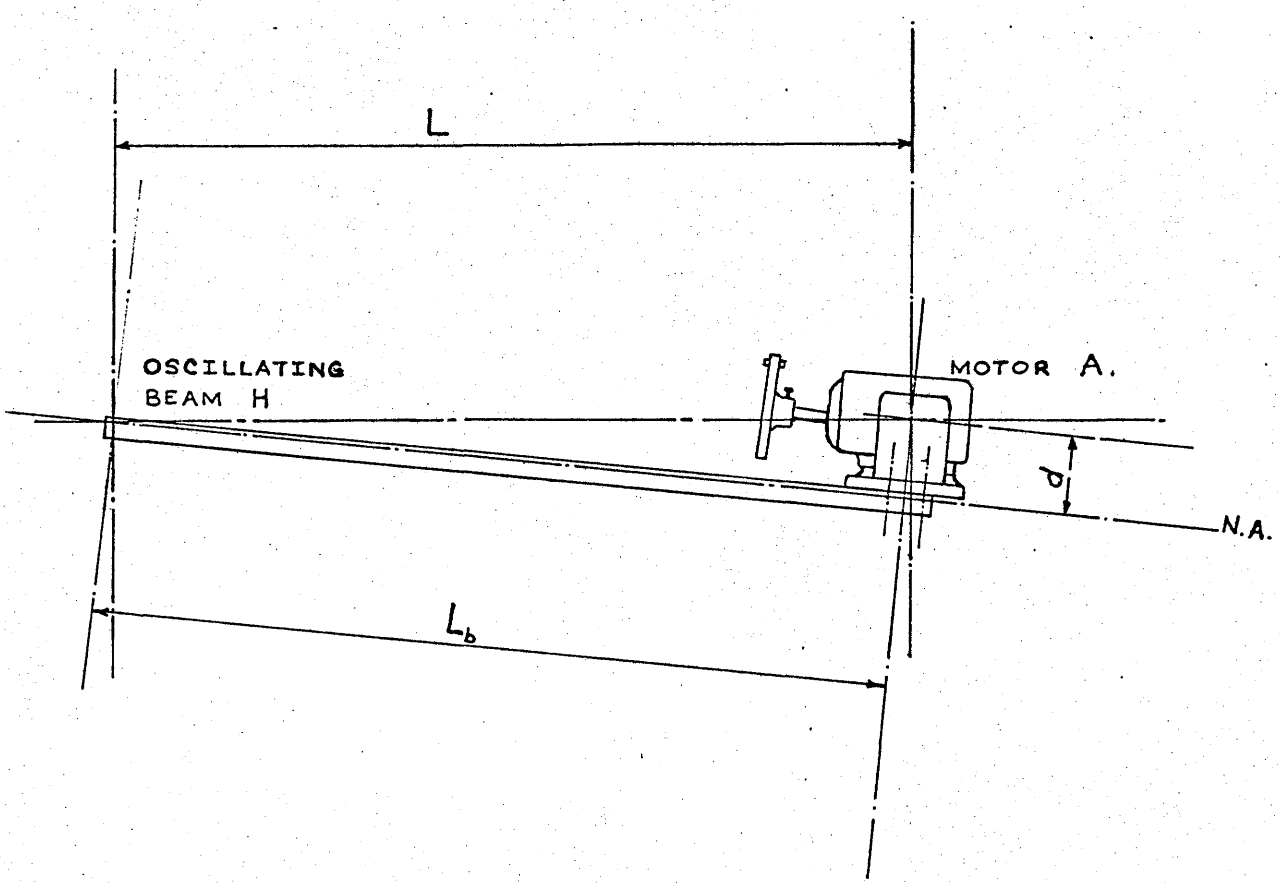


FIG. (7.3-4). SCHEMATIC CONFIGURATION OF THE MAIN INERTIA BODY.

Since the distance d, from the configuration of Fig. (7.3-4), is fixed by the need to align the centroid of the motor with the axis of vibration and also since the centre of gravity of the beam is displaced from the axis, the moment of inertia of the oscillating beam is effectively

$$I_b = I_g + ML_b(\frac{L_b^2}{4} + d^2) \qquad \ldots\ldots\ (7.3,9)$$

which on substitution of equation (7.3,8) for I_g gives after simplification

$$I_b = ML_b(\frac{L_b^2}{3} + F^2) \qquad \ldots\ldots\ (7.3,10)$$

where $F^2 = (\frac{J}{A_x} + d^2)$ is a constant,
and J, A_x and d are fixed dimensions of the system.

If the inertias, arising from the components of the system that generates the disturbing force analysed in the preceding section, about the axis of the bearings are represented by I, the approximate expression for the total inertia of the vibrating system then becomes,

$$I_t = ML_b(\frac{L_b^2}{3} + F^2) + I \qquad \ldots\ldots\ (7.3,11)$$

where it is evident that the only variable contained in the equation is the length of the beam. Hence on substituting the above equation into equation (7.3,5) and with

$$L^2 = (L_b^2 + d^2)$$

$$\left(\frac{Z}{\delta}\right) = \frac{\left\{ ML_b\left(\frac{L_b^2}{3} + F^2\right) + I \right\}}{(L_b^2 + d^2)^{\frac{1}{2}}} \cdot r \qquad \ldots\ldots\ (7.3,12)$$

from which the condition for the optimum beam length, that makes the allowable deflection at its largest value, can be examined by differentiating with respect to L_b. Thus, it can be readily shown that the ratio of any Z value to the deflection is a minimum, and hence the permissible displacement is at a maximum when the beam length satisfies the following equation,

$$1.5\ ML_b^4 + ML_b^2 d^2 - IL_b + MF^2 d^2 = 0 \qquad \ldots\ldots\ (7.3,13)$$

This condition will yield two real roots of identical values but of opposite signs. The plus or minus merely signifies the difference in direction taken for the beam.

Because of the low mass rate per unit length of duralumin, $M = 0.059 \times 10^{-3}$ lbf.s^2./in., a uniform tee-beam of the material serves adequately for the intended purpose. From its sectional dimensions of 1 in. by 1 in. by $\frac{1}{8}$ in., the radius of gyration of the beam section is readily evaluated about the neutral axis that passes through its centroid and which is approximately 0.296 in. from the top. Subsequently with d fixed, (see Fig.(7.3-4)), relative to the position of the vibration axis, the value of F^2 is calculated. Hence, the optimum length of the beam, determined in relation to the size of the whole system from equation (7.3,13), is 9.37 in. and this value will give for any amplitude of Z an approximate maximum displacement δ. The length L_b is taken as

9.4 in. and although the actual length of the beam is slightly longer for the attachment of the cradle, the variation of δ from its maximum value by the approximation of the beam from its optimum is insignificant. Its moment of inertia about the axis of oscillation is 0.0204 lbf.in.s^2.

7.3.(iv). Evaluation of the vertical and transverse spring rates

It is apparent from the equation (7.2,18) that if the limit of a tolerable displacement is given for about one hundred per cent non-linearity, the spring ratio of $\frac{\kappa_2}{\kappa_1}$ is readily evaluated for a suitable transverse spring length ℓ. The total moment of inertia is now known, for having determined the appropriate beam length to be used, the value is easily calculated from equation (7.3,11). Then from an indication of the required vertical stiffness through equation (7.3,6), the actual spring rate of κ_1 is determined from which κ_2 can be fixed.

(a) Vertical stiffness κ_1

For reasons mentioned during the design of the centrifugal disturbing force, the linear natural frequency p is taken as approximately 6Hz. With the actual length of the beam in a region close to the value determined from equation (7.3,13), the moment arm of the linear torque is fixed at L = 9.78 in. and the total moment of inertia, allowing for the weight of the bearing in the attachment of the transverse spring to the beam, is effectively 0.034 lbf.in.s^2. about the axis of vibration. Hence, a suitable spring rate of κ_1 for the designed components is readily evaluated from equation (7.3,6) and it yields for an assumed p = 6Hz a stiffness rate

$\kappa_1 = 0.95$ lbf. per in.

The vertical stiffness is conveniently represented in the form of two linear spring rates of $\frac{\kappa_1}{2}$. Then by allowing the tension of each spring to be adjusted independently, this will facilitate the alignment of the transverse spring and the desired magnitude of the static deflection can also be introduced with reasonable accuracy. Thus, two springs each having a stiffness of about 0.5 lbf. per in. are used.

The measurement of the combined effects of the vertical springs is best achieved through direct calibration when the whole system is assembled without the transverse stiffness κ_2. Since it is not possible to load the springs where they are attached to the beam, the vertical stiffness of the system is determined for an appropriate moment arm and the value is then corrected for the position of the loading.

The loading of the system is by means of a scale-pan suspended from the flange of the tee-beam at a moment arm of 8 in. from the axis of oscillation. Suitable weights in increments of 0.1 lb. are added and the deflection at each instant is recorded at the position of the springs. The graph of Fig.(7.3-5) illustrates the uniformity of the spring rates over any magnitude of the deflection by the direct relationship between the displacement at a moment arm of 9.78 in. radius and the load applied at the above mentioned distance. The stiffness obtained from the plot is 1.44 lbf. per in. and from correcting this value for the position of

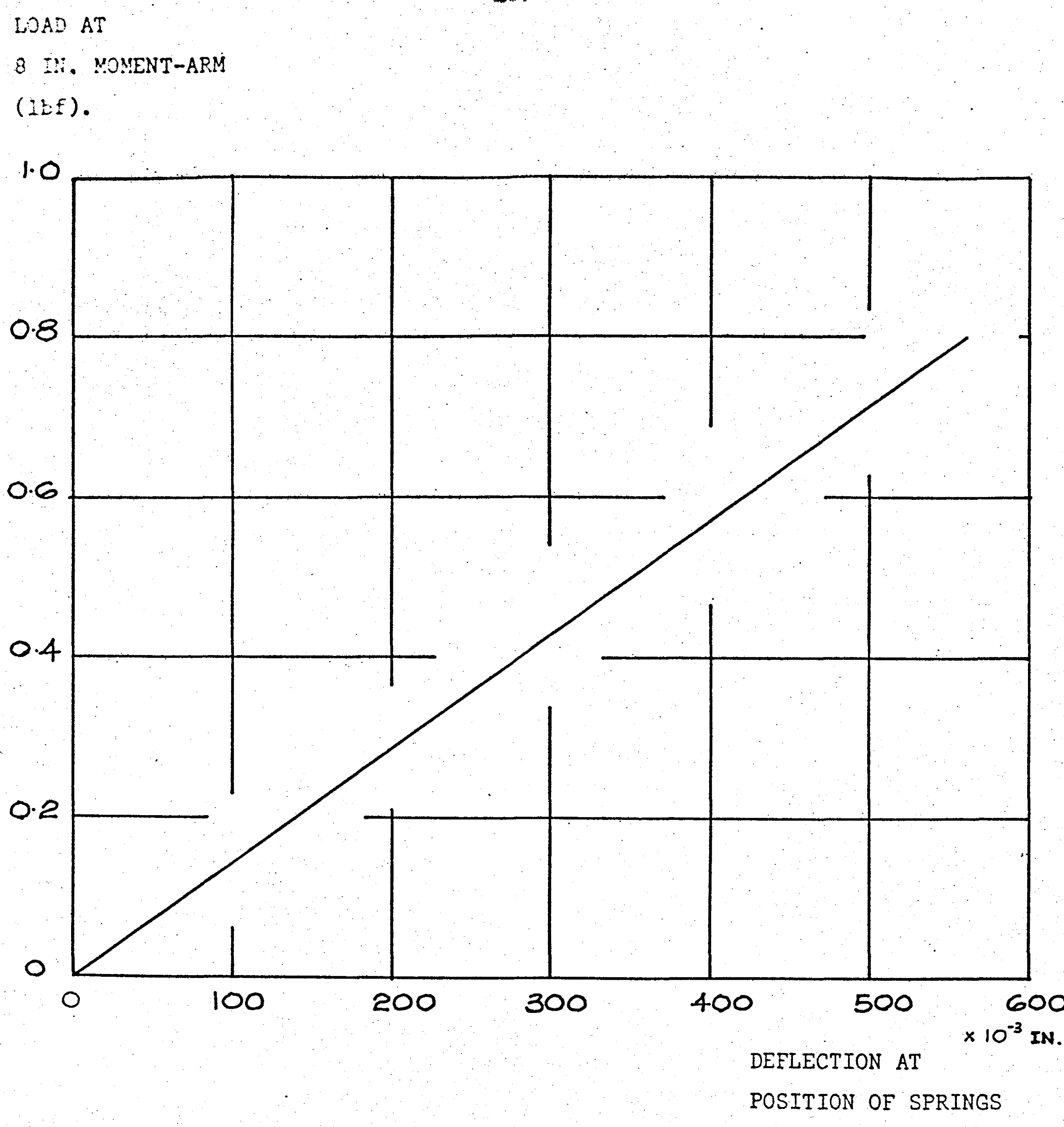


FIG.(7.3-5) THE VERTICAL SPRING RATE

SPRING RATE FROM GRAPH
IS 1.44 lbf/in.

VERTICAL STIFFNESS AT
POSITION OF SPRINGS
K_1 = 1.17 lbf/in.

the load the real spring rate which is present in the system is given by

$$\kappa_1 = 1.27 \text{ lbf. per in.}$$

Since the actual linear torque introduced by the vertical springs will have increased it is apparent that the assumed linear natural frequency will not yield the true dimensionless frequencies of the subharmonic vibration. For the vertical stiffness of the system equation (7.3,6) produces a linear natural frequency of

$$p = 6.6 \text{ Hz}$$

which allows a more accurate forcing speed to be located in the experiment for a given theoretical dimensionless frequency.

(b) Transverse spring stiffness κ_2

In arriving at equation (7.2,16) a reasonable assessment can be made for the required stiffness κ_2 in relation to the other designed components. Since the characteristic of the vertical springs is limited by the consideration for good working speed control and for attaining the desired gravity effect with reasonable accuracy, it is now possible to determine the transverse spring rate for about a one hundred per cent non-linearity at a realizeable displacement.

If the spring rate is evaluated for the case of $\bar{\lambda} = 1.0$, in which the effects of non-linearity are strongest, it will take into account the resulting severness of the vibration, as observed previously from the vibrational characteristics in chapter (6); sections (6.3) and (6.4). This means that the approximations made in the process of deriving equation (7.2,13) in general are applicable to most conditions of the subharmonic vibration. Hence, the transverse stiffness necessary to produce the non-linear effects is determined for $\mu^{\frac{1}{2}}\Delta = 1.0$, where μ is as described by equation (7.2,15), by using equation (7.2,16) as the defining criterion.

This yields for a one hundred per cent non-linearity the following equation

$$1.0 \pm 3\mu\Delta^2 + 3\mu^{\frac{1}{2}}\Delta \left(\frac{\kappa_2}{2\kappa_1}\right)^{\frac{1}{2}} \frac{1}{\ell}\left(\frac{L+\varepsilon}{L}\right)^2 x + \frac{1}{2}\frac{\kappa_2}{\kappa_1}\frac{1}{\ell^2}\left(\frac{L+\varepsilon}{L}\right)^4 x^2$$

...... (7.3,14)

Bearing in mind the approximations of section (7.2)(i), the allowable displacement δ is taken as

$$\delta = x + \Delta = 0.32 \text{ in.}$$

and with a convenient length of the transverse spring $\ell = 2.0$ in. such that the ratios of higher powers of $\frac{\delta}{\ell}$ are insignificant, it is evident that the spring rate κ_2 is readily obtained. In order to ensure attaining the asymmetrical characteristics from the effectiveness of the transverse stiffness over the whole of the subharmonic response, it is considered in the selection of ℓ that the length must not be less than this value. Otherwise the error introduced may exceed, during the predominance of $Q_{\frac{1}{2}}$, the tolerable limit. Hence on substituting for the displacement into equation (7.3,14)

$$1.0 = 3\mu\Delta^2 + 3\mu^{\frac{1}{2}}\Delta \left(\frac{K_2}{2K_1}\right)^{\frac{1}{2}} \frac{1}{\ell} \left(\frac{L+\varepsilon}{L}\right)^2 0.32 - 3\mu\Delta^2 + \frac{1}{2}\frac{K_2}{K_1}\frac{1}{\ell^2}$$

$$\left(\frac{L+\varepsilon}{L}\right)^4 . \; 0.32^2 - 0.64 \left(\frac{K_2}{2K_1}\right)^{\frac{1}{2}} \frac{1}{\ell} \left(\frac{L+\varepsilon}{L}\right)^2 \mu^{\frac{1}{2}}\Delta + \mu\Delta^2$$

.... (7.3,15)

which after simplification gives

$$K_2 = \frac{39}{2} K_1 \ell^2 \left(\frac{\Delta L}{L+\varepsilon}\right)^6 \qquad \text{........ (7.3,16)}$$

Thus, the transverse stiffness for a one hundred per cent non-linearity at the allowable limit of displacement is obtained for the following designed values of the system;

$\ell = 2.0$ in., $K_1 = 1.17$ lbf. per in., $\varepsilon = 0.35$ in., $L = 9.78$ in.

From equation (7.3,16) the upper limit of the transverse spring rate is in the region

$K_2 = 79$ lbf. per in.

A known uniform spring rate of approximately 80 lbf. per in. is used. The actual stiffness K_2 is calibrated by suspending the transverse spring vertically at one end whilst loads are applied at the other. The linear relationship between loading and the corresponding deflection is readily obtained. To reduce the error

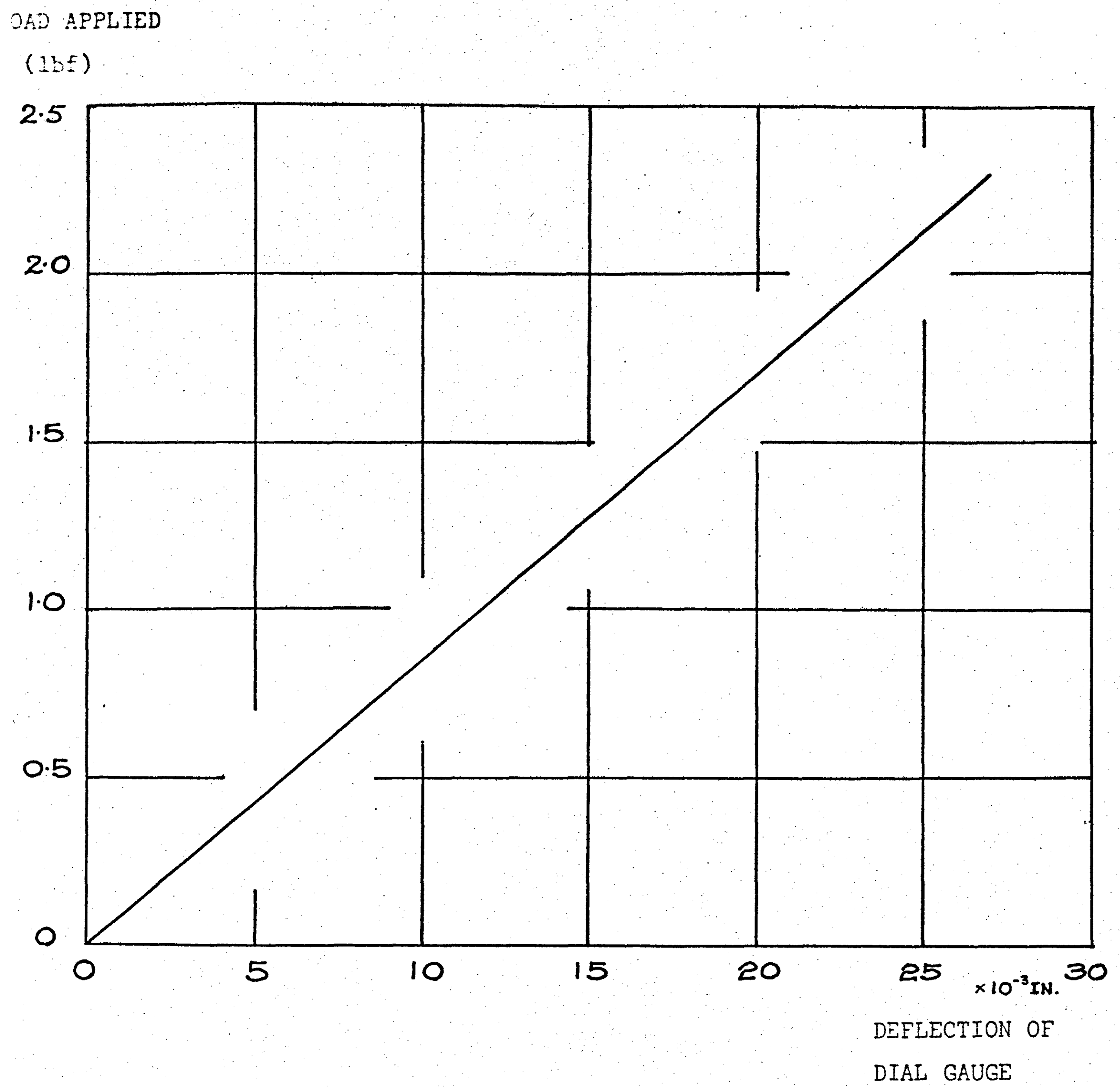


FIG.(7.3-6) THE TRANSVERSE SPRING RATE

TRANSVERSE STIFFNESS K_2 = 86.8 lbf/in.

in the measurement of the deflection, a nitch is made at the centre of the lower side of the scale pan through which the weights are introduced. The gauge probe is then effectively in contact with the pan when the dial gauge is carefully aligned vertically below. The dial gauge recordes effectively the actual deflection of the transverse spring for each instant of the increment of the loading. The loading of the spring is in increments of 0.2 lb. with the alignment frequently checked during the experiment. The readings are recorded at each instant and likewise on unloading. The discrepancy between the corresponding values is almost negligible and it arises probably because of the stricture effects. Hence, from the graph of Fig.(7.3-6) the actual stiffness calculated for the transverse spring for an approximate condition of equation (7.3.14) has a value

$$K_2 = 86.8 \text{ lbf. per in.}$$

It is reasonable to assume that with light weight bearings fitted at either ends of this spring, only tension in line with the spring axis is transmitted.

7.3.(v). Subsidiary consideration

In having determined the parameters of the main components, it is logical to investigate at this stage of the design whether other resonance effects exist that will destroy the vibratory motion under consideration. If such extraneous effects are to arise, the main trouble is likely to be from the beam. The beam natural frequency of lateral vibration is investigated and as the design maximum speed of the excitation is about 30Hz the analysis is limited to the fundamental mode.

With a single degree of freedom in the vertical direction, the tee-beam is considered as a cantilever. For the lateral motion the effective length of the beam is taken as 9.4 in. and the corresponding fundamental mode is easily evaluated by applying the energy method in which the mode curve satisfies both end-conditions. The appropriate natural frequency for the beam is then given by

$$P_{\ell} = 3.5156 \left\{ \frac{EI}{ML_b^4} g \right\}^{\frac{1}{2}} \qquad \ldots\ldots\ldots \quad (7.3,17)$$

where EI is the flexual rigidity in the direction of the vibration, and the fundamental mode is in radians per second.

The inertia I of the beam in the appropriate plane is 0.01056 in.^{4} and from the above equation the value obtained is 272Hz. It is evident therefore that the difference between the region of the excitation frequency and the lateral vibration modes is sufficiently large and the use of the determined parameters is satisfactory.

As for the natural frequency of the beam in the direction of the forced motion, a reasonable indication of the large difference between its value and the operating speed of the motor is obtained by considering the dynamic deflection curve to have the modal shape produced by the combination of the loads due to the vertical spring forces and the mass of the beam at the end. The error of approximation is known to be less than one per cent. The equivalent mass to be added to the end mass to allow for the mass of the beam is approximately $\frac{33}{140} M_b$. The fundamental mode evaluated on the above assumption is 79.5Hz for a linear spring force of 1.2 pounds. It is evident, for the approximate optimum beam length used, that the fundamental modes are sufficiently apart from the excitation frequency.

7.4. The damping system

7.4.(i). Viscous damping

Having designed the main components of the vibrating system, it is appropriate to give due consideration to the viscous damping force. Since the relationship between the amplitude and the angular displacement of the cradle is a directly linear function, the system can be damped through the angular motion of the cradles side-extensions. Fig.(7.3-3) illustrates that one of the arms is longer to facilitate this and the subsequent measurements of the amplitudes.

Viscous damping is introduced, as shown in Fig.(7.1-1)(a), by attaching the oscillating arm to the shaft of a Kinetrol precision dashpot J in which the damping rate remains constant whatever the oscillating frequency when the required value is set. Friction is negligible as the shaft is on ball-races and the shaft seal is a flexible synthetic rubber sleeve that is bonded to both the cover and the shaft. With the air excluded from the silicone fluid, the effect of backlash is eliminated.

By ~~accuracy~~ accurately aligning the dashpot shaft with the axis of vibration to achieve the desired damping condition, the problem of applying the viscous force without increasing the load on the tee-beam is overcome. Also, errors from the measurements of the lever arms to obtain the correct damping rate will not arise. For the arrangement used, an indication of the critical damping rate is readily assessed from the equation

$$C_c = 2L\{\kappa_1 I_t\}^{\frac{1}{2}} \qquad \text{..........} \quad (7.4.1)$$

It is evident that the critical value, as shown to be a function of both the linear spring stiffness and the moments of inertia, is fixed by the design of the system. The approximate range of the dashpot coefficients necessary is then known from the limit set to the value of R.

Negligible friction exists in the simple arm-linkage mechanism for ball-race bearings are used at each of the linkage-joints. The lever arms L_a (see Fig.(7.1-1)(a)) are equal in length and they are splined at the appropriate ends to just fit on to the corresponding shafts. The attachments are held in position through the tightening of the respective fittings as shown in Figs.(7.1-1), (a) and (b). Duralumin is used for the lever arms to reduce the unnecessary oscillating mass.

7.4.(ii). Estimation of the damping rate

An effective method of determining the amount of damping present during the second order subharmonic vibration is to measure the rate of decay of the oscillation. For linear systems lightly damped, the two complex roots to the auxiliary equation yield two particular solutions in the standard elementary expression. Since the sum or the difference of any two solutions multiplied by a constant is also a solution, the solution of the natural oscillation can then be expressed also in the form

$$y = \frac{A}{2}\{ e^{(\alpha + j\beta)t} + e^{(\alpha - j\beta)t} \} = Ae^{\alpha t} \cos \beta t \quad \ldots (7.4,2)$$

where A is an arbitrary constant determined from initial conditions and the terms α and β are defined as

$$\alpha = -Rp \text{ and } \beta = p(R^2 - 1)^{\frac{1}{2}}$$

Hence, the dimensionless coefficient of the resistance to the velocity is conveniently ascertained by allowing the system to execute the free vibration, in which the rate of decay is governed by the term $e^{\alpha t}$, without the transverse spring attached.

As the oscillating amplitude is proportional to the magnitude of the shaft angular displacement, the vibration is described by the characteristics that are recorded from the shaft behaviour. The successive amplitudes of the free vibration are depicted on a trace, and by measuring any two amplitudes y_1 and y_2 at exactly the corresponding time t_1 and t_2 from the curve, the dimensionless coefficient R is readily evaluated from the expression

$$R = \frac{1}{p}\left(\frac{1}{t_2 - t_1}\right) \log_e \frac{y_1}{y_2} \quad \ldots\ldots\ldots \quad (7.4,3)$$

obtained from the solution (7.4,2).

The trace of the relationship between the vertical displacement at the position of the springs and the angular motion of the shaft is recorded by means of an Elliot A.C. Pick-up, P, (see Fig.(7.1-1)(a)) which is a data transmission element energised from an a.c. supply. The unit consists of rotor, bearings, and shaft and it is directly coupled on to the arm of the cradle. Subsequently, through the use of a Rank S.E.429 amplifier and frequency demodulator illustrated in Fig.(7.5-1), a linear output voltage directly proportional to the angular shaft displacement is obtained. There is no phase lag introduced in the mechanical to electrical conversion of the signals. In this manner, principally by eliminating slip rings and brushes, the frictional forces are reduced to a minimum. What inherent friction that exists is small and can be considered during the forced motion as seen in (32) to be viscous.

The evaluation of the damping rate by this means is sufficiently accurate since the experimental results compare favourably with the curves obtained by the theoretical analysis. The accuracy is maintained to within 8 per cent for the worst possible condition, as the sensitivity of the trace is governed from the use of a resistor circuit as illustrated in Fig.(5.1-2) to obtain a satisfactory deflection curve from the U.V. recorder. The error introduced in the measurements is further reduced by regulating the spacing of the successive amplitudes. Through the number of oscillations executed in each instant of a decaying curve recorded, the corresponding coefficient R determined from each set of measurement shows good consistency.

For a stiffness rate K_1 = 1.17 lbf. per in. and a linear natural frequency p = 6.6 Hz, equation (7.4.3) yields, for the experimental condition in which the amplitude of the disturbing

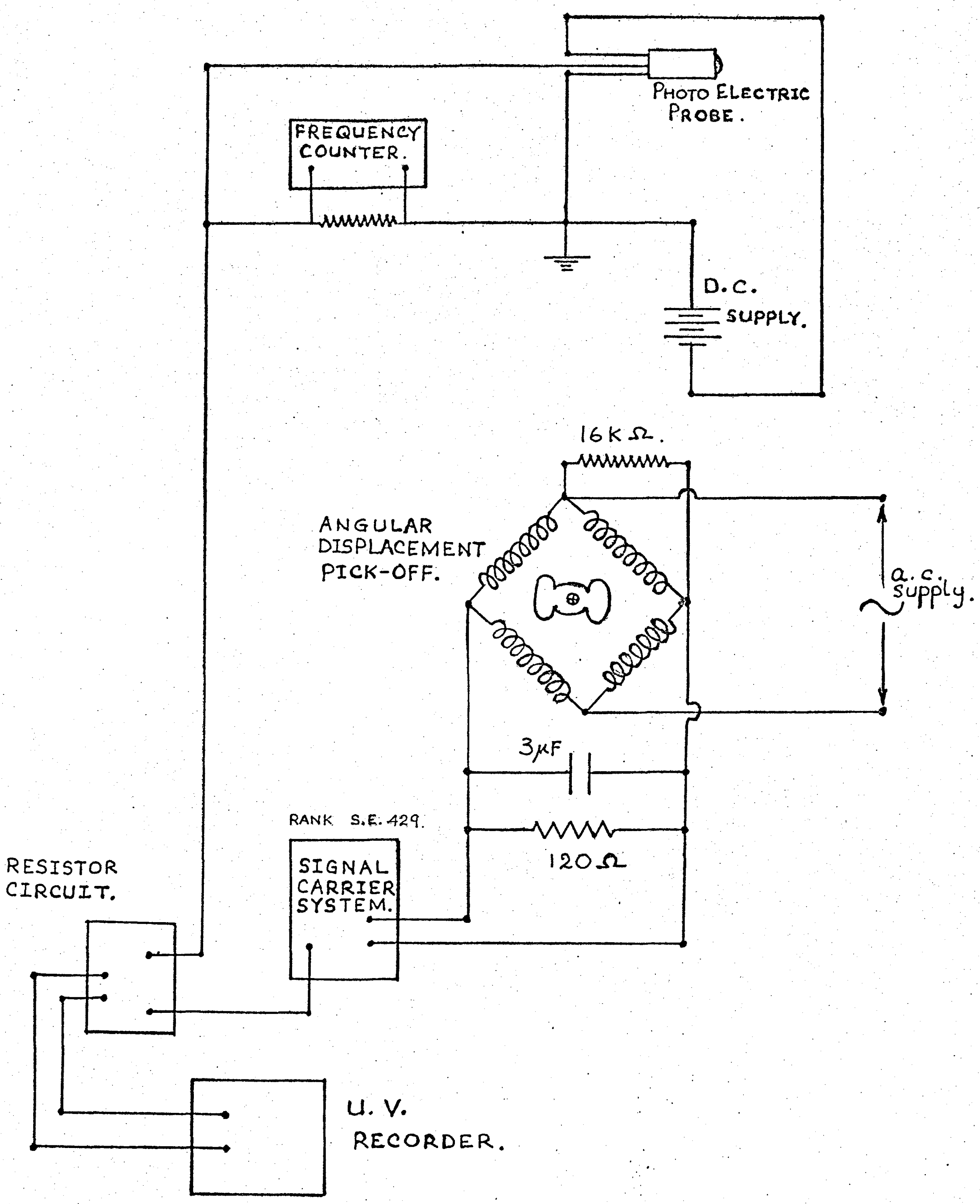


FIG.(7.5.-1). SCHEMATIC CIRCUIT DIAGRAM FOR MEASUREMENT OF DISPLACEMENT AND EXCITATION FREQUENCY.

force $\ddot{Z}$ = 0.0242 and the magnitude of the non-linearity $\bar{A}$ = 0.8, a dimensionless damping ratio

$$R = 0.0086$$

7.5. The finalised system

In having designed the governing parameters of the system, the other salient points are briefly discussed together with the selected equipments for monitoring the respective variables. Figs. (7.1-1)(a) to (c) show photographs of the designed system.

A heavy steel plate S which is finely machined on both sides is used to form a platform for the bearing blocks E on one side and the spring tower T at the other end. The platform is firmly clamped on to a concrete base. The two identical pillars house the bearings D between which the cradle B is suitably mounted. The pillars are welded on to the respective steel plates and when the vibration axis is accurately defined by their positioning, the plates and the platform are machined to form a sliding fit. In this way, the bearing blocks can be adjusted individually as seen from Fig.(7.1-1)(b) and the system is assembled without the difficulty of realigning the axis on each occasion.

As described in section (7.3), one cradles arm is longer than the other and to avoid undue loading of the shaft an extra bearing is housed in the block. It is secured in position by a pressed fit into the retaining cap as illustrated in Fig.(7.3-3). With the motor A suitably mounted on the cradle, the tee-beam H of the determined length is attached below to form an oscillating system.

The linear and non-linear restoring forces are introduced separately as shown in Fig.(7.1-1)(a). In this arrangement, the different magnitudes of the coefficient of non-linearity is achieved by merely changing one stiffness rate with respect to the other. Otherwise the linear and non-linear restoring torques existing simultaneously must be analysed separately.

Two light springs are used to produce the required vertical stiffness. This will facilitate the levelling of the transverse spring and hence it enables the desired magnitude of the gravitational effect to be obtained with a degree of accuracy. The springs are attached above and below the tee-beam and the other respective ends are attached to adjustable holders K, which are also secured in position by means of grub-screws.

The spring tower also consists of two $\frac{3}{4}$ in. bolts of half inch B.S.F. thread to provide for the lateral movement of the horizontal spring holder G. The transverse spring can then be adjusted in length until the component responsible for introducing the non-linear restoring torque has zero initial tension. The design of the spring holder G also allows the transverse spring to be positioned vertically and when it is accurately aligned with the vibration axis, the holder is bolted in position. In the attachment of the transverse spring, light weight bearings are used at both ends to reduce friction losses to a minimum.

It is evident from the preceding chapter that with the amplitude of the subharmonic motion consisting of non-harmonic and harmonic components, the vibration can only be measured effectively through obtaining an experimental wave-form of the non-linear displacement. Harmonic analysis of the curve is then possible to determine the response characteristics under the physical condition described by equation (7.2,14). This means the periodic curves of the resulting motion are recorded independently of the forcing frequency.

The vibratory motion is recorded from the angular displacement of the cradle shaft through the use of an Elliot A.C. pick-up P. The unit is a data transmission element which is energised from an a.c. supply, and the output voltage varies with its rotor position. Hence it produces an instantaneous output voltage proportional to the corresponding angular shaft displacement. The linearity of the operating range is limited to within plus or minus thirty degrees which will more than satisfy the design requirements in view of the approximation for the non-linear stiffness rate. The unit is accurately mounted on a platform as shown in Fig.(7.1-1)(a), and to operate the pick-up within the linear characteristics it may be necessary to adjust the position of the rotor when the respective shafts are attached. Figs.(7.1-1)(a) and (b) show the two arms O which secure the unit in position are easily slackened when required for this purpose. The neutral position of the output voltage to angular displacement is determined by observing the output trace monitored on an oscilloscope. The attachment of the rotor shaft to the oscillating cradle is through a simple coupling Q which allows for any small misalignment of the two shafts.

The output element from the pick-up unit is connected to a Rank S.E.429 carrier-system M to form a balanced bridge network with the circuit elements in the amplifier and demodulator units. The instantaneous change from the transducer unbalances the bridge and correspondingly produces in proportion a modulated-carrier signal. The modulated signal is amplified to an adequate level, suitable for eventual monitoring purposes, before being rectified to produce a polarised voltage proportional to the original instantaneous input signal. In the whole conversion process, no phase is introduced in the effect of mechanical to electrical output.

For measuring the excitation frequency, the established method of having the motor shaft to work as a synchronous switch is not suitable if unintended resistive forces are to be avoided. The speed of the motor is measured as effectively through the use of a miniature photo-electric probe R which is suitably mounted on it. There is no mechanical contact between the rotating shaft and the transducer for the probe operates on the basis of varying intensity of the reflected light into a photo-diode. Hence, if the flywheel is divided into a reasonable number of black and white markings, the changes in intensity of the reflected light as the shaft rotates produce corresponding variations in the output voltage. With the frequency of the output signal proportional to the number of white strips and the motor speed, the signal pulse is readily indicated on a frequency counter. The variation in the rotational speed does not affect the output amplitude for the voltage level is not a function of velocity. However, although there is no fall in output level over the speed range, a resistor circuit is used to obtain a satisfactory deflection from the U.V. recorder X whenever necessary to maintain the high degree of accuracy in the measurement of the traces.

Fig.(7.5-1) illustrates the schematic circuit diagram for measuring the vibration amplitudes and the frequency of the disturbing force. Experimental wave-forms recorded for different particular frequencies of the second order subharmonic motion are shown in Fig.(7.5-2). The traces of the experimental results are

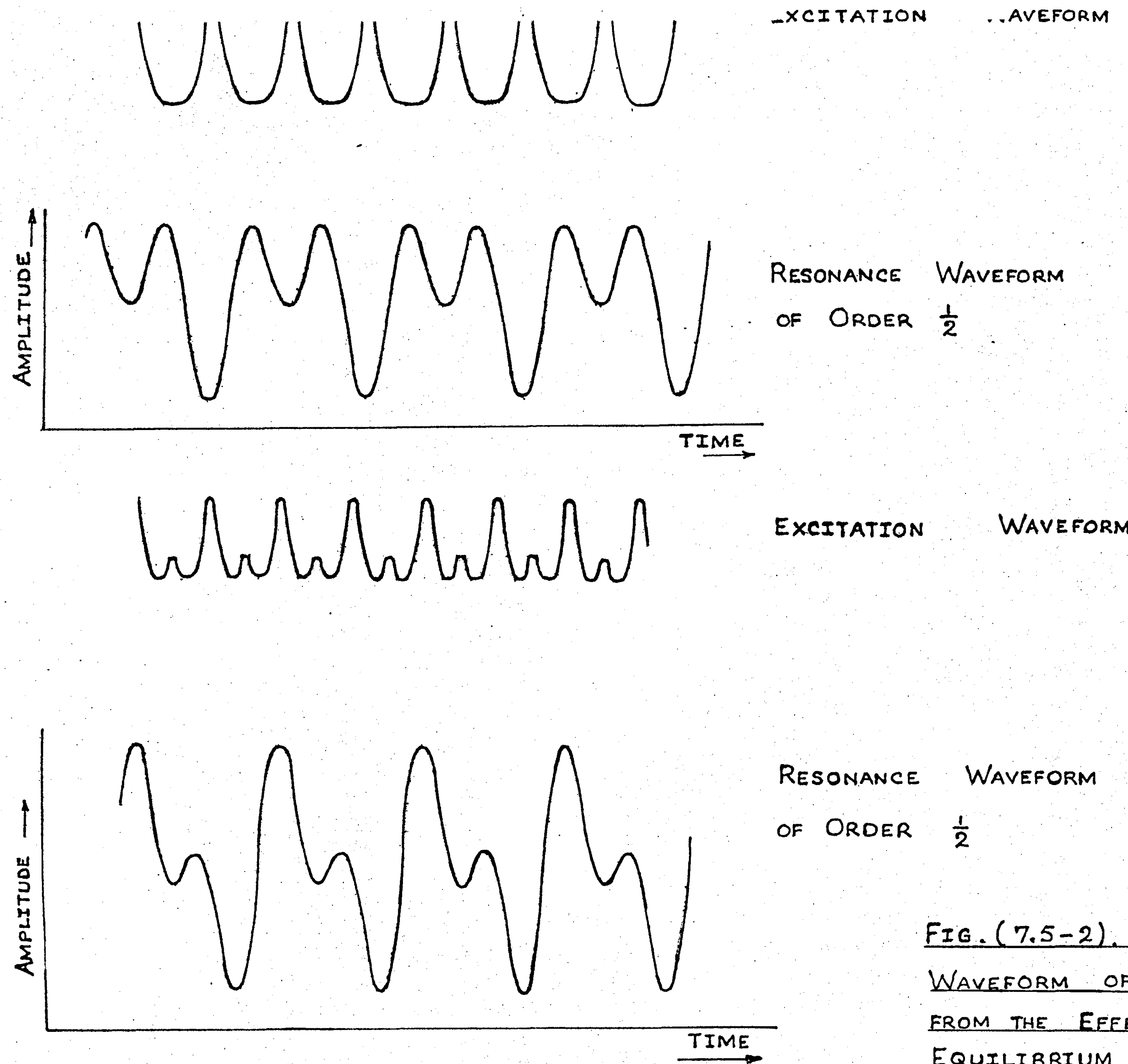


FIG. (7.5-2). EXPERIMENTAL TIME WAVEFORM OF THE DESIGNED SYSTEM FROM THE EFFECT OF GRAVITY ON THE EQUILIBRIUM OF MOTION.

analysed as described in section (5.4). The constant of proportionality between the actual vertical displacement and the corresponding recorded deflection when multiplied by the square root of the coefficient μ gives the conversion factor for reducing the amplitudes into a dimensionless quantity.

7.6. Discussion

It is evident from the resulting motion that is generated within certain frequency limits, the analytical procedure developed for the system in which the effects of gravitational forces on the non-linear suspension are considered yields subharmonic responses lower than the order one-third. In general the physical system is observed to execute the vibratory motion of predominant magnitude for every two cycles of the disturbing force over a wide frequency range. If the effective degree of non-linearity is large in relation to the small magnitude of excitation the subharmonic vibration exists, in precedence, over the frequency region of the odd order.

The designed spring system of the non-linearity defined by the expression of equation (7.2,14) gives satisfactory results when in comparison with the theoretical curves in which the harmonic component is approximated to a function of the disturbing force as given by equation (6.4,6). It is readily seen from the discussion of section (6.4)(iii) that the analysis by this approximation is sufficiently accurate for all magnitudes of the non-linearity, if the effective asymmetry of the non-linear restoring forces is not increased with the vibration. As such characteristics in which the weight of the system transmitted to the supports is increased with resonance does not arise for the small dimensionless amplitude of the excitation that is designed, it is not unreasonable then to compare the experimental results with the values from the simplified calculations. Besides, for the small amplitude of $\bar{Z}$ the harmonic component of the subharmonic vibration has a response in close similarity to the resonance curve of the higher phase where the amplitude is virtually independent of the forcing frequency. The accuracy of the analytical method in the prevalent region of the subharmonic $Q_{\frac{1}{3}}$ is established to be practically identical to that of a five-term solution, whilst in the neighbourhood of minor subharmonic vibration the frequency error is estimated for fixed value of $Q_{\frac{1}{3}}$ to be less than five per cent. The use of the approximated results for comparison is thus apparent.

The main source of experimental error is introduced from measuring the wave-forms of the vibratory motion. Although the pick-up signal is modulated, it is inevitable that from the subsequent large amplification other harmonics appear in the traces as minor irregularities. Because of the difficulty in determining accurately the subharmonic components, the error is reduced by regulating the speed of the trace and hence the x-axis of the curve. This allows a greater number of divisions in a cycle for the harmonic analysis. Inaddition by ensuring that a sufficiently large deflection from the U.V. recorder is obtained for the signal, the accuracy is maintained with the use of the Fharol chart which gives readings to the second decimal place. It is readily seen that in general the design yields experimental results of sufficient accuracy. The relative frequency error of the predominant subharmonic component for the early stages of the vibration is lower than three per cent. This means that the accuracy is still limited to within an error of eight per cent for the worst condition possible, and which for most applications is reasonable.

The subharmonic responses to the varying degree in the effects resulting from the influences of the gravitational force on a system are demonstrated experimentally and the corresponding results are recorded. By considering the state of relatively large $\bar{\Lambda}$ this does not necessarily signify the analysis is restricted in accordance to heavy rigid bodies for achieving the sufficiently large deflection. It is because in practise these physical conditions are likely to be encountered in restoring force characteristics in which the normal stiffness rate for reasonable deflection increases rapidly when the deflection becomes predominant. Often, the pronounced hardening stiffness rate is experienced suddenly and hence the coefficient μ is correspondingly large. This is readily seen from equation (7.2,15). As in the preceding chapter the vibration is illustrated by the behaviour of the more important subharmonic components.

Figs.(7.6-1) to (7.6-11) illustrate the resonance characteristics that are influenced by the existence of static deflection for small magnitudes of the disturbing force. The general similarity of the physical responses with the previous results of chapter VI are obvious and the significant effects due to the various influences are discussed in section (6.3). Since the effective non-linearity is determined mainly by the relationship between $\bar{\Lambda}$ and the amplitude of $\bar{Z}$, it is apparent, if this lies in the region of the subharmonic motion, that whatever the magnitude of excitation the significance of gravity effects on the response is not reduced. The vibratory motion,observed for small amplitude of excitation, is generated without difficulty if the working region of the displacement experiences the pronounced change in the stiffness rate, and as $\bar{\Lambda}$ becomes larger the apparent concern regarding the severity of the physical response and its existence over a greater frequency range is readily evident from Fig. (7.6-1).

An additional area of instability is introduced in the resonance of the subharmonic motion for the small amplitude of the disturbing force. It is experienced on the lower frequency side of the response. In the analysis of chapter IV and the discussion of section (6.2), it is seen that there is real possibility of the non-existence of a periodic subharmonic solution in the presence of damping. An unstable oscillation occurs in the second order region of the subharmonic vibration, and the stability of the phenomena is effectively described by the value of the parameter C_2 in relation to the criterion expressed by equation (6.2,1). If the parameter of the variational equation enters the unstable domain the vibratory motion cannot be sustained for the unstable harmonic has an oscillation of the order one-half and the system is effectively excited with negative damping.

For the designed physical condition, the extent of the instability increases with $\bar{A}$. The reason is principally because of the insignificant shift for the small amplitude of $\bar{Z}$ in the mean dynamic displacement from the position of static equilibrium. This is readily seen from Figs. (7.6-5) to (7.6-8). The existing periodic solution of that domain is for the higher frequency side of the harmonic resonance in which the amplitude for the positions of the frequency response that are influenced by $\bar{A}$ does not vary significantly. This means that although the frequency term of equation (6.2,1) is effectively increased with $\bar{A}$, it is not raised in proportion to the parameter C_2 for small amplitudes of $\bar{Z}$.

As it is common when out-of-balance force is experienced the magnitude of excitation is often relatively small, Fig.(7.6-2) illustrates the response characteristics for such conditions of varying amplitude of $\bar{Z}$. If the non-linearity, which is apparently governed by the relationship between $\bar{A}$ and the degree of excitation since viscous damping principally limits the finite amplitude, only permits subharmonics to exist in the higher frequency side of the response curve, the vibration is usually introduced on reducing the working speed. The resulting motion is often pronounced as the rate of energy dissipation is too low to forestall effectively the continual growth of the vibration with the lowering of the forcing frequency. It is also evident if the subharmonics are generated on the lower frequency side of the response, similar undesireable effects are shown to exist when the influence of gravitational force on a system is considered. The vibration is immediately pronounced, but on reducing the excitation frequency, the vibratory motion reaches a stage in which the large amplitudes cannot be effectively restrained by the lowering of the excitation. Although the frequency has moved away from the region, the prevailing motion remains until the inherent energy is dissipated.

Often the above characteristics are known to exist in practice. However the disagreeable result due to this region of unstability, in view of the results from the preceding chapter, can be avoided other than through considerable increase of damping and hence the cost, or through the raising of the amplitude of $\ddot{Z}$ which in practise is not a positive solution for obvious reasons. It is apparent from that discussion that a domain which describes conditions favourable to the existence of subharmonic motion can be defined by the terms of the independent system parameters. If the critical expressions are derived for the boundary curves, it allows an effective and economical means of suppressing the phenomena by illustrating the relative position of the existing condition of a particular system. The state of non-linearity can then be changed appropriately whenever it becomes necessary to achieve the physical condition that is outside the region. This possibility is investigated in the following chapter.

As also readily seen from the previous discussion that in general the subharmonic vibration is not significantly sensitive to damping. For the small amplitude of excitation, this is just as apparent since the harmonic component in the frequency region of the subharmonic is practically independent of the forcing frequency and the fundamental harmonic is the component by which energy is supplied to the system. If the appropriate Figs.(6.3-18) and (7.6-11) are observed, it is evident that the phase $\phi_{\frac{1}{3}}$ at the optimum amplitude for the latter condition occurs at a higher value. Since the coefficient of the viscous force is fixed and with the amplitude of the fundamental harmonic practically indifferent to changes in frequency, the higher phase values must be principally due to the increased significance from the greater values of $C_{\frac{1}{3}}$ attained. Moreover, the finite limits of the vibrating amplitude are also dependent upon the initial stiffness rate of the non-linear suspension. As it is necessary to keep this stiffness value sufficiently high to lower the coefficient of non-linearity, the uneconomical cost of suppressing the subharmonic phenomena through viscous damping alone is obvious. Fig.(7.6-3) illustrates the magnitude of the response in relation to the increase of the damping coefficient.

Since heavy damping is commonly known to suppress any vibratory motion, it is of interest to mention that in the raising of $\bar{Z}$ to overcome the instability experienced a significant shift occurs to the position of the mean dynamic displacement in the neighbourhood where the subharmonic begins. The characteristic applies for whatever the degree of asymmetry, and the parameter C_2 is reduced correspondingly away from the region of instability. Thus the instability of the lower phase response in due course ceases to exist. The condition for the existence of the subharmonics is readily ascertained from the stability criterion defined by equation (6.2,1).

In the absence of any effective viscous damping, the severity of the resonance in the lower phase is evident. The importance of considering the effects from the influence of gravitational forces on the equilibrium of motion is demonstrated for the physical state of negligible damping, and it is apparent from the results of Fig.(7.6-4) that such consideration is necessary and must not inadvertently be neglected in the analysis however small the amplitude of $\ddot{Z}$. Pronounced second order subharmonic vibration exists in the frequency region of the order one-third. The reason becomes more apparent in the following paragraph. Experimentally and as in most applications, the 'jump' phenomena occurs at a frequency value higher than the theoretical. The difficulty to locate accurately the vertical gradient probably arises from the positions of the preceding subharmonic coefficients that prescribe the initial conditions. Often at these frequency conditions the preceding stable coefficients are in close proximity to the separatrix which defines the two regions of stability. As it is common in practise for small variation in the phase shift this is sufficient to cause the instability to occur earlier than predicted for the theoretical case. Hence, as Fig.(6.5-2) readily illustrates, the resulting vibration is apparent.

The characteristics of the shift in the mean dynamic displacement from the position of static equilibrium with frequency are shown by Figs.(7.6-5) to (7.6-8). For the small amplitudes of $\ddot{Z}$, the deflection during the early stages of $Q_{\frac{1}{2}}$ is seen for both branches of the response to be insignificant and the non-linearity is effectively at a maximum. Consequently, as the predominant term is the second order, the vibratory motion of the order one-half is readily excited if the condition which describes the relationship of the system parameters and frequency lies within the domain favourable for subharmonic motion, and as observed in the above paragraph it is the more important order of motion. Since the harmonic component which is the effective amplitude of the disturbing force is limited, the response of the lower phase will depend upon whether the non-linearity increases the value of the parameter C_2 in overall significance. This has been discussed in an earlier paragraph and the vibration will exist in a pronounced state over the unstable domain if the equilibrium state of the second region can be maintained. In all other aspects, the significant characteristics of the mean dynamic position are similar, for whatever the degree of excitation, to the results of the preceding chapter and have accordingly been discussed.

The changes in the phase of the principal subharmonic component with frequency are illustrated by Figs.(7.6-9) to (7.6-11). As the vibratory motion is not significantly sensitive to viscous force and in view of the degree of excitation, the phase $\phi_{\frac{1}{2}}$ of the finite amplitude limits occurs at higher values due principally to the relatively larger existing $Q_{\frac{1}{2}}$ in relation to the other variables. Otherwise the general behaviour is similar to the results for larger amplitudes of $\bar{Z}$. Where viscous damping is zero, the phase is either zero or $\frac{\pi}{2}$ for the lower and higher frequency side of the response respectively.

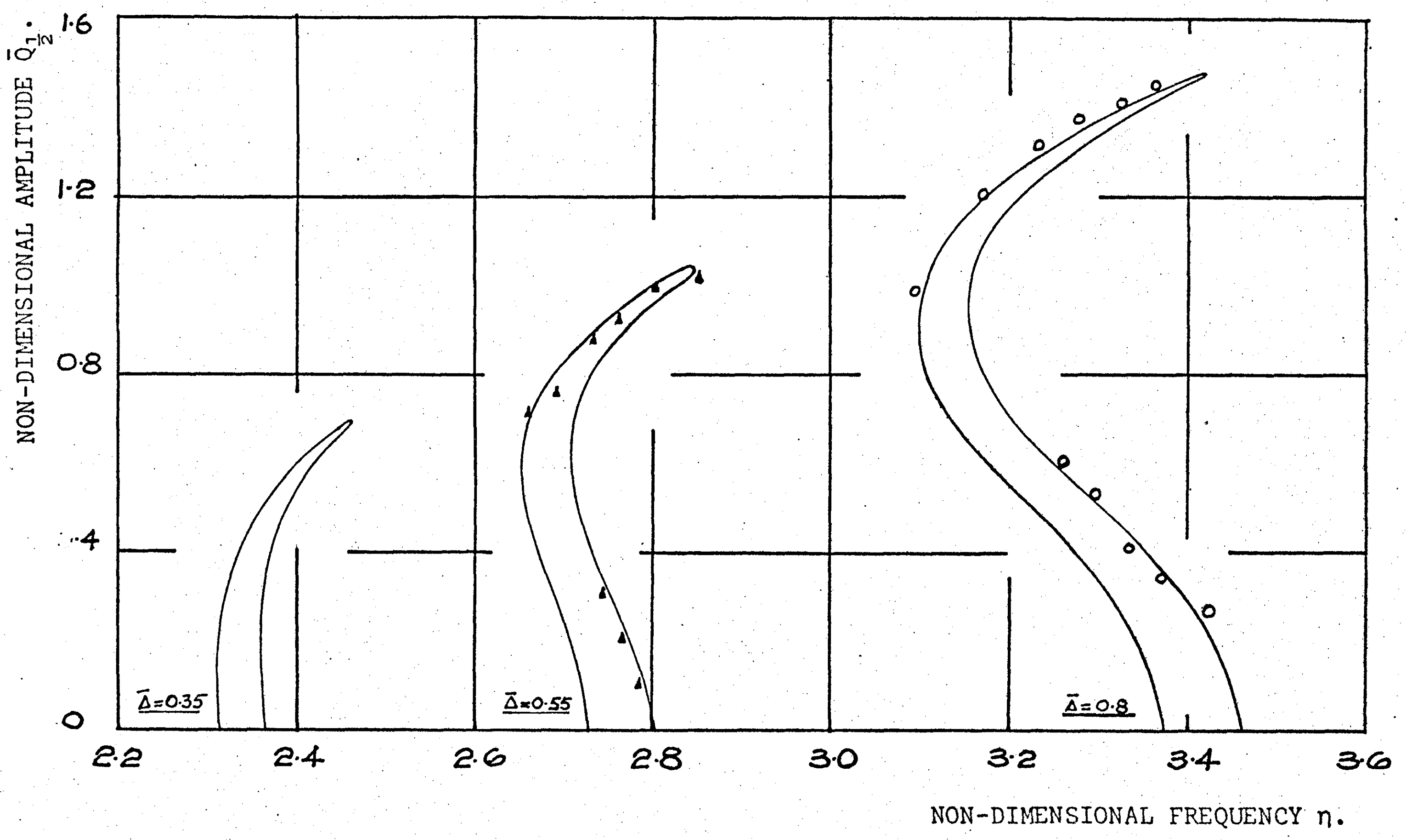


FIG.(7.6-1). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{Z}$ = 0.0242, R = 0.0086.

——— THEORETICAL RESULTS.

▲ o EXPERIMENTAL RESULTS.

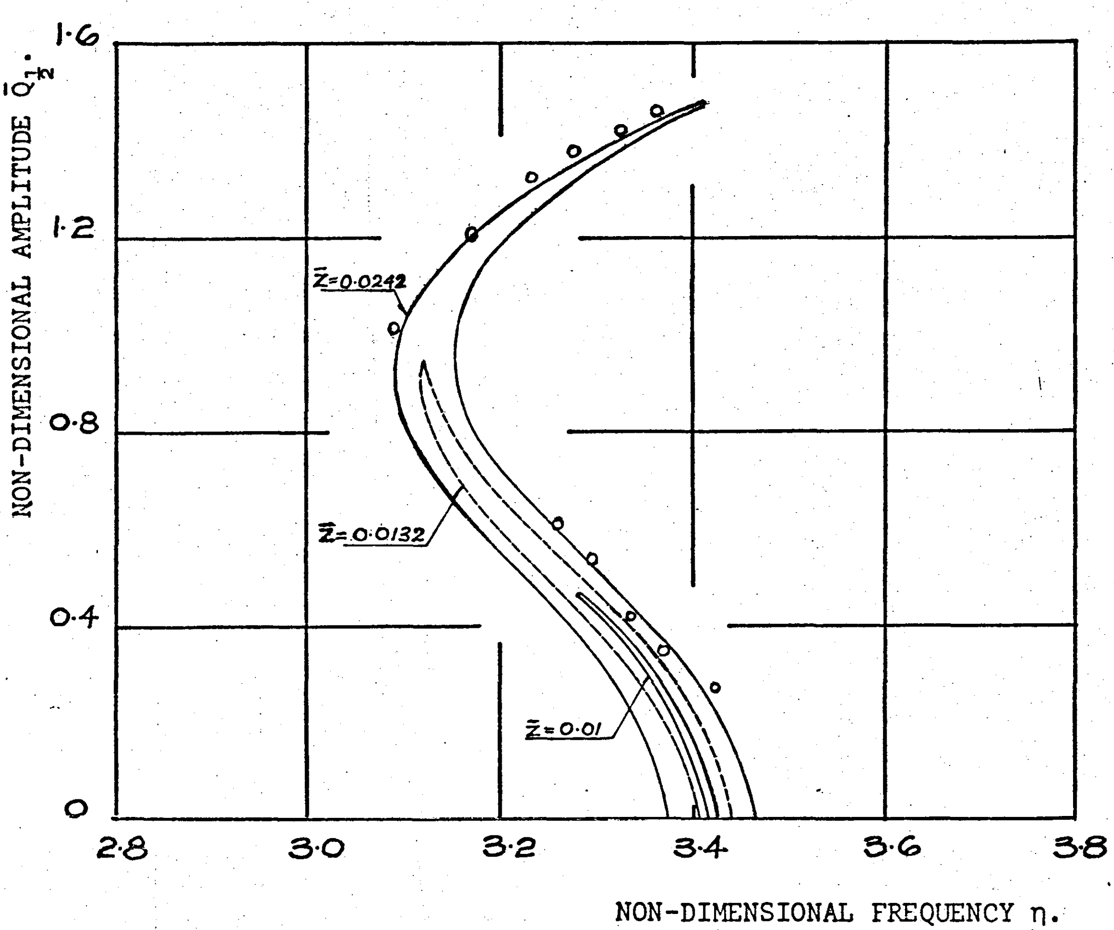
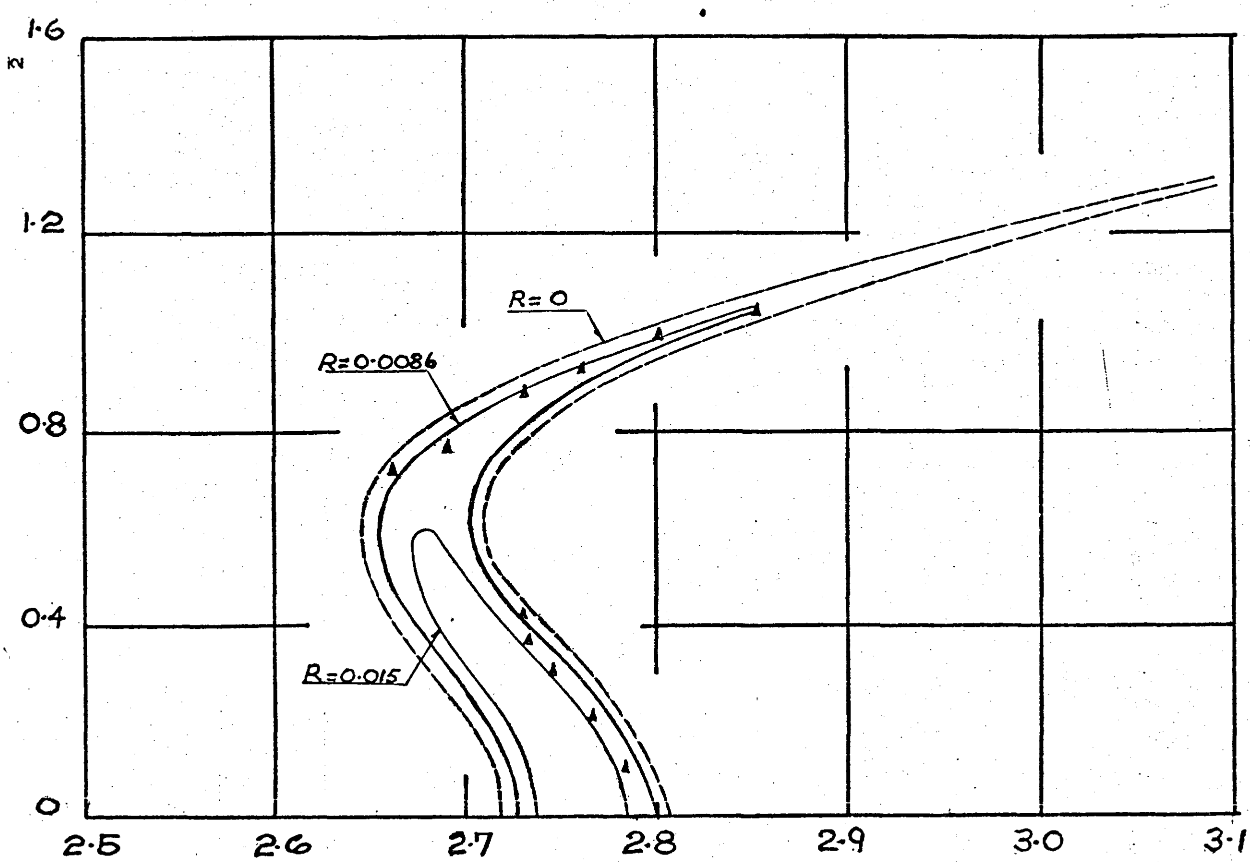


FIG.(7.6-2). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{\Delta} = 0.8$, $R = 0.0086$.

----- ———— THEORETICAL RESULTS.

o $\bar{Z}=0{\cdot}0242$ EXPERIMENTAL RESULTS.



NON-DIMENSIONAL FREQUENCY η.

FIG.(7.6-3). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{\Delta} = 0.55$, $\bar{Z} = 0.0242$.

- - - - - ———— THEORETICAL RESULTS.

▲ EXPERIMENTAL RESULTS.

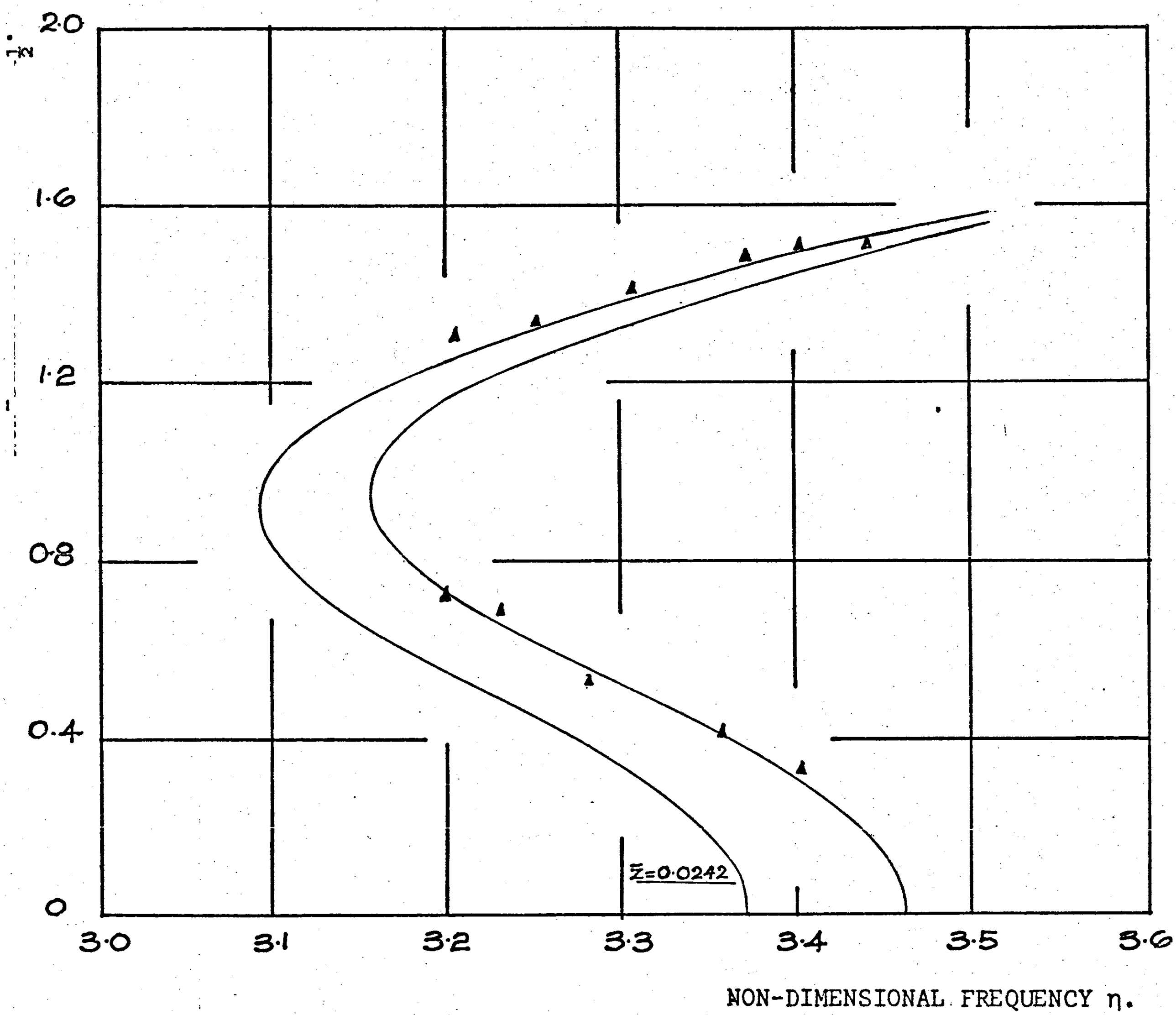


FIG. (7.6-4). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{Q}_{\frac{1}{2}}$. $\bar{\Lambda} = 0.8$, R = 0.

——— THEORETICAL RESULTS.

▲ EXPERIMENTAL RESULTS.

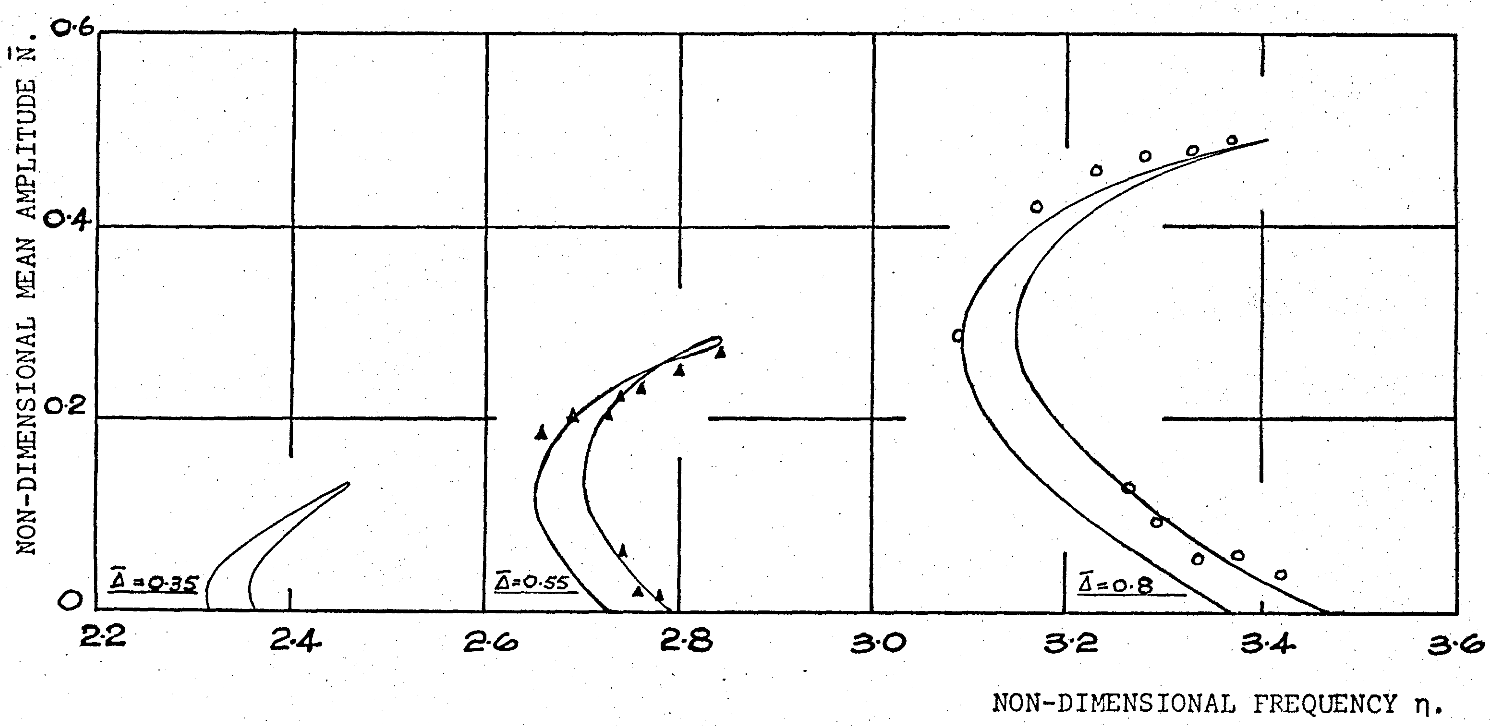


FIG.(7.6-5). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{N}$. $\bar{Z}$ = 0.0242, R = 0.0086.

——— THEORETICAL RESULTS.

▲ o EXPERIMENTAL RESULTS.

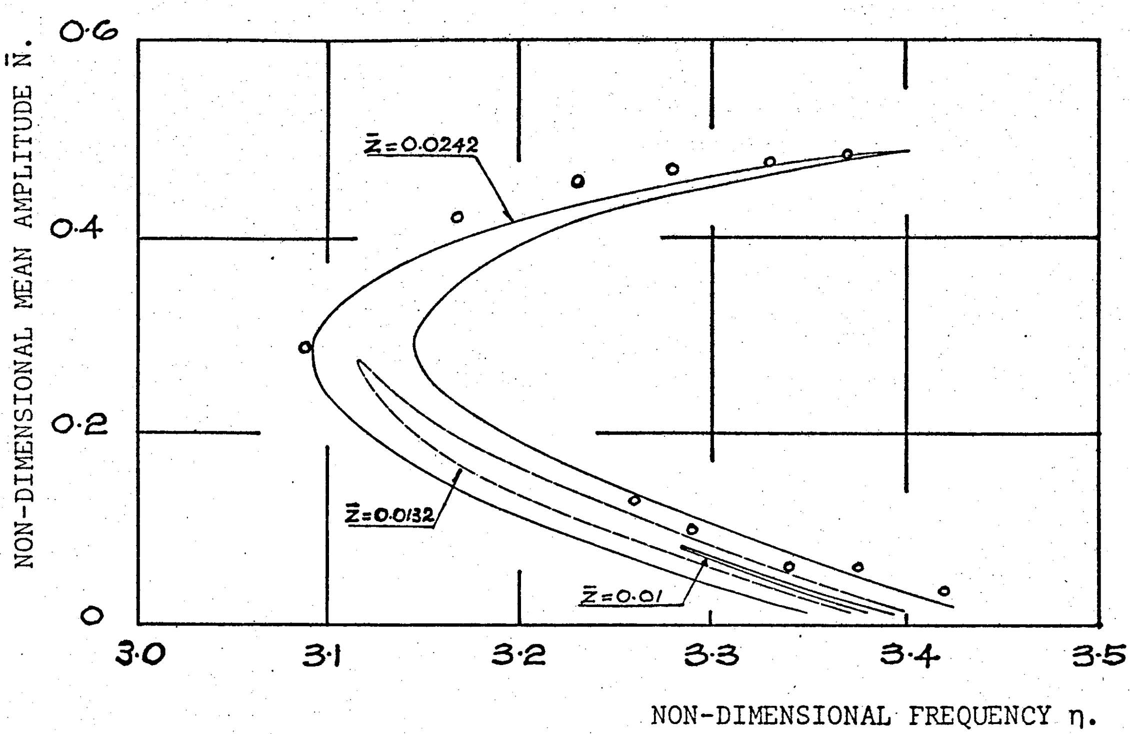


FIG.(7.6-6). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{N}$. $\bar{\Delta}$ = 0.8, R = 0.0086.

——— THEORETICAL RESULTS.

o $\bar{Z}$= 0·0242 EXPERIMENTAL RESULTS.

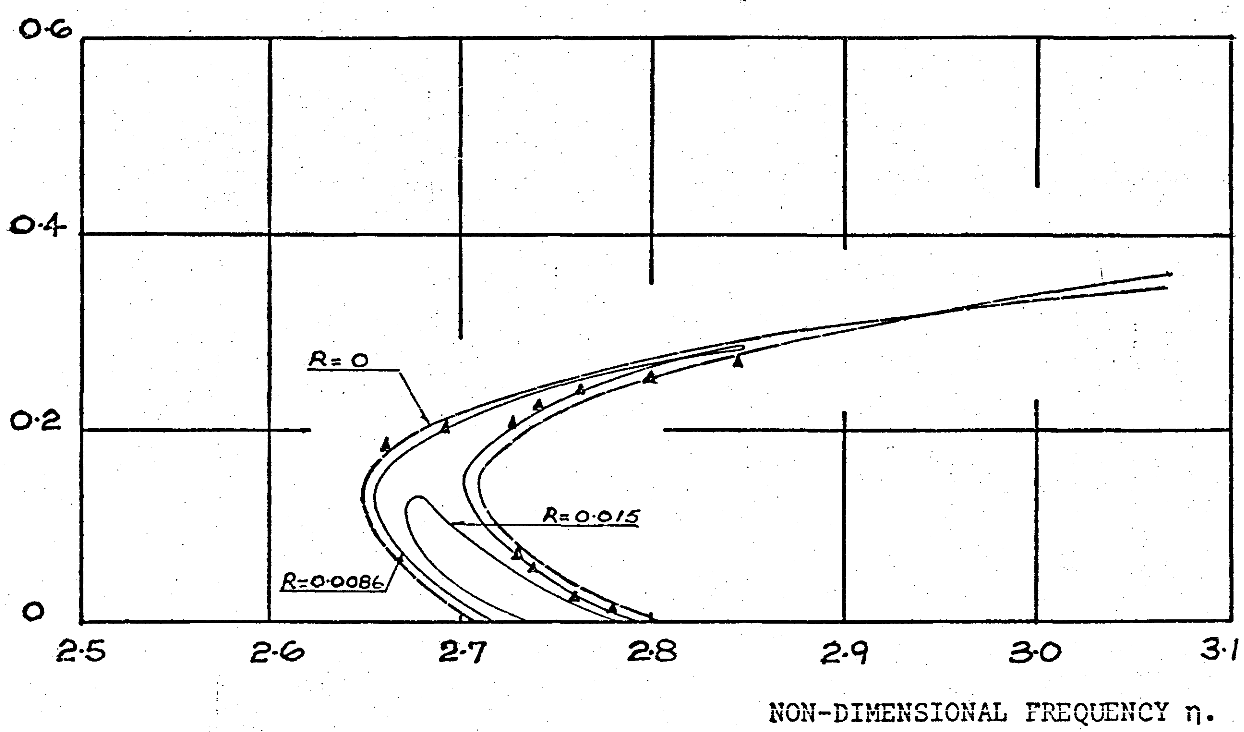


FIG.(7.6-7). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{N}$. $\bar{\Delta} = 0.55$, $\bar{Z} = 0.0242$.

——— ——— THEORETICAL RESULTS.

▲ R=0·0086 EXPERIMENTAL RESULTS.

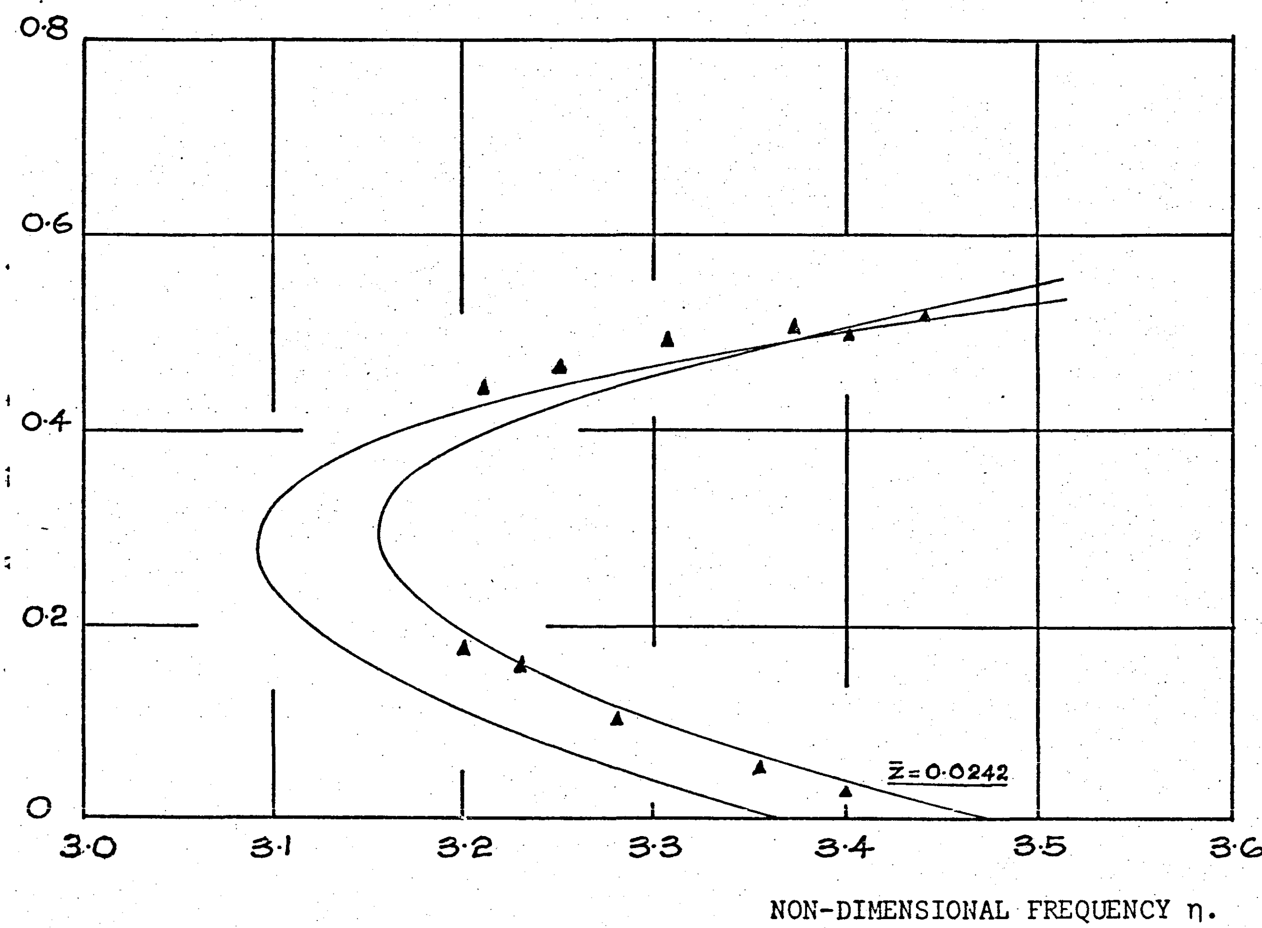


FIG.(7.6-8). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\bar{N}$. $\bar{\Delta} = 0.8$, $R = 0$.

——— THEORETICAL RESULTS.

▲ EXPERIMENTAL RESULTS.

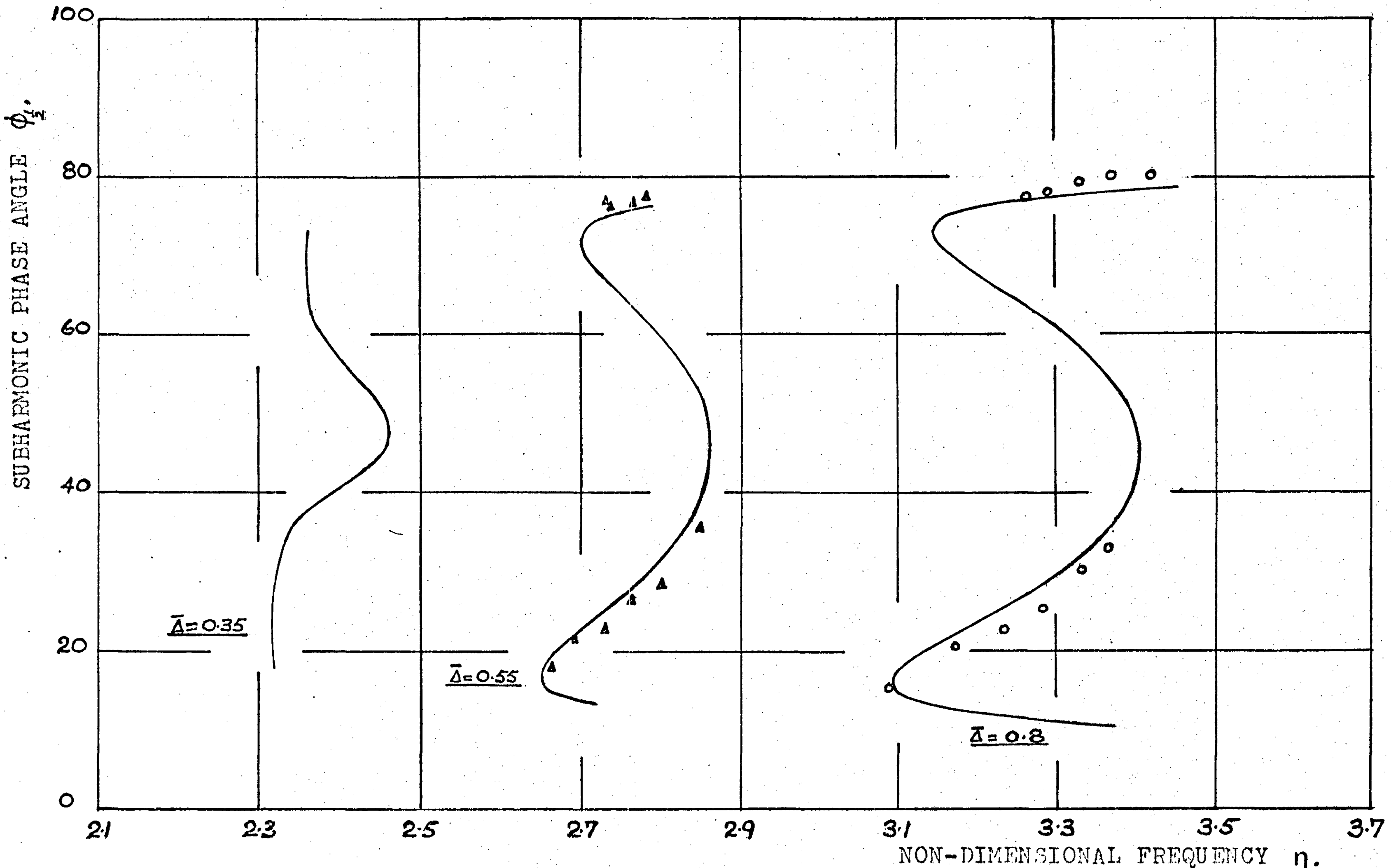


FIG.(7.6-9). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\phi_{\frac{1}{2}}$. $\bar{Z}=0.0242$, $R=0.0086$.

——— THEORETICAL RESULTS.

▲ ○ EXPERIMENTAL RESULTS.

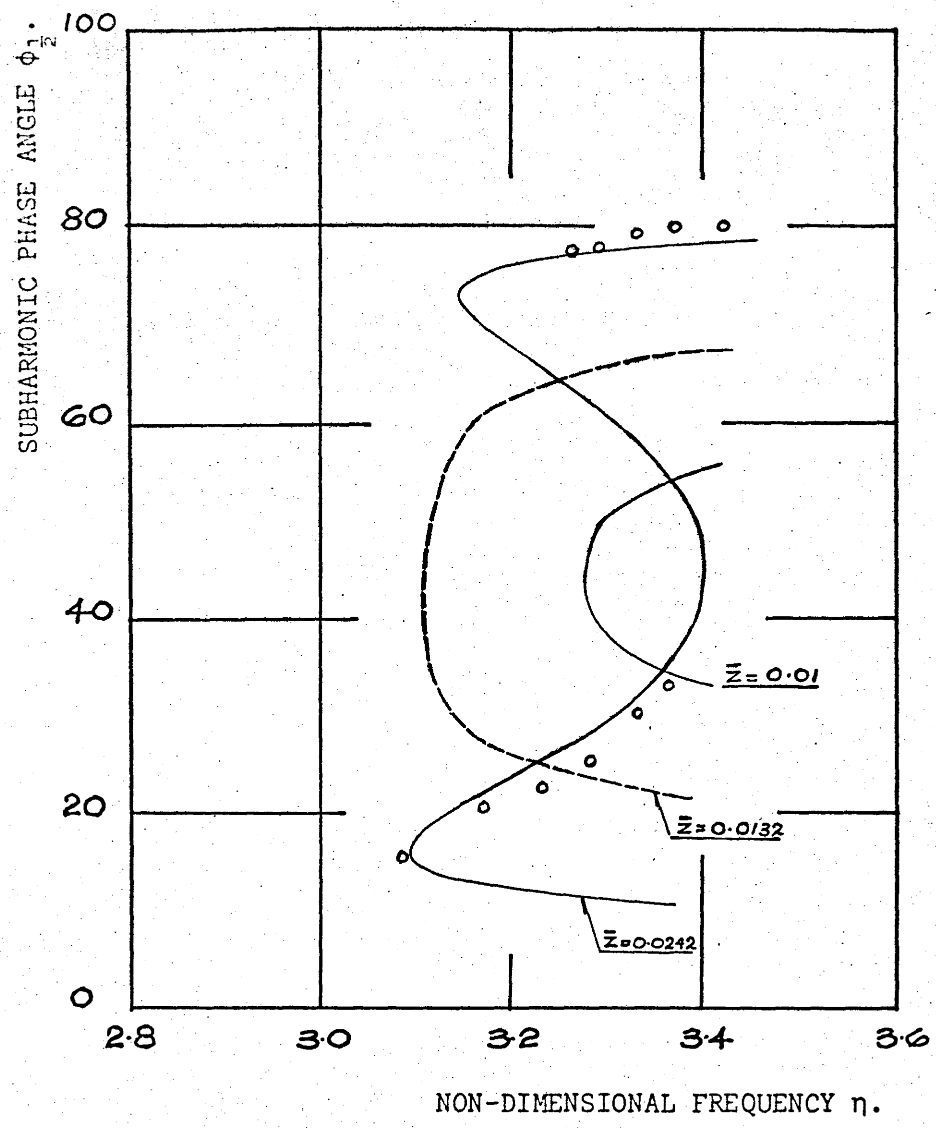


FIG.(7.6-10). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\phi_{\frac{1}{2}}$. $\bar{\Delta}$ = 0.8, R = 0.0086.

- - - - ———— THEORETICAL RESULTS.

o $\bar{Z}$=0·0242 EXPERIMENTAL RESULTS.

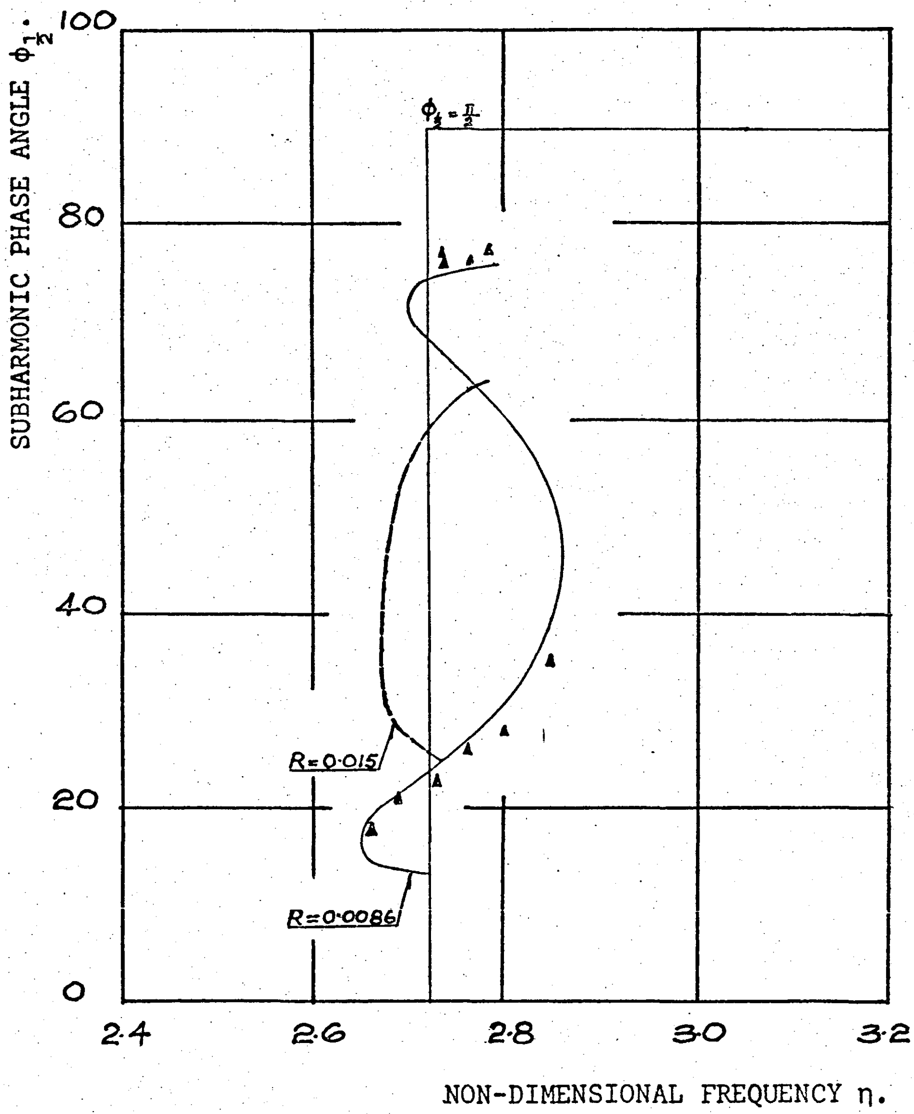


FIG.(7.6-11). SUBHARMONIC SOLUTION WITH $\bar{Q}_1$ APPROXIMATED.

RESPONSE CURVES FOR $\phi_{\frac{1}{2}}$. $\bar{\Lambda} = 0.55$, $\bar{Z} = 0.0242$.

– – – – ———— THEORETICAL RESULTS.

▲ EXPERIMENTAL RESULTS.

Chapter VIII

The extinction of predominantly subharmonic vibration under the effects of gravity.

8.1. The limiting conditions described by the relationship between the independent systems parameters.

From the analysis of the behaviour of the important components of subharmonic vibration, the significance of considering the effects of gravity on the equilibrium of motion is evident. As they are not completely balanced by the static deflection force the predominant subharmonic component is of the second order. It is seen that the magnitude of non-linearity is increased and that the effective degree is determined by the relationship between the system parameters in which the principal factors influencing the magnitude of the vibration response are the amplitudes of $\bar{A}$ and $\bar{Z}$ whilst the coefficient of damping limits the optimum resonance amplitude.

Since in the region of pronounced subharmonics, the frequency values are dependent on the effective non-linearity and as the vibration characteristics of the subharmonic components clearly indicate the critical state of non-linearity in which subharmonics cannot be generated is determined by the relative values of the independent parameters, it is within possibility then to attain the experimental conditions for which the phenomenon is effectually destroyed by defining the limiting inequalities independent of the frequency term and of the dependent variables. Consequently, as the inequality describes the boundary regions for the existence of the subharmonic solution, it allows for suitable permutations of the independent variables possible in the early design stages.

8.1.(i). The inequality.

Although the vibratory motion is not significantly sensitive to viscous damping, a limiting relationship for the degree of asymmetry does exist between the coefficient of the viscous force and the amplitude of $\bar{\bar{Z}}$. However it is considered that because of the resonance magnitude being principally dependent on the latter and on the amplitude of $\bar{A}$, the efficacy of the inequality becomes more apparent by representing the locus of the expression between these two parameters for fixed values of R. Besides, the illustration is in a more tractable form for an assessment of the overall importance of the non-linear coefficient μ and of the degree of asymmetry resulting from the effects of gravity since both parameters can be interpreted through the term static deflection.

Under the effects of gravity, significant changes in the position of the mean dynamic displacement exist over the resonance and the state of the effective restoring force of the system is characterised by its behaviour. As this is illustrated by the response of the constant term N, then on using equation (3.2.13) with the definition expressed by Fig.(8.2-1)(a) yields after simplification

$$4NY(1 + N^2 + 3\Delta Y) + 6Y^2 (Q_{\frac{1}{3}}^2 + Q_1^2) - Q_{\frac{1}{3}} \{ 1 - \frac{\eta^2}{4} + \frac{3}{4}$$

$$(Q_{\frac{1}{3}}^2 + 2Q_1^2) + 3Y^2 \} = 0 \qquad \ldots\ldots \quad (8.1,1)$$

For the resonant state of predominantly large $Q_{\frac{1}{3}}$, the frequency response is reasonably approximated without limiting the generality of the condition for the existence of the component if it is regarded within the region as given by

$$\eta^2 = \{ 4 + 3 (Q_{\frac{1}{3}}^2 + 2Q_1^2) + 12Y^2 \pm 12YQ_1 \} \qquad \ldots\ldots \quad (8.1,2)$$

The plus sign corresponds to the lower phase of the response and the negative sign to the higher frequency side of the curve.

Thus, with the appropriate sign for the predominant region of subharmonic vibration equation (8.1,1) is reduced on using the substitution (8.1,2) to

$$4N^3 + 4N + 12\Delta NY + 6YQ_1^2 = - 3Q_{\frac{1}{3}}^2 (2Y + Q_1) \qquad \ldots\ldots \quad (8.1,3)$$

Although the term N in the presence of viscous damping is always negative whatever the physical state of the system, it is apparent from the above expression that real roots of $Q_{\frac{1}{3}}$ exist for

$$2(2N^2 + 2N + 6ANY + 3YQ_1^2) \gtrless 0 \qquad ..... \quad (8.1,4)$$

as the value of the term on the right of equation (8.1,3) can either be positive or negative. Since Y the effective asymmetry of the restoring force can only be positive, and with the accompanying harmonic Q_1 in the second phase quadrant principally influenced by the amplitude of $\bar{Z}$ and having an approximate value defined by equation (6.4,6), the resulting value whether higher or lower than zero is dependent on whether the amplitude of $\bar{A}$ or the magnitude of the disturbing force has the greater influence in determining the characteristic behaviour of the position of dynamic equilibrium, and hence the effective non-linearity. This corroborates the earlier discussion (see section (6.3)) that generally there are two limiting values of $\bar{Z}$ for a fixed amplitude of $\bar{A}$ since only a single real root of N exists at a given time. The subharmonics can exist over a large difference in the value of the degree of excitation. Thus, for a given magnitude of $\bar{A}$, the subharmonic vibration cannot be generated when equation (8.1,4) equals to zero.

With the constant coefficient $N = (Y - A)$, equation (8.1,4) is reduced in the state of subharmonic resonance not being substained in equilibrium to

$$2Y^2 + Y(3Q_1^2 + 2) - 2(A^2 + A) = 0 \qquad .... \quad (8.1,5)$$

This gives the condition of non-linearity, that results from the effect of gravity influencing the equilibrium of motion, for which subharmonic vibration cannot be generated.

From equation (8.1,3), it is evident that real roots of

$Q_{\frac{1}{3}}$ also cannot be obtained when

$$3Y^2(Q_1^2 - 4R^2) < 2R^2(2 + 3Q_1^2 - 2R^2) \quad \ldots. \quad (8.1,6)$$

The inequality indicates the experimental conditions under the effects of viscous damping. As the frequency region is in the neighbourhood of optimum resonance amplitude, then on substituting for the effective asymmetry the upper limit on the amount of damping a system may have and still be able to generate a subharmonic solution is given by

$$\left(\frac{2R^2(2 + 3Q_1^2 - 2R^2)}{3(Q_1^2 - 4R^2)}\right)^{\frac{3}{2}} + (\tfrac{3}{2}Q_1^2 + 1)\left(\frac{2R^2(2 + 3Q_1^2 - 2R^2)}{3(Q_1^2 - 4R^2)}\right)^{\frac{1}{2}}$$

$$< (\Delta^3 + \Delta) \quad \ldots. \quad (8.1,7)$$

in which Q_1, the accompanying harmonic of the subharmonic vibration, is the only dependent variable.

Generally, if the degree of non-linearity does not increase with the vibration, the use of the approximation of the harmonic component expressed by equation (6.4,6) within the frequency band-width is seen to produce identical vibration characteristics and in the region of predominant $Q_{\frac{1}{3}}$ results of compatible accuracy. The approximated value otherwise is lower than the harmonic coefficient of solution (8.2,1), and if it is substituted to simplify the equations describing the behaviour of the other components it will limit considerably the magnitude of the subharmonic response. However, if it is applied in the neighbourhood of the physical condition that is described by the inequality, it is evident from earlier discussions of the preceding chapters (see section (6.4)) that the approximation of the fundamental harmonic, independent of frequency and as the effective amplitude of $\bar{Z}$, is perfectly reasonable. Besides the optimum amplitude of Q_v under the critical extinction condition occurs in the frequency region which has a value approximately twice the natural frequency, irrespective of the magnitude of the non-linearity. This is readily apparent from equation (6.4,3) in which the variables take the corresponding equivalent values of the subharmonic region.

Thus, substituting for the dependent variable as the effective amplitude of $\bar{Z}$, the inequality (8.1,7) is reduced to

$$2R\left(\frac{3 + 8Z^2}{12Z^2 - 9R^2}\right)^{\frac{3}{2}} (2Z^2 - R^2) < (\Delta^2 + \Delta) \quad \ldots. \quad (8.1,8)$$

The above inequality enables the locus of the critical relationship between the independent describing parameters of a system to be readily illustrated.

As it is expressed completely in terms of the parameters defined in equation (3.1,5), this means the critical condition for which subharmonic vibration is destroyed is obtained through any one of the limiting values in relation to whatever the quantity of the other parameters. The above inequality also allows if it becomes necessary for the value of each parameter to be changed appropriately to satisfy the particular requirements.

The physical significance is more readily apparent by illustrating the inequality (8.1,8) as shown in Fig.(8.1-1). For reasons mentioned in an earlier paragraph, the locus defining the boundary of the limiting relationship is considered between the amplitudes of $\bar{A}$ and of $\bar{Z}$ for fixed coefficients of viscous damping. It is evident from the comparison with the experimental results that the limiting relationship between the respective extinction values is applicable whatever the magnitude of gravity effects influencing the equilibrium of motion.

<u>8.1.(ii). The optimum $\bar{A}$ specifying the minimum extinction value of viscous damping R.</u>

The common practise of eliminating the subharmonic resonance by the increase of damping is obviously not satisfactory since the vibratory motion is not significantly sensitive to viscous force, and in such cases of vibration isolation where the primary function is also to reduce the forces transmitted this will be undesireable for it raises the transmissibility. In regard to reducing the amplitude of $\bar{Z}$, the relative cost of balancing the system to the stage of achieving the extinction is usually high, and often it may not be the most appropriate step to take for correcting the problem. The need is for a suitable compromise.

Although the critical magnitude of $\bar{A}$ for which there is no subharmonic solution is evidently dependent on the values of the other governing parameters, it is however apparent that there exists an optimum state in which subharmonic resonance cannot be

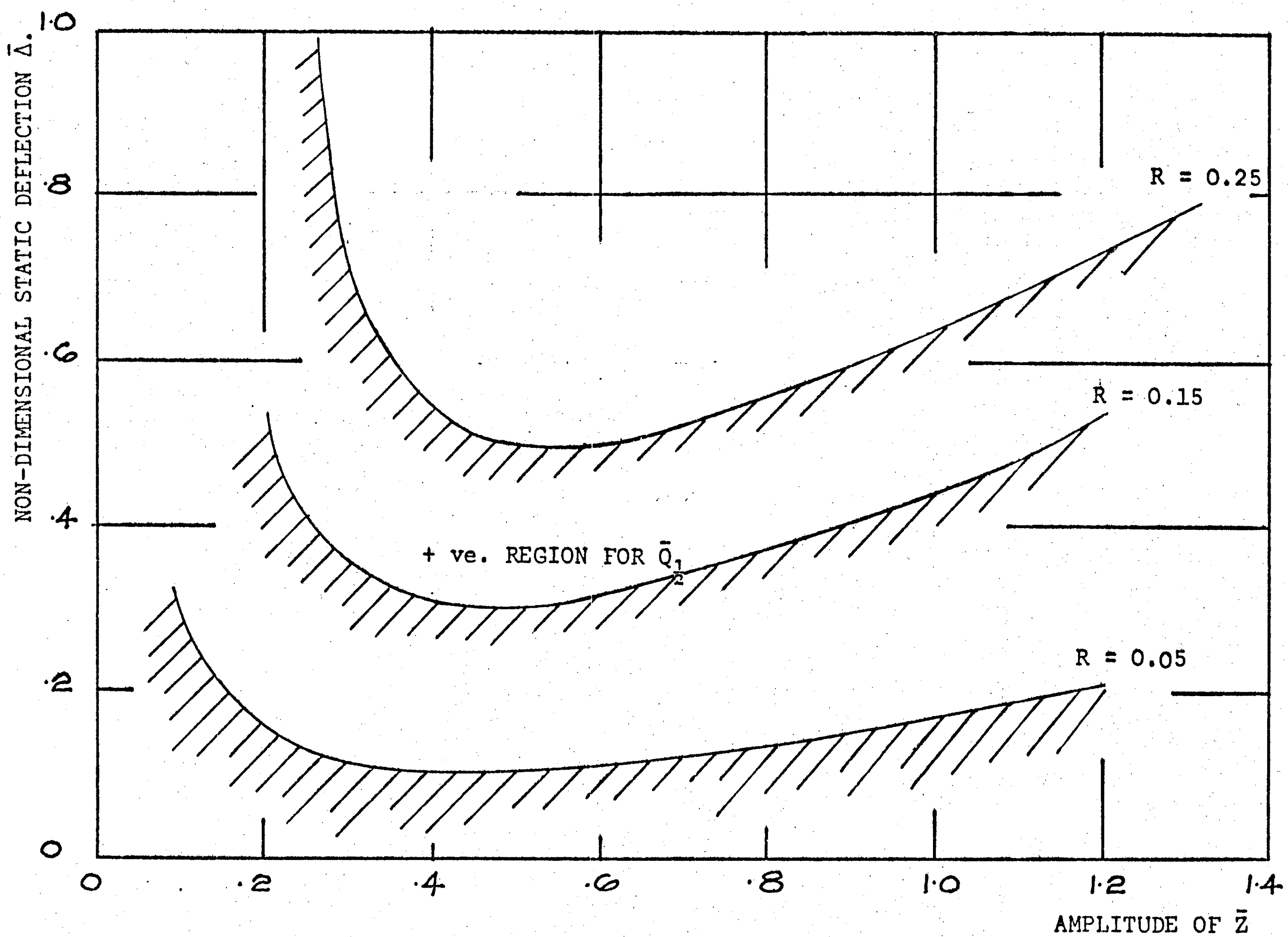


FIG.(8.1-1). THE PHYSICAL CONDITIONS AS DEFINED BY THE INEQUALITY (8.1,8) FOR WHICH THE SUBHARMONICS UNDER THE EFFECTS OF GRAVITY ARE DESTROYED.

generated irrespective of the degree of excitation under the effects of gravity. This physical condition is obtained on determining from equation (8.1,8) the critical amplitude of the disturbing force for a convenient damping rate of the dashpot.

Hence, the evaluation of the magnitude of the disturbing force from which subharmonic vibration cannot be generated because of the extinction condition of $\bar{A}$ is given approximately for a fixed value of viscous damping, after simplification, by

$$Z^4 + (0.1877 - 1.877\,R^2)\,Z^2 - (0.0704 + 0.704R^2) = 0 \qquad \ldots\ (8.1,9)$$

It is readily seen from the above expression that the extinction of the pronounced subharmonic vibration from the inequality (8.1,8) need not depend on the amplitude and frequency of the disturbing force. For a suitable viscous damping rate, the value of excitation which subsequently enables the critical amplitude of $\bar{A}$ to be determined is fixed from equation (8.1,9). The limiting $\bar{A}$ is the optimum value permissible if subharmonic vibration is to be avoided. With this value of $\bar{A}$ the extinction condition is achieved irrespective of the degree of excitation.

The critical degree of the gravity effects influencing the equilibrium of motion specifies the minimum extinction value of viscous damping. Thus, for an agreeable coefficient of extinction R, the vibratory motion is destroyed through either reducing the difference between the two stiffness rates of a hardening suspension or lowering the amplitude of static deflection. If the system is run in the frequency region of the subharmonics, the latter method of destroying the resonance by raising the initial stiffness rate means the response domain is also shifted from the region.

Finally, since the influence of gravitational force on the equilibrium of motion results in a predominant subharmonic component of the second order, the subharmonics of higher order than one-half are effectively non-existent in the presence of the critical damping for the limiting value of $\bar{\lambda}$ determined from equation (8.1,9) and inequality (8.1,8). The effects of gravity in the optimum state permissible for $\bar{\lambda}$ cannot induce subharmonic resonance and the reduction in the amplitude of $\bar{Z}$ need only reach a suitable stage of balancing.

8.1.(iii). The upper limit of viscous damping for a given frequency ratio.

In general the magnitude of the vibration response to the effects of viscous damping is not significantly reduced. If the need arises in which the system must operate at a constant speed in the region of pronounced subharmonic resonance, the suppression of the vibration by damping only is shown in the following paragraph, to require relatively large coefficients when the effective non-linearity is not in the neighbourhood of the boundary described by the inequality (8.1,8). The limiting relationship is derived as a function of frequency to indicate that along the considered frequency region the upper limit of viscous damping the system may have and still have a subharmonic solution can be greater than twice the value from the procedure of the previous section.

It is apparent in reference to equation (8.1,9) that for the existence of subharmonic motion during the predominance of the subharmonic component $Q_{\frac{1}{3}}$,

$$9Q_1^2Y^2 \;>\; R^2(4 + 3Q_{\frac{1}{3}}^2 + 6Q_1^2 + 12Y^2 - 4R^2) \qquad \ldots\ldots \;(8.1,10)$$

For the state of the response in mind, it is not unreasonable to apply the approximation as defined by equation (8.1,2). Hence on substituting with the appropriate sign the above inequality is reduced to

$$R(\eta^2 - (12Y^2 + 4R^2))^{\frac{1}{2}} \;<\; 3Q_1Y \qquad \ldots\ldots \;(8.1,11)$$

If the effective non-linearity of the system is in the neighbourhood of the inequality (8.1,8), the expression for the upper limit of viscous damping may further be reduced to

$$R \;<\; \frac{3Q_1Y}{(\eta^2-12YQ_1)^{\frac{1}{2}}} \qquad \ldots\ldots \;(8.1,12)$$

The dependent variables in both inequalities (8.1,11) and (8.1,12) will have to be expressed completely in terms of the system parameters if the limiting coefficient of R is to have any practical significance. It is shown that for the state of the response under consideration, the harmonic component Q_1 can be approximated satisfactorily and the effective asymmetry of the restoring forces in the critical region is given by

$$Y^3 + \zeta Y(1 + \frac{8}{3} Z^2) - (A^3 + A) = 0 \qquad \ldots\ldots \quad (8.1,13)$$

in which only a single real root is obtained with a pair of conjugate imaginary values.

Thus, with equations (6.4,6) and (8.1,13), the extinction coefficient of viscous damping is readily evaluated from the inequality (8.1,11) for any particular dimensionless frequency ratio in the region of predominant subharmonic vibration. For small deviation from linearity in the characteristics of the restoring forces, the inequality can take the form

$$R < \frac{4ZY}{(\eta^2+16ZY)^{\frac{1}{2}}} \qquad \ldots\ldots \quad (8.1,14)$$

Table (8.1-2) shows that the experimental results in general are in good agreement with the critical values of the inequality (8.1,11). As it is readily illustrated by the form of the inequality, the critical damping rate for subharmonic extinction is increased with the effective degree of non-linearity. The limiting coefficients specified by the inequality are slightly lower for cases in which the effective asymmetry of the restoring forces is large. The reason is principally because the actual value of Q_1 is smaller than that which is given

EFFECTIVE NON-LINEARITY REPRESENTED THROUGH THE AMPLITUDES OF		DIMENSIONLESS FREQUENCY RATIO	THE MINIMUM EXTINCTION VALUE FROM THE INEQUALITY (8.1,11)	EXPERIMENTAL COEFFICIENT
$\bar{\Delta}$	$\bar{Z}$	η		R
1.0	0.4	3.6	0.36	0.41
1.0	0.4	3.8	0.34	0.414
0.8	0.4	2.9	0.32	0.335
0.8	0.4	3.1	0.305	0.36
0.8	0.4	3.2	0.3	0.355
0.8	0.4	3.44	0.295	0.35
0.6	0.4	2.6	0.26	0.3
0.6	0.4	2.5	0.27	0.29
0.6	0.4	2.9	0.25	0.305
0.6	0.4	3.2	0.235	0.255
0.4	0.4	2.4	0.176	0.2
0.4	0.4	2.6	0.17	0.2
0.4	0.4	2.7	0.166	0.19
1.0	0.8	3.2	0.51	0.53
1.0	0.8	3.6	0.49	0.55
1.0	0.8	3.8	0.47	0.57
1.0	0.8	4.5	0.455	0.45
1.0	0.6	3.4	0.46	0.5
1.0	0.6	3.6	0.435	0.51
1.0	0.6	3.8	0.43	0.515

TABLE(8.1-2). THE COMPARISON BETWEEN THE EXPERIMENTAL UPPER LIMITS OF VISCOUS DAMPING AND THE VALUES OF THE INEQUALITY (8.1,11) FOR THE VARIOUS MAGNITUDES OF NON-LINEARITY.

by the approximation (8.1,2). The effective asymmetry substituted into the inequality is correspondingly reduced. This is evident from the characteristic behaviour of the variable concerned. Although the difference, which is insignificant where the deviation from linearity is small, is generally within the second decimal place, it is sufficient since it increases with frequency to cause the apparent limitation of the inequality near the frequency of optimum vibration amplitude. However, the upper limits of viscous damping given by the inequality are in the close vicinity of the experimental results, and the limiting relationship is applicable for all magnitudes of $\bar{\lambda}$ providing the frequency is in the region that satisfies the requirements in the approximation (8.1,2).

It is readily seen from Table (8.1-2) that for the respective sets of parameters the critical coefficients of viscous damping do not differ in any degree over the optimum amplitude region of the values necessary for the extinction of the resonance. This means that the upper limit of R once determined for a particular frequency is applicable in practise over the whole range of the pronouhced subharmonic vibration. Since the relative cost of balancing the system to the stage of achieving the extinction is often high, the apparent advantage from the inequality (8.1,8), in determining the optimum degree in the effects of gravity for which a system may experienced and yet subharmonic resonance is not generated, is evident by the above magnitudes of the viscous damping required in suppressing the vibratory motion. In cases where the effects of gravitational force on the equilibrium of motion are large, this can be more than twice the agreeable compromised value, for apart from the problem of uneconomical cost the large damping factor can also be undesireable in view of the difficulty that is encountered through high transmissibility ratio.

8.2. The frequency ratios defining the domain of subharmonic motion.

Since the subharmonic phenomena exist only in a limited frequency range that is governed by the system describing parameters, the convenience of realizing the extinction of the resonance by determining the two critical boundary frequencies from the limiting relationship between these variables is apparent. As the resonance condition can then be predicted with reasonable accuracy and without undue speculation on the values the dependent coefficients of the solution take in the region, the problem of unknowingly working the system within the frequency band-width need not arise.

8.2.(i). The two lower stable branches of the response.

If the subharmonic domain of the predominant component is described by the lower and upper limits of η, the frequency range in which the subharmonic amplitude $Q_{\frac{1}{2}}$ cannot be generated is expressed by the inequality in the form

$$\eta^2 \lessgtr \{ 4 + 6(Q_{\frac{1}{2}}^2 + 2Y^2) - 8R^2 \} \pm 4 \{ 9Q_{\frac{1}{2}}^2Y^2 - R^2(4 + 6(Q_{\frac{1}{2}}^2 + 2Y^2) - 4R^2 \}^{\frac{1}{2}} \qquad \ldots\ldots (8.2,1)$$

The plus and minus signs signify the upper and lower critical frequencies of the subharmonic band-width respectively. At the critical point just before real roots of $Q_{\frac{1}{2}}$ exist, the harmonic response is the predominant component of motion. Hence, for the frequency range in which $Q_{\frac{1}{2}}$ cannot be obtained, the dependent variables of inequality (8.2,1) are expressed by equations (8.1,1) and (8.1,5) with $Q_{\frac{1}{2}} = 0$. The resulting equations are similar to using an approximate 3-term solution in the analysis of the harmonic resonance.

8.2.(ii). Approximation of the dependent variables.

From the results of the preceding chapters which readily illustrate the characteristics of the effective asymmetry, it is evident that the values of N are relatively small in the neighbourhood of the frequency region under consideration. The terms of N^2 and of N^3 can be discarded without introducing errors of serious proportion, and the considerable simplification in the expression, describing the behaviour of the non-linear restoring force under the effects of gravity, and hence in the resulting inequality is sufficient justification for the reduction in accuracy. Thus, after simplification equation (8.1,5) takes the form

$$Y = A\left(\frac{1 + 3A^2 - 1.5Q_1^2}{1 + 3A^2 + 1.5Q_1^2}\right)^{\frac{1}{2}} \qquad \ldots\ldots \quad (8.2,2)$$

On using the above substitution, the inequality is now reduced to an expression in which the only dependent variable is the harmonic component Q_1. It is shown for the assumptions made in the derivation of equation (6.4,6) that generally where the effective non-linearity does not increase with resonance the approximation of Q_1 yields results of comparable accuracy. However, it is apparent for the intended response that the amplitude Q_1 is as justifiably defined by equation (6.4.6) as the procedure advocated in (20,22), where the accompanying harmonic component of the subharmonic vibration is approximated either for a linear case or from the consideration of non-linearity being small. Since these are shown to be used with acceptable accuracy, then from considering the frequency position of the component in relation to the subharmonic frequency region, the approximation of equation (6.4.4) to equation (6.4.6) if it does not give a closer value will equally be of comparable accuracy. If the approximations of the respective variables are not reasonable, it will become apparent from the limitation of the inequality when the critical frequencies are compared with the experimental results.

8.2.(iii). The limiting relationship between the dimensionless frequency ratio and the systems parameters.

On substituting equations (8.2.2) and (6.4.6), an approximate expression for attaining the extinction of subharmonic resonance through either raising or lowering the frequency is then given in terms of the systems parameters by

$$n^2 \lessgtr a \pm 4(16Z^2Y^2 - R^2a)^{\frac{1}{2}} \qquad \ldots\ldots \quad (8.2.3)$$

where $a = (12 + 32Z^2 + 36Y^2 - 24R^2)$.

and $$Y = A\left(\frac{3 + 9A^2 - 8Z^2}{3 + 9A^2 + 8Z^2}\right)^{\frac{1}{2}}$$

Table (8.2-1) illustrates that generally the experimental results of the limiting frequencies are in agreement with the inequality (8.2.3). Since, in the approximations made, the generality of the expression remains, the inequality is applicable for all magnitudes of non-linearity. However, it must be mentioned that the limiting relationship does not indicate, for cases where the degree of asymmetry becomes larger with resonance, the higher limit of the extinction frequency in the domain of the optimum subharmonic amplitude.

As discussed in section (6.3), the limiting frequency positions are influenced principally for fixed amplitudes of $\bar{Z}$ by the extent of the gravitational effects on the equilibrium of motion. This is readily illustrated by the above inequality in the significance of the term $\bar{A}$ to the quantity of the first expression. The frequency band-width being the difference of the two critical values is governed by the quantity following the plus or minus sign. As the coefficient of viscous damping is relatively small, the domain of subharmonic vibration is dependent on the amplitudes of $\bar{A}$ and of $\bar{Z}$ which corroborates the results of the preceding chapters.

SYSTEM PARAMETERS			THE LIMITING FREQUENCIES FROM INEQUALITY (8.2,3)		EXPERIMENTAL RESULTS	
$\bar{\Delta}$	R	$\bar{Z}$	η_1	η_2	η_1	η_2
0.8	0.15	0.8	2.8	4.19	2.88	4.2
0.6	0.15	0.4	2.4	3.3	2.4	3.27
1.0	0.25	0.8	2.84	4.7	2.97	4.68
1.0	0.25	0.6	2.85	4.59	2.81	4.53
0.8	0.15	0.4	2.7	3.88	2.61	3.85
0.4	0.15	0.4	2.34	2.77	2.32	2.75
1.0	0.25	0.4	3.25	4.36	3.12	4.32
1.0	0.15	0.4	3.14	4.5	2.98	4.46

TABLE(8.2-1). THE COMPARISON OF THE LIMITING FREQUENCIES, BETWEEN THE CRITICAL VALUES FROM THE INEQUALITY (8.2,3) AND THE EXPERIMENTAL RESULTS.

Chapter IX

Conclusion

In having to support the weight of the vibrating system, static deflection is produced and the position of the mean dynamic displacement during the subharmonic vibration does not coincide either with the static equilibrium position or with the origin of the symmetric restoring force characteristics. Appreciable changes in the subharmonic response of the system exist. The effect of gravity is not completely balanced by static deflection force and its influence on the equilibrium of motion is significant. The shift in the position of the mean dynamic displacement varies over the frequency band-width and the effective non-linearity is not necessarily reduced as the resonance progresses. This depends on the relationship between the initial value of the displacement parameter from the origin and the amplitude of $\bar{\bar{Z}}$. In either case, the coefficient of non-linearity is larger, and pronounced second order subharmonic vibration is readily generated.

With the existence of damping in the system, the effective term of non-linearity is $3A\mu x^2$ and the frequency of the subharmonic response is dependent on the amount of weight initially transmitted to the supports. However small $\bar{\lambda}$ is in relation to the amplitude of the disturbing force, the predominant subharmonic vibration is of the order one-half if the physical condition described by the independent parameters lies within the domain for which there are real roots of the subharmonic solution. Although even and odd subharmonic components also exist in the vibratory motion, the second order resonance is the strongest and it predominates over the higher orders. Only the first three components of the vibration can be measured effectively, and complete suppression of the subharmonic responses is achieved for the critical damping coefficient.

For a small degree of out-of-balance excitation, the effective non-linearity is determined expressively by the parameter $\bar{\lambda}$. The shift of the mean dynamic displacement from static equilibrium is insignificant. An additional area of instability exists on the lower frequency branch of the forced response. The extent of the unstable region is reduced when the pronounced influence of the gravitational force on the non-linearity is lowered by the increase in the amplitude of the disturbing force. The instability is readily ascertained from the relative significant increase in the value of the parameter C_2 in the analysis of the variational equation.

In the unstable region of the subharmonic resonance, real roots of the subharmonic components do not exist in the presence of damping. The polynomial simultaneous equations from the application of the two theoretical methods cannot be solved. The vibration is not sustained in an equilibrium state because of the accumulative effect in the accompanying harmonic of the subharmonic vibration. The build-up oscillation occurs in the second order region and it has the same frequency as the predominant subharmonic component $Q_{\frac{1}{2}}$. The vibratory motion cannot exist along with the harmonic resonant oscillation, and the system is effectively excited with negative damping. The procedure of fixing the points of the resulting unstable components, $Q_{\frac{1}{2}}$ and H, is by the numerical method described in section (8.2).

The limiting frequencies of the band-width are significantly affected by the influence of gravity effects on the equilibrium of motion, and are determined principally by the magnitude of the effective asymmetry. The subharmonic response is not appreciably sensitive to viscous damping. Considerable rate of damping is necessary in the suppression of the resonance. In most applications the use of a large viscous force is not practical, and the limiting inequality that is expressed in terms of the response governing parameters readily illustrates the best means of destroying effectively the resonance by altering conveniently the appropriate parameters. For cases in which a large coefficient of the dash-pot rate is not desirable, the extinction is through reducing either the amplitude of the unbalanced force or the degree of gravity effect on the equilibrium of motion. The latter means is the most convenient in view of the high cost of balancing a system. The extinction condition is achieved by raising the initial stiffness rate or by lowering the pronounced difference of the two stiffnesses of the suspension. Equations (8.1,8) and (8.1,9) enable the dash-pot to be fixed at an acceptable rate and the corresponding optimum limit of gravity effect tolerable to be determined. In this physical state of asymmetry the amplitude of $\bar{\bar{Z}}$ has no significant influence on the effective non-linearity as regard to exciting the subharmonic phenomena. Otherwise an appreciable increase of damping is required, especially in the state of non-linearity that increases with subharmonic resonance. The agreeable rate set to the dash-pot for the optimum value of $\bar{\Lambda}$ is the minimum coefficient of damping for complete suppression of the subharmonic responses. The amplitude of the disturbing force is reduced only to a convenient stage of balancing.

The phase difference of the predominant subharmonic component is less than $\frac{\pi}{2}$, and $Q_{\frac{1}{3}}$ exists either within the same phase quadrature of the disturbing force or in the opposite third quadrant. The resulting response depends on the initial conditions.

The accuracy of the two theoretical methods employed in the analysis of a system with asymmetrical restoring force characteristics is generally good. The investigation is not restricted in the magnitude of non-linearity. The theoretical results are in good correlation with the experimental values that are obtained from an actual simulation of the differential equation and from the experimental model. The five-term approximate solution does not introduce any appreciable increase of error with the increase in the degree of non-linearity through the value of the parameter $\bar{\Lambda}$. This is because in the resonance state the predominant subharmonic is considerably larger than the higher order subharmonics.

The subharmonic vibration cannot exist along with the resonant state of Q_1 and the harmonic component is reasonably approximated as the effective amplitude of the disturbing force. The use of the approximation which is independent of frequency allows the critical state for subharmonic extinction to be determined readily. The substitution into the simultaneous equations to calculate the subharmonic response generally yields results of compatible accuracy when the effect of gravity does not cause the effective non-linearity to increase with resonance.

Appendix I

The Ritz-Galerkin Method

The reasons for selecting this as the principal method in the analysis of subharmonic extinction are discussed in chapter II. The equivalence of Ritz minimizing method and Galerkin method is examined in (29), and the essence of the theory that is applied in the investigation is derived from considering the variational equation of a conservative single degree-of-freedom system.

According to Hamilton's principle, the motion of the dynamical system exists in such a manner that

$$\delta I = \delta \int_{t_1}^{t_2} L \, dt = 0 \qquad \ldots\ldots\ldots \quad (AI.1)$$

where for the absolute displacement co-ordinate x and time t, the Largrangian function $L = (T - V)$;

T is the kinetic energy function, $T = T(x, \dot{x})$,

and V is the potential energy function, $V = V(x)$.

On applying the calculus of variation and integration by parts, the above expression gives Largrange's equation of motion for the system, which is

$$\int_{t_1}^{t_2} \{ \frac{\partial L}{\partial x} - \frac{d}{dt} (\frac{\partial L}{\partial \dot{x}}) \} dt = 0 \qquad \ldots\ldots \text{(AI,2)}$$

For an approximate steady state solution in the form

$$\bar{x} = a_o + a_1\phi_1(t) + a_2\phi_2(t_2) + a_3\phi_3(t_3) + \ldots \qquad \ldots \text{(AI,3)}$$

where a_i ($i = 0, 1, 2, \ldots.n$) is the undetermined coefficient.

the Ritz minimizing method proves that, if the functions $\phi_i(t)$ are properly chosen time functions consistent with the physical restraints and satisfy the boundary conditions, the approximate solution $\bar{x}$ converges into a true solution x by increasing the number of terms. Since equation (AI,3) is not an exact solution, the substitution into the resulting equation of (AI,2) will produce an error function. The extent of the error is obviously dependent on the accuracy of the approximation and it is often possible to have reasonable accuracy from merely using the first few terms of equation (AI,3) if the choice of these terms are made appropriately. The minimizing procedure determines for the choice of approximate solution the values of the coefficients for which the error is an absolute minimum.

Thus, with the approximate solution of (AI,3) the conditions for the expression I to be a minimum over each cycle of vibration are that

$$\frac{\partial I}{\partial a_i} = \int_o^T (\frac{\partial L}{\partial \bar{x}} \phi_i + \frac{\partial L}{\partial \dot{\bar{x}}} \dot{\phi}_i) dt = 0 \qquad \ldots\ldots \text{(AI,4)}$$

where $t_1 = 0$ and $t_2 = T$, and T is the periodic time for one complete cycle.

On integrating by parts the second term of equation (AI.4), the above condition can be reduced to the form

$$\frac{\partial I}{\partial a_i} = \int_o^T \{ \frac{\partial L}{\partial \bar{x}} - \frac{d}{dt} (\frac{\partial L}{\partial \dot{\bar{x}}}) \} \phi_i \, dt + [\frac{\partial L}{\partial \dot{\bar{x}}} \phi_i]_o^T = 0 \quad \text{.... (AI.5)}$$

Since in periodic vibration $\phi_i(o) = \phi_i(T) = 0$, the expression within the first bracket of equation (AI.5) is Langranges form of the equation of motion. The rearrangement of the terms into the standard form of the differential equation for the approximate solution $\bar{x}$ yields the error function $E(\bar{x})$. Thus, the conditions of the Ritz-Galerkin method enable the values of the unknown coefficients to be determined by integrating the deficit function for each of the n terms in $\bar{x}$ and subsequently solving the resulting n simultaneous algebraic equations.

The essence of the method is that the deficiency, which is introduced on substituting the approximate solution of the form (AI.3) into the describing equation of motion, must be zero when it is integrated over one complete cycle of the vibration for each of the n terms. The generalised conditions may be expressed in the form

$$\int_o^{4\pi} E(\bar{x}) \, d(\theta) = 0$$

$$\int_0^{4\pi} E(\tilde{X})\ \phi_1\ d(\theta) = 0 \qquad \ldots\ldots \quad (AI,8)$$

$$\int_0^{4\pi} E(\tilde{X})\ \phi_2\ d(\theta) = 0$$

$$\ldots\ldots\ldots\ldots$$

$$\int_0^{4\pi} E(\tilde{X})\ \phi_n\ d(\theta) = 0$$

where the independent variable t is changed to the equivalent corresponding parameter of the system $d\theta$.

The upper limit of integration is 4π since one complete cycle of second order subharmonic vibration occurs in every two cycles of the disturbing force.

Because the error function is often readily derived, it is common practise that the procedure of the above expressions is applied directly. It is more convenient than having to carry out the whole minimizing process.

Appendix II

The simultaneous algebraic equations

In solving the differential equation (3.1.8) by means of the Ritz-Galerkin method, the choice of the approximate solution determines the sensitivity of this procedure.

The approximate solution $\tilde{x}$ in the form of equation (3.2.1) satisfies the 'boundary' condition and the time functions are consistent with the physical restraints. Hence, with

$$a = A \cos \frac{\theta}{2}, \quad b = B \sin \frac{\theta}{2},$$

$$c = C \cos \theta \text{ and } d = D \sin \theta,$$

the substitution of $\tilde{x}$ into the describing equation of motion produces the resulting error function

$$E(\tilde{x}) = \eta^2 \tilde{x}'' + 2F\eta \tilde{x}' + (1 + 3A^2\mu)\,\tilde{x} + \mu(3A\tilde{x}^2 + \tilde{x}^3) - Z\eta^2 \cos \theta \qquad \text{... (A2.1)}$$

which varies from instant to instant.

By applying the conditions of Ritz-Galerkin method, the five simultaneous algebraic equations (3.2,8) to (3.2,12) are obtained from the following integration for each corresponding term in the equation (3.2,1).

For the non-periodicity coefficient, the condition yields

$$\int_0^{4\pi} E(X)\, d\theta = \int_0^{4\pi} \{ - \eta^2 (\tfrac{1}{4} (a + b) + (c + d)) + 2R\eta \, (\tfrac{1}{2}$$

$$(B \cos \tfrac{\theta}{2} - A \sin \tfrac{\theta}{2}) + (D \cos \theta - C \sin \theta))$$

$$+ (1 + 3A^2\mu) (N + a + b + c + d) + 3A\mu$$

$$(N^2 + a^2 + b^2 + c^2 + d^2) + 6A\mu (N (a + b + c + d)$$

$$+ a(b + c + d) + b(c + d) + cd) - 2\eta^2 \cos \theta$$

$$+ \mu (N^3 + a^3 + b^3 + c^3 + d^3) + 3\mu N^2 (a + b + c + d)$$

$$+ 3\mu a^2 (N + b + c + d) + 3\mu b^2 (N + a + c + d)$$

$$+ 3\mu c^2 (N + a + b + d) + 3\mu d^2 (N + a + b + c)$$

$$+ 6\mu N (a (b + c + d) + b(c + d) + cd) + 6\mu$$

$$(ab (c + d) + cd (a + b)) \} \, d\theta$$

$$= (1 + 3A^2\mu) \, 4\pi N + 6A\mu\pi (2N^2 + A^2 + B^2 + C^2 + D^2)$$

$$+ \mu\pi \{ 4N^3 + 3A^2 (2N + C) + 3B^2 (2N - C) + 6N$$

$$(C^2 + D^2) + 6AED \} = 0 \qquad \ldots\ldots \quad (A2,2)$$

The above equation is multiplied by $\mu^{\frac{1}{2}}$ to reduce the coefficients to non-dimensional quantities, and on rearranging yields

$$2\bar{N}\ (1 + \bar{N}^2 + 3\bar{A}\ (\bar{A} + \bar{N})\ ) + 3(\bar{A} + \bar{N})\ (\bar{A}^2 + \bar{B}^2 + \bar{C}^2 + \bar{D}^2)$$

$$+ \frac{3}{2}\ \bar{C}\ (\bar{A}^2 - \bar{B}^2) + 3\bar{A}\bar{B}\bar{D} = 0 \qquad \ldots\ldots \quad (A2,3)$$

The above equation is the same as equation (3.2,8).

Similarly for the periodic function $\cos\frac{\theta}{2}$,

$$\int_0^{4\pi} E(\ddot{X})\cos\frac{\theta}{2}\, d\theta$$

$$= -\eta^2\frac{A}{2}\pi + 2R\eta B\pi + 2(1 + 3A^2\mu)\, A\pi + 6A\mu\pi\ (2NA$$

$$+ AC + BD) + \mu\pi\ (\frac{3}{2}A^3 + 6N^2A + \frac{3}{2}B^2A + 3C^2A + 6NAC$$

$$+ 3D^2A + 6NBD) = 0 \qquad \ldots\ldots \quad (A2,4)$$

The rearrangement of the terms of the above equation gives

$$\bar{A}(1 - \frac{\eta^2}{4}) + R\eta\bar{B} + \frac{3}{4}\ \bar{A}\ (\bar{A}^2 + \bar{B}^2) + \frac{3}{2}\ \bar{A}\ (\bar{C}^2 + \bar{D}^2) + 3\bar{A}$$

$$(\bar{A} + \bar{N})^2 + 3(\bar{A} + \bar{N})\ (\bar{A}\bar{C} + \bar{B}\bar{D}) = 0 \qquad \ldots\ldots \quad (A2,5)$$

which is identical to equation (3.2,8).

For the periodic function $\sin\frac{\theta}{2}$, the condition of Ritz-Galerkin yields

$$\int_0^{4\pi} E(\bar{x}) \sin\frac{\theta}{2}\, d\theta$$

$$= -\eta^2 \frac{B}{2}\pi - 2R\eta A\pi + 2(1 + 3\Delta^2\mu) B\pi + 6\Delta\mu\pi\,(2NB + AD$$

$$- BC) + \mu\pi\,(\frac{3}{2} B^3 + 6N^2B + \frac{3}{2} A^2B + 3C^2B + 3D^2B + 6NAD$$

$$- 6NBC) = 0 \qquad \ldots\ldots \quad (A2,6)$$

which on converting into the dimensionless form the simultaneous algebraic expression (A2,6) becomes

$$\bar{B}\,(1 - \frac{\eta^2}{4}) - R\eta\bar{A} + \frac{3}{4}\bar{B}\,(\bar{A}^2 + \bar{B}^2) + \frac{3}{2}\bar{B}\,(\bar{C}^2 + \bar{D}^2) + 3\bar{B}(\bar{\Delta} + \bar{N})^2$$

$$+ 3(\bar{\Delta} + \bar{N})\,(\bar{A}\bar{D} - \bar{B}\bar{C}) = 0 \qquad \ldots\ldots \quad (A2,7)$$

Equation (A2,7) is the same as equation (3.2,9).

Similarly for the periodic function cos θ,

$$\int_0^{4\pi} E(\bar{x}) \cos\theta\, d\theta$$

$$= -2\eta^2\pi\,(C + Z) + 4R\eta D\pi + 2(1 + 3\Delta^2\mu)\, C\pi + 3\Delta\mu\pi\,(A^2$$

$$- B^2 + 4NC) + \mu\pi\,(\frac{3}{2} C^3 + 6N^2C + 3A^2N + 3A^2C - 3B^2N$$

$$+ 3B^2C + \frac{3}{2} D^2C) = 0 \qquad \ldots\ldots \quad (A2,8)$$

which also gives from the rearrangement of the terms of the simultaneous algebraic expression (A2,8),

$$-\eta^2\,(\bar{C} + \bar{Z}) + 2R\eta\bar{D} + \bar{C} + \frac{3}{4}\bar{C}\,(\,2\,(\bar{A}^2 + \bar{B}^2) + (\bar{C}^2 + \bar{D}^2)\,)$$

$$+ 3\bar{C}\,(\bar{A} + \bar{N})^2 + \frac{3}{2}\,(\bar{A} + \bar{N})\,(\bar{A}^2 - \bar{B}^2) = 0 \qquad \ldots\ldots (A2,9)$$

Finally, for the periodic function sin θ, the application of the Ritz-Galerkin method gives,

$$\int_0^{4\pi} E(\bar{X}) \sin\theta \, d\theta$$

$$= - 2\eta^2\pi D - 4R\eta C\pi + 2(1 + 3A^2\mu)\, D\pi + 3A\mu\pi\,(4ND + 2AB)$$

$$+ \mu\pi(\frac{3}{2}\,D^3 + 6N^2D + 3A^2D + 3B^2D + \frac{3}{2}\,C^2D + 6NAB) = 0 \;..(A2,10)$$

The expression becomes in the dimensionless form

$$\bar{D}\,(1 - \eta^2) - 2R\eta\bar{C} + 3\bar{D}\,(\bar{A} + \bar{N})^2 + 3(\bar{A} + \bar{N})\,\bar{A}\bar{B} + \frac{3}{4}\,\bar{D}\,(\bar{D}^2 + \bar{C}^2)$$

$$+ \frac{3}{2}\,\bar{D}\,(\bar{A}^2 + \bar{B}^2) = 0 \qquad \ldots\ldots (A2,11)$$

In the simplification of the simultaneous algebraic equations, the following identities are used:

$$\bar{Q}_3^2 = \bar{A}^2 + \bar{B}^2 , \quad \bar{Q}_1^2 = \bar{C}^2 + \bar{D}^2 ,$$

$$\bar{A} = \bar{Q}_3 \cos\phi_3 , \quad \bar{C} = \bar{Q}_1 \cos\phi_1 , \qquad \ldots\ldots (A2,12)$$

$$\bar{B} = \bar{Q}_3 \sin\phi_3 , \quad \bar{D} = \bar{Q}_1 \sin\phi_1 .$$

The substitution of these identities into equations (3.2,8) to (3.2,13) gives

$$2N\,\{\,1 + N^2 + 3A\,(A + N)\,\} + 3(A + N)\,(Q_3^2 + Q_1^2) + \frac{3}{2}\,Q_1 Q_3^2$$

$$\cos (\phi_1 - 2\phi_{\frac{1}{2}}) = 0 \qquad \ldots\ldots \quad (A2,13)$$

$$\{ (1 - \frac{\eta^2}{4}) + \frac{3}{4} (Q_{\frac{1}{2}}^2 + 2Q_1^2) + 3(A + N)^2 \} Q_{\frac{1}{2}} \cos \phi_{\frac{1}{2}} + R\eta Q_{\frac{1}{2}}$$

$$\sin \phi_{\frac{1}{2}} + 3(A + N) Q_1 Q_{\frac{1}{2}} \cos (\phi_1 - \phi_{\frac{1}{2}}) = 0 \quad \ldots\ldots \quad (A2,14)$$

$$\{ (1 - \frac{\eta^2}{4}) + \frac{3}{4} (Q_{\frac{1}{2}}^2 + 2Q_1^2) + 3(A + N)^2 \} Q_{\frac{1}{2}} \sin \phi_{\frac{1}{2}} - R\eta Q_{\frac{1}{2}}$$

$$\cos \phi_{\frac{1}{2}} + 3(A + N) Q_1 Q_{\frac{1}{2}} \sin (\phi_1 - \phi_{\frac{1}{2}}) = 0 \quad \ldots\ldots \quad (A2,15)$$

$$\{ (1 - \eta^2) + 3(A + N)^2 + \frac{3}{4} (2Q_{\frac{1}{2}}^2 + Q_1^2) \} Q_1 \cos \phi_1 - \eta^2 Z$$

$$+ 2R\eta\, Q_1 \sin \phi_1 + \frac{3}{2} (A + N) Q_{\frac{1}{2}}^2 \cos 2\phi_{\frac{1}{2}} = 0 \ldots\ldots (A2,16)$$

$$\{ (1 - \eta^2) + 3(A + N)^2 + \frac{3}{4} (2Q_{\frac{1}{2}}^2 + Q_1^2) \} Q_1 \sin \phi_1 - 2R\eta Q_1$$

$$\cos \phi_1 + \frac{3}{2} (A + N) Q_{\frac{1}{2}}^2 \sin 2\phi_{\frac{1}{2}} = 0 \qquad \ldots\ldots (A2,17)$$

The bars above the respective terms are omitted for convenience.

From equations (A2,14) and (A2,15) it is evident that for the subharmonic solution, $Q_{\frac{1}{2}} \neq 0$. Thus, on multiplying equation (A2,14) by $\sin \phi_{\frac{1}{2}}$ and equation (A2,15) by $\cos \phi_{\frac{1}{2}}$ and substracting gives

$$3(\bar{A} + \bar{N}) \bar{Q}_1 \sin (\phi_1 - 2\phi_{\frac{1}{2}}) - R\eta = 0 \qquad \ldots\ldots \quad (A2,18)$$

Equation (A2,18) is identical to equation (3.2,14).

From multiplying equation (A2,14) by $\cos \phi_{\frac{1}{2}}$ and equation (A2,15) by sin $\phi_{\frac{1}{3}}$ and then adding also gives

$$3(\bar{A} + \bar{H})\, Q_1 \cos(\phi_1 - 2\phi_{\frac{1}{3}}) + (1 + \frac{\eta^2}{4}) + \frac{3}{4}(\bar{Q}_{\frac{1}{3}}^2 + 2\bar{Q}_1^2) + 3(\bar{A} + \bar{H})^2 = 0 \qquad \ldots\ldots \text{(A2,19)}$$

This expression is the same as equation (3.2,15).

Similarly, in obtaining equation (3.2,16) equation (A2,16) is multiplied by sin ϕ_1 and equation (A2,17) by cos ϕ_1 and then on substituting yields

$$R(4\bar{Q}_1^2 + \bar{Q}_{\frac{1}{3}}^2) - 2\eta\bar{Z}\bar{Q}_1 \sin\phi_1 = 0 \qquad \ldots\ldots \text{(A2,20)}$$

Finally, for equation (3.2,17), this is obtained by multiplying equation (A2,16) by cos ϕ_1 and equation (A2,17) by sin ϕ_1. Subsequently on adding yields

$$2\bar{Q}_1^2\left((1 - \eta^2) + 3(\bar{A} + \bar{H})^2 + \frac{3}{4}(2\bar{Q}_{\frac{1}{3}}^2 + \bar{Q}_1^2) \right) - \bar{Q}_{\frac{1}{3}}^2\left((1 - \frac{\eta^2}{4}) + 3(\bar{A} + \bar{H})^2 + \frac{3}{4}(\bar{Q}_{\frac{1}{3}}^2 + 2\bar{Q}_1^2) \right) - 2\eta^2\bar{Z}\bar{Q}_1 \cos\phi_1 = 0$$

$$\ldots\ldots \text{(A2,21)}$$

Equation (A2,21) is identical to equation (3.2,17) of section (3.2).

Appendix III

Trigonometric Identities

The principle of the analytical method that is employed in section (3.3) is the equilibrium of the forces, and the periodicity in the vibratory motion enables the displacement to be approximated to the form of solution

$$\tilde{X} = N + A \cos \frac{\theta}{2} + B \sin \frac{\theta}{2} + C \cos \theta + D \sin \theta \quad \ldots \text{(A3,1)}$$

which is consistent with the physical restraints and which satisfies the boundary requirements.

Thus, for the substitution of the approximate solution into the differential equation of motion to equate the cos and sin functions of the respective frequencies to zero, the squared and cubes of $\tilde{X}$ are as follows:

$$\text{Let} \quad a = A \cos \frac{\theta}{2}$$

$$b = B \sin \frac{\theta}{2} \qquad \ldots\ldots \text{(A3,2)}$$

$$c = C \cos \theta$$

$$d = D \sin \theta$$

then

$$\tilde{X}^2 = (N^2 + a^2 + b^2 + c^2 + d^2) + 2N(a + b + c + d) + 2a(b + c + d) + 2b(c + d) + 2cd \qquad \ldots\ldots \text{(A3.3)}$$

and

$$\tilde{X}^3 = (N^3 + a^3 + b^3 + c^3 + d^3) + 3N^2(a + b + c + d) + 3a^2(N + b + c + d) + 3b^2(N + a + c + d) + 3c^2(N + a + b + d) + 3d^2(N + a + b + c) + 6Na(b + c + d) + 6Nb(c + d) + 6ab(c + d) + 6cd(a + b) + 6Ncd \qquad \ldots\ldots \text{(A3.4)}$$

The above resulting periodic functions are simplified with the following trigonometric identities:

$$a^2 = \frac{A^2}{2}(1 + \cos\theta) \; ,$$

$$ac = \frac{AC}{2}\left(\cos\frac{\theta}{2} + \cos\frac{3\theta}{2}\right) \; ,$$

$$b^2 = \frac{B^2}{2}(1 - \cos\theta) \; ,$$

$$ad = \frac{AD}{2}\left(\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right) \; ,$$

$$c^2 = \frac{C^2}{2}(1 + \cos 2\theta) \; ,$$

$$bc = \frac{BC}{2}\left(\sin\frac{3\theta}{2} - \sin\frac{\theta}{2}\right) \; ,$$

$$d^2 = \frac{D^2}{2}(1 - \cos 2\theta) \; ,$$

$$bd = \frac{BD}{2}\left(\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right) \; ,$$

$$ab = \frac{AB}{2} \sin \theta \ ,$$

$$cd = \frac{CD}{2} \sin 2\theta \ ,$$

$$a^3 = A^3 \left(\frac{3}{4} \cos \frac{\theta}{2} + \frac{1}{4} \cos \frac{3\theta}{2}\right) \ ,$$

$$a^2b = \frac{A^2B}{4} \left(\sin \frac{\theta}{2} + \sin \frac{3\theta}{2}\right) \ ,$$

$$b^3 = B^3 \left(\frac{3}{4} \sin \frac{\theta}{2} - \frac{1}{4} \sin \frac{3\theta}{2}\right) \ ,$$

$$a^2c = \frac{A^2C}{4} (1 + 2 \cos \theta + \cos 2\theta) \ ,$$

$$c^3 = C^3 \left(\frac{3}{4} \cos \theta + \frac{1}{4} \cos 3\theta\right) \ , \qquad \ldots\ldots \ (A3.5)$$

$$a^2d = \frac{A^2D}{4} (2 \sin \theta + \sin 2\theta) \ ,$$

$$d^3 = D^3 \left(\frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta\right) \ ,$$

$$b^2a = \frac{B^2A}{4} \left(\cos \frac{\theta}{2} - \cos \frac{3\theta}{2}\right) \ ,$$

$$b^2c = \frac{B^2C}{4} (2 \cos \theta - 1 - \cos 2\theta) \ ,$$

$$abc = \frac{ABC}{4} \sin 2\theta \ ,$$

$$b^2d = \frac{B^2D}{4} (2 \sin \theta - \sin 2\theta) \ ,$$

$$abd = \frac{ABD}{4} (1 - \cos 2\theta) \ ,$$

$$c^2a = \frac{C^2A}{4} \left(2 \cos \frac{\theta}{2} + \cos \frac{3\theta}{2} + \cos \frac{5\theta}{2}\right) \ ,$$

$$cda = \frac{CDA}{4} \left(\sin \frac{5\theta}{2} + \sin \frac{3\theta}{2}\right) \ ,$$

$$c^2b = \frac{C^2B}{4} \left(2 \sin \frac{\theta}{2} - \sin \frac{3\theta}{2} + \sin \frac{5\theta}{2}\right) \ ,$$

$$cdb = \frac{CDB}{4} (\cos \frac{3\theta}{2} - \cos \frac{5\theta}{2}) \; ,$$

$$c^2d = \frac{C^2D}{4} (\sin \theta + \sin 3\theta) \; ,$$

$$d^2a = \frac{D^2A}{4} (2 \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} - \cos \frac{5\theta}{2}) \; ,$$

$$d^2b = \frac{D^2B}{4} (2 \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} - \sin \frac{5\theta}{2}) \; ,$$

$$d^2c = \frac{D^2C}{4} (\cos \theta - \cos 3\theta)$$

On retaining only the constant, the first and second subharmonic components, $\tilde{x}^2$ and $\tilde{x}^3$ become:

$$\tilde{x}^2 = (N^2 + \frac{A^2}{2} + \frac{B^2}{2} + \frac{C^2}{2} + \frac{D^2}{2}) + (2NA + AC + BD) \cos \frac{\theta}{2} + (2NB + AD - BC) \sin \frac{\theta}{2} + (\frac{1}{2} (A^2 - B^2) + 2NC) \cos \theta + (2ND + AB) \sin \theta \qquad \ldots\ldots\ldots \quad (A3,6)$$

$$\tilde{x}^3 = N (N^2 + \frac{3}{2} (A^2 + B^2) + \frac{3}{2} (C^2 + D^2)) + \frac{3}{4} C (A^2 - B^2) + \frac{3}{2} ABD + (A (\frac{3}{4} (A^2 + B^2) + \frac{3}{2} (C^2 + D^2) + 3N^2) + 3N (AC + BD)) \cos \frac{\theta}{2} + (B (\frac{3}{4} (A^2 + B^2) + \frac{3}{2} (C^2 + D^2) + 3N^2) + 3N(AD - BC)) \sin \frac{\theta}{2} + (C (\frac{3}{4} (C^2 + D^2) + \frac{3}{2} (A^2 + B^2) + 3N^2) + 3N(A^2 - B^2)) \cos \theta + (D (\frac{3}{4} (C^2 + D^2) + \frac{3}{2} (A^2 + B^2) + 3N^2) + 3NAB) \sin \theta$$

$$\ldots\ldots \quad (A3,7)$$

The procedure of equating the coefficients of the respective periodic functions separately to zero, when the approximate solution is substituded into the non-linear differential equation, gives identical simultaneous algebraic equations that are obtained by the Ritz-Galerkin method.

Bibliography

(1) Reif, Z.F., 'The effect of static deflection on the harmonic resonance of a system with a hardening non-linear spring'. The Aeronautical Journal of the Royal Aeronautical Society, January 1970.

(2) Reif, Z.F., 'Effect of static deflection on the natural frequency of a system with a hardening non-linear spring'. Journal of Mechanical Engineering Science, vol.9, No.3, 1967.

(3) Reif, Z.F., 'Effect of static deflection on the undamped vibration of a non-linear system', London University, PH.D. thesis, 1967.

(4) Ludeke, C.A., Pong, W., 'The extinction of predominantly subharmonic oscillations in non-linear systems', Journal of

Applied Physics, vol.24, No.1, January 1953.

(5) Ludeke, C.A., 'Predominantly subharmonic oscillations', Journal of Applied Physics, vol.22, No.11, November 1951.

(6) Klotter, K., 'Non-linear vibration problems treated by the averaging method of W.Ritz', U.S. Natt. Congress of Applied Mechanics, 1951, pg. 125 - 131.

(7) Hayashi, C., Nishikawa, Y., Abe, M., 'Subharmonic oscillations of order half', I.R.E. Transactions on circuit theory, June 1960.

(8) Ludeke, C.A., 'Non-linear phenomena', Transaction of the A.S.M.E., April 1957.

(9) Levenson, M.E., 'Harmonic and Subharmonic response for the Duffing's Equation', Journal of Applied Physics, Vol.20, November 1949.

(10) Cunningham, W.J., 'Even order subharmonic oscillations', Journal of

Applied Physics, vol.27, No.11, November 1956.

(11) Caughey, T.K., 'The existence and stability of ultraharmonics and subharmonics in forced non-linear oscillations', Journal of Applied Mechanics, December 1954.

(12) Bush, G.L., 'Low frequency unsymmetrical steady-state responses to Duffing's equation', Yale University, PH.D. dissertation, 1966.

(13) Atkinson, C.P., Heflinger, L.O., 'Subharmonic and superharmonic oscillations of a bilinear vibrating system', Journal of Franklin Institute, September 1955.

(14) Burgess, J.C., 'Harmonic, superharmonic and subharmonic response for single degree of freedom systems of Duffing type', Technical Report No.27, 1954. Stanford University.

(15) Whittaker, E.T., Watson, G.N., 'A course of modern analysis', Cambridge University Press, 1965.

(16) Crossley, F.E., 'Bounds of stability of subharmonic oscillations of a bi-stable oscillator', Symp. Theory of Machines, 1970.

(17) Hayashi, C., Nishikawa, Y., 'Initial conditions leading to different types of periodic solutions for Duffing's equation', International Union Theoret. Appl. Mech., Symp. Non-linear Oscillations, Kiev, 1961.

(18) Minorsky, N., 'Non-linear Oscillations', D. Van Nostrand Comp., Inc., 1962.

(19) Kryloff, N., Bogoliuboff, N., 'Introduction to non-linear mechanics', Princeton University Press, 1949.

(20) Stoker, J.J., 'Non-linear vibrations', Interscience publishers, 1950.

(21) Klotter, K., 'Non-linear Oscillations', Applied Mechanics Review, vol.10, no.11, 1957.

(22) Hayashi, C., 'Non-linear oscillations in physical systems', McGraw-Hill, 1964.

(23) Ayre, J., 'Engineering Vibrations', McGraw-Hill, 1958.

(24) Jacobsen, L.S., Jespersen, H.J., 'Steady forced vibrations of single mass systems with symmetrical as well as unsymmetrical non-linear restoring elements', Journal of Franklin Inst., October 1935.

(25) Hartog, J.P., Heiles, R.M., 'Forced vibration in non-linear systems with various combinations of linear springs', Journal of Applied Mechanics, 1936.

(26) Young, D., 'Shock spectra for non-linear spring-mass systems and their applications to design', A.I.A.A. Journal, vol.1, No.7, July, 1963.

(27) Reif, Z.F., 'Approximate methods for the solution of non-linear vibration equations', Bull. Mech. Engng. Educ., vol.9, 1970.

(28) Billett, R.A., 'Experimental studies of the effects of a non-linear bearing stiffness on shaft whirl', Inst.

Mech. Eng. Appl. Mech. Convention, 1966.

(29) Newland, D.E., 'On the methods of Galerkin, Ritz, and Krylov-Bogoliubov in the theory of non-linear vibrations', Inter. Journal of Mechanical Science, Vol.7, 1965.

(30) Carnegie, W., Reif, Z., Wee, R.S.S., 'A numerical technique for the solution of some non-linear systems', The Aeronautical Journal of the Royal Aeronautical Society, August 1969.

(31) Ludeke, C.A., 'A method of equivalent linearization for non-linear oscillatory systems with large non-linearity', Journal of Applied Physics, Vol.20, July 1949.

(32) Ludeke, C.A., 'An electro-mechanical device for solving non-linear differential equations', Journal of Applied Physics, Vol.20, June, 1949.

(33) Brock, J.E., 'An iterative numerical method

for non-linear vibrations', A.S.M.E., March, 1951.

(34) Molyneux, W.G., 'Design of non-linear spring supports for vibration isolation at very low frequencies', The Engineer, Technical Contributors Section, June, 1951.

(35) Newland, D.E., 'Non-linear aspects of the performance of centrifugal pendulum vibration absorbers', Journal of Engineering for Industry, August, 1964.

(36) Young, D., Hess, F.N., 'On the stability of harmonic solutions of a modified form of Duffing's equation', Proc. 2nd. U.S. National Congress of Applied Mechanics, 1954.

(37) Pipes, L.A., Harvill, L.R., 'Applied mathematics for engineers and physicists', McGraw-Hill, 1970.

(38) Manley, R.G., 'Waveform Analysis', Chapman and Hall, 1945.

(39) Ludeke, C.A., 'An experimental investigation of forced vibrations in a mechanical system having a non-linear restoring force', Journal of Applied Physics, vol.17, July 1946.

(40) Clauser, F.H., 'The behaviour of non-linear systems', Journal of the Aeronautical Sciences, May, 1956.

(41) Rosenberg, R.M., 'Non-linear oscillations', A.M.R., November 1961.

(42) Davey, A.B., Payne, A.R., 'Rubber in engineering practise', Maclaren and Son, 1965.

(43) Hayashi, C., 'Stability investigation of the non-linear periodic oscillations', Journal of Applied Physics, vol.24, No.3, 1953.

(44) Mclachlan, N.W., 'Theory and application of Mathieu functions', Dover Publications Inc., 1964.

(45) Routh, E.J., 'Advanced Rigid Dynamics', The Macmillan Company, New York,

(45) Steinman, D.B., 'Simple formula solves all higher-degree equations', Civil Engineering, February 1951.