The Public Good index for games with several levels of approval in the input and output

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Abstract

The Public Good index is a power index for simple games introduced by Holler and later axiomatized by Holler and Packel, so that some authors also speak of the Holler–Packel index.[1] A generalization to the class of games with transferable utility was given by Holler and Li. Here we generalize the underlying ideas to games with several levels of approval in the input and output – so-called (j,k) simple games. Corresponding axiomatizations are also provided.

Keywords: Public Good index, Public Good value, (j,k) simple games, simple games, TU games, values, axiomatization

1   Introduction

Assume that you are submitting a paper to a computer science conference (or some other scientific discipline with a similar reviewing convention). You paper is usually send to several reviewers, which are typically chosen by the programming committee or assign themselves in some kind of bidding procedure. Unattached the selection process, for each paper there exists a set N of reviewers. The task of the reviewers is to read and to evaluate the submitted paper. Besides some comments and remarks in free text, a summarizing evaluation according to a certain predefined scale is requested. A typical scale consists e.g. of the possible answers "strong accept", "accept", "weak accept", "borderline", "weak reject", "reject", and "strong reject". After every reviewer has announced his or her evaluation, these individual opinions are summarized to a group decision, where we assume that only the outcomes

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[1]The paper is dedicated to the occasion of the 75th birthday of Manfred J. Holler.

"accept" or "reject" are possible. Of course, this oversimplifies the practical setting where we may have discussion rounds between the reviewers with the possibility to adjust their evaluations or some kind of interaction with the authors of the paper. Such a decision rule v may be formalized as follows: For some set of agents N and a set of levels of approval for the input J, each vector in $J^{|N|}$ is mapped to an element of the set of levels of approval in the output K. In our example we have $|J| = 7$ and $|K| = 2$, but may also consider an output set K of cardinality three by distinguishing between a lecture, a poster presentation, or rejection. If the options in J can be mapped to a numerical score, like e.g. $+3, +2, +1, 0, -1-, 2, -3$ in our example, then such a decision rule might be simply given by some threshold τ. I.e., accept all papers with mean of the scores at least τ. However, rules might be more complicated including extra conditions like e.g. requiring that no paper with at least one "strong reject" is accepted. Given a specific decision rule v one might ask for the "influence" of a specific agent $i \in N$ on the group decision. Having only homogeneous agents in mind this question does not seem to make too much sense. However, agents may also be heterogeneous. In our example the reviewers may have different levels of expertise, which is indeed a common query to the reviewer when writing his or her evaluation. Of course, we as the author of the paper usually do not have the details to determine the influence of the individual reviewers and should have little interest to do so, but the author of the one day is the organizer of a huge conference the other day and possibly in charge to design the details of the decision rules.

Taking our exemplifying story aside, we can clearly imagine situations where the individual opinions of $|N|$ agents from an ordered set J of inputs are mapped to an output from an ordered set K. To this end $(|J|, |K|)$ simple games have been introduced, see e.g. [Freixas and Zwicker, 2003, Freixas and Zwicker, 2009], and we remark that simple games are in one-to-one correspondence to $(2, 2)$ simple games with $J = \{0, 1\}$ and $K = \{0, 1\}$. Measurements of influence for simple games are also called power indices and the Public Good index, introduced in [Holler, 1982], is a particular example. The question of this paper is whether a measure in the vein of the Public Good index can be defined for the class of (j, k) simple games. We motivate a few variants and give a axiomatizations. An axiomatization of the Public Good index for simple games was given in [Holler and Packel, 1983], so that some people also speak of the Holler–Packel index, and the generalization to TU games was axiomatized in [Holler and Li, 1995]. A different axiomatization, for both cases and based on potential functions, was given in [Haradau and Napel, 2007]. For $(j, 2)$ simple games a Public Good index was recently introduced in [Courtin and Tchantcho, 2020] along

with two axiomatizations.

The remaining part of this paper is structured as follows. In Section 2 we summarize some necessary preliminaries from the literature before we discuss different generalizations of the Public Good index and corresponding axiomatizations to the class of (j,k) simple games in Section 3.

2 Preliminaries

Let $N = \{1, 2, ..., n\}$ be a finite set of agents or voters. Any subset S of N is called a coalition and the set of all coalitions of N is denoted by the power set 2^N. For given integers $j, k \geq 2$ we denote by $J = \{0, \ldots, j-1\}$ the possible input levels and by $K = \{0, \ldots, k-1\}$ the possible output levels, respectively. We write $x \leq y$ for $x, y \in \mathbb{R}^n$ if $x_i \leq y_i$ for all $1 \leq i \leq n$. For each $\emptyset \subseteq S \subseteq N$ we write x_S for the restriction of $x \in \mathbb{R}^n$ to $(x_i)_{i \in S}$. As an abbreviation, we write $x_{-S} = x_{N \setminus S}$. Instead of $x_{\{i\}}$ and $x_{-\{i\}}$ we write x_i and x_{-i}, respectively. Slightly abusing notation we write $\mathbf{a} \in \mathbb{R}^n$, for the vector that entirely consists of a's, e.g., $\mathbf{0}$ for the all zero vector.

Definition 1. *Let $j, k \geq 2$ and $n \geq 0$ be integers. A* (j,k) simple game *is a mapping $v \colon J^n \to K$ satisfying $v(\mathbf{0}) = 0$ and $v(x) \leq v(y)$ for all $x, y \in J^n$ with $x \leq y$.*[2]

Example 1. *For $n = j = k = 3$ let the $(3,3)$ simple game v be defined via*

$$v(x) = \begin{cases} 0 & : & 3x_1 + 2x_2 + x_3 < 7 \\ 1 & : & 7 \leq 3x_1 + 2x_2 + x_3 < 12 \\ 2 & : & x_1 = x_2 = x_3 = 2 \end{cases}$$

for all $x \in \{0, 1, 2\}^3$.

Definition 2. *A* simple game *is a mapping $v \colon 2^N \to \{0, 1\}$ that satisfies $v(\emptyset) = 0$, $v(N) = 1$, and $v(S) \leq v(T)$ for all $\emptyset \subseteq S \subseteq T \subseteq N$, where the finite set N is called the* player set *or* set of players.[3]

Let v be a simple game with player set N. A subset $S \subseteq N$ is called *winning coalition* if $v(S) = 1$ and *losing coalition* otherwise. A winning coalition $S \subseteq N$ is called

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[2]Some authors also require $v(\mathbf{j} - \mathbf{1}) = k-1$, which would clash with the potential function approach as it is the case for simple games. Note that we have reversed the order of the input levels of approval compared to [Freixas and Zwicker, 2003].

[3]In some papers $v(S) \leq v(T)$ is dropped in the definition of a simple game and they speak of *monotonic simple games* is it is additionally assumed. For the potential function approach we will drop the condition $v(N) = 1$ later on, while it is indeed necessary for the normalized Public Good index.

minimal winning coalition if all proper subsets $T \subsetneq S$ of S are losing. The set of minimal winning coalitions is denoted by MWC(v).

Example 2. *For player set $N = \{1, 2, 3\}$ let v be the simple game defined by $v(S) = 1$ iff $w(S) := \sum_{i \in S} w_i \geq 3$ and $v(S) = 0$ otherwise for all $S \subseteq N$, where $w_1 = 3$, $w_2 = 2$, and $w_3 = 1$.*

The winning coalitions of the simple game from Example 2 are given by $\{1\}$, $\{2, 3\}$, $\{1, 2\}$, $\{1, 3\}$, and $\{1, 2, 3\}$. Only $\{1\}$ and $\{2, 3\}$ are minimal winning coalitions.

In order to embed a given simple game $v\colon 2^N \to \{0, 1\}$ as a $(2, 2)$ simple game $\hat{v}$ with $J = \{0, 1\}$ and $K = \{0, 1\}$, we assume $N = \{1, \ldots, n\}$. To each coalition $S \subseteq N$ we assign the vector $x^S \in \{0, 1\}^n$ with $x_i^S = 1$ iff $i \in S$ and $x_i^S = 0$ otherwise. Given a vector $x \in \{0, 1\}^n$ the corresponding coalition is given by $S = \{i \in N \mid x_i = 1\}$, so that $v(S) = \hat{v}(x^S)$.

The (raw) Public Good index for a simple game v with player set N and a player $i \in N$ is given by

$$\text{PGI}_i(v) = |\{S \in \text{MWC}(v) \mid i \in S\}|. \tag{1}$$

With this, the (normalized) Public Good index is given by

$$\overline{\text{PGI}}_i(v) = \frac{\text{PGI}_i(v)}{\sum_{j \in N} \text{PGI}_j(v)} \tag{2}$$

and is e.g. *efficient*, i.e., $\sum_{i \in N} \overline{\text{PGI}}_i(v) = 1$. Note that for the normalized version it is important to assume that $v(N) = 1$ since MWC(v) is empty otherwise, so that $\overline{\text{PGI}}_i(v)$ would be undefined.

A generalization of simple games, without the monotonicity assumption, are games with transferable utility – so-called TU games.

Definition 3. *A TU game is a mapping $v\colon 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$, where the finite set N is called the* player set *or* set of players.

If we additionally assume $v(S) \leq v(T)$ for all $\emptyset \subseteq S \subseteq T \subseteq N$, we speak of a *monotone TU game* or a *capacity*.

The analog of minimal winning coalitions in the context of TU games are *minimal crucial coalitions*, see e.g. [Haradau and Napel, 2007] or *real gaining coalitions*, see [Holler and Li, 1995]. To this end, we call a player $i \in S \subseteq N$ *crucial* in a TU game v if $v(S) > v(S \backslash i)$. A coalition S in which every player i is crucial is called *minimal crucial coalition* and the set of minimal crucial coalitions is denoted by MCC(v). A

coalition $S \subseteq N$ is called a real gaining coalition if $v(S) - v(T) > 0$ for all proper subsets $\emptyset \subseteq T \subsetneq S$ of S. The set of all real gaining coalitions of v is denoted by RGC(v). Note that for monotone TU games there is no difference between a minimal crucial and a real gaining coalition, i.e., MCC(v) = RGC(v). With these generalized notions, the Public Good value for a TU game v with player set N and a player $i \in N$ is given by

$$\text{PGV}_i(v) = \sum_{S \in \text{MCC}(v), i \in S} v(S),^4 \tag{3}$$

so that $\text{PGI}_i(v) = \text{PGV}_i(v)$ if v is a simple game. For the rest of the article, we will refer to MCC(v) as the minimal critical coalitions of v.

Let Γ be a subclass of all TU games. A *value* on Γ is a function Ψ that maps each game $v \in \Gamma$ to $\mathbb{R}^{|N|}$, where N is the player set of v. An example of a value is the Public Good value PGV, defined componentwise in Equation (3). A *potential* on Γ is a function P that maps each game $v \in \Gamma$ to a real number $P(v)$.

Definition 4. *A value Ψ on Γ admits a potential function if there exists a potential $P \colon \Gamma \to \mathbb{R}$ such that*

$$\Psi_i(v) = P(v) - P(v_{-i}) \tag{4}$$

for all $v \in \Gamma$ and all $i \in N$, where N is the player set of v and v_{-i} is the TU game with player set $N \backslash \{i\}$ defined by $v_{-i}(S) = v(S)$ for all $\emptyset \subseteq S \subseteq N \backslash \{i\}$.

Note that the subclass Γ of TU games has to be closed with respect to taking subgames v_{-i} in order to apply this definition. So, from a technical point of view we either have to include the game $v_\emptyset$ with empty player set in the set of TU games and subclasses of TU games Γ or define $P(v_\emptyset) := 0$ separately (which is the usual choice).[5] As shown in [Haradau and Napel, 2007, Proposition 1] the Public Good value PGV admits a potential P on the class Γ of (monotone) TU games, where

$$P(v) = \sum_{S \in \text{MCC}(v)} v(S). \tag{5}$$

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[4]Note that the authors from [Holler and Li, 1995] used the definition $\text{PGV}_i(v) = \sum_{S \in \text{RGC}(v), i \in S} v(S)$, while the authors from [Haradau and Napel, 2007] used $\text{PGV}_i(v) = \sum_{S \in \text{MCC}(v), i \in S} v(S)$. As already mentioned, there is no difference for monotone TU games. Also the axiomatization of the Public Good value from [Haradau and Napel, 2007] can be slightly adjusted by replacing the notion of minimal critical coalitions by real gaining coalitions in their definition of $\pi(v, N)$ and the corresponding axiom of *distributing the worths* of MCCs.

[5]If we do not set $P(v_\emptyset) = 0$, then the potential of a value is only determined up to an additive constant.

Note that each minimal critical coalition S in v with $i \notin S$ is also a minimal critical coalition in v_{-i} and vice versa. Analogously, that each real gaining coalition S in v with $i \notin S$ is also a real gaining coalition in v_{-i} and vice versa.

We say that a value Ψ on Γ *distributes the sum of the worths of the minimal critical coalitions for all players* in v iff

$$\sum_{i \in N} \Psi_i(v) = \sum_{i \in N} \sum_{S \in \mathrm{MCC}(v), i \in S} v(S) = \sum_{S \in \mathrm{MCC}(v)} |S| \cdot v(S) \tag{6}$$

for all $v \in \Gamma$, where N is the player set of v. With this, [Haradau and Napel, 2007, Proposition 2] states that the Public Good value PGV is the unique value that admits a potential and distributes the sum of the worths of the minimal critical coalitions for all players on the class of monotone TU games. The great advantage of an axiomatization via a potential is that this also gives an axiomatization for all subclasses Γ' of TU games that are closed with respect to taking subgames v_{-i}. So, if we relax the condition $v(N) = 1$ of a simple game, we also obtain an axiomatization for simple games. Note that while $v(N) = 1$ it may happen that $v_{-i}(N\backslash\{i\}) \neq 1$, i.e., v_{-i} does not contain a winning coalition, which happens if player i is a so-called *vetoer*.

Another common property of values is linearity. To this end we note that TU games form an $\mathbb{R}$-vector space with sum $(v + v')(S) := v(S) + v'(S)$ and scalar multiplication $(\lambda \cdot v)(S) := \lambda \cdot v(S)$ for all TU games v, v' with the same player set N, all $\lambda \in \mathbb{R}$, and all $S \subseteq N$. With this, a value Ψ is called *linear* if $\Psi(v + v') = \Psi(v) + \Psi(v')$ and $\Psi(\lambda \cdot v) = \lambda \cdot \Psi(v)$. From Equation (3) we can directly conclude that the Public Good value PGV is linear. If only the first property, on the sum of two TU games holds, then one speaks of *additivity*. Since the sum of two simple game (considered as TU games) does not need to be a simple game, the so-called *transfer axiom* was introduced by Dubey [Dubey, 1975]:

$$\Psi(v \wedge v') + \Psi(v \vee v') = \Psi(v) + \Psi(v'),$$

where $(v \wedge v')(S) = \min\{v(S), v'(S)\}$ and $(v \vee v')(S) = \max\{v(S), v'(S)\}$ for all simple games v, v' with the same player set N and all coalitions $S \subseteq N$. Note that the definition of $\wedge$ and $\vee$ might also be applied to general TU games. In our context we only use $v \oplus v' := v \vee v'$ for two simple or TU games v, v'. Two simple games v and v' are called *mergeable* if $S \in \mathrm{MWC}(v)$ and $S' \in \mathrm{MWC}(v')$ implies $S \nsubseteq S'$ and $S' \nsubseteq S$. The identity $\mathrm{PGI}_i(v \oplus v') = \mathrm{PGI}_i(v) + \mathrm{PGI}_i(v')$ for the raw Public Good index for two mergeable simple games was used in [Holler and Packel, 1983] to axiomatize

the normalized Public Good index. Similarly, for two (j, k) games v and v' we define $(v \oplus v')(x) = \max\{v(x), v(x')\}$ for all $x \in J^n$, where n is the number of players of v and v'.

3   Generalizing the Public Good index to (j, k) simple games

The first question we have to answer is that for a suitable generalization of the concept of a minimal winning coalition in a simple game to an arbitrary (j, k) simple game. Having the definition of minimal critical and real gaining coalitions for TU games in mind, we propose:

Definition 5. *Let v be a (j, k) simple game with player set $N = \{1, \ldots, n\}$ and $J = \{0, 1, \ldots, j-1\}$. A vector $x \in J^n$ is called* minimal critical *if $v(x) > v(x')$ for all $x' \in J^n$ with $x' \leq x$ and $x' \neq x$. The set of minimal critical vectors of v is denoted by* $\mathrm{MCV}(v)$.

Note that for $j = 2$ and $k = 2$ each minimal critical vector x corresponds to a minimal winning coalition $S = \{1 \leq i \leq n \mid x_i = 1\}$ in the corresponding simple game. For $j = 2$ and arbitrary $k \geq 2$ we can embed a $(2, k)$ simple game v as a TU game $\hat{v}$, so that the minimal critical vectors of v are in 1-to-1 correspondence with the minimal critical coalitions of $\hat{v}$.

Let Γ be a subclass of all (j, k) simple games, where $j \geq 2$ and $k \geq 2$ are arbitrary but fixed. A *value* on Γ is a function Ψ that maps each game $v \in \Gamma$ to $\mathbb{R}^{|N|}$, where N is the player set of v. A *potential* on Γ is a function P that maps each game $v \in \Gamma$ to $\mathbb{R}$.

Definition 6. *A value Ψ on a subclass Γ of (j, k) simple games admits a potential function if there exists a potential $P \colon \Gamma \to \mathbb{R}$ such that*

$$\Psi_i(v) = P(v) - P(v_{-i}) \tag{7}$$

for all $v \in \Gamma$ and all $i \in N$, where N is the player set of v and v_{-i} is the (j, k) simple game with player set $N \setminus \{i\}$ defined by $v_{-i}(x) = v(y)$ for all $x \in J^{N \setminus \{i\}}$ and $y \in J^N$ with $y_i = 0$ and $y_j = x_j$ for all $j \in N \setminus \{i\}$.[6] Moreover, we set $P(v_\emptyset) := 0$ for a game $v_\emptyset$ with empty player set.

Again, the subclass Γ of (j, k) simple games has to be closed with respect to taking subgames v_{-i} in order to apply this definition. We observe that each minimal critical vector x of v with $x_i = 0$ is also a minimal critical vector of v_{-i} if we remove the entry for x_i (so that it is a vector in $J^{N \setminus \{i\}}$) and vice versa. We say that a value Ψ on a subclass

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[6]By A^B we denote the set of all mappings from B to A whose cardinality is $|A|^{|B|}$.

Γ of (j,k) simple games *distributes the sum of the worths of the minimal critical vectors for all players* in v iff

$$\sum_{i=1}^{n} \Psi_i(v) = \sum_{i=1}^{n} \sum_{x \in \text{MCV}(v), x_i \neq 0} v(x) = \sum_{x \in \text{MCV}(v)} v(x) \cdot \left| \{1 \leq i \leq n \mid x_i \neq 0\} \right| =: \Lambda(v) \quad (8)$$

for all $v \in \Gamma$, where $N = \{1, \ldots, n\}$ is the player set of v.

Theorem 1. *Let $j, k \geq 2$ be integers. Then, there exists a unique value Ψ on the class Γ of all (j,k) simple games that admits a potential function and distributes the sum of the worths of the minimal critical vectors for all players. We have*

$$\Psi_i(v) = \sum_{x \in \text{MCV}(v), x_i \neq 0} v(x) \quad (9)$$

for all $v \in \Gamma$ and all i in the player set $\{1, \ldots, n\}$ of v. The potential function is given by

$$P(v) = \sum_{x \in \text{MCV}(v)} v(x) \quad (10)$$

for all $v \in \Gamma$.

Proof. First we assume that the potential is given by Equation (10). Since Ψ admits a potential function we have

$$\begin{aligned}
\Psi_i(v) &= P(v) - P(v_{-i}) = \sum_{x \in \text{MCV}(v)} v(x) - \sum_{x \in \text{MCV}(v_{-i})} v_{-i}(x) \\
&= \sum_{x \in \text{MCV}(v)} v(x) - \sum_{x \in \text{MCV}(v_{-i})} v(x) \\
&= \sum_{x \in \text{MCV}(v), x_i \neq 0} v(x)
\end{aligned}$$

for all $v \in \Gamma$ and all i in the player set of v, where we have used the relation between the minimal critical vectors of v and those of v_{-i}. Thus, Equation (9) is valid. With this we have

$$\sum_{i=1}^{n} \Psi_i(v) = \sum_{i=1}^{n} \sum_{x \in \text{MCV}(v), x_i \neq 0} v(x) = \sum_{x \in \text{MCV}(v)} v(x) \cdot \left| \{1 \leq i \leq n \mid x_i \neq 0\} \right| = \Lambda(v),$$

i.e., Ψ distributes the sum of the worths of the minimal critical vectors for all players and so satisfies both axioms.

For the other direction we assume that Ψ admits a potential $\tilde{P}$, so that

$$\Lambda(v) = \sum_{i=1}^{n} \Psi_i(v) = \sum_{i=1}^{n} \left(\tilde{P}(v) - \tilde{P}(v_{-i}) \right) = n \cdot \tilde{P}(v) - \sum_{i=1}^{n} \tilde{P}(v_{-i}),$$

which is equivalent to

$$\tilde{P}(v) = \frac{\Lambda(v) + \sum_{i=1}^{n} \tilde{P}(v_{-i})}{n} \tag{11}$$

for each $v \in \Gamma$, where $N = \{1, \dots, n\}$ is the player set of v. For each $S \subseteq N$ we denote by v_S the (j,k) simple game with player set S defined by $v_S(x) = v(y)$ for all $x \in J^S$, where $y \in J^N$ with $y_j = x_j$ for all $j \in S$ and $y_j = 0$ otherwise. E.g. $v_{-1} = v_{N\setminus\{i\}}$ and $v_N = v$. Since $(v_S)_T = v_T$ for all $\emptyset \subseteq T \subseteq S \subseteq N$ Equation (11) can be generalized to

$$\tilde{P}(v_S) = \frac{\Lambda(v_S) + \sum_{i \in S} \tilde{P}(v_{S\setminus\{i\}})}{|S|}$$

for all $\{i\} \subseteq S \subseteq N$. So, starting from $\tilde{P}(v_\emptyset) = 0$, we can recursively compute $\tilde{P}(v_S)$ for all $\emptyset \neq S \subseteq N$, so that especially $\tilde{P}(v) = \tilde{P}(v_N)$ is uniquely defined. ∎

We call the value Ψ for (j,k) simple games defined by Equation (9) *Public Good value* (for (j,k) simple games). For the $(3,3)$ simple game v from Example 1 the minimal critical vectors are $(1,1,2)$, $(1,2,0)$, $(2,0,1)$, $(2,1,0)$, and $(2,2,2)$, where $v(x) = 1$ for all $x \in \mathrm{MCV}(v)\setminus\{(2,2,2)\}$ and $v((2,2,2)) = 2$. With this we compute

$$\Psi_1(v) = 6, \quad \Psi_2(v) = 5, \quad \text{and } \Psi_3(v) = 4$$

for the value Ψ characterized in Theorem 1.

We would like to remark that we also may motivate a different definition for a Public Good value for (j,k) simple games. To this end we define the vector $y = x \downarrow i \in J^n$ for each $x \in J^n$ with $x_i \neq 0$ by $y_j = x_j$ for all $j \neq i$ and $y_i = x_i - 1$. Assume that agent i has strictly increasing costs in i and that the rewards are strictly increasing in $v(x)$.[7] As in the process of a coalition forming member by member we may imagine that starting from $x = \mathbf{0}$ the final vector x forms step by step via the inverse operation of $\downarrow$.[8] So, similarly, as one can argue that only minimal winning coalitions will be formed, we deduce that under the described model for every finally formed vector

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[7]For $(2,2)$ simple games represented as simple games this means that entering a coalition comes at a certain cost while a coalition gets a reward iff it is a winning coalition.

[8]More precisely, for each $x \in J^n$ with $x_i \neq j - 1$ we can define the vector $y = x \uparrow i \in J^n$ by $y_j = x_j$ for all $j \neq i$ and $y_i = x_i + 1$.

$x \in J^n$ with $v(x) \neq 0$ we have $x \in \mathrm{MCV}(v)$. Now what is the contribution of a player i to a minimal critical vector x with $x_i \neq 0$ to the worth $v(x)$? If the answer is $v(x)$, then we end up with the value characterized in Theorem 1. However, if we have a look at the minimal critical vector $x = (2, 2, 2)$ in the $(3, 3)$ simple game v from Example 1, then $v(1, 2, 2) = v(2, 1, 2) = v(2, 2, 1) = 1$ may justify the assumption that every player contributes just a surplus of 1 to the worth of vector x. Thus, we would obtain a value defined by

$$\Psi_i(v) = \sum_{x \in \mathrm{MCV}(v), x_i \neq 0} \big(v(x) - v(x \downarrow i)\big). \tag{12}$$

Note the similarity to the Banzhaf index. For simple games the difference is that we sum over all minimal winning instead of all winning coalitions. For the $(3, 3)$ simple game v from Example 1 we would obtain

$$\Psi_1(v) = 5, \quad \Psi_2(v) = 4, \quad \text{and } \Psi_3(v) = 3.$$

We observe that there is no difference between both variants if $k = 2$. And indeed, they match the variant introduced in [Courtin and Tchantcho, 2020]. For all (j, k) simple games not identically mapping to zero we define the normalized version

$$\overline{\Psi}_i(v) = \frac{\Psi_i(v)}{\sum_{j=1^n} \Psi_j(v)}. \tag{13}$$

Excluding the (j, k) simple games $v \equiv 0$, we speak of *non-trivial* (j, k) simple games. Our next aim is an axiomatization for $\overline{\Psi}$. To this end we propose a generalization of mergeability for simple games:

Definition 7. *Two (j, k) simple games v and v' with the same player set $\{1, \ldots, n\}$ are mergeable if*

(1) $\mathrm{MCV}(v) \cap \mathrm{MCV}(v') = \emptyset$;

(2) $x \in \mathrm{MCV}(v), x' \in \mathrm{MCV}(v'), x \leq x' \implies v(x) < v'(x')$; and

(3) $x \in \mathrm{MCV}(v), x' \in \mathrm{MCV}(v'), x \geq x' \implies v(x) > v'(x')$.

Note that (2) and (3) imply (1). Since $v(x) > 0$ for all $x \in \mathrm{MCV}(v)$ the definition for $(2, 2)$ simple games goes in line with the definition for simple games. Actually, we have $v(x) = 1$ for every minimal critical vector of some $(j, 2)$ simple game. If $k > 2$, then we have to distinguish the critical vectors according to their output value

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$v(x)$. Next we study the relation of the minimal critical vectors of the sum of two mergeable (j, k) simple games with those of their "summand games".

Lemma 1. *Let v be a (j, k) simple game with player set $\{1, \dots, n\}$. For each vector $x \in J^n$ with $v(x) > 0$ there exists a vector $x' \le x$ with $v(x') = v(x)$ and $x' \in \mathrm{MCV}(v)$.*

Proof. If $x \in \mathrm{MCV}(v)$, then the statement is true for $x' = x$. Otherwise there exists a player $1 \le i \le n$ with $x_i \ne 0$ such that $v(x) = v(x \downarrow i)$. If $x \downarrow i \in \mathrm{MCV}(v)$, then we can set $x' = x \downarrow i$ and are done. Otherwise we iteratively apply the operator $\downarrow$ (which terminates since the number of players and output levels is finite). ∎

We remark that the minimal critical vector x' does not need to be unique. To this end we may slightly adjust the $(3, 3)$ simple game v from Example 1 by setting $v(x) = 1$ for $x = (2, 2, 2)$.

Lemma 2. *Let v and v' be two (j, k) simple games with the same player set $\{1, \dots, n\}$ that are mergeable. Then, we have*

$$\mathrm{MCV}(v \oplus v') = \mathrm{MCV}(v) \cup \mathrm{MCV}(v').$$

Proof. Consider $x \in v \oplus v'$. Since $(v \oplus v')(x) = \max\{v(x), v'(x)\}$ we assume $(v \oplus v')(x) = v(x)$ and $v'(x) \le v(x)$ w.l.o.g. If $x \notin \mathrm{MCV}(v)$, then there exists a player $1 \le i \le n$ with $x_i \ne 0$ such that $v(x \downarrow i) = v(x)$. However, this implies $(v \oplus v')(x \downarrow i) \ge v(x \downarrow i) = v(x) = (v \oplus v')(x)$, which is a contradiction. Thus, we have $\mathrm{MCV}(v \oplus v') \subseteq \mathrm{MCV}(v) \cup \mathrm{MCV}(v')$.

Consider $x \in \mathrm{MCV}(v)$. First we show $v(x) > v'(x)$. To this end we apply Lemma 1 to conclude the existence of a vector $x' \in J^n$ with $x' \le x$ and $v'(x') = v'(x)$. Now the stated inequality is implied by Definition 7.(3) and we have $(v \oplus v')(x) = v(x)$. Assume $x \notin \mathrm{MCV}(v \oplus v')$ for a moment. Let $1 \le i \le n$ be a player with $(v \oplus v')(x \downarrow i) = (v \oplus v')(x)$. Since

$$(v \oplus v')(x \downarrow i) = \max\{v(x \downarrow i), v'(x \downarrow i)\} \le \max\{v(x \downarrow i), v'(x)\} < v(x) = (v \oplus v')(x),$$

we obtain a contradiction. Thus, $\mathrm{MCV}(v) \subseteq \mathrm{MCV}(v \oplus v')$ and, by symmetry, also $\mathrm{MCV}(v') \subseteq \mathrm{MCV}(v \oplus v')$, so that $\mathrm{MCV}(v) \cup \mathrm{MCV}(v') \subseteq \mathrm{MCV}(v \oplus v')$. ∎

Note that $\mathrm{MCV}(v) \cap \mathrm{MCV}(v') = \emptyset$, i.e., we have the disjoint union $\mathrm{MCV}(v \oplus v') = \mathrm{MCV}(v) \uplus \mathrm{MCV}(v')$.

We say that a minimal critical vector $x \in \mathrm{MCV}(v)$ is *critical for player i and output level τ* if $v(x) \ge \tau$ and $v(x \downarrow i) < \tau$. So, a given minimal critical vector $x \in \mathrm{MCV}(v)$ (with

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$x_i \neq 0$) is critical for $v(x) - v(x \downarrow i)$ output levels. Denoting the number of pairs (x, τ) such that $x \in \text{MCV}(v)$ with $x_i \neq 0$ is critical for player i with output level τ by $c_i(v)$, we have

$$c_i(v \oplus v) = c_i(v) + c_i(v') \qquad (14)$$

for two mergeable (j, k) simple games v, v' with player set $\{1, \dots, n\}$ and $1 \leq i \leq n$.

Definition 8. *Let v be a (j, k) simple game with player set $\{1, \dots, n\}$. A player $1 \leq i \leq n$ is called a* null player *if we have $v(x) = v(x')$ for all $x, x' \in J^n$ with $x_j = x'_j$ for all $j \neq i$.*

Note that we have $x_i = 0$ for every null player i and every minimal critical vector $x \in \text{MCV}(v)$. The analog for simple games is that no null player is part of a minimal winning coalition.

Definition 9. *Let v be a (j, k) simple game with player set $N := \{1, \dots, n\}$ and $\pi \colon N \to N$ be a permutation, i.e., a bijection. The (j, k) simple game πv is defined by $(\pi v)(x) = v(x')$ for all $x \in J^n$ where $x'_i = x_{\pi(i)}$ for all $1 \leq i \leq n$.*

A value Φ on the class of (non-trivial) (j, k) simple games is called anonymous *if for each permutation $\pi \colon N \to N$ we have $\overline{\Psi}_i(\pi v) = \overline{\Psi}_{\pi(i)}(v)$, where N is the player set of an arbitrary (non-trivial) (j, k) simple game v and $i \in N$ an arbitrary player.*

Theorem 2. *The value $\overline{\Psi}$ defined in Equation (13) and Equation (12) is the unique value for non-trivial (j, k) simple games that satisfies the axioms:*

(A1) i is a null player in v $\quad \Rightarrow \quad$ $\overline{\Psi}_i(v) = 0$.

(A2) $\overline{\Psi}$ is efficient, i.e., $\sum_{i=1}^n \overline{\Psi}_i(v) = 1$.

(A3) If $\text{MCV}(v) = \{x\}$ for a game v, then $\overline{\Psi}_i(v) = \overline{\Psi}_j(v)$ for all players i, j with $x_i, x_j \neq 0$.

(A4) For all mergeable (j, k) simple games v, v' with player set N we have

$$\overline{\Psi}_i(v \oplus v') = \frac{c(v) \cdot \overline{\Psi}_i(v) + c(v') \cdot \overline{\Psi}_i(v')}{c(v) + c(v')}$$

for all $i \in N$, where $c(\tilde{v}) = \sum_{j \in N} c_j(\tilde{v})$ for every non-trivial (j, k) simple game $\tilde{v}$ with player set N.

Proof. It is immediate that the value $\overline{\Psi}$ defined in Equation (13) and Equation (12) satisfies the axioms (A1), (A2), and (A3). For (A4) we first note $\Psi_i(\tilde{v}) = c_i(\tilde{v})$ for every

(j, k) simple game $\tilde{v}$ and every player i in $\tilde{v}$. Using the mergeability of v and v' we compute

$$\overline{\Psi}_i(v \oplus v') = \frac{c_i(v \oplus v')}{c(v \oplus v')} = \frac{c_i(v) + c_i(v')}{c(v) + c(v')} = \frac{c(v) \cdot \overline{\Psi}_i(v) + c(v') \cdot \overline{\Psi}_i(v')}{c(v) + c(v')}.$$

Conversely, given any value Φ on the class of non-trivial (j, k) simple games satisfying the axioms (A1) through (A4) we proceed as follows. First we consider an arbitrary non-trivial (j, k) simple game v with $|\text{MCV}(v)| = 1$ and let x be the unique minimal critical vector. From (A1), (A2), and (A3) we conclude

$$\Phi_i(v) = \begin{cases} 1 / |\{j \mid x_j \neq 0\}| & \text{if } x_i \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now consider any non-trivial (j,k) simple game $\tilde{v}$ with player set N and minimal critical vectors enumerated as $\text{MCV}(\tilde{v}) = \{x^1, \ldots, x^m\}$. Denoting the non-trivial (j, k) simple game with unique minimal critical vector x^h by v^h, where $1 \leq h \leq m$, we can write

$$\tilde{v} = v^1 \oplus v^2 \oplus \cdots \oplus v^m.$$

Note that the v^h are sequentially mergeable in the sense that v^{h+1} and $v^1 \oplus \cdots \oplus v^h$ are mergeable for each $h = 1, 2, \ldots, m - 1$. We can extend (A4) inductively to a sum of such games to obtain for each player $i \in N$

$$\Phi_i(\tilde{v}) = \sum_{h=1}^{m} c(v^h)\Phi_i(v^h) / \sum_{h=1}^{m} c(v^h).$$

Thus, the axioms (A1)-(A4) allow us to compute $\Phi_i(\tilde{v})$ for each non-trival (j, k) simple game $\tilde{v}$ and each player i of $\tilde{v}$, i.e., there is at most one value satisfying axioms (A1)-(A4). So, given our first observation on $\overline{\Psi}$, we conclude $\Phi = \overline{\Psi}$. ∎

We remark that the axioms (A1) and (A2) mimic similar axioms for simple or TU games that are used frequently in the literature. For axiom (A4) we refer to the discussion in [Holler and Packel, 1983] noting that the proof of Theorem 2 is rather similar to the one of [Holler and Packel, 1983, Section III]. Note that for $k = 2$ output levels axiom (A3) can be replaced by anonymity, see Definition 9. However, for $k > 2$ we need some kind of stronger axiom in order to uniquely define the value of non-trivial (j, k) simple games with a unique minimal critical vector. Of course,

axiom (A3) might be considered to be too demanding for the cases where $x_i, x_j \neq 0$ and $x_i \neq x_j$. There is an ongoing discussion about properties that a reasonable power index or value should have, see e.g. [Allingham, 1975, Kurz, 2020]. We would also like to point the reader to the two axiomatizations of the Public Good index for $(j, 2)$ simple games in [Courtin and Tchantcho, 2020], which share several axioms on the one hand and use a few different on the other hand.

Another approach to motivate the definition of a value for (j, k) simple games is pursued in [Kurz et al., 2021] for the Shapley value.

Definition 10. *Let v be an arbitrary (j, k) simple game with player set $N = \{1, \ldots, n\}$. The average game, denoted by $\widetilde{v}$, associated to v is defined by*

$$\widetilde{v}(S) = \frac{1}{j^n(k-1)} \sum_{x \in J^n} \left[v((\mathbf{j} - \mathbf{1})_S, x_{-S}) - v(\mathbf{0}_S, x_{-S}) \right] \tag{15}$$

for all $S \subseteq N$.

For the $(3, 3)$ simple game v from Example 1 the average game $\tilde{v}$ is given by $\tilde{v}(\emptyset) = 0$, $\tilde{v}(\{1\}) = \frac{1}{2}$, $\tilde{v}(\{2\}) = \frac{5}{18}$, $\tilde{v}(\{3\}) = \frac{1}{6}$, $\tilde{v}(\{1, 2\}) = \frac{2}{3}$, $\tilde{v}(\{1, 3\}) = \frac{2}{3}$, $\tilde{v}(\{2, 3\}) = \frac{1}{2}$, and $\tilde{v}(\{1, 2, 3\}) = 1$. Note that $\tilde{v}$ always is a TU game taking values between 0 and 1.

In [Kurz et al., 2021, Theorem 4.1] it was shown that the Shapley value of a (j, k) simple game v, as defined in e.g. [Freixas, 2005], equals the Shapley value of the TU game $\tilde{v}$. Unfortunately there is no such nice relation between the Public Good value and our analogs for (j, k) simple games since for the $(3, 3)$ simple game from Example 1 and the corresponding average TU game $\tilde{v}$ we have

$$\mathrm{PGV}_1(\tilde{v}) = \frac{51}{18}, \quad \mathrm{PGV}_2(\tilde{v}) = \frac{44}{18}, \quad \text{and } \mathrm{PGV}_3(\tilde{v}) = \frac{42}{18}.$$

To sum up, we have seen that different generalizations of the Public Good value for TU games or the normalized Public Good index for simple games to the class of (non-trivial) (j, k) simple games, including axiomatizations, are possible. As anticipated e.g. in [Freixas, 2012], a power index for simple games can admit more than one reasonable extension for (j, k) simple games. From our personal point of view, Theorem 1 provides the most convincing variant. But this may be just a matter of taste or might depend on the application. The question of the public good properties of the proposed values is not touched at all. As done in [Courtin and Tchantcho, 2020] for $(j, 2)$ simple games, other power indices based on *Riker's Size Principle* [Riker, 1962, p. 32] may be treated similarly.

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