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General method for including Stueckelberg fields

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Abstract A systematic procedure is proposed for inclusion of Stueckelberg fields. The procedure begins with the involutive closure when the original Lagrangian equations are complemented by all the lower order consequences. The Stueckelberg field is introduced for every consequence included into the closure. The generators of the Stueckelberg gauge symmetry begin with the operators generating the closure of original system. These operators are not assumed to be a generators of gauge symmetry of any part of the original action, nor are they supposed to form an on shell integrable distribution. With the most general closure generators, the consistent gauge invariant theory is iteratively constructed, without obstructions at any stage. The Batalin–Vilkovisky form of inclusion of the Stueckelberg fields is worked out and the existence theorem for the Stueckelberg action is proven.

1 Introduction

In 1938, Stueckelberg proposed [1] to reformulate the Proca action for the massive vector field in a gauge invariant way by introducing the scalar field. Since then, the general idea attributed to Stueckelberg has been widely used to equivalently reformulate the original non-gauge theory in a gauge invariant way by introducing some extra fields. Historical review of ideas about, and applications of the Stueckelberg method can be found in the article [2].

The most often used scheme of introducing the Stueckelberg fields follows the pattern of the original work [1] implying that Lagrangian includes the gauge invariant part, and the non-gauge invariant part. In the case of massive vector field, the gauge invariant part is the Maxwell Lagrangian, while the non-invariant part is the massive term. Then, the finite gauge transformation defining the symmetry of the invariant part is made of the fields in the entire Lagrangian. This makes the Lagrangian depending on the gauge parameters. After that, the gauge parameters are treated as the Stueckelberg fields. Under this scheme, the Stueckelberg gauge transformations of the original fields are the same as in the theory without non-invariant part of the Lagrangian. In this sense, the broken gauge invariance of the original fields is restored. Once the Stueckelberg fields are included as the parameters of gauge transformations of the invariant part of the action, their own gauge symmetry is defined as the composition of original gauge transformations. In this way, the gauge transformation of Stueckelberg fields compensates the change of the non-invariant part of the Lagrangian caused by the transformation of original fields. The equivalence to the original theory is established by imposing the gauge conditions fixing the Stueckelberg fields to zero. In this gauge, the Stueckelberg theory reduces to the original one. This pattern of inclusion the Stueckelberg fields, sometimes referred to as the "Stueckelberg trick", is summarized and studied in very general form in the article [3] where one can also find a review of the vast contemporary literature on this topic.

The "Stueckelberg trick" works well in many models, but it does not seem consistent as a general method as it is more art than science. The division of the Lagrangian into invariant and non-invariant parts is rather arbitrary, as is the predefined choice of gauge transformations. For example, given any transformations of the fields that form the Lie group, any invariant of the group can be added to and subtracted from any Lagrangian. Thus, one can get a division into invariant and non-invariant parts with respect to any transformation. Thereafter, the above pattern can be applied, resulting in a model with almost any gauge symmetry that is not necessarily relevant to the original dynamics. Even the number of gauge parameters can be any within this approach. In the case of the Proca action, one could shift the original vector field not by a gradient of a scalar, but – for example – by a divergence of anti-symmetric second rank tensor. In this case, another part of the Lagrangian – the square of divergence of original vector field – is considered as an invariant, and the rest as a non-invariant part. The method works, and

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results in the gauge invariant theory of massive spin one with reducible gauge symmetry parameterized by antisymmetric tensor.

Another commonly used scheme of inclusion of Stueckelberg fields is the method of conversion of the Hamiltonian second class constrained systems into the first class ones. The first version of the method [4–6] implied to extend the phase space of the original system by new canonical variables whose number coincides with the number of second class constraints. After that, the constraints and Hamiltonian are continued into the extended space to convert the system into the first class. Once the gauge conditions are imposed killing the conversion variables, the system reduces to the original second class system. The conversion variables can be viewed as Hamiltonian version of Stueckelberg fields. The original proposal for conversion implied to linearly include the conversion variables into the effective first class constraints. In general, the gauge symmetry, being generated by the effective first class constraints, is non-abelian. Later on, the abelian conversion scheme has been proposed [7–9]. Under this scheme, the original second class constraints are extended by a power series in the conversion variables to become abelian first class constraints. Any original phase space function, including Hamiltonian, is extended by the conversion variables to Poisson commute with the effective abelian first class constraints. The existence theorem of the abelian conversion is proven by the homological perturbation theory (HPT) tools in the article [9]. In the article [10], the conversion method is extended to the general second class systems on symplectic manifolds, with the constraints not necessarily being scalar functions, but sections of a bundle over the symplectic base. In this setup, the conversion is proven to exist, though not necessarily abelian. This conversion scheme allows one to extend Fedosov deformation quantisation to the second class constrained systems.

Notice important distinctions of the Hamiltonian conversion method from the described above “Stueckelberg trick” which is widely applied in Lagrangian formalism. The starting point of the Hamiltonian conversion is a complete system of the second class constraints, including primary and secondary ones. The Hamiltonian equations are first order, while primary constraints are zero order. All these equations are variational. The secondary constraints are zero order differential consequences of the variational equations, and they are not variational by themselves. It is the completion of the original equations by the lower order consequences which allows one to explicitly count the degree of freedom (DoF) number. The same applies not only to the constrained Hamiltonian equations, but to any system of field equations. The completion of the system by the lower order consequences is known as the involutive closure. Given the involutive closure of the equations, one can count degree of freedom number in a covariant manner, not appealing to the $1+(d-1)$ decomposition. Simple and explicitly covariant DoF number counting recipe is worked out in the article [11] for any involutively closed system of field equations, not necessarily Lagrangian. For the involutive closure of Lagrangian system, the general recipe (see relation (8) of the article [11]) can be further simplified. The covariant DoF count is explained in the Appendix of this article. The Hamiltonian scheme of including Stueckelberg fields proceeds from involutive closure of variational equations, in particular the number of conversion variables coincides with the number of constraints. The Lagrangian pattern of the Stueckelberg trick does not account for the structure of involutive closure, even the number of the Stueckelberg fields is unrelated to the number of lower order consequences of the Lagrangian equations. Notice one more essential distinction between the pattern of Lagrangian “Stueckelberg trick” and the Hamiltonian conversion procedure. The first one proceeds from certain integrable distribution on the space of fields, which is considered as gauge symmetry of “invariant part” of the action. The second class constraints are unrelated, in general, to any integrable distribution, and the Hamiltonian conversion methods do not employ any predefined transformations of the original fields.

One of the motivations for introducing the Stueckelberg fields is the idea to provide consistent inclusion of interactions by controlling compatibility of Stueckelberg gauge symmetry when the free theory is deformed. This idea works well in various examples, see [3] and references therein. However, it does not seem a consistent general scheme, as it controls just algebraic consistency of the Stueckelberg symmetry, not the number of propagating DoF’s, while the artificial symmetry is not necessarily reasonably related to the structure of the dynamics. In the article [11], the method is proposed to consistently include interactions proceeding from the involutive closure of field equations and without introducing Stueckelberg fields. Proceeding from the free field equations brought to the involutive form, the method allows one to iteratively find all the consistent vertices in the equations. Even though the original non-involutive equations are Lagrangian, the involutive closure is not a Lagrangian system. Therefore, not all the vertices are necessarily Lagrangian. For applications of this scheme of inclusion of interactions, see, for example, [11–15]. While non-Lagrangian vertices may have their own advantages, in particular they can be stable in higher derivative field theories [14,16], it seems interesting to have a method of identifying all the consistent Lagrangian vertices. The way to construct all the consistent Lagrangian vertices is briefly noticed in the next section as a side remark.

The main subject of this article is to work out a method of inclusion the Stueckelberg fields which proceeds from the involutive closure of Lagrangian equations. In this sense, the method can be considered as the Lagrangian counterpart of the conversion method for the Hamiltonian second class constrained systems.

Notation

DeWitt's condensed notation [17,18] is adopted, where the indices cover both space time-points x and a set of numerical labels. As a rule, all the indices are understood as condensed, the exceptions are clear from the context. The derivatives ∂_i are variational w.r.t. the fields ϕ^i. Summation over condensed indices includes integration over x. Sign $\approx$ is the on shell equality.

2 Involutive closure of the Lagrangian equations and the Stueckelberg fields

In condensed notation, given the action $S(\phi)$, Lagrangian equations read

$$\partial_i S(\phi) = 0. \tag{1}$$

Suppose the action admits no gauge symmetry. This means that matrix of second derivatives of the action does not have on-shell kernel. Inclusion of Stueckelberg fields in the case with gauge symmetry can be considered along the similar lines, though it is slightly more complex. It will be addressed elsewhere.

Let us complement equations (1) with the consequences

$$\tau_\alpha(\phi) = -\Gamma^i_\alpha(\phi)\partial_i S(\phi), \tag{2}$$

such that the system

$$\partial_i S(\phi) = 0, \qquad \tau_\alpha = 0 \tag{3}$$

is involutive. Here, the involution means that the system does not admit any lower order consequence which is not already included. For a review of the involution concept in the partial differential equation (PDE) theory, and various applications, one can consult the book [19]. If the original Lagrangian equations are not involutive, by adding consequences, it will be brought to the involution. This can be considered as Lagrangian analogue of Dirac–Bergmann algorithm of constrained Hamiltonian formalism with the consequences τ_α being analogues of the second class constraints. The difference is that the order of derivatives is considered w.r.t. any space-time coordinate. Much like the Hamiltonian conversion, the Lagrangian procedure of inclusion Stueckelberg fields begins with involutive closure of equations of motion.

In principle, one can include the consequences of the higher order than the original Lagrangian equations. The only requirement is that the system of Lagrangian equations and their consequences (3) are involutive — i.e. any lower order consequence is already contained among these equations[1]. Example of this sort is considered in the end of this section.

The consequences τ_α are supposed independent. This also means that the generators Γ^i_α, being the local differential operators, are also independent. The specific conditions of independence are explained in the next section. The overcomplete set of generators Γ would lead to the reducible Stueckelberg gauge symmetry. This case is not considered in the article, though it can be of interest for some models.

The involutive system (3) admits gauge identities

$$\Gamma^i_\alpha(\phi)\partial_i S(\phi) + \tau_\alpha(\phi) \equiv 0, \tag{4}$$

which can be viewed as a rephrasing of the definition of consequences τ_α (2). The involutively closed system (3) is non-Lagrangian as such though it is equivalent to the original Lagrangian system (1). In non-Lagrangian systems, the second Noether theorem does not apply, so the gauge identities are not necessarily related to a gauge symmetry. Since the original Lagrangian equations (1) do not have gauge symmetry, then their involutive closure (3) will not be gauge invariant either, because they define the same mass shell.

The involutive closure (3) of the Lagrangian system is characterised by three types of numbers which determine the DoF number: (i) the orders of original Lagrangian equations (1); (ii) the orders of the consequences τ_α included into involutive closure (3) of the system; (iii) the orders of differential operators Γ^i_α generating consequences of Lagrangian equations included into the involutive closure. The DoF number is a certain linear combination of these three types of integers. The DoF counting is detailed in the Appendix, see relation (65) and corresponding explanations.

Let us make a side remark on consistent inclusion of interactions, proceeding from the involutive closure (3) of the Lagrangian system. Given the free Lagrangian without gauge symmetry, the problem is to find all the vertices such that the DoF number remains unchanged upon inclusion of interactions. The procedure is quite simple. At first, one has to bring the system of the free Lagrangian equations into the involutive form, by complementing them with all the lower order consequences. Given the involutive closure, the DoF number is fixed. The second step to consistent inclusion of interactions is to *simultaneously* deform the free action S *and* the

[1] This also has a counterpart in the constrained Hamiltonian formalism. Given the Lagrangian with the first order derivatives, one can construct the Hamiltonian formalism as if the acceleration were included. Then, the phase space would include auxiliary coordinates, absorbing velocities, and also extra momenta. These extra variables are suppressed by the second class constraints. In principle, these constraints can be converted into the first class, by usual conversion procedure. This conversion procedure, at Lagrangian level, would correspond to the involutively closed system constructed by inclusion of the higher order consequences of original Lagrangian equations.

generators Γ^i_α of the consequences,

$$S(\phi)=\frac{1}{2}M_{ij}\phi^i\phi^j \;\mapsto\; \mathsf{S}_{int}=\sum_{k=0}\overset{(k)}{\mathsf{S}}(\phi),$$
$$\overset{(k)}{\mathsf{S}}=\frac{1}{k+2}M_{i_1\dots i_{k+2}}\phi^{i_1}\dots\phi^{i_{1+2}}\,; \tag{5}$$
$$\Gamma^i_\alpha \;\mapsto\; \mathsf{G}^i_\alpha(\phi)=\sum_{k=0}\overset{(k)}{\mathsf{G}}{}^i_\alpha, \qquad \overset{(k)}{\mathsf{G}}{}^i_\alpha=G^i_{\alpha\, j_1\dots j_k}\phi^{j_1}\dots\phi^{j_k}, \tag{6}$$

where the vertices $M_{i_1\dots i_{k+2}}$, $G^i_{\alpha\, j_1\dots j_k}$ are field-independent poly-differential operators. The first operator M_{ij} is the same as in the free theory, and $\overset{(0)}{\mathsf{G}}{}^i_\alpha$ coincides with the generator of the consequences included into the involutive closure at the free level. Given the vertices in the action (5) and the generators of consequences (6), one gets the deformation of the consequences included into the involutive closure (3):

$$\tau_\alpha=-\Gamma^i_\alpha\partial_i S \;\mapsto\; -\mathsf{G}^i_\alpha\partial_i\mathsf{S}_{int}=\mathsf{T}_\alpha=\sum_{k=0}\overset{(k)}{\mathsf{T}}_\alpha,$$
$$\overset{(k)}{\mathsf{T}}_\alpha=\mathsf{T}_{\alpha i_1\dots i_{k+1}}\phi^{i_1}\dots\phi^{i_{k+1}}\,; \tag{7}$$
$$\mathsf{T}_{\alpha i_1\dots i_{k+1}}=\sum_{l=0}^{k+1}G^i_{\alpha(j_1\dots j_l}M_{j_{l+1}\dots j_{k+1})i}. \tag{8}$$

Here, the round brackets mean symmetrization of corresponding indices. Upon inclusion of interaction, the deformed action (5) and consequences (7) by construction obey the gauge identity:

$$\mathsf{G}^i_\alpha\partial_i\mathsf{S}_{int}+\mathsf{T}_\alpha\equiv 0. \tag{9}$$

One can see once again that from the perspective of algebraic consistency, any interaction is admissible for the field equations without gauge symmetry. The consistency of interactions is provided not just by algebraic consistency but also the DoF number should remain unchanged upon deformation. This condition can be easily controlled making use of the involutive form of field equations. The vertices (5), (6) will be consistent if the following two conditions are met: (i) the system

$$\partial_i\mathsf{S}_{int}=0, \qquad \mathsf{T}_\alpha=0 \tag{10}$$

remains involutive; (ii) the DoF number (65) for Eq. (10) remains the same as it is for the free system. Upon inclusion of interaction, the orders of Lagrangian equations, generators, and consequences, being ingredients of the DoF number count, can increase, or remain unchanged. If they do not increase, the DoF number obviously remains unchanged. These three orders can increase, however, in the correlated way without changing the DoF number, once relation (65) still holds true. Therefore, the interaction can be consistent, in principle, even if the higher derivatives are involved.

Below in this section, the iterative procedure is described for inclusion of Stueckelberg fields. Under this procedure, the Stueckelberg field ξ^α is assigned to every consequence $\tau_\alpha(\phi)$ included into the involutive closure (3) of original Lagrangian equations. The Stueckelberg action is sought for as a power series in the fields ξ^α:

$$\mathcal{S}_{St}(\phi,\xi)=\sum_{k=0}^{\infty}\mathcal{S}_k\,,$$
$$\mathcal{S}_k(\phi,\xi)=W_{\alpha_1\dots\alpha_k}(\phi)\,\xi^{\alpha_1}\cdots\xi^{\alpha_k}, \quad k>0, \tag{11}$$

where $\mathcal{S}_0(\phi)$ is the original action $S(\phi)$, and the first expansion coefficient W_α is defined by τ_α (2):

$$W_\alpha(\phi)=\frac{\partial\mathcal{S}_{St}(\phi,\xi)}{\partial\xi^\alpha}\Big|_{\xi=0}=\tau_\alpha. \tag{12}$$

Hence, at $\xi=0$, the field equations for the Stueckelberg action reproduce the involutive closure (3) of the original Lagrangian equations.

The equivalence of the Stueckelberg theory to the original one is provided by the gauge symmetry of the action (11) such that the fields ξ^α can be gauged out, with $\xi^\alpha=0$ being admissible gauge fixing condition. This means, the number of gauge parameters should coincide with the number of consequences τ_α (2) included into involutive closure of the original Lagrangian system. The gauge transformations are iteratively sought for order by order of the Stueckelberg fields

$$\delta_\epsilon\phi^i=R^i_\alpha(\phi,\xi)\epsilon^\alpha, \quad \delta_\epsilon\xi^\gamma=R^\gamma_\alpha(\phi,\xi)\epsilon^\alpha,$$
$$R^i_\alpha(\phi,\xi)=\sum_{k=0}\overset{(k)}{R}{}^i_\alpha, \quad R^\gamma_\alpha(\phi,\xi)=\sum_{k=0}\overset{(k)}{R}{}^\gamma_\alpha, \tag{13}$$
$$\overset{(k)}{R}{}^i_\alpha(\phi,\xi)=R^i_{\alpha\beta_1\dots\beta_k}(\phi)\xi^{\beta_1}\dots\xi^{\beta_k},$$
$$\overset{(k)}{R}{}^\gamma_\alpha(\phi,\xi)=R^\gamma_{\alpha\beta_1\dots\beta_k}(\phi)\xi^{\beta_1}\dots\xi^{\beta_k}. \tag{14}$$

The Stueckelberg action (11) is supposed to be invariant with respect to the gauge transformations (13). The gauge symmetry (13) of the action is equivalent to the Noether identities

$$\delta_\epsilon\mathcal{S}_{St}=0, \quad \forall\epsilon^\alpha \quad \Leftrightarrow \quad R^i_\alpha\partial_i\mathcal{S}_{St}+R^\gamma_\alpha\frac{\partial\mathcal{S}_{St}}{\partial\xi^\gamma}\equiv 0. \tag{15}$$

These identities can be expanded in Stueckelberg fields,

$$\left(R^\gamma_\alpha(\phi,\xi)\frac{\partial}{\partial\xi^\gamma}+R^i_\alpha(\phi,\xi)\frac{\partial}{\partial\phi^i}\right)\mathcal{S}_{St}$$
$$\equiv\sum_{k=0}\sum_{m=0}^{k}\left(\overset{(k-m)}{R}{}^\gamma_\alpha\frac{\partial\mathcal{S}_{(m+1)}}{\partial\xi^\gamma}+\overset{(k-m)}{R}{}^i_\alpha\frac{\partial\mathcal{S}_{(m)}}{\partial\phi^i}\right)\equiv 0. \tag{16}$$

Once the Noether identities are valid for every order in ξ, each term in the sum over k should vanish separately. In this way, the requirement of the gauge symmetry results in the sequence of relations

$$\sum_{m=0}^{k}\left(\overset{(k-m)}{R}{}^{\gamma}_{\alpha}\frac{\partial\mathcal{S}_{(m+1)}}{\partial\xi^{\gamma}}+\overset{(k-m)}{R}{}^{i}_{\alpha}\frac{\partial\mathcal{S}_{(m)}}{\partial\phi^{i}}\right)\equiv 0, \tag{17}$$

k=0,1, 2,.... For $k = 0$, given the boundary condition (12), the above relations reduce to the identities between consequences τ and Lagrangian equations,

$$\overset{(0)}{R}{}^{\gamma}_{\alpha}\tau_{\gamma}+\overset{(0)}{R}{}^{i}_{\alpha}\partial_i S\equiv 0. \tag{18}$$

Any identity between τ_α and $\partial_i S$ reduces[2] to the linear combination of the identities in the closure of the original system (4). This means, the identity (18) reads

$$\overset{(0)}{R}{}^{\gamma}_{\alpha}\left(\tau_{\gamma}+\Gamma^{i}_{\gamma}\partial_i S\right)\equiv 0, \tag{19}$$

where $\overset{(0)}{R}{}^{\gamma}_{\alpha}$ can be any non-degenerate matrix. Below, we stick to the simplest choice

$$\overset{(0)}{R}{}^{\gamma}_{\alpha}=\delta^{\gamma}_{\alpha}. \tag{20}$$

This choice does not restrict the generality for two reasons. First, as demonstrated below, it admits consistent inclusion of Stueckelberg fields for any action and any generating set of the consequences included into the involutive closure of original Lagrangian system (3). Second, any other choice of non-degenerate $\overset{(0)}{R}{}^{\gamma}_{\alpha}$ can be absorbed by the change of the gauge parameters ϵ^α.

Given relation (19), the choice (20) defines zero order of the Stueckelber gauge symmetry (13) for the original fields:

$$\overset{(0)}{R}{}^{i}_{\alpha}=\Gamma^{i}_{\alpha}. \tag{21}$$

Given zero order of the expansion for the gauge transformations (13) in ξ, and zero and first order in the Stuekelberg action (11),

$$\begin{aligned}\mathcal{S}_{St}(\phi,\xi)&=S(\phi)+\tau_\alpha(\phi)\xi^\alpha+\ldots,\\ &\delta_\epsilon\phi^i=\Gamma^i_\alpha(\phi)\epsilon^\alpha+\ldots,\ \delta_\epsilon\xi^\alpha=\epsilon^\alpha+\ldots,\end{aligned} \tag{22}$$

all the higher orders are iteratively defined by relations (17), both for the action and for the gauge transformations. The procedure of resolving relation (17) with certain k for $\mathcal{S}_{(k+1)}$ and $\overset{(k)}{R}_{\alpha}$ is inductive. Relations (18), (20), (21) solve Eq. (17) for $k = 0$. Substituting this solution into (17) with $k = 1$, we get the equation for $\mathcal{S}_{(2)}$ and $\overset{(1)}{R}_{\alpha}$. This equation is labeled by index α, and it is linear in ξ^β. Once the relation has to be met for any ξ^β, the equation is a square matrix. The symmetric part of the matrix defines the structure coefficient $W_{\alpha\beta}$ of $\mathcal{S}_{(2)}$, while the anti-symmetric part defines the structure coefficients of $\overset{(1)}{R}_{\alpha}$. The solution involves certain ambiguity related to the fact that gauge generators are defined modulo on-shell vanishing contributions, and up to a linear combination. It is the ambiguity which is common for any gauge theory. Given $\overset{(l)}{R}_{\alpha}$, $\mathcal{S}_{(l+1)}$, $l = 0, 1, \ldots, k$, they all are substituted into Eq. (17) of the order $k + 1$ that defines $\overset{(k+1)}{R}_{\alpha}$ and $\mathcal{S}_{(k+2)}$. This iterative procedure is unobstructed at any stage. This is seen from the algebraic consequences of the gauge identity studied in the next section. The formal proof of consistency of the procedure for inclusion the Stueckelberg fields is provided in Sect. 4 by the HPT tools.

3 Gauge algebra of the involutive closure of Lagrangian system

In this article, the class of field theories is considered such that the action does not admit gauge symmetry[3], while the Lagrangian equations are not involutive. The involutive closure (3) is non-Lagrangian as such, though it is equivalent to the original Lagrangian system (1). Since the original equations (1) are complemented by their consequences τ_α (2), the extended system (3) admits gauge identities (4). These identities are unrelated to any gauge symmetry. This is possible because the involutively closed system (3) is non-Lagrangian, so the second Noether theorem does not apply. The procedure of inclusion Stueckelberg fields described in the previous section is intended to convert the identities (4) into Noetherian ones by introducing the new fields ξ and gauge symmetry such that the generators of the identities are converted into gauge generators (22).

The gauge identities (4) turn out having a sequence of consequences of their own. The gauge identities and their consequences are understood as the gauge algebra of the involutive closure (3). It is the algebra which is behind existence of the solution to the conversion equations (17) in every order. This algebra is considered in this section.

Let us begin with the remarks concerning equivalence relations for the generators of consequences Γ^i_α (2). First, notice that the original action $S(\phi)$ is assumed having no gauge symmetry. Hence, if an identity occurs between

[2] Accurate formulation of completeness of the identities (4) is provided in the next section, see in particular relations (26), (27). Here we proceed from the intuitive understanding that the consequences τ, being defined as *independent* linear combinations of the equations by relations (2), cannot admit any other dependency with $\partial_i S$ besides the one following from the definition.

[3] Inclusion of Stueckelberg fields in the models enjoying another gauge symmetry can be considered along the same lines, though the procedure would need some adjustments.

Lagrangian equations, the identity generator is trivial,

$$\kappa^i \partial_i S \equiv 0 \quad \Leftrightarrow \quad \exists E^{ij} = -E^{ji} : \kappa^i = E^{ij} \partial_j S. \tag{23}$$

Since the consequences τ_α (2) are supposed independent, any identity between them is trivial in the similar sense:

$$\kappa^\alpha \tau_\alpha \equiv 0 \quad \Leftrightarrow \quad \exists E^{\alpha\beta} = -E^{\beta\alpha} : \kappa^\alpha = E^{\alpha\beta} \tau_\beta. \tag{24}$$

The generators of the involutive closure Γ^i_α (2) are considered equivalent if they result in the same consequences τ_α. Hence, the difference between equivalent generators Γ^i_α and Γ'^i_α is a trivial generator of the identity between the original equations:

$$\begin{aligned} &\Gamma^i_\alpha(\phi)\partial_i S(\phi) - \Gamma'^i_\alpha(\phi)\partial_i S(\phi) \equiv 0 \\ &\quad \Leftrightarrow \quad \Gamma^i_\alpha - \Gamma'^i_\alpha = E^{ij}_\alpha \partial_j S, \quad E^{ij}_\alpha = -E^{ji}_\alpha. \end{aligned} \tag{25}$$

Once the consequences τ_α (2) are independent, any set of identities (we label the set elements by the index A) among the involutive equations (3) is spanned by the identities (4):

$$\begin{aligned} &\Lambda^i_A \partial_i S + \Lambda^\alpha_A \tau_\alpha \equiv 0 \quad \Leftrightarrow \quad \exists U^\alpha_A : \\ &\Lambda^i_A \partial_i S + \Lambda^\alpha_A \tau_\alpha \equiv U^\alpha_A \left(\Gamma^i_\alpha \partial_i S + \tau_\alpha \right). \end{aligned} \tag{26}$$

The expansion coefficients U^α_A define the generators of the identities Λ_A modulo natural ambiguity,

$$\begin{aligned} &\Lambda^i_A = U^\alpha_A \Gamma^i_\alpha + E^{ij}_A \partial_j S + E^{i\alpha}_A \tau_\alpha, \\ &\Lambda^\alpha_A = \quad U^\alpha_A - E^{i\alpha}_A \partial_i S + E^{\alpha\beta}_A \tau_\beta, \; E^{ij}_A = -E^{ji}_A, \; E^{\alpha\beta}_A = -E^{\beta\alpha}_A, \end{aligned} \tag{27}$$

with E_A being arbitrary. All the relations above are valid, in principle, for any regular system of field equations admitting irreducible generating set for gauge identities. These relations do not imply that the Eq. (3) follow from Lagrangian equations (1).

Now, let us exploit the fact that the original field equations are Lagrangian to deduce the consequences of the gauge identities (4). Consider the action of variational vector field $\Gamma_\alpha = \Gamma^j_\alpha \partial_j$ onto the consequence τ_β (2). On shell, this amounts to the second variation of the action S along the field Γ_α. Since the variational derivatives commute, the second variations of the action along variational vector fields commute on shell

$$\Gamma^i_\alpha(\phi)\partial_i \tau_\beta(\phi) \approx \Gamma^i_\alpha \Gamma^j_\beta \partial^2_{ij} S = \Gamma^i_\beta \Gamma^j_\alpha \partial^2_{ij} S. \tag{28}$$

Once the matrix $\Gamma^i_\alpha(\phi)\partial_i \tau_\beta$ is symmetric on shell, off shell the symmetry can be broken by the contributions proportional to τ_α and $\partial_i S$:

$$\Gamma^i_\alpha(\phi)\partial_i \tau_\beta = W_{\alpha\beta} + R^i_{\alpha\beta} \partial_i S + R^\gamma_{\alpha\beta} \tau_\gamma, \quad W_{\alpha\beta} = W_{\beta\alpha}. \tag{29}$$

The matrix $W_{\alpha\beta}$ is off shell symmetric. On shell, $W_{\alpha\beta} \approx \Gamma^i_\alpha \Gamma^j_\beta \partial^2_{ij} S$. The structure coefficients $R^\gamma_{\alpha\beta}$, $R^i_{\alpha\beta}$ do not have certain symmetry w.r.t. the lower labels. Consider antisymmetric part of relations (29), and use the definition of τ_α (2)

$$\left(\Gamma^i_{[\alpha}(\phi)\partial_i \Gamma^j_{\beta]} - R^j_{[\alpha\beta]} \right) \partial_j S - R^\gamma_{[\alpha\beta]} \tau_\gamma \equiv 0. \tag{30}$$

The relation above is the identity between the original Lagrangian equations $\partial_i S$ and their consequences τ_α. Any identity of this type reduces to the basic identity (4) according to relation (26). The coefficients in this identity are connected with each other by relation (27). Applying (27) to the specific identity (30) we arrive at the following relations defining the set of structure functions $R^i_{\alpha\beta}$, $R^\gamma_{\alpha\beta}$ involved in (30) in terms of a single independent structure coefficient $U^\gamma_{\alpha\beta}$ and arbitrary structure functions E:

$$\begin{aligned} &\Gamma^j_\alpha \partial_j \Gamma^i_\beta - \Gamma^j_\beta \partial_j \Gamma^i_\alpha - U^\gamma_{\alpha\beta} \Gamma^i_\gamma - R^i_{\alpha\beta} + R^i_{\beta\alpha} \\ &\quad - E^{ji}_{\alpha\beta} \partial_j S - E^{i\gamma}_{\alpha\beta} \tau_\gamma = 0; \end{aligned} \tag{31}$$

$$U^\mu_{\alpha\beta} - R^\mu_{\alpha\beta} + R^\mu_{\beta\alpha} + E^{j\mu}_{\alpha\beta} \partial_j S - E^{\mu\nu}_{\alpha\beta} \tau_\nu = 0. \tag{32}$$

Let us briefly comment on relations (31), (32). The first of them demonstrates that the generators of the consequences Γ^i_α do not necessarily form an on-shell integrable distribution, unlike the generators of gauge symmetry. These generators commute on shell to their linear combinations with the structure coefficients $U^\gamma_{\alpha\beta}$, while $R^i_{\alpha\beta}$ describes deviation from integrability of the distribution. Both $U^\gamma_{\alpha\beta}$ and $R^i_{\alpha\beta}$ are defined modulo natural off-shell ambiguity, given the original generators Γ^i_α. Relations (32) identify the anti-symmetric part of structure function $R^\gamma_{\alpha\beta}$ involved in the relation (28) as the involution coefficient $U^\gamma_{\alpha\beta}$ in commutation relations of generators (31). Symmetric parts of $R^\gamma_{\alpha\beta}$, $R^i_{\alpha\beta}$ can be absorbed by on-shell vanishing part of symmetric structure function $W_{\alpha\beta}$ defined by relation (29).

Notice that relations (29), (31), (32) are the immediate consequences of the identity (4). They follow from the fact that the set of identities (4) is complete and irreducible. In their turn, the identities (4) follow from the definition of the functions τ as independent linear combinations (2) of the l.h.s. of Lagrangian equations (1). Once the original equations are consistent, all their consequences, including relations (29), (31), (32) cannot be inconsistent.

Once the structure coefficients $R^\gamma_{\alpha\beta}$, $R^i_{\alpha\beta}$, $W_{\alpha\beta}$ are found from relations (29), (31), (32) modulo natural ambiguity, they define the first order of the Stueckelberg gauge symmetry generators and the second order of Stueckelberg action,

$$\overset{(1)}{R}{}^i_\alpha = R^i_{\alpha\beta}\xi^\beta, \quad \overset{(1)}{R}{}^\gamma_\alpha = R^\gamma_{\alpha\beta}\xi^\beta, \quad \mathcal{S}_{(2)} = \frac{1}{2} W_{\alpha\beta}\xi^\alpha\xi^\beta. \tag{33}$$

Notice that if the generators Γ^i_α of consequences included into involutive closure of Lagrangian equations form integrable distribution, $\overset{(1)}{R}{}^i_\alpha$ will vanish, as $R^i_{\alpha\beta} = 0$. In this case, the Stueckelberg symmetry does not mix up the original fields with the Stueckelberg ones, much like it happens in the "Stueckelberg trick". If the distribution generated by Γ is not integrable, i.e. $R^i_{\alpha\beta} \neq 0$, the deviation from integrability is included into the generator of Stueckelberg symmetry at the first order in ξ, compensating non-commutativity of zero order term. As demonstrated in the next section, the iterative procedure consistently continues in the higher orders, and it inevitably results in the gauge transformations with on-shell integrable distribution.

Concluding this section, let us mention that relations (29), (31), (32) can have further consequences involving higher structure functions. These higher structures contribute to the higher orders in the Stueckelberg action and gauge generators. All these higher relations should be also consistent as the original Lagrangian equations are supposed having no contradictions, and hence any inconsistency is impossible in their consequences.

4 BV master equation for the Stueckelberg gauge symmetry

In this section, the BV formalism is rearranged to serve as a tool for consistent inclusion of Stueckelberg fields.

If the Stueckelberg action $\mathcal{S}_{St}(\phi, \xi)$ (11) and the corresponding gauge generators $R^i_\alpha(\phi, \xi)$, $R^\beta_\alpha(\phi, \xi)$ (13), (15) would have been known from the outset, it could be considered as the usual Lagrangian gauge theory without any distinction between Stueckelberg fields ξ^α and original fields ϕ^i. Then, the master action can be constructed for the gauge system along the usual lines of the BV formalism [20,21]. The ghosts C^α are assigned to all the gauge parameters ϵ^α, and the anti-fields are introduced for all fields, including ghosts. The usual Grassmann parity and ghost number gradings are imposed on the fields and anti-fields[4]:

$$
\begin{aligned}
&\epsilon(\phi^i) = \epsilon(\xi^\alpha) = 0, \quad \epsilon(C^\alpha) = 1, \quad gh(\phi^i) = gh(\xi^\alpha) = 0, \\
&\qquad gh(C^\alpha) = 1; \\
&\epsilon(\phi^*_i) = \epsilon(\xi^*_\alpha) = 1, \quad \epsilon(C^*_\alpha) = 0, \quad gh(\phi^*_i) = gh(\xi^*_\alpha) = -1, \\
&\qquad gh(C^*_\alpha) = -2. \qquad (34)
\end{aligned}
$$

The BV action is sought for as a solution to the master equation

$$
(S_{BV}, S_{BV}) = 0, \qquad gh(S_{BV}) = 0, \quad \epsilon(S) = 0, \qquad (35)
$$

where $(\cdot, \cdot)$ is the anti-bracket

$$
\begin{aligned}
&(A, B) = \frac{\partial^R A}{\partial \varphi^I} \frac{\partial^L B}{\partial \varphi^*_I} - \frac{\partial^R A}{\partial \varphi^*_I} \frac{\partial^L B}{\partial \varphi^I}, \\
&\quad \varphi^I = (\phi^i, \xi^\alpha, C^\alpha), \qquad \varphi^*_I = (\phi^*_i, \xi^*_\alpha, C^*_\alpha). \qquad (36)
\end{aligned}
$$

To solve the master equation, the usual setup of the BV formalism for irreducible gauge systems would imply imposing the boundary condition on the action,

$$
S_{BV}(\varphi, \varphi^*) = \mathcal{S}_{St}(\phi, \xi) + C^\alpha \left(R^i_\alpha(\phi, \xi)\phi^*_i + R^\gamma_\alpha(\phi, \xi)\xi^*_\gamma \right) + \dots . \qquad (37)
$$

Here, the first term is the gauge invariant action, and the second one includes generators of the gauge symmetry of the action multiplied by corresponding ghosts C^α and anti-fields ϕ^*_i, ξ^*_γ. This term has anti-ghost degree 1. The dots stand for the terms of higher anti-ghost degrees. The anti-ghost degree is imposed in usual way [22]:

$$
\begin{aligned}
&agh(\phi^*_i) = agh(\xi^\alpha) = 1, \quad agh(C^*_\alpha) = 2; \\
&agh(\phi^i) = agh(\xi^\alpha) = agh(C^\alpha) = 0. \qquad (38)
\end{aligned}
$$

Given the boundary condition (37), the higher terms can be iteratively found from the master equation (35) by expansion with respect to the anti-ghost degree. The unique existence of the solution can be proven by the usual HPT tools [21,22].

From the perspective of including Stueckelberg fields in the BV formalism, the boundary condition (37) is unsuitable, because neither the Stueckelberg action $\mathcal{S}_{St}$ is known from the outset, nor are the gauge generators R^β_α, R^i_α. The action is known up to the first order in Stueckelberg fields (12), while the gauge generators are known only at $\xi^\alpha = 0$, see (20), (21). Hence, the known part of the boundary condition (37) reads

$$
S_{BV}(\varphi, \varphi*) = S(\phi) - \tau_\alpha(\phi)\xi^\alpha + C^\alpha \Gamma^i_\alpha(\phi)\phi^*_i + C^\alpha \xi^*_\alpha + \dots, \qquad (39)
$$

where the dots stand for the higher orders of anti-fields *and* Stueckelberg fields. So, to find the solution for the master equation (35) of the Stueckelberg theory one has to proceed from the boundary condition (39) iterating the solution order by order w.r.t. anti-fields and Stueckelberg fields. This means, another resolution degree has to be imposed instead of the anti-ghost number (38) such that would be nonzero for Stueckelberg fields. The boundary condition (39) should be, at maximum, of the first order in the resolution degree, so ξ has to be assigned the weight 1. So, we impose the following resolution degree:

$$
\begin{aligned}
&deg(\xi^\alpha) = deg(\xi^*_\alpha) = deg(\phi^*_i) = 1, \\
&deg(C^*_\alpha) = 2, \quad deg(C^\alpha) = deg(\Phi^i) = 0. \qquad (40)
\end{aligned}
$$

The solution to the master equation (35) is sough for as the expansion of the action $S_{BV}(\varphi, \varphi^*)$ w.r.t. the resolution

[4] For simplicity, we consider the case with even gauge symmetries and even original fields. Adjustments are made to the odd case by inserting known sign factors.

degree,

$$S_{BV}(\varphi,\varphi^*)=\sum_{k=0}\overset{(k)}{S},\qquad deg\overset{(k)}{S}=k. \tag{41}$$

Once the solution is found in all the orders resolution degree, the complete Stueckelberg action is extracted as zero order w.r.t. to the anti-ghost number (i.e. with switched off anti-fields), while the Stueckelberg gauge generators are defined by the first order of S_{BV} w.r.t. the anti-ghost degree (i.e. as the coefficients at ξ^*_γ and ϕ^*_i).

Consider the master action up to the next order of the resolution degree after the boundary condition (39),

$$\begin{aligned}S_{BV}(\varphi,\varphi^*)=\,&S(\phi)-\tau_\alpha\xi^\alpha+C^\alpha\left(\Gamma^i_\alpha(\phi)\phi^*_i+\xi^*_\alpha\right)\\&+\frac{1}{2}W_{\alpha\beta}\xi^\alpha\xi^\beta+C^\alpha\left(R^\gamma_{\alpha\beta}\xi^\beta\xi^*_\gamma+R^i_{\alpha\beta}\xi^\beta\phi^*_i\right)\\&+\frac{1}{2}C^\beta C^\alpha\left(U^\gamma_{\alpha\beta}C^*_\gamma+\phi^*_j\phi^*_iE^{ij}_{\alpha\beta}+\xi^*_\mu\phi^*_iE^{\mu i}_{\alpha\beta}\right.\\&\left.+\xi^*_\mu\xi^*_\nu E^{\mu\nu}_{\alpha\beta}\right)+\dots,\end{aligned} \tag{42}$$

where all the structure coefficients are supposed to be functions of the original fields. The first line in this expression is the boundary condition (39) defined by the original action, and by the generators Γ^i_α of the consequences τ_α included in the involutive closure of the system. The next lines include the most general expression with the resolution degree 2, and ghost number 0. The structure coefficient involved in $\overset{(2)}{S}$ are defined by the master equation. Let us expand the l.h.s. of the master equation w.r.t. the resolution degree up to the first order. Notice that $\overset{(k)}{S}$, $k>2$ cannot contribute to zero and first orders of the expansion, so (42) is sufficient at this level

$$(S_{BV},S_{BV})_0=2(\Gamma^i_\alpha\partial_iS+\tau_\alpha)C^\alpha\equiv0, \tag{43}$$

$$\begin{aligned}(S_{BV},S_{BV})_1=\,&2\xi^\gamma(\Gamma^i_\alpha\partial_i\tau_\gamma-R^i_{\alpha\gamma}\partial_iS-R^\beta_{\alpha\gamma}\tau_\beta-W_{\gamma\alpha})C^\alpha\\&-C^\alpha C^\beta\big(\phi^*_i(\Gamma^j_\alpha\partial_j\Gamma^i_\beta-\Gamma^j_\beta\partial_j\Gamma^i_\alpha\\&-U^\gamma_{\alpha\beta}\Gamma^i_\gamma-R^i_{\alpha\beta}+R^i_{\beta\alpha}-E^{ji}_{\alpha\beta}\partial_jS-E^{i\gamma}_{\alpha\beta}\tau_\gamma)\\&-\xi^*_\mu(U^\mu_{\alpha\beta}-R^\mu_{\alpha\beta}+R^\mu_{\beta\alpha}+E^{j\mu}_{\alpha\beta}\partial_jS\\&-E^{\mu\nu}_{\alpha\beta}\tau_\nu)\big)=0.\end{aligned} \tag{44}$$

Relation (43) is valid, given the gauge identity (4). The first order of the master equation (44) holds by virtue of identities (29), (31), (32) upon identification of the structure coefficients in the expansion (42) with corresponding structure functions in the mentioned identities.

As we have seen, the solution to (35) exists up to the second order w.r.t. resolution degree (40). Let us consider the general order k. Substitute the expansion (41) into the master equation and take k-th order. It has the following structure:

$$(S_{BV},S_{BV})_k=\delta\overset{(k+1)}{S}+B_k(S,\overset{(1)}{S},\dots,\overset{(k)}{S}), \tag{45}$$

where B_k involves only $\overset{(l)}{S}$, $l\le k$, and the operator δ reads:

$$\begin{aligned}\delta O=\,&-\frac{\partial^RO}{\partial\phi^*_i}\partial_iS-\frac{\partial^RO}{\partial\xi^*_\alpha}\tau_\alpha+\frac{\partial^RO}{\partial C^*_\alpha}\left(\phi^*_i\Gamma^i_\alpha+\xi^*_\alpha\right)\\&+\frac{\partial^RO}{\partial\xi^\alpha}C^\alpha.\end{aligned} \tag{46}$$

By virtue of identity (4), the operator δ squares to zero,

$$\delta^2O=\frac{\partial^RO}{\partial C^*_\alpha}\left(\Gamma^i_\alpha\partial_iS+\tau_\alpha\right)\equiv0, \tag{47}$$

so it is a differential. Obviously, δ decreases the resolution degree by one,

$$deg(\delta)=-1. \tag{48}$$

Notice that δ is acyclic in the strictly positive resolution degree because the identities (3) are independent, i.e.

$$\delta X=0,\ deg(X)>0\quad\Leftrightarrow\quad\exists Y:\ X=\delta Y. \tag{49}$$

By Jacobi identity $(S,(S,S))\equiv0$, $\forall S$. Expanding the identity w.r.t. the resolution degree, one can see that B_k of relation (45) is δ-closed,

$$\delta B_k=0,\quad k>0. \tag{50}$$

Then, because of (49), B_k is δ-exact,

$$\exists Y_{k+1}:\ \ B_k=\delta Y_{k+1},\quad deg(Y_{k+1})=k+1. \tag{51}$$

Substituting (51) into (45) we arrive at the relation

$$\delta\left(\overset{(k+1)}{S}+Y_{k+1}\right)=0. \tag{52}$$

This provides solution for $\overset{(k+1)}{S}$,

$$\overset{(k+1)}{S}=-Y_{k+1}+\delta Z_{k+2},\qquad deg(Z_{k+2})=k+2. \tag{53}$$

The solution is unique modulo natural δ-exact ambiguity.

In this way, one can iteratively find the master action of the Stueckelberg theory, given the original action, generators Γ^i_α of consequences τ_α (2) included into the involutive closure (3) of Lagrangian system. The solution is unobstructed at any order. Once the master action is found, its zero order w.r.t. the anti-ghost number defines complete Stueckelberg action, while the Stueckelberg gauge generators are defined by the first order of agh, in accordance with (37).

In the end of this section, notice some similarity between the BV formalism for the Stueckelberg embedding described above, and the BV formalism for the field theories with unfree gauge symmetry[5] [23]. In the latter case, the BV formalism

[5] By unfree gauge symmetry, we mean the case when the gauge parameters are constrained by differential equations. The most known example of the unfree gauge symmetry is the unimodular gravity where the gauge parameters are constrained by transversality equation.

[23] also involves the compensatory fields ξ with ghost number 0 and resolution degree 1. These fields compensate, in a sense, the constraints imposed on the gauge parameters making the effective gauge symmetry parameters unconstrained. These variables can be viewed, in a broad sense, as Stueckelberg fields. In the case of unfree gauge symmetry, however, there is no pairing between the gauge symmetry parameters and the compensator fields, unlike the case of Stueckelberg symmetry considered in this article.

5 Concluding remarks

Let us make concluding remarks and discuss further perspectives. At first, let us sum up the proposed scheme of including Stueckelberg fields.

Proposed inclusion of Stueckelberg fields proceeds from involutive closure of Lagrangian system (3) which includes the consequences (2) of the original Lagrangian equations. The involutive system (3) does not admit any lower order consequence, while the original Lagrangian equations are not necessarily involutive. The Hamiltonian analogue of the involutive closure is the completion of the original equations by secondary constraints, being zero order consequences of the original primary constraints and the first order Hamiltonian equations. The involutive closure of Lagrangian equations allows explicitly covariant degree of freedom count, see relations (61), (65) in the Appendix. The explicit control of the DoF number in the involutive closure allows one to consistently include interactions in the covariant form for the second class systems, see (5), (6), (7). This method has been first proposed in the article [11] (for applications of the method in specific models see, for example, [12–15]). The original proposal of [11] allows one to find all the consistent vertices in the field equations, including non-Lagrangian one, without distinctions between variational and non-variational interactions. In this articles we added a side remark which allows one to identify all the consistent Lagrangian interactions. In this article, however, the involutive closure is considered not for its own sake, but as a starting point for inclusion of Stueckelberg fields.

The proposed procedure implies to introduce the Stueckelberg field ξ^{α} for every consequence (2) added to the Lagrangian equations to form the involutive closure (3). The Stueckelberg action (11) and gauge symmetry generators (13) are sought for as the power series in Stueckelberg fields proceeding from the requirement of gauge symmetry. The boundary conditions (22) for the action and gauge generators are defined by the original action and generators Γ^{i}_{α} of consequences τ_{α} included into the involutive closure (3). Given the boundary conditions, relations (17), being the expansion coefficients of the Noether identities for Stueckelberg gauge symmetry, allow one to iteratively find order by order all the expansion coefficients of the action and gauge generators. This iterative procedure of inclusion Stueckelberg fields is unobstructed at every stage. Upon inclusion of Stueckelberg fields, the theory has the same DoF number (see in the Appendix), while the gauge fixing $\xi = 0$ is admissible. This means the equivalence of the Stueckelberg theory to the original one. It is worth to mention that existence of the Stueckelberg embedding of the original system does not impose any restriction on the generators Γ^{i}_{α} (2) besides independence of consequences τ_{α} included into involutive system (3). In particular, the generators of consequences Γ, being the leading terms in Stueckelberg gauge symmetry transformations of original fields (22), are not required to form the integrable distribution, even on shell. This contrasts to the logic of the most common form of "Stueckelberg trick" which implies that the Stueckelberg gauge transformations of the original fields begin with the gauge generators of gauge symmetry of a part of original action. Being the generators, they inevitably commute to each other, at least on shell. As is seen from this article, the integrability of the distribution generated by Γ's is unnecessary for existence of consistent Stueckelberg embedding. The algebraic structure behind the consistency of the proposed Stueckleberg embedding is described in Sect. 3. This algebra follows from the gauge identities (4) of the involutive closure (3) much like the gauge algebra of the gauge invariant system follows from the Noether identities. In the case of Noether identities of gauge invariant system, the structure relations of the gauge algebra are deduced proceeding from the three factors: (i) dependence of the equations, (ii) independence of the generators of identities, and (iii) the gauge symmetry of the equations. It is the third factor which does not apply to the gauge identity (4), while the first two ones do. Therefore, similar structure relations follow from the first two factors, while the integrability of the gauge distribution, being a consequence of the third factor is not required. The structure relations of the gauge algebra of the closure of Lagrangian system, being described in Sect. 3, are equivalent to the equations of Sect. 2 defining the Stueckelberg action and Stueckelberg gauge generators. Once the gauge algebra is consistent (as it is a consequence of the non-contradictory identities (4)), this explains why the Stueckelberg embedding is unobstructed at all the stages.

In this article, also BV formalism is proposed to perform the Stueckelberg embedding proceeding from the involutive closure of Lagrangian system (3). There are three key distinctions from the conventional BV-formalism [20,21]. The first difference is that the boundary conditions (39) imply to specify, besides the action, also the generators of consequences Γ^{i}_{α} and the consequences τ_{α} included into the involutive closure of Lagrangian equations (3). The second difference is the set of variables. Once the original action is non-gauge invariant, no ghost would be introduced under the usual BV procedure [20]. For inclusion of the Stueckelberg embedding, the ghosts are assigned to every generator of consequence Γ^i_α, and the Stueckelberg field ξ^α is introduced for every consequence τ_α (34). The anti-fields are introduced for all the fields, including ghosts and Stueckelberg fields. The third distinction is that the solution of the master equation is iterated w.r.t. a different resolution degree (40) which counts the orders both anti-fields and Stueckelberg fields. The solution of master equation exists at every order of the iterative procedure, no obstruction can arise. Once the master action is found, the anti-field independent part gives the Stueckelberg action, while the first order in the anti-fields gives the generators of Stueckelberg gauge symmetry. This gives another way to construct the Stueckelberg embedding of the original theory.

Concerning further perspectives, two directions can be mentioned for developing the proposed method. The first direction is related to the possibilities of applying this method to specific models of field theory. From this perspective the potential advantage of the method is that it allows one to control the degree of freedom number in an explicitly covariant fashion at every stage, be it inclusion of interactions, on Stueckelberg embedding. Among the models of current interest, various generalizations of gravity can be mentioned where the original Lagrangian equations are not involutive, and hence the second class constraints arise. On the other hand, these models do not offer any obvious hint for the "Stueckelberg trick" which would correspond to the conversion of the second class constraints at Hamiltonian level. In particular, this concerns generalizations of unimodular gravity [24,25] and "new massive gravity" in $3d$ [26–28]. The proposed method seems an appropriate tool to study dynamics in the models of this type. These models, being complex by themselves, are not suitable, however, as touchstones for the first article on the general method, so the applications will be studied elsewhere.

Another aspect of possible future developments concerns generalizations of the method as such. With this regard, the inclusion of the case of gauge invariant actions whose Lagrangian equations admit the lower order consequences. The extension of the procedure for inclusion Stueckelberg fields seems straightforward. One more issue concerns the case where the reducible set of consequences is included into the involutive closure (3) of Lagrangian system. Notice that the involutive closure is not unique, as the higher order consequences can be added without breaking involutivity, and also the generating set of consequences can be chosen over-complete. Therefore, the same action can admit different involutively closed extensions of Lagrangian equations. The set of Stueckelberg fields is defined by specific involutive closure of Lagrangian equations. That is why, the multiple choice is possible for the set of Stueckelberg fields for the same Lagrangian, including, in principle, reducible sets of gauge generators. One more potential aspect of interest in this story is related to gauge fixing. Hypothetically, there can be the case when the gauge fixing is admissible which kills the original fields while the physical degrees of freedom are described by the Stueckelberg fields. This would allow to construct dual formulations of the same dynamics connecting them by different schemes of inclusion Stueckelberg fields and different gauge fixing.

Finally, let us mention the issue of locality of the Stueckelberg embedding procedure proposed in this article. The existence of the embedding is proven in Sect. 4 in terms of condensed notation that could potentially hide the obstructions related to locality. There is, however, a non-rigorous reason to believe that the locality problem should not arise. Once the action and the generators of consequences (2) are of the finite order, the structure functions of the algebra (see Sect. 3), being consequences of the identities (4), should involve the finite number of the derivatives. The theory involves a natural invertible operator $W_{\alpha\beta}$ (29) which defines the kinetic term for Stueckelberg fields. The inverse can be non-local, but the embedding procedure does not need to invert W.

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Appendix

Degree of freedom count in involutive closure of Lagrangian systems

In this Appendix, the relations are provided for DoF number count in the involutively closed systems. First these relations have been obtained in the article [11] for general involutive field equations[6], not necessarily being the involutive closure of any Lagrangian system. Here, these relations are slightly re-arranged to be more convenient for making simplifications related to the systems being the involutive closure of Lagrangian equations (3).

In the Appendix, all the indices are by default understood as numerical labels, not condensed ones. Exceptions are specially reported in each case.

The degree of freedom number is counted in terms of orders of equations, gauge identities and gauge symmetries. Let us explain the definitions of these orders.

Consider a system of field equations

$$E_I(\phi, \partial\phi, \dots, \partial^{n_I}\phi) = 0, \tag{54}$$

where n_I is the maximal order of partial derivatives involved in the equation with the label I. The number n_I is considered as the order of equation n_I. The partial derivatives by all space-time coordinates are treated on the equal footing, and the order n_I accounts for the mixed derivatives. For example, the order of the equation $\frac{\partial^2\phi}{\partial x \partial t} = 0$ is 2.

A system of equations is considered involutive if it does not admit such consequences of a lower order that are not yet included in the original system. If the system is not involutive, it can be always complemented by the lower order consequences, to make it involutive. In principle, higher order consequences can be also added, to make the system involutive. Completion of the system by the consequences is understood as involutive closure of the system, if no new lower order consequences exist. The involutive closure has the same solutions as the original system. In this sense, the involutive closure is equivalent to the original system.

Let the Eq. (54) admit gauge identities

$$L^I_A E_I \equiv 0, \tag{55}$$

where L^I_A are the differential operators with the field depending coefficients,

$$L^I_A = \sum_{k=0}^{l^I_A} L^{I\,\mu_1\dots\mu_k}_A(\phi, \partial\phi, \partial^2\phi \dots)\partial_{\mu_1}\dots\partial_{\mu_k}, \qquad l^I_A \in \mathbb{N}. \tag{56}$$

Operator L^I_A is called the generator of gauge identity, and l^I_A is the order of the operator. The order i_A of the gauge identity (55) with the label A is defined by the following rule:

$$i_A = \max_{\{I\}}\left(l^I_A + n_I\right), \tag{57}$$

i.e. it is the maximal aggregate order of the identity generator and the equation it acts on.

[6] Certain regularity assumptions are made for deducing these relations, see in [11]. Here, the regularity issues are not addressed.

Suppose the involutive system (54) admits gauge symmetry transformations,

$$\delta_\epsilon \phi^i = R^i_\alpha \epsilon^\alpha, \qquad \delta_\epsilon E_I \approx 0, \quad \forall \epsilon^\alpha, \tag{58}$$

where the gauge generators R^i_α are the differential operators with the field-depending coefficients,

$$R^i_\alpha = \sum_{k=0}^{r^i_\alpha} R^{i\,\mu_1\dots\mu_k}_\alpha(\phi, \partial\phi, \dots)\,\partial_{\mu_1}\cdots\partial_{\mu_k}. \tag{59}$$

The gauge variation (58), by definition, leaves the mass shell invariant.

The order r_α of the gauge transformation with specific parameter ϵ^α is defined as the maximal number of derivatives acting on the parameter in the transformation of any field,

$$r_\alpha = \max_{\{i\}}\left(r^i_\alpha\right). \tag{60}$$

Given the involutive equations with the complete set of gauge identities and gauge symmetries, the DoF number is counted by the rule

$$N_{DoF} = \sum_I n_I - \sum_A i_A - \sum_\alpha r_\alpha. \tag{61}$$

So, the DoF is computed as follows: the total order of the identities and the total order of gauge symmetries are subtracted from the total order of the equations. Notice two interesting features of this counting recipe. First, it does not involve the number of fields. Second, zero order relations (of any sort, be it equation, identity, or gauge symmetry) do not contribute to the DoF number. This relation for N_{DoF} has been first deduced in the article [11] in a slightly different form. The recipe (61), being equivalent to the original one, is more convenient for simplifications accounting for specifics of involutive closure of Lagrangian systems.

Let us apply the DoF number counting recipe (61) first to the involutive closure of Lagrangian system (3). Denote as n_α the order of consequence τ_α (2), and let g^i_α be the order of the differential operator Γ^i_α generating the consequence, and n_i is the order of Lagrangian equation (1). By construction (2), the order of consequence τ_α cannot exceed the maximal aggregate order of the generator of consequence and original Lagrangian equations (1),

$$n_\alpha \le \max_{\{i\}}\left(g^i_\alpha + n_i\right). \tag{62}$$

If only lower-order consequences are included, then this is a strict inequality. With the consequences whose order is higher than the original equations, the equality is possible. The maximum of the l.h.s. inequality (62) is reached at certain i, which is denoted $\bar{i}$,

$$\max_{\{i\}}\left(g^i_\alpha + n_i\right) = g^{\bar{i}}_\alpha + n_{\bar{i}}. \tag{63}$$

The order of the corresponding operator $\Gamma^{\bar{i}}_{\alpha}$ is unique for given α. This order is denoted just g_α. In the identity (4), the coefficient at τ_α has zero order. Given the inequality (62), the order of identity (4), being defined by the rule (57), reads

$$i_\alpha = g_\alpha + n_{\bar{i}}. \tag{64}$$

As a result, the DoF number of any Lagrangian system without gauge symmetry[7] can be counted making use of the orders related to its involutive closure (3):

$$N_{DoF} = \sum_i n_i + \sum_\alpha (n_\alpha - i_\alpha) = \sum_{i\neq\bar{i}} n_i + \sum_\alpha (n_\alpha - g_\alpha). \tag{65}$$

The above formula allows one to count DoF number of any Lagrangian system in explicitly covariant way. In this form, it works for the case without gauge symmetry (second class constrained systems, from Hamiltonian perspective). To account for the gauge symmetry, one has to subtract the total order of gauge symmetry generators (see (61)) in the irreducible case. Further adjustments can be made for the reducible gauge symmetry.

Now, let us discuss the DoF number upon inclusion Stueckelberg fields. Consider the Stueckelberg action and gauge symmetry transformations,[8]

$$\begin{aligned} &\mathcal{S}_{St} = S(\phi) + \tau_\alpha \xi^\alpha + \frac{1}{2} W_{\alpha\beta}\xi^\alpha \xi^\beta + \cdots , \\ &\delta_\epsilon \xi^\alpha = \epsilon^\alpha + \cdots , \quad \delta_\epsilon \phi^i = \Gamma^i_\alpha \epsilon^\alpha + \cdots , \end{aligned} \tag{66}$$

where $\cdots$ stand for the higher orders in ξ. From the perspective of the DoF number counting, there are two main distinctions between the involutive closure (3) of the original Lagrangian system, and the Stueckelberg theory (66). First, the consequences τ_α are replaced by Lagrangian equations for ξ^α

$$\tau_\alpha(\phi) \quad \mapsto \quad \frac{\partial \mathcal{S}_{St}}{\partial \xi^\alpha} = \tau_\alpha + W_{\alpha\beta}\xi^\beta + \cdots , \tag{67}$$

where the indices are condensed. Denote $\bar{n}_\alpha$ the order of the equation for Stueckelberg field ξ^α. Notice that the operator $W_{\alpha\beta}$ results from the action of the operator Γ^i_α on τ_β. This means that the total order of the equations $\frac{\partial \mathcal{S}_{St}}{\partial \xi^\alpha}$ exceeds the total order of the equations τ_β by the total order of operators Γ, i.e.

$$\sum_\alpha \bar{n}_\alpha = \sum_\alpha (n_\alpha + g_\alpha) . \tag{68}$$

Hence, inclusion of Stueckelberg fields increases the positive contribution to the DoF number (65) by $\sum_\alpha g_\alpha$. The second relevant distinction of the Stueckelberg theory (66) from the involutive closure of original Lagrangian equations (3) is the gauge symmetry. The order of the gauge transformation is g_α, at least in the first approximation in Stueckelberg fields. Hence, the negative part of the DoF count changes to the same number $\sum_\alpha g_\alpha$. These two changes cancel each other, so the DoF number remains unchanged.

Below, we illustrate involutive closure, inclusion of Stueckelberg fields and DoF counting by two simple examples: mechanical toy model, and Proca Lagrangian.

Example 1. Mechanical toy model

As a toy example, consider $L = \frac{1}{2}\dot{x}^2$. The Lagrangian equation, $\ddot{x} = 0$, is involutive, as it does not admit any lower order differential consequence. However, one could add a higher differential consequence, being just a derivative of the equation. This extension is also involutive system, as no lower order consequences exist. The system (3) in this case reads

$$\frac{\delta S}{\delta x} = -\ddot{x} = 0, \qquad \tau = \dddot{x} = 0, \tag{69}$$

and the generator of the consequence Γ is just $\frac{d}{dt}$. The identity (4) between the Eq. (3) in this case reads

$$\tau + \Gamma \frac{\delta S}{\delta x} \equiv \dddot{x} - \dddot{x} \equiv 0, \tag{70}$$

Let us exemplify the DoF count formula (65) by the toy model (69). There is one Lagrangian equation of the second order $\ddot{x} = 0$, i.e. $n_i = 2$. There is one consequence of the third order $\dddot{x}$, i.e. $n_\alpha = 3$. There is one generator of consequence $\Gamma = \frac{d}{dt}$, it is of the first order, i.e. $g = 1$. Substituting these numbers into (65), one gets $N_{DoF} = 2 + 3 - (2 + 1) = 2$, as it should be. Now, given the involutive system of Lagrangian equation and its consequence (69), let us include the Stueckelberg fields following the procedure of Sect. 2. The operator W (29) in this case is just $\frac{d^4}{dt^4}$, so the Stueckelberg Lagrangian (modulo total derivative) and gauge symmetry transformations (66) read

$$L_{St} = \frac{1}{2}\left(\dot{x} + \ddot{\xi}\right)^2 , \qquad \delta_\epsilon \xi = \epsilon, \quad \delta_\epsilon x = -\dot{\epsilon}. \tag{71}$$

The degree of freedom count adds one to the positive part of the sum (65) as the third order consequence $\dddot{x} = 0$ is replaced by the fourth order equation for ξ. Simultaneously, 1 is added to the negative contribution, because the first order gauge symmetry is switched on. Obviously, the Noether identity for Lagrangian (71) at $\xi = 0$ reproduces the identity (70) of the system (69). The equivalence of the Stueckelberg Lagrangian to the original one is obvious, as the gauge fixing condition $\xi = 0$ is admissible, and the Stueckelberg equations reduce to (69) in this gauge.

Let us comment on this elementary example from the perspective of conversion method, for the Hamiltonian sec-

[7] If the Lagrangian system had gauge symmetry, one should additionally subtract the total order of gauge symmetry transformations, see (59), (60)

[8] Here, the labels are understood as condensed indices.

ond class constrained systems. The Lagrangian $L = \frac{1}{2}\dot{x}^2$ could be considered as including accelerations. Then, the Ostrogradski method should be applied. The velocity would become an auxiliary canonical coordinate, whose conjugate momentum should vanish due to the primary constraint. Conservation of the primary constraint results in the secondary constraint which connects the canonical momentum of the original coordinate with the velocity. The pair of primary and secondary constraints is of the second class. If they are converted into the first class, we arrive at gauge invariant Hamiltonian action. All the momenta can be excluded by the inverse Legendre transformation in this action, and we arrive at Lagrangian (71). This example demonstrates that the covariant procedure of inclusion Stueckelberg fields proposed in Sect. 2 directly corresponds to the Hamiltonian conversion of the second class systems. It also illustrates the fact that the method works well proceeding from any extension of Lagrangian system by the consequences, including the higher order ones, if the starting point is an involutively closed system.

Example 2. Proca action

Let us exemplify DoF count recipe (65) by the Proca equations for massive vector field,

$$\frac{\delta S_{Proca}}{\delta A_\mu} \equiv \left(\delta^\mu_\nu(\Box - m^2) - \partial^\mu\partial_\nu\right) A^\nu = 0. \qquad (72)$$

The Proca equations is a system of four equations of the second order, so $n_i = 2, i = 1, 2, 3, 4$. Therefore, the total order of Lagrangian equations is 8. The Proca equations are not involutive as they admit the first order consequence:

$$\tau = -\partial^\mu \frac{\delta S_{Proca}}{\delta A^\mu} \equiv m^2 \partial_\mu A^\mu. \qquad (73)$$

So, we have one first order consequence τ, to be added for the sake of involutive closure. This means, $n_\alpha = 1$. The generator $\Gamma^\mu = \partial^\mu$ of the single consequence is the operator of divergence. It has the order one, $g = 1$. No other equations and consequences are included in the involutive closure (72), (73) of Proca system. The gauge identity (4) for the Proca system reads,

$$\partial^\mu \frac{\delta S_{Proca}}{\delta A^\mu} + \tau \equiv 0\,. \qquad (74)$$

The identity is of the order 3, as the first order operator ∂ acts on the second order Proca equations. Now, let us calculate the DoF number applying relation (65). The total order of the equations in the involutive closure of the Proca system is 9: there are 4 second order Proca equations (72), plus one first order consequence, $\sum_i n_i + \sum_\alpha n_\alpha = 4 \times 2 + 1 = 9$. There is also one third order gauge identity (74), $\sum_\alpha i_\alpha = 3$. Substituting these numbers into the formula (65), we get $9 - 3 = 6$. So, it is 6 DoF's by the phase space count, that corresponds to 3 by configuration space count, as it should be for the massive spin 1 in $d = 4$.

Given the involutive closure of the Proca equations, let us introduce the Stueckelberg field following the prescription of Sect. 2. The matrix W (29) in this case is the d'Alembertian, and no higher order corrections appear. So, we arrive to the standard Stueckelberg equations, gauge symmetries, and Noether identities:

$$\frac{\delta\mathcal{S}}{\delta\xi} = m^2(\partial_\mu A^\mu - \Box\xi) = 0,$$
$$\frac{\delta\mathcal{S}}{\delta A_\mu} = \left(\delta^\mu_\nu\Box - \partial_\nu\partial^\mu\right) A^\nu + m^2(A_\mu - \partial_\mu\xi) = 0; \qquad (75)$$

$$\delta_\epsilon\xi = \epsilon, \quad \delta_\epsilon A_\mu = \partial_\mu\epsilon\,; \qquad \frac{\delta\mathcal{S}}{\delta\xi} - \partial_\mu\frac{\delta\mathcal{S}}{\delta A_\mu} \equiv 0. \qquad (76)$$

Let us apply to this system the DoF number count recipe (61). There are five second order equations (75), so the total order of the equations is ten. There is one first order gauge symmetry, and one third order gauge identity. So, the DoF number is $10 - 3 - 1 = 6$, as it should be.

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