On the Perron solution of the caloric Dirichlet problem: an elementary approach

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Abstract. By an easy trick taken from caloric polynomial theory, we construct a family $\mathscr{B}$ of *almost regular* domains for the caloric Dirichlet problem. $\mathscr{B}$ is a basis of the Euclidean topology. This allows to build, with a basically elementary procedure, the Perron solution to the caloric Dirichlet problem on every bounded domain.

1. Introduction and key theorem

As it is very well known today, the construction of the Perron solution to the Dirichlet problem for the heat equation on a general bounded domain only rests upon three basic principles: the *caloric maximum principle*; the *convergence principle*, i.e., the closure of the sheaf of the caloric functions with respect to the uniform convergence; the *solvability principle*, i.e., the solvability of the caloric first boundary value problem on the open sets of a basis of the Euclidean topology.

The first one of these principles is very elementary, and the second one is a simple consequence of some good properties of the Gauss–Weierstrass kernel, the fundamental solution of the heat equation. On the contrary, the proof of the solvability principle has been always considered in the literature a difficult task, requiring Volterra integral equation theory, or double layer potential method, or an involved procedure based on a reflection principle (see, e.g., respectively, Bauer [2, Chapter 1, Section 2], Watson [6, Chapter 2, Section 2.2], Constantinescu and Cornea [4, Chapter 3, Section 3.3]).

The aim of this note is to draw attention to an easy trick—borrowed from caloric polynomial theory—allowing to construct with a very elementary procedure, a basis of open sets on which the first boundary value problem for the heat equation is solvable. The crucial point of this procedure is Theorem 1.1 below, in which w denotes the polynomial

$$w(z) = w(x,t) := t - |x|^2.$$

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Here and in what follows

$$z = (x, t), \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R},$$

denotes the point of $\mathbb{R}^{N+1}$; $|x|$ is the Euclidean norm of x.

We will use the notation H to denote the *heat operator*

$$H := \Delta - \partial_t,$$

where $\Delta := \sum_{j=1}^N \partial_{x_j}^2$ is the Laplacian in $\mathbb{R}^N$. We call *caloric* the smooth functions solutions to $Hu = 0$. If $\Omega \subseteq \mathbb{R}^{N+1}$ is open, we will denote by $\mathscr{C}(\Omega)$ the linear space of the caloric functions in Ω.

To our aim, it is convenient to fix some more notations. If $\alpha = (\alpha_1, \ldots, \alpha_N, \alpha_{N+1})$ is a multi-index with non-negative integer components, we let

$$|\alpha|_c = caloric\ height\ of\ \alpha := \alpha_1 + \cdots + \alpha_N + 2\alpha_{N+1}.$$

A polynomial in $\mathbb{R}^{N+1}$ is a function of the kind

$$p(z) = \sum_{|\alpha|_c \leq m} a_\alpha z^\alpha, \quad a_\alpha \in \mathbb{R} \text{ for every } \alpha,$$

where $m \in \mathbb{Z}$, $m \geq 0$. In this case, we say that p has caloric degree $\leq m$; we will say that p has caloric degree equal to m if $\sum_{|\alpha|_c = m} a_\alpha z^\alpha$ is not identically zero.

Here is the key theorem of our note.

Theorem 1.1. *Let p be a polynomial in $\mathbb{R}^{N+1}$. Then, there exists a unique polynomial q in $\mathbb{R}^{N+1}$ such that*

$$H(wq) = -Hp \tag{1.1}$$

Proof. Let us denote by m the caloric degree of $-Hp$ and by $\mathcal{P}_m$ the linear space of the polynomials in $\mathbb{R}^{N+1}$ having caloric degree less than or equal to m. Since w has caloric degree two, then $H(wq) \in \mathcal{P}_m$ if $q \in \mathcal{P}_m$; therefore,

$$q \longmapsto T(q) := H(wq)$$

maps $\mathcal{P}_m$ in $\mathcal{P}_m$. To prove that (1.1) has a unique solution q it is (more than) enough to show that T is surjective and injective. To this end, since $\mathcal{P}_m$ is a linear space of finite dimension and T linearly maps $\mathcal{P}_m$ in $\mathcal{P}_m$, we only have to prove that T is injective. Let $q \in \mathcal{P}_m$ be such that $T(q) = 0$. Then,

$$u := wq$$

is caloric in $\mathbb{R}^{N+1}$; hence, in particular, in the open region

$$P := \{(x, t) \in \mathbb{R}^{N+1} : t > |x|^2\}. \tag{1.2}$$

Moreover, since $w = 0$ on

$$\partial P = \{(x, t) \in \mathbb{R}^{N+1} \ : \ t = |x|^2\}, \tag{1.3}$$

$u|_{\partial P} = wq|_{\partial P} = 0$. Then, by the caloric maximum principle (see, e.g., [3, Theorem 8.2]) $u = 0$ in P. Since $w > 0$ in P, this implies $q = 0$ in P, hence in $\mathbb{R}^{N+1}$. We have so proved that $q = 0$ if $T(q) = 0$, that is the injectivity of T, completing the proof.

$\square$

From the previous theorem, one immediately obtains the following corollary, in which ∂P is the closed set in (1.3), boundary of the region P in (1.2).

Corollary 1.2. *Let p be a polynomial in $\mathbb{R}^{N+1}$. Then, there exists a unique polynomial u_p in $\mathbb{R}^{N+1}$ such that*

$$\begin{cases} Hu_p = 0 & \text{in } \mathbb{R}^{N+1}, \\ u_p = p & \text{on } \partial P. \end{cases} \tag{1.4}$$

Proof. Let q be a polynomial satisfying (1.1). Then, $u_p = wq + p$ solves (1.4). Moreover, if v is any polynomial solving (1.4), then $v - u_p$ is caloric in $\mathbb{R}^{N+1}$ - hence in P - and $v - u_p = 0$ on ∂P. The caloric maximum principle implies $v - u_p = 0$ in P, hence in $\mathbb{R}^{N+1}$. Then, $v = u_p$, that is the uniqueness part of the Corollary.

2. A proof of the solvability principle: a basis of H-almost regular domains
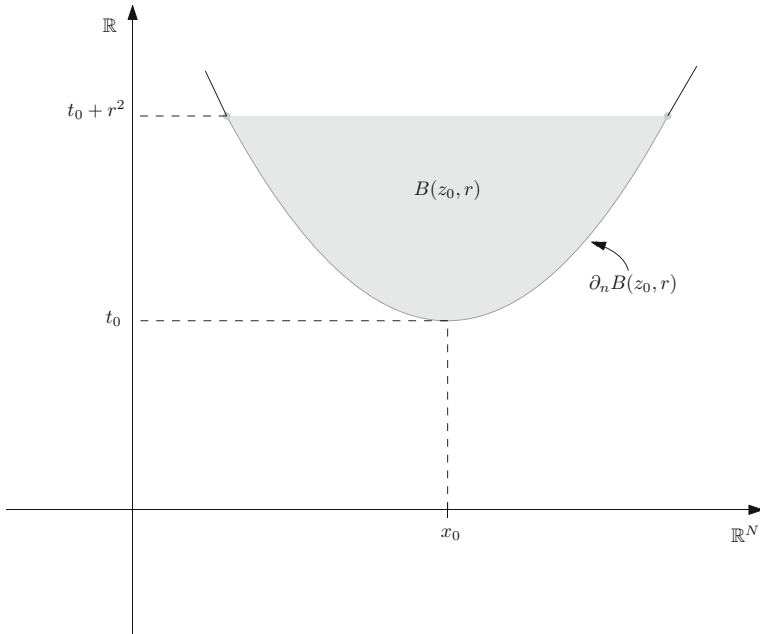
Let $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$ and let $r > 0$. We call:

(i) *caloric bowl* of *bottom z_0* and *opening r* the open set

$$B(z_0, r) := \{(x, t) \in \mathbb{R}^{N+1} \ : \ |x - x_0|^2 < t - t_0 < r^2\};$$

(ii) *normal* or *caloric boundary* of $B(z_0, r)$ the subset of $\partial B(z_0, r)$

$$\partial_n B(z_0, r) := \{(x, t) \in \mathbb{R}^{N+1} \ : \ |x - x_0|^2 = t - t_0, \ 0 \le t - t_0 \le r^2\}.$$

Obviously,

$$\mathscr{B} := \{B(z_0, r) \ : \ z_0 \in \mathbb{R}^{N+1}, \ r > 0\}$$

is a basis of the Euclidean topology.

The aim of this section is to prove the following Theorem 2.1. We summarize the content of this theorem by saying that every caloric bowl is H-*almost regular*. Indeed, actually, we prove that the first boundary value problem for the heat equation on every caloric bowl B is uniquely solvable if the boundary data are only given on the normal boundary $\partial_n B$. We would like to explicitly stress that the novelty of Theorem 2.1 is not in its content but in its elementary, direct and simple proof.

Theorem 2.1. *Let $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$ be arbitrarily fixed and let $B = B(z_0, r)$ be the caloric bowl of bottom z_0 and opening r. Then, for every $\varphi \in C(\partial_n B, \mathbb{R})$ there exists a unique solution to the boundary value problem*

$$\begin{cases} Hu = 0 & in \ B, \\ u = \varphi & on \ \partial_n B. \end{cases} \tag{2.1}$$

Precisely: there exists a unique function u_φ^B caloric in B and continuous up to $B \cup \partial_n B$ such that

$$u_\varphi^B(z) = \varphi(z) \ for \ every \ z \in \partial_n B.$$

Moreover, $u_\varphi^B \geq 0$ if $\varphi \geq 0$.

Proof. The uniqueness of u_φ^B and its positivity when φ is positive is a direct consequence of the caloric maximum principle (see, e.g., [3, Theorem 8.2]). To prove the existence of u_φ^B, we may, and do, assume $z_0 = (0, 0)$, since H is left translation invariant and

$$B(z_0, r) = z_0 + B(0, r), \quad 0 \in \mathbb{R}^{N+1}.$$

Let $\varphi \in C(\partial_n B, \mathbb{R})$ and let $(p_k)_{k \in \mathbb{N}}$ be a sequence of polynomials in $\mathbb{R}^{N+1}$ uniformly convergent to φ on $\partial_n B$. By using the notation of Corollary 1.2, we let

$$u_k := u_{p_k}, \quad k \in \mathbb{N}.$$

Then, u_k is caloric in B since it is caloric in $\mathbb{R}^{N+1}$. Moreover, $u_k = p_k$ on ∂P in (1.3), hence, on $\partial_n B$. Therefore, by the caloric maximum principle,

$$\max_{\overline{B}} |u_k - u_h| = \max_{\partial_n B} |u_k - u_h| = \max_{\partial_n B} |p_k - p_h| \longrightarrow 0 \text{ as } k, h \longrightarrow \infty.$$

From this, one gets the existence of a function $u \in C(\overline{B}, \mathbb{R})$ which is caloric in B—thanks to the convergence principle (Theorem A.2 below)—and such that

$$u|_{\partial_n B} = \lim_{k \to \infty} u_k|_{\partial B} = \lim_{k \to \infty} p_k|_{\partial B} = \varphi.$$

Thus, the function u is the requested function u_φ^B. $\hfill\square$

Remark 2.2. From the above proof, it follows that the solution u_φ of (2.1) actually is continuous up to $\overline{B}$.

For completeness reasons, and also to stress its elementary character, in Appendix A we will give a simple proof of the convergence principle. In Appendix B, we sketch how to construct the caloric Perron solutions starting from Theorem 2.1.

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Declarations

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Appendix A. The convergence principle

• *The fundamental solution of H*

The Gauss–Weierstrass kernel, i.e., the function

$$\Gamma : \mathbb{R}^{N+1} \longrightarrow \mathbb{R}, \quad \Gamma(x,t) = \begin{cases} 0 & \text{if } t \leq 0, \\ (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right) & \text{if } t > 0, \end{cases}$$

is the fundamental solution with pole at the origin of the heat operator in $\mathbb{R}^{N+1}$. Γ is smooth in $\mathbb{R}^{N+1} \smallsetminus \{(0,0)\}$ and locally summable in $\mathbb{R}^{N+1}$. Its crucial property is the following one:

$$\varphi(z) = -\int_{\mathbb{R}^{N+1}} \Gamma(z-\zeta) H\varphi(\zeta) \, d\zeta \tag{A.1}$$

for every $z \in \mathbb{R}^{N+1}$ and for every $\varphi \in C_0^\infty(\mathbb{R}^{N+1}, \mathbb{R})$.

• *Caloric norm and caloric disks*

If $z = (x,t) \in \mathbb{R}^{N+1}$, we let

$$\|z\| = caloric\, norm\, of\, z := (|x|^4 + t^2)^{\frac{1}{4}}.$$

We call *caloric disk* of center $z_0 \in \mathbb{R}^{N+1}$ and radius $r > 0$ the open set

$$D(z_0, r) := \{z \in \mathbb{R}^{N+1} \; : \; \|z - z_0\| < r\}.$$

• *A representation formula for caloric functions*

An easy consequence of property (A.1) is the following representation theorem

Theorem A.1. *Let $\Omega \subseteq \mathbb{R}^{N+1}$ be open. For every caloric disk $D = D(z_0, r)$ such that $\overline{2D} := \overline{D(z_0, 2r)} \subseteq \Omega$, there exists a function*

$$(z, \zeta) \longmapsto K_D(z, \zeta)$$

of class C^∞ in an open set containing $\overline{D} \times \overline{2D}$ such that

$$u(z) = \int_{2D} K_D(z, \zeta) u(\zeta) \, d\zeta \quad \text{for every } z \in D$$

and for every u caloric in Ω.

Proof. Let $\psi \in C_0^\infty(2D, \mathbb{R})$ be such that $\psi \equiv 1$ in a neighborhood of $\overline{D}$. Then, $\psi u \in C_0^\infty(2D, \mathbb{R})$ and $u = \psi u$ in D. As a consequence, by (A.1),

$$u(z) = -\int_{2D} \Gamma(z-\zeta) H(\psi u)(\zeta) \, d\zeta$$

$$= -\int_D \Gamma(z-\zeta)\, (u(\zeta) H\psi(\zeta) + 2\langle \nabla u(\zeta), \nabla \psi(\zeta)\rangle) \, d\zeta \text{ for every } z \in D,$$

where ∇ and $\langle\ ,\ \rangle$ denote, respectively, the gradient and the inner product in $\mathbb{R}^N$. Integrating by parts the second summand at the last right hand side, we find

$$u(z) = \int_{2D} K_D(z, \zeta) u(\zeta) \, d\zeta,$$

where, denoting by Δ the Laplacian with respect to the spatial variables,

$$K_D(z, \zeta) = -\Gamma(z - \zeta) H\psi(\zeta) - 2\langle\nabla\Gamma(z - \zeta), \nabla\psi(\zeta)\rangle + \Gamma(z - \zeta)\Delta\psi(\zeta).$$

Then, K_D is a smooth function in a neighborhood of $D \times 2D$ and we are done. □

• *The convergence principle*

As recalled in the Introduction, the convergence principle is the statement of the following theorem.

Theorem A.2. *Let (u_k) be a sequence of caloric functions in an open set $\Omega \subseteq \mathbb{R}^{N+1}$. Suppose (u_k) uniformly convergent to a function $u : \Omega \longrightarrow \mathbb{R}$ on every compact subset of Ω. Then,*

$$u \in C^\infty(\Omega, \mathbb{R}) \quad and \quad Hu = 0 \ in \ \Omega.$$

Proof. It is enough to show that u is smooth and caloric in every caloric disk D such that $\overline{2D} \subseteq \Omega$. So, let D be such a disk. Then, by Theorem A.1,

$$u_k(z) = \int_{2D} K_D(z, \zeta) u_k(\zeta) \, d\zeta \text{ for every } z \in D \tag{A.2}$$

and for every $k \in \mathbb{N}$. Since (u_k) is uniformly convergent on $\overline{2D}$, letting k go to infinity in (A.2), we get

$$u(z) = \int_{2D} K_D(z, \zeta) u(\zeta) \, d\zeta \text{ for every } z \in D.$$

The smoothness of the kernel K_D implies $u \in C^\infty(D, \mathbb{R})$. To show that u is caloric in D we argue as follows. Denoting

$$H^* := \Delta + \partial_t$$

the formal adjoint of H, for every $\varphi \in C_0^\infty(D, \mathbb{R})$ we have

$$\int_D (Hu)\varphi \, dz = \int_D u \, H^*\varphi \, dz = \lim_{k\to\infty} \int_D u_k \, H^*\varphi \, dz = \lim_{k\to\infty} \int_D (Hu_k) \, \varphi \, dz = 0,$$

since u_k is caloric for every $k \in \mathbb{N}$. Hence,

$$\int_D (Hu) \, \varphi \, dz = 0 \quad \text{for every } \varphi \in C_0^\infty(\Omega, \mathbb{R}).$$

This implies $Hu = 0$ in D, completing the proof. □

• *A consequence of the convergence principle*

A family $\mathscr{F}$ of real functions in an open set $\Omega \subseteq \mathbb{R}^{N+1}$ is said *directed-upward.* if for every $u, v \in \mathscr{F}$ there exists $w \in \mathscr{F}$ such that

$$u \le w, \quad v \le w.$$

Then, Theorem A.2 and a Real Analysis lemma imply the following result.

Theorem A.3. *Let $\mathscr{F}$ be a directed-upward family of caloric functions in an open set $\Omega \subseteq \mathbb{R}^{N+1}$. Let $u : \Omega \longrightarrow] -\infty, \infty]$,*

$$u := \sup \mathscr{F}.$$

If u is bounded above on every compact subset of Ω, then

$$u \in C^{\infty}(\Omega, \mathbb{R}) \quad and \quad Hu = 0 \ in \ \Omega.$$

Proof. A Real Analysis Lemma (see, e.g., [1, Lemma 3.7.1]) implies the existence of a monotone increasing sequence (u_k) of functions in $\mathscr{F}$ such that

$$\lim_{k \longrightarrow \infty} u_k = u \quad \text{pointwise in } \Omega.$$

Now, we can argue as in the proof of Theorem A.2. Let D be a caloric disk such that $\overline{2D} \subseteq \Omega$. Then, identity (A.2) holds for every function $u_k, k \in \mathbb{N}$. Since

$$u_1 \le u_k \quad \text{and} \quad \sup_{2D} u_k \le \sup_{2D} u < \infty,$$

by Lebesgue Dominated Convergence Theorem, letting k go to infinity in (A.2), one obtains

$$u(z) = \int_{2D} K_D(z, \zeta) \, u(\zeta) \, d\zeta \qquad \forall z \in D.$$

The smoothness of the kernel K implies the smoothness of u in D so that, since (u_k) is increasing, by Dini Theorem (u_k) converges uniformly on every compact subset of D. As a consequence, by Theorem A.2, u is caloric in D. This completes the proof of Theorem A.3 since D is any caloric disk such that $\overline{2D} \subseteq \Omega$. $\qquad \square$

• *The operator $f \longmapsto h_f^B$*

Let B be a caloric bowl. As an application of Theorem A.3, we show how to extend the operator

$$C(\partial_n B, \mathbb{R}) \ni \varphi \longmapsto u_{\varphi}^B \in \mathscr{C}(B),$$

defined in Theorem 2.1, to the bounded above lower semicontinuous functions.

Let B be a caloric bowl and let

$$f : \partial_n B \longrightarrow] -\infty, \infty[$$

be a bounded above lower semicontinuous function. Define

$$\mathscr{F}(B, f) := \{u_\varphi^B \; : \; \varphi \in C(\partial_n B, \mathbb{R}), \; \varphi \le f\}$$

and

$$h_f^B := \sup \mathscr{F}(B, f).$$

Obviously, if the function f is continuous, then $h_f^B = u_f^B$, so that $f \longmapsto h_f^B$ is an extension of $f \longmapsto u_f^B$. By using the caloric maximum principle, it is easy to show that $\mathscr{F}(B, f)$ is directed-upward and that

$$h_f^B \le m \quad \text{if } m = \sup_{\partial_n B} f.$$

Then, by Theorem A.3,

$$h_f^B \in \mathscr{C}(B).$$

Appendix B. The caloric Perron solution

• *Mean Value Theorem for caloric functions*

For every $z_0 \in \mathbb{R}^{N+1}$ and every $r > 0$, we let

$$\Omega_r(z_0) = \text{Pini–Watson ball with pole at } z_0 \text{ and radius } r$$
$$:= \left\{ z \in \mathbb{R}^{N+1} \; : \; \Gamma(z_0 - z) > (4\pi r)^{-\frac{N}{2}} \right\}$$

and

$$W(z) = W(x, t) = \text{Watson kernel}$$
$$:= \frac{1}{4} \left(\frac{|x|}{t} \right)^2.$$

We also denote by $M_r(u)(z_0)$ the average operator

$$M_r(u)(z_0) = \left(\frac{1}{4\pi r} \right)^{\frac{N}{2}} \int_{\Omega_r(z_0)} u(\zeta) \, W(z - \zeta) \, d\zeta.$$

Then, the following theorem holds.

Theorem B.1. *Let $\Omega \subseteq \mathbb{R}^{N+1}$ be open and let $u \in C(\Omega, \mathbb{R})$. The following statements are equivalent:*

(i) *For every Pini–Watson ball, $\Omega_r(z)$ with closure contained in Ω*

$$u(z) = M_r(u)(z).$$

(ii) *For every $z \in \Omega$, there exists $r(z) > 0$ such that*

$$u(z) = M_r(u)(z) \quad \text{for } 0 < r < r(z).$$

(iii) $u \in C^\infty(\Omega, \mathbb{R})$ *and*

$$Hu = 0 \text{ in } \Omega.$$

Proof. See Watson [6, Chapter 1 and Chapter 2] and see also Evans [5, Section 2.3].
□

• *Supercaloric and subcaloric functions*
  Let $\Omega \subseteq \mathbb{R}^{N+1}$ be open and let

$$u : \Omega \longrightarrow \mathbb{R}$$

be a lower semicontinuous function. We say that u is *supercaloric in Ω* if for every $z \in \Omega$ there exists $r(z) > 0$ such that

$$u(z) \geq M_r(u)(z) \quad \text{for } 0 < r < r(z).$$

We say that u is *subcaloric in Ω* if $-u$ is supercaloric. With $\overline{\mathscr{C}}(\Omega)$ ($\underline{\mathscr{C}}(\Omega)$), we denote the family of the supercaloric (subcaloric) functions in Ω. It can be elementarily proved that a sufficiently smooth function u is supercaloric (subcaloric) in an open set Ω if and only if

$$Hu \leq 0 \text{ in } \Omega \quad (Hu \geq 0 \text{ in } \Omega).$$

• *A caloric Perron-type regularization*
  To begin with, we fix some notation. If B is a caloric bowl, we denote

$$\hat{B} := \overline{B} \smallsetminus \partial_n B.$$

Equivalently

$$\hat{B} = B \cup \text{top}(B),$$

where

$$\text{top}(B) := \partial B \smallsetminus \partial_n B.$$

Let us consider a bounded above supercaloric function u in an open set $\Omega \subseteq \mathbb{R}^{N+1}$. If $B = B(z_0, r)$ is a caloric bowl such that $2B := B(z_0, 2r) \subseteq \Omega$, we define

$$u_B : \Omega \longrightarrow \mathbb{R}$$

as follows:

$$u_B(z) = u(z) \quad \text{if } z \notin \hat{B},$$

and

$$u_B(z) = h_f^{2B}(z) \quad \text{if } z \in \hat{B},$$

where $f = u|_{\partial_n 2B}$.

We want to explicitly remark that u_B is *caloric* in B and *continuous* up to $\hat{B}$. The function u_B is what we call *caloric Perron-type regularization of u in B*. It satisfies all the crucial properties of the classical harmonic Perron regularization. Precisely, the following theorem holds.

Theorem B.2. *Let u be a bounded above supercaloric function in an open set* $\Omega \subseteq \mathbb{R}^{N+1}$ *and let B be a bowl such that* $2B \subseteq \Omega$. *Then,*

(i) $u_B \in \overline{\mathscr{C}}(\Omega)$;

(ii) $u_B \leq u$;

(iii) u_B *is caloric in B and continuous in* $\hat{B}$;

(iv) *if* $v \in \overline{\mathscr{C}}(\Omega)$ *and* $v \leq u$ *then* $v_B \leq u_B$.

Proof. The proof of this theorem follows basic standard lines in harmonic and caloric Potential Theory. It uses in a crucial way the properties of the operator $f \longmapsto u_f^{2B}$ and the minimum principle for supercaloric functions (for this principle we directly refer to Watson's monograph [6, Theorem 3.11]. □

We close this appendix with the following point.

• *The caloric Perron solution*

Let $\Omega \subseteq \mathbb{R}^{N+1}$ be open and bounded and let $\varphi \in C(\partial\Omega, \mathbb{R})$. We let

$$\overline{\mathcal{U}}_\varphi^\Omega := \{u \in \overline{\mathscr{C}}(\Omega) \; : \; u \text{ bounded above, } \liminf_{x \to y} u(x) \geq \varphi(y) \; \forall y \in \partial\Omega\},$$

and

$$\overline{H}_\varphi^\Omega := \inf \overline{\mathcal{U}}_\varphi^\Omega.$$

We also let

$$\underline{H}_\varphi^\Omega := -\overline{H}_{-\varphi}^\Omega.$$

From the quoted above minimum principle for supercaloric functions, one easily gets

$$m \leq \underline{H}_\varphi^\Omega \leq \overline{H}_\varphi^\Omega \leq M,$$

where

$$m = \min_{\partial\Omega} \varphi \quad \text{and} \quad M = \max_{\partial\Omega} \varphi.$$

Actually, a stronger result holds

Theorem B.3. *For every* $\varphi \in C(\partial\Omega, \mathbb{R})$, *one has*

(i) $\overline{H}_\varphi^\Omega$ is caloric in Ω;

(ii) $\overline{H}_\varphi^\Omega = \underline{H}_\varphi^\Omega$.

This is the caloric version of the celebrated Perron–Wiener Theorem for harmonic functions. It can be proved with a standard procedure in which the caloric Perron-type regularization plays the crucial rôle.

Due to this theorem,

$$H_\varphi^\Omega := \overline{H}_\varphi^\Omega = \underline{H}_\varphi^\Omega$$

is called the Perron solution of the caloric Dirichlet problem on the open set Ω with boundary data φ. It coincides with the classical solution if a classical solution exists.

REFERENCES

[1]   D. H. Armitage and S. J. Gardiner. *Classical potential theory. Springer Monographs in Mathematics*. Springer, London, 2001.

[2]   H. Bauer. *Harmonische Räume und ihre Potentialtheorie*. Ausarbeitung einer im Sommersemester 1965 an der Universität Hamburg gehaltenen Vorlesung. Lecture Notes in Mathematics, No. 22. Springer-Verlag, Berlin-New York, 1966.

[3]   S. Biagi and A. Bonfiglioli. *An introduction to the geometrical analysis of vector fields with applications to maximum principles and Lie groups*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2019.

[4]   C. Constantinescu and A. Cornea. *Potential theory on harmonic spaces*. Springer-Verlag, New York-Heidelberg, 1972. With a preface by H. Bauer, Die Grundlehren der mathematischen Wissenschaften, Band 158.

[5]   L. Evans. *Partial Differential Equations*. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010.

[6]   N. A. Watson. *Introduction to Heat Potential Theory*. Mathematical Surveys and Monographs vol.182, Amer. Math. Soc., Providence RI, 2012.

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