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"MONETARY AND FISCAL POLICY IN A NONLINEAR MODEL OF PUBLIC DEBT"

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Monetary and fiscal policy in a nonlinear model of public debt

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Abstract

In this paper we study the dynamic relationship between the public debt ratio and the real interest rate. Specifically, by means of a macroeconomic model of simultaneous difference equations - one for the debt ratio and the other for the real interest rate - we focus on the role of monetary policy, fiscal policy and risk premium in affecting the stability of the debt ratio and the existence of steady states, if any. We show that, in a dynamic framework, fiscal rules may not be enough to control the pattern of the debt ratio, and the adoption of a monetary policy, in the form of an interest rate rule, is necessary to control the pattern of the debt ratio for assuring its sustainability over time. Notably, the creation or disappearance of steady states, or periodic (stable) cycles, can generate scenarios of multistability. While we obtain clear evidence that an active monetary policy has a stabilizing effect on both the real interest rate and the debt ratio, we also find that, in some scenarios, fiscal policy is not sufficient to avoid explosive patterns of the debt ratio.

Keywords: public debt, interest rate, instability, chaos.

1 Introduction

A long tradition in macroeconomic literature focuses on the monetary consequences of fiscal policies (Silber, 1970; Solow and Blinder, 1973; Barro 1974). These contributions introduce a public budget constraint in a IS-LM framework to assess the impact of public expenditure or taxation on real interest rate and real income. Mainly, such a framework shows that the multiplier effects of a fiscal policy strictly depend on how the budget deficit is financed by means of taxes, money or public debt (Christ, 1968).

As far as we know, less attention has been devoted to study the conditions under which public bond sales will, or will not, (de)stabilize the debt-to-GDP ratio (debt ratio hereinafter) and the real interest rate. This issue has become even more important in the light of the current public debt outbreak consequence of the Covid-19 health emergency.

Our aim is to study this issue by integrating the analysis of a growing public debt ratio with the responses of the real interest rate to its variations. Specifically, our goal is to identify potential thresholds, if any, beyond which an indebted economy can drive towards the default, even moving from an initial stable macroeconomic equilibrium with public debt.

To develop our analysis we start from the standard linear dynamic model of the government's debt constraint (Barro, 2008; Blanchard, 2017, Gordon, 2006; Mankiw, 2014) which requires that current public spending plus the cost of servicing current debt equals current tax revenues plus the issuance of new debt. We propose a number of changes to this intertemporal model to highlight the potential *nonlinearity* of the relationship linking the debt ratio to the real interest rate. To the best of our knowledge, the nonlinearity of such a relationships are still not studied. But, in our opinion, they may be crucial to explain the so-called public debt sustainability.

In the presence of a government budget constraint one would expect that an increase in bond supply, *ceteris paribus*, would raise the real interest rate since investors must be induced to raise the holding of bonds relative to money. Additionally, for an indebted economy an increase in bond supply could also raise the risk of bankruptcy, pushing up the *risk premium* of the interest rate. This additional effect impinges upon the path of the debt ratio because the higher the cost of debt, the higher the debt ratio, given the level of GDP.

The post Keynesian economics has extensively discussed the importance of fiscal policy to stabilize public debt (Arestis and Sawyer, 2003; Hein, 2018;

Setterfiled, 2007). This point of view has even been echoed by IMF (2013) that does ”mea culpa” on the size of fiscal multipliers: during a deep recession, with monetary policy made powerless by the liquidity trap and near-zero interest rates, fiscal policy is far from ineffective, and the debt ratio can decrease because GDP growth is greater than debt, and because of a positive long-term impact on potential GDP growth. Nonetheless, alternatives on the use of fiscal and monetary policy, even from a post Keynesian perspective persist. For example, Ribiero and Lima (2019), using a neo-Kaleckian framework, have recently shown that a fiscal rule setting a limit for government spending (excluding the payment of interests) may not ensure a non-explosive trajectory of the public debt ratio.

However, less attention has been devoted to study how monetary policy can affect the public debt ratio and its change over time. So, this is the focus of the present paper. Precisely, we show that, in a dynamic framework, fiscal rules may not be enough to control the dynamics of the debt ratio and that the adoption of an active monetary policy, in the form of an interest rate rule, is necessary to control the pattern and sustainability of the debt ratio. Notably, this result provides a renewed perspective to the traditional Keynesian view that fiscal policy is preferable to monetary policy. Rather, we show that in a dynamic framework, and in presence of an increasing risk premium, the stabilization of the debt ratio requires a mix of monetary and fiscal policies to balance the (potential) explosive dynamics of deficit and spread.

Accordingly, we present a stripped-down macroeconomic model of public debt sustainability with two first-order difference equations, one for the debt ratio and the other for the real interest rate. However, in order not to stray too far from the standard linear model, we employ the traditional first-order difference equation of the public debt ratio as a benchmark for useful comparisons. Our idea here is that the *spread* between the actual indebtedness of an economy and the level of it considered as “normal” or sustainable by investors can be seen as a proxy of the risk premium.

Therefore, in our model the debt ratio and the real interest rate change together. Importantly, shocks in some of the policy parameters, or in one of the boundary conditions, can cause nonlinear responses of the economy, even chaotic ones, which make it extremely likely and alarming that the debt ratio exceeds some critical threshold value.

To simplify our analysis, we consider the existence of only one real interest rate on government bonds, i.e. the rate on the composite bond of the

Keynesian model. We leave out questions relating to yield curve or capital returns, which react to the uncertainty of the financial markets and to changes in the debt ratio. We reserve to study these issues in a future work.

In order to make the model as flexible as possible, we also assume that the GDP growth varies in response to changes in both the budget deficit and the real interest rate, as in a IS-LM model. The combined effect of fiscal and monetary policies determines the path of the economy and the deviation of the GDP growth rate from its trend assumed as exogenous (Solow, 1957).

We achieve three main results. First, we find multiple steady states with the coexistence of stable and unstable equilibria, and even coexistence of several attractors, each with its own basin of attraction, i.e.: "corridor stability". Second, we get steady states in which the government can be either a lender or a borrower. But, from the global analysis emerges that exogenous small shocks can shift the economy away from the stable equilibrium. In this case, active interest rate policy helps to stabilize the debt-to-GDP ratio more than restrictive fiscal policy aimed at deficit control. Finally, from our analysis emerges that history is important: shocks trace the path of the economy with the crucial implication that the higher the initial debt ratio the higher the probability of pushing the economy towards the default regions.

As well known, debt sustainability can be regarded as a short-, medium-, or long-term concept. However, although we are interested in its intertemporal dynamics we do not make any assumptions about "Ponzi games", namely the possibility for the government to run a policy that uses the issuance of ever increasing new debt to repay old debt and to finance interest payments. We only assume that the public debt is financed with bond supply. This allows us to focus on cases where budget deficit persists in the medium run increasing the debt ratio. Our main concern is with short- and medium-run dynamics since every time a disequilibrium occurs the increase in the real interest rate can be lethal to the sustainability of the public debt.

As said, using this framework we achieve a wide range of scenarios. Some of our equilibria are stable but so close to the boundary of instability, or chaos, that negligible shocks in monetary and fiscal policies or in behavioral parameters can lead the economy towards a default.

The paper is organized as follows. In the next section we resume the literature on the topic. We will shortly analyze the standard intertemporal linear public debt constraint as a benchmark. Then, endogenizing the real interest rate, the budget constraint and the GDP growth rate we derive a nonlinear two-dimensional discrete dynamic system that governs the evolution of the variables in the model. Changes in fiscal and monetary policies are considered as well, and conflicting scenarios are studied to show that a long run equilibrium can be reached, but small shocks can easily destabilize the steady state so that the debt ratio may exhibit erratic time patterns, pushing the economy towards bankruptcy.

These results are obtained through a qualitative study of the nonlinear two-dimensional discrete dynamical system described in section 3. The sufficient conditions for the existence of one or two equilibria, together with some benchmark cases, characterized by constant primary budget deficit or absence of monetary and fiscal policy will be considered in section 4, whereas in section 5 some numerical explorations of the combined effects of fiscal and monetary policies will be performed. Finally, in the last section some conclusions and suggestions for further studies are given.

2 Literature

The problem of fiscal sustainability dates back to authors like Smith and Ricardo, who discussed the role of public debt in affecting national income and employment. Their concerns were about the comparison between tax and public expenditure. Specifically, Ricardo stressed the intergenerational distribution of the debt burden, called government debt neutrality, later named "Ricardian equivalence theorem" by Barro (1974).

Thereafter, in the view that markets are unable to get full employment, Keynes (1936) argued that public expenditure and debt finance were necessary to ensure an adequate level of aggregate demand, given current investments. Afterwards, Keynesians shared the Ricardian view that public debt is shouldered by the generation that issues the debt. Shortly, he argued that the current generation pays the cost of financing the debt, while debt service is only a transfer from taxpayers to bondholders *within* the respective country. Hence, Keynes concluded, a raising debt ratio does not pose particular problems.

This conclusion was debated in many subsequent contributions. Barro (1979) showed that in an intertemporal model of welfare maximization the government (a benevolent social planner) should keep the tax rate constant. This happens because in his model the tax smoothing policy reduces the distortion of a large tax rate variation in one single period, which has a welfare cost larger than the net present value of several small variations caused by a tax smoothing policy. Further, government spending today – in form of public investment – can benefit future generations. But, current resources often cannot cover the full cost of the public investment. Therefore, public debt allows to spread the high cost of a long-term investment also on future generations, which will reap the full benefits of public investment contributing to bear the cost of that project. Hence, the legitimacy of debt finance for public investment was increasingly recognized.

The discussion on public debt sustainability was revived in the 1980s. In that years, public finance came into focus because of a growing public sector and increasing budget deficit in numerous OECD economies. For the first time, the sustainability of public debt was questioned, that is the risk for governments not to honour their debt to investors. In fact, public debt is sustainable when the government can continue servicing it without requiring an unrealistically large correction to its future revenue or primary expenditure path. In practice, debt sustainability is assessed by asking if the current course of fiscal policy – tax and expenditure measures – can be sustained without facing an exploding debt path.

This issue has received a considerable attention after the financial crisis of 2008, and it is newly at the core of the current economic debate on how to fight the global health emergency related to Covid-19 virus. In particular, in the eurozone the structural growth of the debt ratio remains an alarming phenomenon. These facts raise questions about optimal "active" and "passive" fiscal and monetary policies aimed to stabilize the debt ratio (Eichengreen and Wyplosz, 1998; Bohn, 1998; Fincke and Greiner, 2016; Galì and Perotti, 2003; Beqiraj et al., 2018).

As said, the role played by active and passive fiscal policy for the sustainability of public debt has been largely analysed by the post Keynesian literature (Arestis and Sawyer 2003; Hein 2018; Setterfield 2007). Using a dynamic framework, Ribeiro and Lima (2019) consider the effects of the adoption of a government expenditure ceiling on the trajectory of public debt. They show that a fiscal rule that merely sets a limit for public spending, excluding interest payments, may not be enough to ensure a non-explosive trajectory of the public debt to output ratio.

In this vein, we focus on the capability of monetary and fiscal policies to stabilize the debt ratio in the presence of a risk premium that may change over time in response to the debt ratio evolution. Accordingly, a simplified "Taylor's rule" of monetary policy (Peersman and Smets, 1999; Woodford, 2001; Orphanides, 2003) is used to evaluate whether active monetary measures are corrective or rather let the debt ratio grow driving the economy towards the default. In our framework monetary and fiscal policies interact over time affecting both the debt ratio and the GDP growth. The model allows for complex dynamics with stable and unstable equilibria that can even generate chaos and explosive patterns.

2.1 The linear model of debt sustainability

The starting point of our analysis is the intertemporal linear government's budget constraint. It states that, with money supply fixed, the change in public debt at any time t must be equal to the public deficit of that period

$$B_t - B_{t-1} = (G_t - T_t) + rB_{t-1} \tag{1}$$

B_t denotes government debt issued at time t, G_t denotes the government spending in the same period, while T_t denotes tax revenues. The interest rate is denoted by r_t. All variables are in real terms. Now, let $b_t = \frac{B_t}{Y_t}$ and $d_t = \frac{G_t - T_t}{Y_t}$ be, respectively, the ratios of government debt issuance and primary budget deficit with respect to the the real GDP denoted by Y_t, and let be g_t the growth rate of output. The equation (1) can be written as

$$b_t = (1 + r_t - g_t)\, b_{t-1} + d_t \tag{2}$$

where the approximation $\frac{(1+r_t)}{(1+g_t)} \simeq (1 + r_t - g_t)$ has been used, provided that $r_t \ll 1$ and $g_t \ll 1$. Equation (2) implies that the debt ratio b_t increases over time if the government runs a deficit and, at the same time, the real interest rate exceeds the real GDP growth rate.

It is worth noting that equation (2) can be seen as a first-order linear difference equation in b_t if r, g and d are assumed to be *exogenous* parameters (Barro, 1974, 1979; Alesina and Perotti, 1995; Neck and Strum, 2008; Blanchard, 2017). Their size determines the long-run dynamics of the linear model (2) whose equilibrium point $\frac{d}{g-r}$ is asymptotically stable whenever $r < g$, i.e. with either a lender ($d < 0$) or borrower ($d > 0$) government. The solutions are instead unstable when $r > g$.

Now, we consider the budget constraint formalized in the following way

$$\Delta b_t = b_t - b_{t-1} = (r - g)\, b_{t-1} + d \tag{3}$$

Equation (3) states that to reduce the public debt at the end of the period t, the primary surplus must be larger than debt servicing, which can be expressed as

$$-d > (r - g)\, b_{t-1} \tag{4}$$

In other words, the debt ratio will increase during the period t if r exceeds g unless d is in sufficient surplus.

Bohn (1995) shows, however, that in the presence of uncertainty the condition $g > r$ cannot assure converge towards a stable equilibrium. Moreover, the endogenous nature of the variables in equation (3) can render the dynamics of the debt ratio much more unpredictable, so that the *ex ante* evaluation of a stabilizing fiscal and monetary policy is not trivial and does not assure the sustainability of public debt *a priori*.

The literature on the public debt ratio has provided numerous empirical studies to evaluate the efficiency of fiscal policies to control the evolution of the public debt ratio (Balassone and Franco, 2000; Chalk and Hemming, 2000; Beqiraj et al., 2018). Still, these quantitative analyses do not address the issue of *nonlinearity* between the debt ratio and the real interest rate. Consequently, many crucial issues related to the sustainability of public debt remain on the margins of both the empirical and theoretical analyses.

This paper attempts to fill this gap. We extend the existing theoretical models and study the effectiveness of fiscal and monetary policy to control the debt ratio in the presence of *nonlinearity* and *endogeneity* in the variables entering the equation (2). In such a scenario, the problem of sustainability becomes much more slippery and complex.

3 Interest rate and debt dynamics

With respect to the previous literature, a key novelty of our analysis is related to the dynamics of the real interest rate r_t, that depends not only on monetary policy decisions, but also on public debt evolution and its endogenous risk premium.

3.1 Interest rate dynamics

We start by modeling the impact of monetary policy on the real interest rate as follows:

$$r_t = r_{t-1} + \lambda \left(r^* - r_{t-1}\right)^{2n+1} \tag{5}$$

Equation (5) captures the impact of monetary policy on the real interest rate through the expression $\lambda\left(r^{*}-r_{t-1}\right)^{2n+1}$, where the parameter $\lambda>0$ measures the speed of adjustment of financial markets in response to a change in monetary policy. The odd exponent $2n+1$ provides the cost of adjustment of monetary policy. When $n=0$ the cost is linear, whereas for $n=1$ the cost is a cubic function of its fundamental. In the latter case, the reaction of the central bank to small deviations of the real interest rate from the target r^{*} takes place in a smooth manner to avoid the destabilization of both financial markets and debt ratio. In what follows, the assumption $n=1$ assures that the adjustment cost is convex for $r_t<r^{*}$ and concave for $r_t>r^{*}$.

The term $\lambda\left(r^{*}-r_{t-1}\right)^{3}$ can also be seen as a basic "Taylor's rule": it says how central bank should alter the real interest rate in response to changes in economic conditions, for the short-run stabilization of the economy while still maintaining long-run growth. Therefore, this monetary rule makes the recommendation that the central bank should decrease the actual interest rate when it is too high relative to the target r^{*}. Conversely, when the interest rate is low, the current interest rate should be raised (Taylor, 1993; Judd and Rudebush, 1998).

To model the relationship between the real interest rate and public debt, we start by the empirical evidence. Figure 1 shows the scatter between the real interest rate for some European countries from 2000 to 2017 (relative to the average value in the eurozone) and the risk premium proxied by the *spread* between the actual indebtedness of any single country and the average public debt level of the Euro area.

The empirical analysis suggests the existence of a *nonlinear relation* between the two variables. For this reason, we assume that the interest rate r_t is a nonlinear function of the difference between the debt ratio b_{t-1} and the "normal" level of it that, in a Keynesian perspective, is exogenously determined by market expectations.

Thus, we assume that the real interest rate r_t follows the dynamic equation

$$r_t=r_{t-1}+\alpha\arctan\left(b_{t-1}-b^{*}\right)+\lambda\left(r^{*}-r_{t-1}\right)^{2n+1} \tag{6}$$

The dynamics of r_t depends on the risk premium required by investors to hold public bonds in their portfolio, and on the monetary policy which affects the pattern of the real interest rate. b^{*} is an exogenous measure of what investors intend by "normal" level of debt ratio, and r^{*} is the monetary target of the central bank.

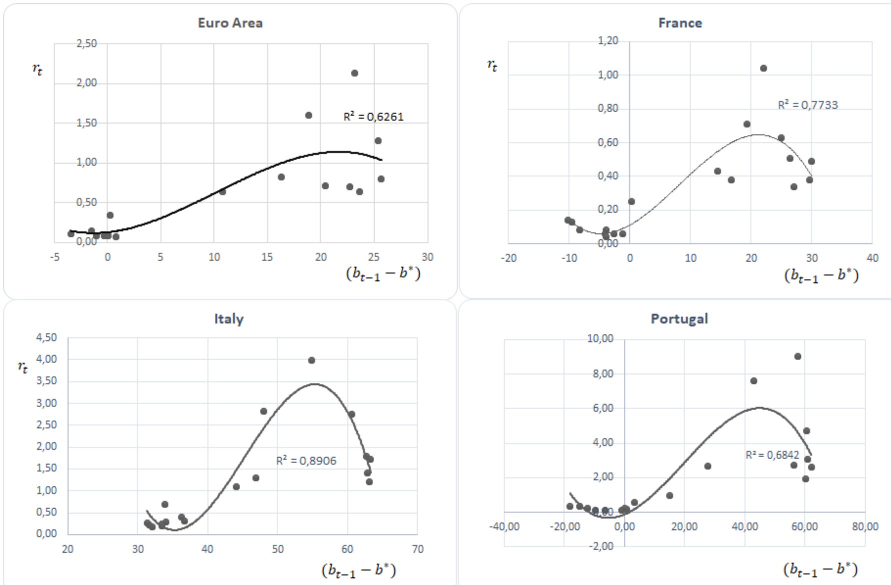

Figure 1: **The relationship between real interest rate and risk premium.**

In this equation the spread $(b_{t-1} - b^*)$ is a measure of the risk premium, and $\alpha > 0$ is the average coefficient of the relative risk aversion.

3.2 Public debt dynamics

To complete the model we employ an IS-LM framework. Specifically, we assume that

$$g_t = \bar{g} + \beta d_t - h r_t \tag{7}$$

where $\bar{g}$ is the secular component of the GDP growth, while β and h represent the sensitivity of the economy to changes in fiscal and monetary policy, respectively. As in a standard Keynesian framework, real output increases in the presence of a positive primary deficit and decreases when monetary policy is restrictive.

While in the linear version of the debt ratio model the budget deficit is given, the endogenous nature of our variables implies that the deficit can change over time, and that the government manages to control the debt ratio.

We assume that the "active" fiscal policy is described by the relation

$$d_t = \frac{\sigma}{1 + \delta b_{t-1}} \tag{8}$$

where $\sigma > 0$ is the budget deficit.[1] The parameter $\delta \geq 0$ represents the sensitivity of the fiscal policy to the previous-period debt ratio. The higher δ, the smaller the budget deficit d_t, given its (initial) value $\sigma > 0$.

By substituting the equations of g_t and d_t into (2), we get the time evolution of the debt ratio

$$b_{t+1} = \left(1 - \bar{g} - \frac{\beta\sigma}{1 + \delta b_t}\right) b_t + (1 + h)\, r_t b_t + \frac{\sigma}{1 + \delta b_t} \tag{9}$$

3.3 The model

The time evolution of both the debt ratio and the real interest rate is obtained by the iteration of a two-dimensional nonlinear map $T : (b_t, r_t) \to (b_{t+1}, r_{t+1})$

$$T : \begin{cases} b_{t+1} = \left(1 - \bar{g} - \frac{\beta\sigma}{1+\delta b_t}\right) b_t + (1 + h)\, r_t b_t + \frac{\sigma}{1+\delta b_t} \\ r_{t+1} = r_t + \alpha \arctan\left(b_t - b^*\right) + \lambda \left(r^* - r_t\right)^{2n+1} \end{cases} \tag{10}$$

We study the dynamic properties of the map (10), and explore the behavior of the model for economically meaningful parameter values. Since we are interested in the sustainability of the debt ratio, we will focus on the case of $b_t \geq 0$, even if the dynamic model (10) is feasible for $b_t < 0$ as well. We will highlight the role of some local and global bifurcations that explain the qualitative changes and evolution of the economic system, including the occurrence of different kinds of instability in the debt ratio and fluctuations in the real interest rate, with worrying default scenarios.

Moreover, benchmark cases with $\lambda = 0$ (no monetary policy), $\alpha = 0$ (no uncertainty), and $\delta = 0$ (no fiscal policy) will be studied in the following. These cases provide a basic mathematical structure of our model and may constitute useful economic scenarios for comparison.

4 Fixed points and local stability analysis

Equilibrium (or stationary) situations are obtained from (10) by setting $b_{t+1} = b_t = b$ in the first dynamic equation and $r_{t+1} = r_t = r$ in the second

[1] In this paper $\sigma > 0$ because we study the case of budget deficit.

one. From the first equation we get

$$r = r_1(b) = \frac{1}{1+h}\left(\bar{g} + \frac{\sigma\,(\beta b - 1)}{b\,(1+\delta b)}\right) \tag{11}$$

and from the second equation

$$r = r_2(b) = r^* + \left[\frac{\alpha}{\lambda}\arctan\left(b - b^*\right)\right]^{\frac{1}{2n+1}} \tag{12}$$

Equilibrium points are located at the intersections of these two curves, whose graphs for $\sigma > 0$ are represented in fig. 2. A typical situation is characterized by the presence of two equilibrium points with $b > 0$, say $E_L = (b_L, r_L)$ (lower equilibrium) and $E_U = (b_U, r_U)$ (upper equilibrium) with $0 < b_L < b_U$ and $r_L < r_U$. However, cases with more than two equilibria as well as only one or no equilibria at all may exist, depending on the values of the parameters. Moreover, these equilibrium points with positive debt $b > 0$ are associated with either negative or positive corresponding equilibrium values of the real interest rate, i.e. $r < 0$ or $r > 0$, as shown in fig. 2(a) and 2(b), respectively.

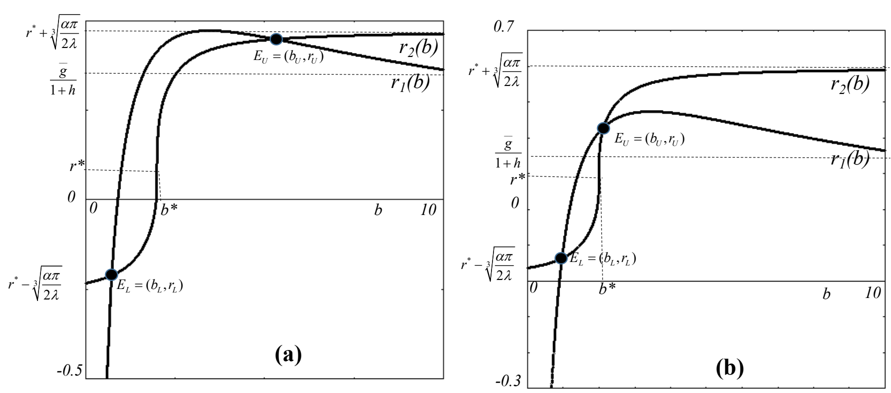

Figure 2: **Public debt and the coexistence of two equilibria.** Interest rate may be negative (a) or positive (b)

Fig. 2 shows an emblematic situation with two equilibria E_L and E_U where $0 < b_L < b^*$, $b_U > b^*$ and the interest rates are $r_L < r^*$ and $r_U > r^*$, respectively. In panel (a), obtained with the set of parameters $\bar{g} = 0.1$, $\beta = 1$,

$\sigma = 1$, $\delta = 0.2$ and $h = 0.1$ in the first dynamic equation, $b^* = 2$, $r^* = 0.1$, $\alpha = 0.1$, $\lambda = 3$ and $n = 1$ in the second one, the two equilibrium points E_L and E_U are characterized by negative and positive equilibria of the real interest rates, respectively. In panel (b), where $r^* = 0.3$ and $\lambda = 6$, both equilibria provide a positive real interest rate.

The local stability of each equilibrium can be determined through the usual linearization procedure based on the Jacobian matrix of the map (10), given by

$$DT(b,r) = \begin{bmatrix} 1 - \bar{g} - \frac{(\beta+\delta)\sigma}{(1+\delta b)^2} + (1+h)\,r & (1+h)\,b \\ \frac{\alpha}{1+(b-b^*)^2} & 1 - (2n+1)\,\lambda\,(r-r^*)^{2n+1} \end{bmatrix} \tag{13}$$

computed at the equilibrium point, and the localization in the complex plane of its eigenvalues. In our model an analytical computation of the coordinates of the equilibrium points is not possible in general, hence the eigenvalues can only be computed numerically. For example, for the set of parameters of fig. 2(a), we get $E_L = (0.724, -0.211)$ and $E_U = (5.294, 0.449)$, and the corresponding eigenvalues are real given by $z_1(E_L) = -0.316$, $z_2(E_L) = 0.196$, and $z_1(E_U) = -0.136$, $z_2(E_U) = 1.150$. Hence, E_L is stable (a stable node) whereas E_U is a saddle point, located along the boundary of the basin of attraction of E_L.

Figures 3(a) and 3(b) show the basins of attraction for the two parameters' constellations used in fig. 2(a) and 2(b), respectively. In fig. 3(a) the basin of attraction of the stable equilibrium E_L is represented by the white region, whereas the grey shaded region represents the basin of divergent trajectories, i.e. the set of initial conditions (b_0, r_0) that generate time evolutions leading to bankruptcy.

The frontier (or watershed) that separates these two basins is formed by the stable set of the saddle point E_U (see e.g. Mira et al., 1996). The distance between the two equilibria E_L and E_U constitutes a proxy of the extension of the basin of E_L, a measure of the robustness of this equilibrium as resilience to exogenous shocks. For the set of parameters used in fig. 2(b) the two equilibria are closer, with $E_L = (0.964, 0.063)$ and $E_U = (2.130, 0.4291)$. Consequently, the representation of the corresponding dynamic scenario of fig. 3(b) shows that the basin of attraction of E_L is smaller than the one of the previous scenario, even if the eigenvalues are quite similar, namely $z_1(E_L) = -0.181$, $z_2(E_L) = 0.292$, and $z_1(E_U) = 0.259$, $z_2(E_U) = 1.223$.

A sufficient condition for the standard situation of just two equilibrium

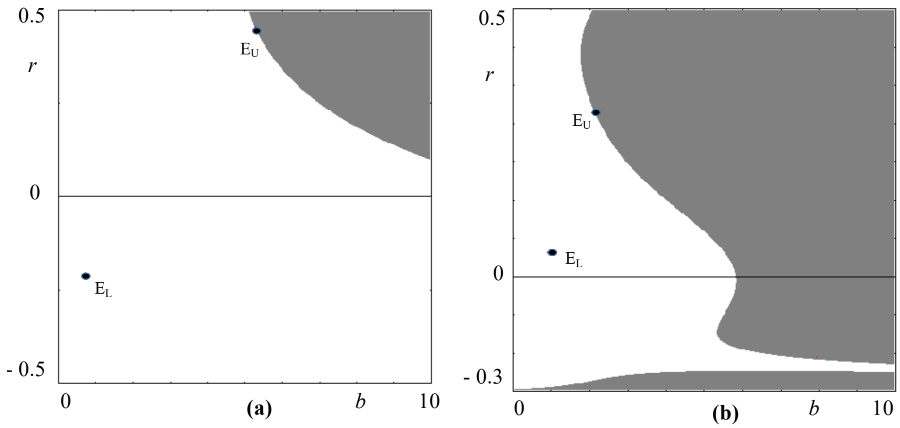

Figure 3: **Phase portraits and basins of attractions.** Panel (a) refers to fig. 2(a), panel (b) to fig. 2(b).

points described above is given by $r_1(b^*) > r_2(b^*) = r^*$, i.e.

$$r^* - \frac{\sigma(\beta b^* - 1)}{(1+h)\, b^* (1 + \delta b^*)} < \frac{\bar{g}}{1+h} < r^* + \left(\frac{\alpha\pi}{2\lambda}\right)^{\frac{1}{2n+1}} \tag{14}$$

As $\frac{\bar{g}}{1+h}$ increases, the whole curve $r_1(b)$ translates upwards, and the upper equilibrium E_U shifts upwards and to the right, until $b_U \to +\infty$ as $\frac{\bar{g}}{1+h}$ reaches the threshold value $r^* + \left(\frac{\alpha\pi}{2\lambda}\right)^{\frac{1}{2n+1}}$. This means that the basin of attraction of E_L gradually enlarges covering the whole phase space where E_L remains the unique globally attracting equilibrium for $\frac{\bar{g}}{1+h} \geq r^* + \left(\frac{\alpha\pi}{2\lambda}\right)^{\frac{1}{2n+1}}$.

By properly tuning the parameters of the model, local bifurcations can occur leading to different stability properties of the equilibrium points. In particular, E_L can undergo period doubling bifurcations that open the well known period doubling cascade that constitutes a bifurcation scenario leading to the creation of chaotic attractors. Moreover, the creation or disappearance of equilibria or periodic cycles can give rise to situations of multistability, i.e. coexistence of different attractors each with its own basin of attraction. Such dynamic situations give rise to severe path dependence, corresponding to a crucial role of initial conditions, so that any exogenous perturbation may cause a different long-run behavior, i.e. the convergence to a different attractor. Some of these dynamic scenarios will be illustrated in section

4, by means of numerical explorations, guided by some of the analytically determined conditions on the parameters.

4.1 Some benchmark cases

In this section we study some benchmark cases of the dynamic model (10) to get the analytical conditions used later as starting points of the numerical explorations.

4.1.1 $\lambda = 0$: no monetary policy

Let's assume $\lambda = 0$. In this scenario, there is no monetary policy, and the equilibrium condition $r_{t+1} = r_t$ imposed in equation (10) provides the unique equilibrium value of debt $b_e = b^*$. From the first dynamic equation, under equilibrium condition $b_{t+1} = b_t$, we get the corresponding equilibrium interest rate

$$r_e = r_1(b^*) = \frac{1}{1+h}\left(\bar{g} + \frac{\sigma\left(\beta b^* - 1\right)}{b^*\left(1+\delta b^*\right)}\right) \tag{15}$$

The Jacobian matrix (13) computed at this unique equilibrium point becomes

$$DT(b^*, r_e) = \left[\begin{array}{cc} 1 - \frac{\sigma}{b^*} & (1+h)\, b^* \\ \alpha & 1 \end{array}\right]$$

where $\lambda = 0$ and the equilibrium value in (15) have been used. For this benchmark case the Schur stability conditions can be easily applied. One of the well known three stability conditions (see e.g. Gandolfo, 2010, or Medio and Lynes, 2001) is $1 - Tr + Det > 0$, where $Tr = 2 - \frac{\sigma}{b^*}$ and $Det = 1 - \frac{\sigma}{b^*} - \alpha b^*\left(1+h\right)$ are the trace and the determinant of the Jacobian matrix, respectively. The stability condition becomes $-\alpha b^*\left(1+h\right) > 0$, which is never satisfied. This means that an eigenvalue is always greater than 1, hence the unique equilibrium (b^*, r_e) is always unstable.

Thus, When $\lambda = 0$, i.e. no monetary policy is applied, then the unique equilibrium of the model (10) is always unstable: this crucial result claims that monetary policy matters to stabilize of the economy, that is to provide sustainable levels of both the real interest rate and the debt ratio.

4.1.2 $\alpha = 0$: no risk premium

We now study the case $\alpha = 0$. The equilibrium condition in the second dynamic equation gives $r_e = r^*$ and from the first equation we get the corresponding two equilibrium values of the debt

$$b_e^{1,2} = \frac{(1+h)\,r^* - \bar{g} - \beta\sigma \pm \sqrt{((1+h)\,r^* - \bar{g} - \beta\sigma)^2 - 4\delta\sigma\,((1+h)\,r^* - \bar{g})}}{2\delta\,(\bar{g} - (1+h)\,r^*)}$$

i.e. two equilibrium points exist, characterized by the same equilibrium interest rate and different public debt levels, provided that $((1+h)\,r^* - \bar{g} - \beta\sigma)^2 \geq 4\delta\sigma\,((1+h)\,r^* - \bar{g})$. This condition is satisfied if $\bar{g} \geq (1+h)\,r^*$, i.e. by a sufficiently high GDP growth rate. In fact, the dynamics of the model requires that the GDP growth must compensate the real interest rate and the sensitivity of the economy to monetary policy, as described by equation (7).

The local stability of these two equilibrium points is determined by the Jacobian matrix (13) that, for $\alpha = 0$, becomes a triangular matrix, whose eigenvalues are therefore the diagonal entries. Moreover, at an equilibrium $E_i = (b_e^i, r^*)$ it becomes

$$DT\left(b_e^i, r^*\right) = \begin{bmatrix} 1 - \bar{g} - \frac{(\beta+\delta)\sigma}{(1+\delta b_e^i)^2} + (1+h)\,r^* & (1+h)\,b_e^i \\ 0 & 1 \end{bmatrix}$$

So, an eigenvalue is always 1 and the only stability condition can be expressed as

$$-2 < (1+h)\,r^* - \bar{g} - \frac{(\beta+\delta)\,\sigma}{(1+\delta b_e^i)^2} < 0. \tag{16}$$

Again, if $\bar{g} \geq (1+h)\,r^*$ the right hand inequality is satisfied, and the only condition for stability is given by the left hand inequality. If one of the parameters is changed, so that threshold value is crossed, then the equilibrium loses stability through a flip (or period doubling) bifurcation. This may occur for small values of δ (a fiscal policy that does not sufficiently react to previous period public debt), small values of h (when the long-run GDP growth is less sensitive to the monetary policy), and large values of β and σ (with a high reaction to fiscal policies, and high public deficit, respectively).

Thus, when $\alpha = 0$ two equilibria exist, provided a sufficiently high GDP growth rate. The stability condition is given by equation (16). The negativity of this expression represents a force of attraction towards equilibria. However, excessive negativity causes an overshooting of both the interest rate and the debt ratio.

A typical situation with $\alpha = 0$ is shown in fig. 4, obtained with the same set of parameters of fig. 2(b) and 3(b). The distance between the two equilibria increases and the size of the basin of attraction of the stable equilibrium becomes larger. Thus, if uncertainty is neglected, the stable equilibrium is more robust, and less vulnerable to exogenous shocks.

4.1.3 $\delta = 0$: no fiscal policy

When we add the assumption $\delta = 0$, the equilibrium condition becomes a first degree equation with the unique equilibrium

$$b_e = \frac{\sigma}{\bar{g} + \beta\sigma - (1+h)\, r^*}$$

From equation (16), we get that the stability of the system is favored by large values of h and/or by small values of σ, β, or $\bar{g}$.

Finally, our model reduces to the standard linear setup for a specific parameters' set. If $\lambda = 0$, $\alpha = 0$, and $\delta = 0$ we get the one dimensional linear model of section 2. When $\lambda = 0$ and $\alpha = 0$ then the second dynamic equation, governing the time evolution of r_t, becomes $r_{t+1} = r_t$, i.e. the interest rate is stationary at its initial value $r_t = r_0$ for each $t \geq 0$. This implies that the first dynamic equation that governs the time evolution of the debt b_t becomes

$$b_{t+1} = (1 - \bar{g} + \beta\sigma - (1+h)\, r_0)\, b_t + \sigma$$

which is identical to (2) (after suitable rescaling of the parameters). Therefore, the linear model appears as a specific case of our general setting.

5 Numerical simulations

In this section we provide some numerical simulations to study the effects of different economic conditions and/or economic policies on the long-run evolution of the economic system.

The bifurcation diagram, using the same set of parameters of fig. 3(a) and taking α as bifurcation parameter, is described in fig. 5(a). It shows that for small values of α the lower equilibrium E_L is stable (with its own basin of attraction, as we already know). Then, it loses stability through a period-doubling bifurcation after which the long run evolution of the model is characterized by stable oscillations of period two with values around the equilibrium values $E_L = (b_L, r_L)$. If α increases further, then the usual

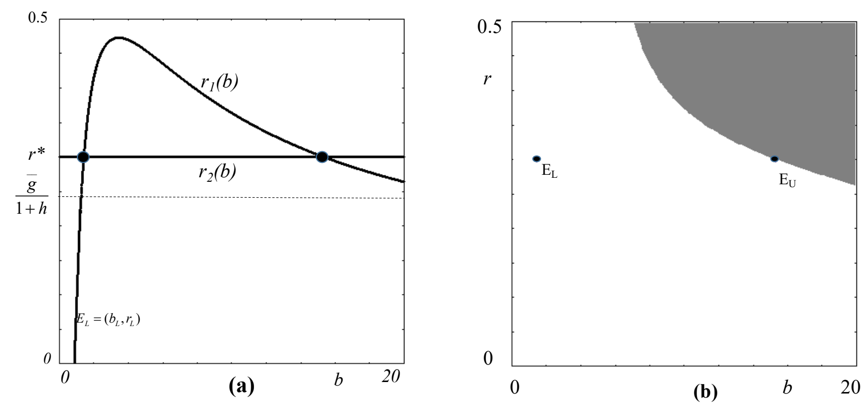

Figure 4: **Equilibria and phase portraits with no uncertainty** $(\alpha = 0)$ The set of parameters is as figures 2(b) and 3(b).

period-doubling cascade occurs, leading to stable periodic cycles of increasing period, a classical route leading to deterministic chaos. The dark vertical segments (as usual intermingled with periodic windows) represent chaotic time patterns of the dynamic variables along which high sensitivity with respect to small shocks of the initial conditions makes predictions quite difficult.

However, the bifurcation diagram of fig. 5(a) also shows an additional form of uncertainty. In a range of the parameter α around 0.5, two different attractors coexist, which can be reached for the same values of α but starting from different initial conditions. This form of coexistence of several attracting sets, each with its own basin of attraction, is also denoted as *multistability* (see e.g. Bischi and Kopel, 2003).

The phase portrait of fig. 5(b), obtained with the same parameters' constellation as the bifurcation diagram and with $\alpha = 0.5$, allows to analyse how the phase plane (b, r) of the dynamical system is shared by the different basins of attractions, and consequently to study how the long run evolution of the system is determined by assigning different initial conditions (or exogenous perturbations that cause shifts of the initial state of the system).

In this picture the white region represents the basin of the chaotic attractor around the lower equilibrium E_L (represented by black pints in the picture). As in fig. 3, the grey region represents the basin of diverging tra-

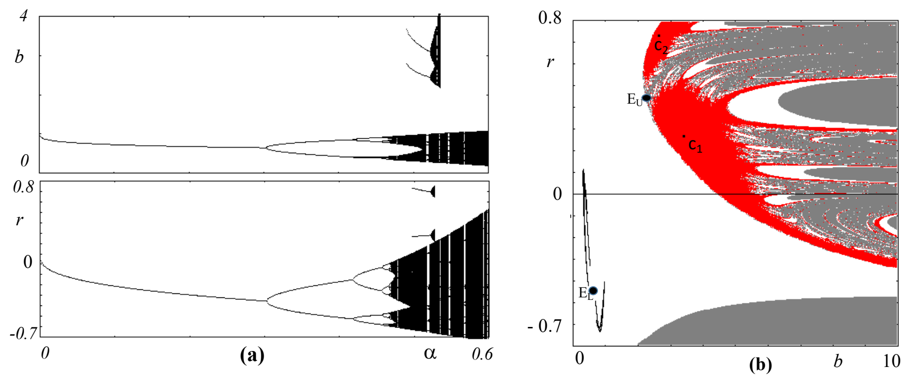

Figure 5: **Uncertainty and transition to chaos: bifurcation diagram (a) and phase portrait (b)**The set of parameters is as figures 2(a) and 3(a).

jectories (or basin of infinity, i.e. trajectories leading to system collapse) whereas the red region represents the basin of attraction of a stable cycle of period 2, represented by the two points denoted as C_1 and C_2 in the figure. The topological structure of the basins represents another form of complexity of our model. The lower equilibrium E_L is surrounded by a chaotic trap for the system, characterized by low or negative interest rates and low public debt, while the initial conditions in the region around the upper equilibrium E_U, unstable, generate trajectories leading to C_1 and C_2, with higher debt and interest rates (red region) or to bankruptcy (grey region).

Therefore, high levels of the parameter α generate instability. Both r and B varies between negative and positive values. Indeed, the larger the perceived uncertainty, the larger the interest rate required by financial markets, the larger the cost of debt service.

In fig. 6 we show the bifurcation diagram obtained by taking the parameter λ as bifurcation parameter, i.e. the intensity of the monetary policy. As we already know from the analysis of the benchmark cases, when $\lambda = 0$ no stable equilibrium exists. Hence, starting from the parameters' constellation used in fig. 3(b), we study the effect of increasing values of λ in the range $[0, 1]$. As expected, severe instability is observed for small values of λ, but stability of the lower equilibrium E_L is obtained by a period halving (or backward flip) bifurcation. Afterwards, the equilibrium remains stable, even with values $\lambda > 1$ (not shown in the diagram). Notice that, for large values of λ, the equilibrium interest rate r_L increases. From the numerical simulation of fig. 3(b), we know that for $\lambda = 6$, a stable equilibrium with $r_L = 0.06$ is obtained. The stabilizing effect of λ is confirmed by bifurcation diagrams with different parameters' constellations.

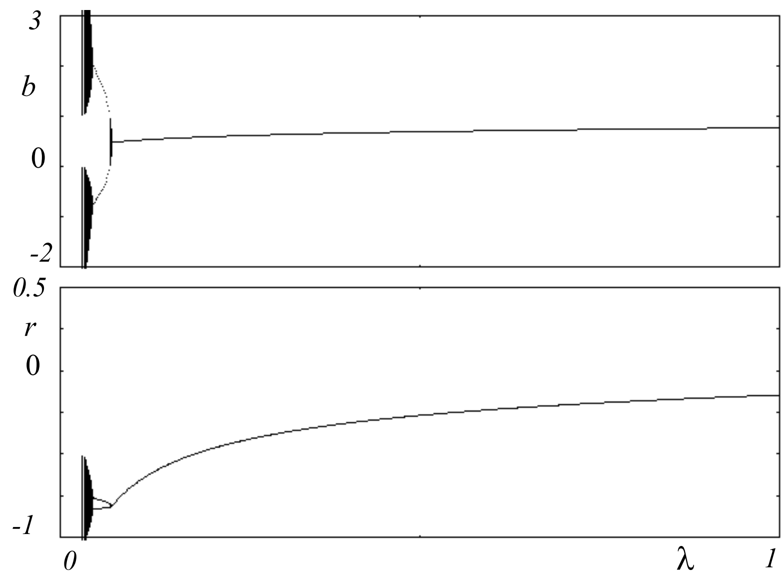

Figure 6: **The stabilizing effect of monetary policy: bifurcation diagrams.** The set of parameters is as figures 2(b) and 3(b).

Finally, we focus on the effects of increasing values of the parameter δ, that represents the sensitivity of the fiscal policy to the previous-period debt ratio.

The bifurcation diagram of fig. 7(a), obtained with the usual set of parameters, but with $\lambda = 0.65$ and $\alpha = 0.4$, shows that for small values of δ both the equilibrium points are unstable (the numerical computation of the corresponding eigenvalues shows that both are saddle points) and a stable cycle of period 2 is the unique attractor at which all the feasible trajectories converge. At $\delta = \delta_F = 0.9$ the lower equilibrium E_L becomes stable through a subcritical flip bifurcation, and for $\delta > \delta_F$, two attractors characterize the long-run evolution of the system: namely the stable equilibrium E_L and a

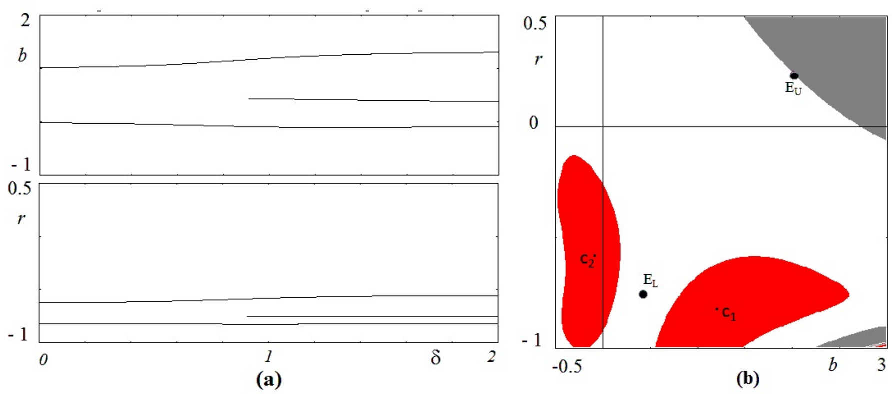

Figure 7: **The effects of fiscal policy: bifurcation diagram (a) and phase portrait (b)** The set of parameters is as figures 2(b) and 3(b).

stable cycle of period two, each with its own basin of attraction. The latter are represented in fig. 7(b) by the white region and the red one, respectively.

The particular structure of the basins shown in fig. 7(b) shows the important feature of nonlinear dynamic models with coexisting attractors, called corridor stability (Leijonhufvud, 1973; Dohtani et al., 2007). This literature argues that nonlinear dynamic models may have the property that small perturbations are recovered as far as they are confined inside the basin of attraction of a locally stable equilibrium, whereas larger perturbations lead to time evolutions that further depart from the equilibrium and go to the coexisting attractor in the long run.

Note that in fig. 7(b), even if the basin of the stable equilibrium E_L is quite large (the whole white region), such equilibrium is very vulnerable to exogenous shocks, due to the narrow corridor around it. Thus, from our numerical experiments we claim that enhancing the role of the fiscal policy can lead to a stability gain of the equilibrium. However, such stability may be vulnerable due to the coexistence of stable oscillatory patterns. This latter result suggests that the sensitivity of fiscal policy to previous-period debt is a key element to avoid explosive patterns of debt ratio and interest rate, but it may be not enough to assure stable equilibria.

6 Conclusions

In this paper we study the non-linear relationship between the public debt ratio and the real interest rate. By means of a macroeconomic model of simultaneous difference equations – one for the debt ratio and the other for the real interest rate – we focus on the role of monetary policy, fiscal policy and uncertainty in affecting the stability and the existence of multiple equilibria, if any. Mainly, the non-linearities, that link the debt ratio to the real interest rate, are important novelties compared to the standard model of intertemporal linear government's budget constraint where the real interest rate, the GDP growth and the primary deficit are assumed to be *exogenous* parameters.

Our augmented model provides some crucial insights on monetary and fiscal policies related to the issue of public debt sustainability. First, we show that an indebted economy can easily shift towards repulsive regions even for negligible and transitory shocks in some of its policy instruments and behavior parameters. Accordingly, the creation or disappearance of equilibria or periodic (stable) cycles can generate situations of multistability, i.e. coexistence of different attractors each with its own basin of attraction. These dynamic scenarios give rise to severe path dependence so that any exogenous shock may cause a different kind of long-run behavior, i.e. the convergence to a different attractor.

While we obtain a clear evidence that, in a dynamic context, an active monetary policy has a stabilizing effect both on the real interest rate and the debt ratio, we also find that a fiscal policy is not a sufficient instrument to avoid explosive patters of the economy. This analytical finding has a crucial regulatory implication. Mixed policies are necessary to stabilize the debt ratio. Indeed, in a globalized and decentralized economy, as the one we study, the role of the Central Bank appears more effective and immediate than that of Governments. Specifically, the model's numerical simulations show that financial markets respond promptly to monetary policies aimed at stabilizing the real interest rate, and thereby bring the GDP growth back around its long-term growth rate. Therefore, the "Taylor rule" employed by the Central Bank is a compass that influences real and financial markets by stabilizing the output growth in the medium run, even in presence of a certain degree of uncertainty. However, the higher the uncertainty about the debt sustainability in financial markets, the larger the chance the economy shifts towards instability. This finding suggests that a crucial goal of Institutions is to reduce noise in financial markets by stabilizing the expectations of the risk-averse investors, thereby reducing the spread and the real interest rate that directly affects the GDP ratio and the public deficit.

Finally, in this paper we study only the case where the budget deficit is financed by means of public debt. It is, however, possible to assume that money supply and taxes can finance the current budget deficit. This analysis can be done, providing additional instruments to affect the dynamics of both the debt ratio and economy, but the system of equations must satisfy more complicated conditions. Thus, this paper is just a starting point for further studies related to the many possible extensions and applications of the model. These studies are left for our future research.

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