Quasi-normal mode of a Planck star

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In this work we present a calculation of the quasi-normal modes of a Planck star, a supposed state in the life of a large mass star in which quantum effects should reverse the collapse causing the star to explode. In order to solve the Regge-Wheeler equations and to calculate the quasi-normal modes, we apply the shooting method.

Keywords: Physics of black holes; Gravitational-wave astrophysics; Waves generation and sources

I. INTRODUCTION

In the context of loop quantum cosmology, in [1], it was demonstrated that the initial singularity of the big bang could be avoided thanks to quantum mechanics effects. In [2] it was conjectured that the same might happen during the collapse of a high mass star: quantum effects might avoid the formation of the singularity and of a black hole by reversing the collapse and leading to the explosion of the star. This final stage in the life of a massive star was called *Planck star*. While in the reference system of the collapsing star this process is very brief, for an observer far from the source the explosion might take millions of years, depending on the mass of the object, because of the general relativistic slowing down of time close to high gravitational fields. More studies that deal with the possible existence of Planck stars are [3–6].

Since the publication of [2], many papers have been published that look for observable effects that might lead to the discovery of Planck stars: in [7] the possibility that the fast radio burst might be due to the explosion of a low-mass Planck star is considered; in [8] the main spectral characteristic of the signal are discussed; finally in [9] the possibility that the excess of GeV photons coming from the center of the Galaxy could be explained by the explosion of a Planck star. For more studies on Planck stars see the review [10].

The final object produced by the merging of black holes (or Planck stars) is in general perturbed and distorted: this leads to the emission of ring-down radiation consisting of quasi normal modes (QNM). These modes have a complex frequency, whose real part is the oscillation frequency, while the imaginary part is the inverse of the damping time. Since the proposed model of a Planck star is different from a Schwarzschild black hole, the QNMs are also different; this might be another method to establish the existence of these objects, therefore, in this work we calculate the QNMs for a perturbed Planck star and compare the results with those of a Schwarzschild black hole.

In Section II we review the model for the metric of a Planck star first described in [3]; in Section III we calculate the Regge-Wheeler equation for our system; in Section IV we apply the shooting method to solve numerically the Regge-Wheeler equations in order to calculate the QNM of a Schwarzschild black hole; in Section V we calculate the QNM of the Planck star and compare the results with the QNM of a Schwarzschild black hole; finally in Section VI we conclude our discussion.

We use the natural units system in which $G = c = \hbar = 1$ so that masses are measured in meters and frequency in m^{-1}.

II. A MODEL FOR A BOUNCING BLACK HOLE

We follow [3] and assume that the system is spherical symmetric (and therefore non rotating), we consider a spherical shell of null matter disregarding the thickness of the shell, we assume time reversal symmetry; we also assume that quantum mechanics effects (which lead to the violation of the Einstein equations) are important in a limited region of spacetime close to the central object, so that at large radii and in the far past and future the metric is again classic; finally, we assume that there is no event horizon. With these assumptions we consider a bouncing metric for the infalling shell divided in three regions, all of which are described by a metric of the form:

$$ds^2 = -F(u,v)dudv + r^2(u,v)(d\theta^2 + \sin^2\theta d\phi^2) \tag{1a}$$

See figure 1 for a Penrose diagram of the spacetime of a Planck star. Region I is the internal part of the black hole and is a portion of a Minkowski spacetime described by the choice

$$F(u_I, v_I) = 1 \qquad r(u_I, v_I) = \frac{v_I - u_I}{2} \tag{1b}$$

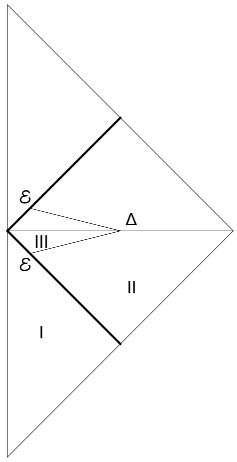

Figure 1: A Penrose diagram of the spacetime of a Planck star. See also [3].

Region II is a portion of Kruskal spacetime in which

$$F(u_{II}, v_{II}) = \frac{32m^3}{r} \exp\left(-\frac{r}{2m}\right) \tag{1c}$$

and r is defined implicitly by:

$$\left(1 - \frac{r}{2m}\right) \exp\left(\frac{r}{2m}\right) = u_{II} v_{II}$$

which can be written in terms of the Lambert W function [15]:

$$r(u, v) = 2m \left[1 + W\left(-\frac{u_{II} v_{II}}{e}\right)\right] \tag{1d}$$

Finally, Region III is the portion of spacetime in which quantum effects are important; in [3] the authors chose the ansatz:

$$F(u_{III}, v_{III}) = \frac{32m^3}{r} \exp\left(-\frac{r}{2m}\right) \qquad r = \frac{v_{III} - u_{III}}{2} \tag{1e}$$

The three regions must be matched with one another. Region I is bounded by the past light cone of the origin $v_I = 0$ and region II is bounded by the null line $v_{II} = v_0$; this implies that the matching condition between region I and region II is:

$$u_{II}(u_I) = \frac{1}{v_0} \left(1 + \frac{u_I}{4m}\right) \exp\left(\frac{u_I}{4m}\right) \tag{2}$$

The matching between the Kruskal region II and the quantum region III is obtained by imposing that the coordinates (u_{III}, v_{III}) are equal to the coordinate (u_{II}, v_{II}) on the boundary.

There are two important points in the above metric: the point $\mathcal{E}$ in which the infalling shell reaches the quantum region and Δ which is the maximal extension of the region in which the Einstein equations are violated [3]. Following [3] we impose that $\mathcal{E}$ has coordinates $(-2\epsilon, 0)$ and that the point Δ is in the point with $t = 0$ and $r = 2m + \delta$.

The above metric is described by the three parameters v_0, δ and ϵ; all of them are functions of the mass of the black hole m [3]:

$$\epsilon \approx \left(\frac{m}{m_p}\right)^{\frac{1}{3}} l_p \qquad v_0 \approx \exp\left(-k \frac{m}{2l_p}\right) \qquad \delta \approx \frac{m}{3} \tag{3}$$

where k is a number depending on the curvature radius of the spacetime and on the bouncing time and whose expression can be found in [3].

Now that we have a model for the infalling shell, we can calculate the Regge-Wheeler equations.

III. REGGE-WHEELER EQUATION

Regge-Wheeler (RW) equations can be calculated starting from the perturbed Einstein equations. We consider a metric of the form (see for example [13]):

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \tag{4}$$

In [14] we have calculated the general expression for the perturbation of the Einstein equations:

$$\begin{aligned}\delta G_{\mu\nu} = &\frac{1}{2}g_{\mu\nu}h^{\rho\sigma}R_{\rho\sigma} - \frac{1}{2}h_{\mu\nu}R - \frac{1}{2}h^{\rho}{}_{\rho;\mu;\nu} + \frac{1}{2}h_{\nu}{}^{\rho}{}_{;\mu\rho} + \frac{1}{2}h_{\mu}{}^{\rho}{}_{;\nu;\rho} + \\ &-\frac{1}{2}h_{\mu\nu}{}^{;\rho}{}_{;\rho} + \frac{1}{4}g_{\mu\nu}g^{\rho\sigma}h^{\tau}{}_{\tau;\rho;\sigma} - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}h_{\rho}{}^{\tau}{}_{;\sigma;\tau} + \frac{1}{4}g_{\mu\nu}g^{\rho\sigma}h_{\rho\sigma}{}^{;\tau}{}_{;\tau}\end{aligned} \tag{5}$$

We notice that, since part of the metric is in a region in which quantum effects are non negligible, the actual equations describing the system are not Einstein equations, but some effective equations; we however use Einstein equations as a first approximation, since it is also of interest to study what they imply in this situation.

Using the Regge-Wheeler gauge, the axial part of the perturbation field $h_{\mu\nu}$ can be written as [13]:

$$h^{axial}_{\mu\nu} = \sum_{l=2}^{+\infty}\sum_{m=-l}^{+l} h^{Bt}_{lm}(u,v)(t^{Bt}_{lm}(\theta,\phi))_{\mu\nu} + \sum_{l=1}^{\infty}\sum_{m=-l}^{+l} h^{B1}_{lm}(u,v)(t^{B1}_{lm}(\theta,\phi))_{\mu\nu} \tag{6}$$

where $(t^{Bt}_{lm}(\theta,\phi))_{\mu\nu}$ and $(t^{B1}_{lm}(\theta,\phi))_{\mu\nu}$ are the Zerilli tensor harmonics whose expression can be found in [13].

We are interested in the axial perturbations, so we have to choose the components $(\mu,\nu) = (u,\phi),(v,\phi),(\theta,\phi)$ in equation (5). Substituting the general expression for metric (1a) in equation (5), we find the following three equations for the two perturbation fields $h^{Bt}_{lm}(u,v)$ and $h^{B1}_{lm}(u,v)$:

$$\begin{aligned}&-\partial_u^2 h^{B1}_{lm} + \partial_u\partial_v h^{B0}_{lm} - \frac{2}{r}(\partial_v r)(\partial_u h^{B0}_{lm} + \frac{(\partial_u F)}{F}(\partial_u h^{B1}_{lm}) - \frac{(\partial_u F)}{F}(\partial_v h^{B0}_{lm}) + \frac{2}{r}(\partial_u r)(\partial_v h^{B0}_{lm}) + \\ &+ 2h^{B0}_{lm}P_1 + 2h^{B1}_{lm}P_2 = 0\end{aligned} \tag{7a}$$

$$\begin{aligned}&+\partial_u\partial_v h^{B1}_{lm} - \partial_v^2 h^{B0}_{lm} - \frac{(\partial_v F)}{F}(\partial_u h^{B1}_{lm}) + \frac{(\partial_v F)}{F}(\partial_v h^{B0}_{lm}) + \frac{2}{r}(\partial_v r)(\partial_u h^{B1}_{lm}) - 2\frac{(\partial_u r)}{r}(\partial_v h^{B1}_{lm}) + \\ &+ 2h^{B1}_{lm}Q_1 + 2h^{B0}_{lm}Q_2 = 0\end{aligned} \tag{7b}$$

$$(\partial_v h^{B0}_{lm}) = -(\partial_u h^{B1}_{lm}) \tag{7c}$$

in the above equations we have suppressed the dependence on the coordinates in order to have a cleaner notations and we have defined:

$$P_1(u,v) = \frac{(\partial_u F)(\partial_v F)}{F^2} + \frac{(\partial_v r)(\partial_u F)}{Fr} - \frac{(\partial_u r)(\partial_v r)}{r^2} - \frac{(\partial_u\partial_v F)}{F} - 3\frac{(\partial_u\partial_v r)}{r} + \frac{F}{4r^2}l(l+1) \tag{8}$$

$$P_2(u,v) = -\frac{(\partial_u F)(\partial_v r)}{Fr} + \frac{(\partial_u r)^2}{r^2} + \frac{(\partial_u^2 r)}{r} \tag{9}$$

$$Q_1(u,v) = \frac{(\partial_u F)(\partial_v F)}{F^2} + \frac{(\partial_v F)(\partial_u r)}{Fr} - \frac{(\partial_u r)(\partial_v r)}{r^2} - \frac{(\partial_u\partial_v F)}{F} - 3\frac{(\partial_u\partial_v r)}{r} + \frac{F}{4r^2}l(l+1) \tag{10}$$

$$Q_2(u,v) = -\frac{(\partial_v F)(\partial_u r)}{Fr} + \frac{(\partial_v r)^2}{r^2} + \frac{(\partial_v^2 r)}{r} \tag{11}$$

By substituting the expression of the functions $F(u,v)$ and $r(u,v)$ into the definition of the potentials, we get:

- Region I: $F(u,v) = 1$ and $r(u,v) = \dfrac{v-u}{2}$

$$P_1(u,v) = \frac{l(l+1)}{(v-u)^2} + \frac{1}{(v-u)^2} \tag{12}$$

$$P_2(u,v) = \frac{1}{(v-u)^2} \tag{13}$$

$$Q_1(u,v) = \frac{l(l+1)}{(v-u)^2} + \frac{1}{(v-u)^2} \tag{14}$$

$$Q_2(u,v) = \frac{1}{(v-u)^2} \tag{15}$$

- Region II $F(u,v)$ is given by (1c) and $r(uv)$ given by (1d):[1]

$$P_1(u,v) = \frac{8m^3}{r}\exp\left[-\frac{r}{2m}\right]\frac{l(l+1)}{r^2} + \left(-3\frac{(\partial_u r)(\partial_v r)}{r^2} - \frac{(\partial_u r)(\partial_v r)}{2mr} + \frac{(\partial_u\partial_v r)}{2m} - 2\frac{\partial_u\partial_v r}{r}\right) \tag{16}$$

$$P_2(u,v) = \frac{(\partial_u r)(\partial_v r)}{r^2} + \frac{(\partial_u r)^2}{r^2} + \frac{(\partial_u r)(\partial_v r)}{2mr} + \frac{(\partial_u^2 r)}{r} \tag{17}$$

$$Q_1(u,v) = \frac{8m^3}{r}\exp\left[-\frac{r}{2m}\right]\frac{l(l+1)}{r^2} + \left(-3\frac{(\partial_u r)(\partial_v r)}{r^2} - \frac{(\partial_u r)(\partial_v r)}{2mr} + \frac{(\partial_u\partial_v r)}{2m} - 2\frac{\partial_u\partial_v r}{r}\right) \tag{18}$$

$$Q_2(u,v) = \frac{(\partial_u r)(\partial_v r)}{r^2} + \frac{(\partial_v r)^2}{r^2} + \frac{(\partial_u r)(\partial_v r)}{2mr} + \frac{(\partial_v^2 r)}{r} \tag{19}$$

- Region III $F(u,v)$ is given by (1c) and $r(u,v) = \dfrac{v-u}{2}$:

$$P_1(u,v) = -\frac{256m^3}{(v-u)^5}\exp\left(-\frac{r}{2m}\right)l(l+1) + \frac{28m+v-u}{4m(v-u)^2} \tag{20}$$

$$P_2(u,v) = \frac{16m+v-u}{4m(v-u)^2} \tag{21}$$

$$Q_1(u,v) = -\frac{256m^3}{(v-u)^5}\exp\left(-\frac{r}{2m}\right)l(l+1) + \frac{28m+v-u}{4m(v-u)^2} \tag{22}$$

$$Q_2(u,v) = \frac{16m+v-u}{4m(v-u)^2} \tag{23}$$

We notice that P_1 and Q_1 are equal in all regions.

We have now to solve the two coupled differential equations (7) with the potentials given above. We start by using equation (7c) in the terms:

$$\begin{aligned}
\partial_u h^{B1}_{lm}\frac{(\partial_u F)}{F} - \partial_v h^{B0}_{lm}\frac{(\partial_u F)}{F} &= -2\partial_v h^{B0}_{lm}\frac{(\partial_u F)}{F} \\
-\partial_u h^{B1}_{lm}\frac{(\partial_v F)}{F} + \partial_v h^{B0}_{lm}\frac{(\partial_v F)}{F} &= -2\partial_u h^{B1}_{lm}\frac{(\partial_v F)}{F} \\
-\partial_u^2 h^{B1}_{lm} &= \partial_u\partial_v h^{B0}_{lm} \\
-\partial_v^2 h^{B0}_{lm} &= \partial_u\partial_v h^{B1}_{lm}
\end{aligned}$$

We now sum and subtract the first two equations (7) and introduce the functions $\Sigma_{lm} = h^{B0}_{lm} + h^{B1}_{lm}$ and $\Delta_{lm} = h^{B0}_{lm} - h^{B1}_{lm}$ and we get (using the fact that P_1 and Q_1 are equal):

$$\begin{aligned}
&2\partial_u\partial_v\Sigma_{lm} - 2\frac{(\partial_v r)}{r}\partial_u\Sigma_{lm} - 2\frac{(\partial_u r)}{r}\partial_v\Sigma_{lm} - \frac{(\partial_u F)}{F}\partial_v\Sigma_{lm} - \frac{(\partial_v F)}{F}\partial_u\Sigma_{lm} + \\
&+ \frac{(\partial_u F)}{F}\partial_v\Delta_{lm} - \frac{(\partial_v F)}{F}\partial_u\Delta_{lm} + (2P_1 + P_2 + Q_2)\,\Sigma_{lm} + (Q_2 - P_2)\Delta_{lm} = 0
\end{aligned} \tag{24a}$$

$$\begin{aligned}
&2\partial_u\partial_v\Delta_{lm} - 2\frac{(\partial_v r)}{r}\partial_u\Delta_{lm} - 2\frac{(\partial_u r)}{r}\partial_v\Delta_{lm} - \frac{(\partial_u F)}{F}\partial_v\Delta_{lm} - \frac{(\partial_v F)}{F}\partial_u\Delta_{lm} + \\
&+ \frac{(\partial_u F)}{F}\partial_v\Sigma_{lm} - \frac{(\partial_v F)}{F}\partial_u\Sigma_{lm} + (2P_1 + P_2 + Q_2)\,\Delta_{lm} - (Q_2 - P_2)\Sigma_{lm} = 0
\end{aligned} \tag{24b}$$

Finally, we make the following ansatz for the solution of the two equations above:

$$\Sigma_{lm}(u,v) = R(u,v)\sqrt{F(u,v)}\phi(u,v) \qquad \Delta_{lm}(u,v) = R(u,v)\sqrt{F(u,v)}\psi(u,v) \tag{25}$$

[1] Note that it depends on the product of the coordinates u and v.

where $F(u,v)$ has the form described in the previous section in the three regions and $R(u,v)$ is defined as:

$$R(u,v) = \begin{cases} \dfrac{v-u}{2} & \text{in Region I} \\ r(uv) & \text{in Region II} \\ \dfrac{v-u}{2} & \text{in Region III} \end{cases} \tag{26}$$

where $r(uv)$ is defined in equation (1d). By substituting this ansatz in (24), after some algebra, we find that the equations reduce to the coupled differential equations system:

$$\partial_u\partial_v\phi + \frac{1}{2}\left[(\partial_u\psi)\frac{(\partial_v F)}{F} - (\partial_v\psi)\frac{(\partial_u F)}{F}\right] + W(u,v)\phi + V\psi = 0 \tag{27a}$$

$$\partial_u\partial_v\psi + \frac{1}{2}\left[(\partial_u\phi)\frac{(\partial_v F)}{F} - (\partial_v\phi)\frac{(\partial_u F)}{F}\right] + W(u,v)\psi - V\phi = 0 \tag{27b}$$

where the potentials are given by:

$$\begin{aligned} W(u,v) = P_1 + \frac{Q_2+P_2}{2} + \\ -\frac{3}{4}\frac{(\partial_u F)(\partial_v F)}{F} - \frac{1}{2}\frac{(\partial_u r)(\partial_v F)}{rF} - \frac{1}{2}\frac{(\partial_v r)(\partial_u F)}{rf} - 2\frac{(\partial_u r)(\partial_v r)}{r^2} + \frac{1}{2}\frac{\partial_u\partial_v F}{F} + \frac{\partial_u\partial_v r}{r} \end{aligned} \tag{28a}$$

$$V(u,v) = \frac{Q_2-P_2}{2} + \frac{1}{2}\left(\frac{(\partial_u F)(\partial_v r)}{rF} - \frac{(\partial_u r)(\partial_v F)}{Fr}\right) \tag{28b}$$

It can be checked, that with the definition of F given in section II the second term in equation (28b) vanishes.

IV. APPLICATION TO THE SCHWARZSCHILD BLACK HOLE

In this section we apply the shooting method as described in [11, 12] in order to calculate the QNM of the Schwarzschild black hole.

In the Schwarzschild black hole case, we deal with only the Region II extended to the whole spacetime. We can make the ansatz that the $\phi(u,v)$ and $\psi(u,v)$ function have the following form:

$$\phi(u,v) = A_1(u+v)B_1(r(uv)) \qquad \psi(u,v) = A_2(u+v)B_2(r(uv)) \tag{29}$$

and we impose that $t = \dfrac{u+v}{2}$, so that the system (27) reduces to:

$$\begin{aligned} A_1(t)B_1'(r) + 2t\dot{A}_1(t)B_1'(r)\exp\left(-\frac{r}{2m}\right) + \ddot{A}_1(t)B_1(r) + uvA_1(t)B_1''(r) + W(t,r)A_1(t)B_1(r) + \\ + \frac{1}{2}\left[(\partial_u A_2(t)B_2(r))\frac{(\partial_v F)}{F} - (\partial_v A_2(t)B_2(r))\frac{(\partial_u F)}{F}\right] + V(u,v)A_2(t)B_2(r) = 0 \end{aligned} \tag{30a}$$

$$\begin{aligned} A_2(t)B_2'(r) + 2t\dot{A}_2(t)B_2'(r)\exp\left(-\frac{r}{2m}\right) + \ddot{A}_2(t)B_2(r) + uvA_2(t)B_2''(r) + W(t,r)A_2(t)B_2(r) + \\ + \frac{1}{2}\left[(\partial_u A_1(t)B_1(r))\frac{(\partial_v F)}{F} - (\partial_v A_1(t)B_1(r))\frac{(\partial_u F)}{F}\right] - V(u,v)A_1(t)B_1(r) = 0 \end{aligned} \tag{30b}$$

where a dot indicates a derivative with respect to the time t and a prime a derivative with respect to the coordinate

r. The potentials are given by:

$$W(t,r) = \exp\left(-\frac{r}{2m}\right)\left[-\frac{20m^4}{r^4} + \frac{30m^3}{r^3} - \frac{8l(l+1)}{r^3} - \frac{7m^2}{r^2} + \frac{m}{2r}\right] - \frac{8m^3}{r^3}\exp\left(-\frac{r}{2m}\right)t^2 \tag{31}$$

$$V(t,r) = \exp\left(-\frac{r}{m}\right)\frac{4m^3}{r^3}(v^2-u^2) \tag{32}$$

$$\frac{(\partial_u F)}{F} = \frac{1}{u}\left(\frac{4m^2}{r^2} - 1\right) \tag{33}$$

$$\frac{(\partial_v F)}{F} = \frac{1}{v}\left(\frac{4m^2}{r^2} - 1\right) \tag{34}$$

$$\tag{35}$$

By substituting the above potentials in the system (30), one can see that the first line of each equation has the following form:

$$\begin{aligned}&\left[\ddot{A}_i(t) + 2t\frac{B_i'(r)}{B(r)} - \frac{8m^3}{r^3}\exp\left(-\frac{r}{2m}\right)t^2 A_i(t)\right]B_i(r)+\\ &+\exp\left(-\frac{r}{2m}\right)\left[\left(1-\frac{r}{2m}\right)r^{-2}\,B_i''(r) - 2m\,r^{-3}\,B_i'(r) + \left(-\frac{20m^4}{r^4} + \frac{30m^3}{r^3} - \frac{8l(l+1)}{r^3} - \frac{7m^2}{r^2} + \frac{m}{2r}\right)\right]A_i(t)\end{aligned} \tag{36}$$

which is separable; we therefore assume that the second lines of equations (30) are a small perturbation[2], so that we can decouple the system and solve separately for $A_i(t)$ and $B_i(r)$. The separated equations have the form:

$$\begin{aligned}&\ddot{A}_i(t) + 2t\frac{B_i'(r)}{B(r)}\exp\left(-\frac{r}{2m}\right) - \frac{8m^3}{r^3}\exp\left(-\frac{r}{2m}\right)t^2 A_i(t) = -\omega^2 A_i(t)\\ &\exp\left(-\frac{r}{2m}\right)\left[\left(1-\frac{r}{2m}\right)\frac{1}{r^2}B_i''(r) - \frac{1}{r^3}B_i'(r) + \left(-\frac{20m^4}{r^4} + \frac{30m^3}{r^3} - \frac{8m^3 l(l+1)}{r^3} - \frac{7m^2}{r^2} + \frac{m}{2r}\right)\right] = \omega^2 B_i(r)\end{aligned} \tag{37}$$

By looking at the the first equation, we note the presence of the exponentials which kill off the last two terms on the left hand side for large distances, so we are left with the equation:

$$\ddot{A}_i(t) = -\omega^2 A_i(t) \tag{38}$$

which has solutions of the form $A_i(t) \propto \exp(\pm i\omega t)$, so that we see that the separation constant ω is the quasi normal mode frequency.

By introducing the reduced radial coordinate $R = r/(2m)$, the second equation can be rewritten:

$$B_i''(R)R^2(R-1) + RB_i'(R) = \left(-\frac{5}{4} + \frac{15}{4}R - l(l+1)R - \frac{7}{4}R^2 + \frac{1}{4}R^3\right)B_i(R) + R^4\exp(R)\omega^2 B_i(R) \tag{39}$$

This equation must be solved numerically for each value of the angular momentum l. We solved it with shooting method and calculated the values of ω by matching the two equations at the radial coordinate $R = 10$. As discussed in [11], with the shooting method only the lowest modes can be reliably computed. Our results are reported in the third column of table I for $l = \{2, 3, 4\}$ along with the QNM calculated in [12], reported in the second column: we see that the modes calculated with our method are in agreement with those of Chandrasekhar-Detweiler to within a few percent, this means that our derivation of the Regge-Wheeler equation and the assumption made in order to simplify it are correct.

V. PLANCK STAR

In order to find the quasi normal mode for the Planck star, we have to solve the differential equations (27) separately in the three regions and match the resulting functions. We shall now derive the solution to the system (27) in each region.

[2] This can be verified only at the end of the calculation, by substituting the expressions for the first order functions $A_i(t)$ and $B_i(r)$ or by comparing the results with classical ones.

A. Region I

In the region I, the spacetime is described by a Minkowski metric. Here we have $F(u,v) = 1$, $r = \dfrac{v-u}{2}$ and $t = \dfrac{v+u}{2}$. We look for functions of the form:

$$\begin{aligned} \phi(u,v) &= A_1^M(u+v)B_1^M(v-u) = A_1^M(t)B_1^M(r) \\ \psi(u,v) &= A_2^M(u+v)B_2^M(v-u) = A_2^M(t)B_2^M(r) \end{aligned} \tag{40}$$

By substituting these expression into (27), we get:

$$-A_1^M(t)B_1^{M\prime\prime}(r) + \ddot{A}_1^M(t)B_1^M(r) - \frac{1}{4}\left(-\frac{4}{r^2} - \frac{1}{m^2} + \frac{l(l+1)}{r^2}\right)A_1^M(t)B_1^M(r) = 0 \tag{41a}$$

$$-A_2^M(t)B_2^{M\prime\prime}(r) + \ddot{A}_2^M(t)B_2^M(r) - \frac{1}{4}\left(-\frac{4}{r^2} - \frac{1}{m^2} + \frac{l(l+1)}{r^2}\right)A_2^M(t)B_2^M(r) = 0 \tag{41b}$$

which are separable; we find:

$$A_i^M(t) \propto \exp(\pm i\omega t) \tag{42}$$

$$B_i^{M\prime\prime}(r) + \frac{1}{4}\left(-\frac{4}{r^2} - \frac{1}{m^2} + \frac{l(l+1)}{r^2}\right)B_i^M(r) = \omega^2 B_i^M(t) \tag{43}$$

B. Region II

In region II the metric is that of Scharzschild. We have already solved the spatial differential equation in the previous section. The time differential equation is given by:

$$\ddot{A}_i^S(t) + 2t\frac{B_i^{S\prime}}{B_i^S}\exp\left(-\frac{r}{2m}\right)\dot{A}_i^S(t) - \frac{8m^3}{r^3}\exp\left(-\frac{r}{2m}\right)A_i^S(t) = -\omega^2 A_i^S(t) \tag{44}$$

We can again assume that the second an third elements are small for large r, so the solution is again oscillating. The radial part has the same form of that reported in equation (39).

C. Region III

In region III $F(u,v)$ is given by equation (1c), while $r = \dfrac{v-u}{2}$ and $t = \dfrac{v+u}{2}$. We look for solutions of the form

$$\begin{aligned} \phi(u,v) &= A_1^Q(u+v)Q_1^Q(v-u) = A_1^Q(t)Q_1^Q(r) \\ \psi(u,v) &= A_2^Q(u+v)B_2^Q(v-u) = A_2^Q(t)Q_2^Q(r) \end{aligned} \tag{45}$$

The system (27) reduces to:

$$\ddot{A}_1^Q(t)B_1^Q(r) - A_1^Q(t)B_1^{Q\prime\prime}(r) + \left[\frac{256m^2 - 32mr + 4r^2}{256m^2r^2} - \exp\left(-\frac{r}{2m}\right)\frac{l(l+1)}{4r^2}\right]A_1^Q(t)B_1^Q(r) = 0 \tag{46a}$$

$$\ddot{A}_2^Q(t)B_2^Q(r) - A_2^Q(t)B_2^{Q\prime\prime}(r) + \left[\frac{256m^2 - 32mr + 4r^2}{256m^2r^2} - \exp\left(-\frac{r}{2m}\right)\frac{l(l+1)}{4r^2}\right]A_1^Q(t)B_1^Q(r) = 0 \tag{46b}$$

Both of the above equations are separable. The solution for the time equation is again of the form $A_i^Q(t) \propto \exp(\pm i\omega t)$.

The differential equation for the spatial function is (again introducing the reduced radial coordinate $R = r/(2m)$):

$$B_i^{Q\prime\prime}(R) = \left[\frac{1}{16} + \frac{1}{R^2} - \frac{1}{4R} - \exp(-R)\frac{l(l+1)}{4R^2}\right]B_i^Q(r) + \omega^2 B_i^Q(r) \tag{47}$$

D. Calculation of quasi normal frequencies

The three solutions reported above must be matched at the points in which the three patches of the spacetime metric meet. We saw in section II (see also [3]) that the Minkowski and Schwarzschild region match when $r(u,v) = r(u_I, v_I)$ where:

$$v_I = v_0 \qquad u = \frac{1}{v_0}\left(1 + \frac{u_I}{4m}\right)\exp\left(\frac{u_I}{4m}\right) \tag{48}$$

while the quantum region has as boundary the point between $\mathcal{E} = (\pm 2\epsilon, 0)$ and Δ which is located at the point with $r = 2m + \delta$ and $t = u + v = 0$. δ and v_0 are defined in section II (see also [3]).

We assume that the observer is located at infinity in the point $u = 0$, so we have to match the solutions in the quantum and Schwarzschild region at the point $\mathcal{E}$ with

$$r_0 = \epsilon = \left(\frac{m}{m_p}\right)^{1/3} l_p \tag{49}$$

We also calculate the QNMs at the point Δ as a double check of our results, so, in this second case,

$$r_0 = 2m + \frac{m}{3}. \tag{50}$$

n	Chandrasekhar Detweiler [12]	Schwarzschild	Planck star at $\mathcal{E}$	Planck star at Δ
		$l = 2$		
0	0.74734 + 0.17792 i	0.74629 + 0.18245 i	0.82053 + 0.00181 i	0.81542 + 0.00082 i
1	0.69687 + 0.54938 i	0.72099 + 0.57401 i	0.44531 + 0.23963 i	0.43821 + 0.22852 i
		$l = 3$		
n	Chandrasekhar Detweiler [12]	Schwarzschild	Planck star at $\mathcal{E}$	Planck star at Δ
0	1.19889 + 0.18541 i	1.19000 + 0.18500 i	0.54405 + 0.11844 i	0.19201 + 0.53348 i
1	1.16402 + 0.56231 i	1.60577 + 0.56177 i	0.36380 + 0.14967 i	0.57942 + 0.15785 i
2	0.85257 + 0.74546 i	0.85071 + 0.74871 i	0.51967 + 0.24554 i	0.49875 + 0.25784 i
		$l = 4$		
n	Chandrasekhar Detweiler [12]	Schwarzschild	Planck star at $\mathcal{E}$	Planck star at Δ
0	1.61835 + 0.18832 i	1.67507 + 0.19514 i	0.45484 + 0.10130 i	0.45879 + 0.12486 i
1	1.59313 + 0.56877 i	1.59305 + 0.56892 i	0.45627 + 0.19286 i	0.45794 + 0.18727 i
2	1.12019 + 0.84658 i	0.92223 + 0.82797 i	0.54294 + 0.26059 i	0.53482 + 0.24879 i

Table I: In the second column we report the QNM calculated by Chandrasekhar and Detweiler in [12], in the third and in the fourth columns we report our results: in the third our QNM for a Schwarzschild black hole and in the fourth those for a Planck star at the point $\mathcal{E}$ and in the fifth those for a Planck star at Δ. We only report the lowest modes, because we use the shooting method, see main text and [11].

The values of the QNM calculated for a Planck star are reported in the fourth column of table I. By comparing the QNMs of a Schwarzschild black hole to the ones of a Planck star, it is possible to notice that imaginary parts are smaller for the star, so that the damping time is longer; this means that the QNMs of a Planck star die off more slowly with time than the one of a Schwarzschild black hole. From a comparison between the fourth and fifth column, we see that the QNM calculated at $\mathcal{E}$ and at Δ are similar (except for the cases $l = 3$ $n = \{0, 1\}$ where the difference is significant), proving our method is robust.

VI. CONCLUSIONS

In this work we have considered the model presented in [3] in order to calculate the QNMs of a Planck star with a method similar to that of Chandrasekhar and Detweiler described in [12]. The application of our method based on the use of the shooting method to the Schwarzschild black hole case has shown that it gives the correct QNM up to a few percent, thus validating our derivation of Regge-Wheeler equations. With this method we have also calculated the QNM for a Planck star, finding that the imaginary part of the QNMs of a Planck star is smaller than that of a Schwarzschild black hole: this means that the QNMs of a Planck star die off more slowly with time with respect to Schwarzschild black hole QNMs. As far as the real part is concerned, we notice that the values for a Planck star are very different from the Scwarzschild case: this might be due to the use of the classical Einstein equations in place of some effective quantum equations in the quantum region. It might be of interest to see the change in the QNM calculated using effective equations: we leave this for a future work.

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