THE NEUTRON STARS STRUCTURE IN METRIC THEORIES OF GRAVITATION

C.O. Loustó[1] y M.A. Castagnino[2]

1 IAFE

2 IFR , IAFE

RESUMEN: Partiendo de un principio variacional de la energía interna total de una estrella de neutrones además de algunas restricciones sobre la forma de la métrica, se hallan las ecuaciones de equilibrio de las estrellas de neutrones. Estas ecuaciones se resuelven para hallar la masa máxima que una estrella de neutrones puede soportar sin que se produzca el colapso total. Finalmente se conectan estos resultados teóricos con los datos observacionales.

ABSTRACT: From the variational principle for the total internal energy of a neutron star and some restrictions on the form of the metric coefficients, we have found equations of equilibrium which are valid for every metric theory of gravitation. We also present some simple solutions of the equations to find the neutron stars maximum mass.

1. INTRODUCTION

There is a great theoretical as well as experimental interest, in the determination of the maximum mass that neutron stars may have, M^{*n}_{max} , before a total gravitational collapse would occur leading to a possible formation of a black hole.

The main doubts to determine M^{*n}_{max} are:

a) The equation of state: M^{*n}_{max} is very much affected by the equation of state chosen for densities greater than the nuclear one, i.e. $3.10^{14} gr/cm^3$.

b) The rotation: most of the analysis show that the rotation does not increase M^{*n}_{max} in more than 20%[1]).

c) The theory of gravitation: although the different gravitational theories may have the post-Newtonian limit near to the one of General Relativity, their predictions for the regimen of intense field of neutron stars structure may greatly differ from those of General Relativity.

Some theories do not even predict the existence of black holes[2]).

In this paper we shall consider a fix equation of state and stars where the rotation is not important in order to examine the variation of M^{*n}_{max} as a function of gravitational theories. Then, the observed value of M^{*n}_{max} could eventually be used as a proof to test gravitational theories.

This problem was explored[3]) by means of the post-Newtonian parametrization formalism, but for the study of M^{*n}_{max} it is necessary to reach central densities which break down the assumptions of P-P-N, i.e. 10^{15} gr/cm^3. Thus, we believe that our formalism could be useful to face this kind of problems.

2. EQUILIBRIUM EQUATIONS

We shall suppose that the star is static and sphericallly symmetric, described by a perfect fluid stress-energy tensor. We shall take stellar configurations with uniform entropy per nucleon s, and chemical composition.

The star interior metric can be described[4]:

$$ds^2 = -Exp(2\Phi(r))dt^2+Exp(2\Psi(r))dr^2+r^2(d\theta+\sin^2\theta\, d\phi^2)$$

The mass M_o that the matter of the star would have if it is dispersed to infinity is simply $M_o = km_oN/4\pi$, where $K = 4\pi G/c^2$, $m_o = 1.66\times10^{-24}$gr is the rest mass of a nucleon and N is the total number of nucleons in the star.

The nucleon number is given by:

$$N = \int \sqrt{-g}\, J_N^o\, dr d\theta d\phi = \int Exp(\Phi+\Psi) J_N^o r^2 dr d\theta d\phi,$$

where J_N^o is the conserved nucleon number current and g is the determinant of the metric.

The nucleon number density measured in a locally inertial reference frame at rest in the star is:

$n = -U_\mu J_N^\mu = Exp(\Phi)J_N^o$ (where $U_\mu = (-Exp(\Phi),0,0,0)$ the four-velocity of a fluid in a static star).

$$\text{Then:}\quad N = 4\pi \int_0^R Exp(\Psi(r))n(r)r^2dr \qquad (1)$$

The internal energy of the star is given by:

$E = (M - Km_oN/4\pi)c^2$ (M is the total mass of the star).

The equilibrium of a configuration will be stable with respect to total energy density oscillations if and only if M, or equivalently E, is a minimum with respect to all such variations.

Using the Lagrange multiplier method: M will be stationary with respect to all variations that leave N fixed if and only if there exists a constant λ for which $M-\lambda KN$ is stationary with respect to all the above mentioned variations.

At this point, we must choose some qualitative behavior for Ψ. We do so, based in the following requirements:

a) Ψ must coincide with General Relativity up to post-Newtonian order.

b) Ψ must incorporate General Relativity up to all order as a particular case.

Let be $m(r) = K \int_0^r \varepsilon(r) r^2 dr$, (2)

where $\varepsilon(r)$ is the total energy density of the fluid, thus $m(R) = M$ (R is the star radius).

As $\Psi(r)$ is a function of $m(r)/r$, up to all order, in General Relativity, we shall suppose that it is also the case in our formalism. This will satisfy b) and also a) with the proper Ψ. Thus $\Psi = \Psi(m(r)/r)$.

From (1) and (2): $\delta M - \lambda K \delta N = K \int_0^R r^2 \; \delta\varepsilon(r) dr -$

$$\lambda K \int_0^R r^2 \, Exp(\Psi) \; \delta n(r) dr - \lambda K \int_0^R dr r^2 n(r) \, Exp(\Psi) \partial\Psi/\partial m(r) \; \delta m(r) \qquad (3)$$

From (1) $\delta m(r) = K \int_0^r r^2 \delta\varepsilon \, dr$. Besides, these variations are not supposed to change the entropy per nucleon s[5].

So, $\delta s = 0 = \delta(\varepsilon/n) + p\,\varepsilon(1/n) \Rightarrow \delta n(r) = n\,\delta\varepsilon/(p+\varepsilon)$, where p is the pressure of the fluid.

Then replacing it in (3):

$$\delta M - \lambda K \delta N = K \int_0^R r^2 dr \left(1 - \lambda n(r) Exp(\Psi(r))/p(r) + \varepsilon(r)\right) -$$

$$\lambda K \int_r^R d\tilde{r}\tilde{r}^2 n(\tilde{r}) Exp(\Psi(\tilde{r})) \, \partial\Psi/\partial m(\tilde{r})) \delta\varepsilon(r),$$

where we have interchanged the r and $\tilde{r}$ integrals in the last term. $\delta M - \lambda K \delta N$ will vanish for all $\delta\varepsilon(r)$ if and only if:

$$1/\lambda = n(r)Exp(\Psi(r))/(p(r)+\varepsilon(r)) + K \int_r^R d\tilde{r}\tilde{r}^2 n(\tilde{r})Exp(\Psi(\tilde{r}))\partial\Psi/\partial m(\tilde{r})$$

λ must be independent of r, then differentiating with respect to r, ($' = d/dr$):

$$0=(n'(p+\varepsilon)-n(p'+\varepsilon')/(p+\varepsilon)^2)+n/(p+\varepsilon)d\Psi/dr-Kr^2n\,\partial\Psi/\partial m(r) \quad (4)$$

The condition of uniform entropy per nucleon gives:

$ds/dr=0=d(\varepsilon/n)/dr+pd(1/n)/dr$ and, therefore, $n'=n\varepsilon'/(p+\varepsilon)$ replacing it in (4):

$$0=-p'/(\varepsilon+p)^2+1/(p+\varepsilon)d\Psi/dr-Kr^2\partial\Psi/\partial m(r)$$

then,

$$dp/dr=(\varepsilon+p)(d\Psi/dr-Kr^2(p+\varepsilon)\partial\Psi/\partial m(r). \quad (5)$$

From (2) $dm/dr=Kr^2\varepsilon$ and from $\Psi=\Psi(m(r)/r)$ we can put eq. (5) as: $dp/dr= -(\varepsilon+p)(m(r)/r+Kr^2p)\partial\Psi/\partial m(r)$.

But $\Psi(r)$ is the interior metric and we want to relate the equation to the exterior one.

Let be $g_{rr}(r)=A(r)=Exp(\tilde{A}(r))$ the exterior metric coefficient, which could be known via the study of test particles orbits or as a vacuum solution of a particular theory of gravitation.

Matching the interior and exterior metric at $r=R$ and demanding it to be valid for every R we find (choosing an ansatz): $\Psi(m(r)/r)=\tilde{A}(M/r))\big|_{M \leftrightarrow m(r)}$

Then our final solution will be:

$$dp/dr = -(\varepsilon+p)(m(r)/r+Kr^2p)\,\partial\tilde{A}/\partial M\big|_{M\leftrightarrow m(r)} =$$

$$-1/2(\varepsilon+p)(m(r)/r+Kr^2p)\partial\ln(A)/\partial M\big|_{M\leftrightarrow m(r)}, \quad (6)$$

where $M \leftrightarrow m(r)$ means that after differentiating partially with respect to the total mass M, we must replace it by m(r) given by (2).

For the Schwarzschild metric $A(r)=(1-2M/r)^{-1}$, replacing it in equation (6) we find the general relativistic equation of equilibrium for neutron stars:
$dp/dr = -(\varepsilon+p)(m(r)/r^2+Krp)/(1-2m(r)/r)$.

3. COMPUTATIONS

We shall present now some examples of application:
a) Constant energy density: $\varepsilon = \varepsilon_0$=constant

Integrating (2): $m(r) = K\varepsilon_o r^3/3$. Replacing it in (6) $2dp/dr = -(\varepsilon_o+p)Kr(\varepsilon_o/3+p)(1/AdA/dr)|_{M\leftrightarrow m(r)}$

Also as $d(m(r)/r) = 2K\varepsilon_o rdr/3$
$2dp/(K(\varepsilon_o+p)(\varepsilon_o/3+p)) = -3/(2K\varepsilon_o)dA/A|_{M\leftrightarrow m(r)}$

Equation which can be easily integrated:

$$p(r)=\varepsilon_o(A^{-1/2}(R)-A^{-1/2}(r))/(A^{-1/2}(r)-3A^{-1/2}(R))$$

Let's see that $p_c=p(0)=$
$$=\varepsilon_o(A^{-1/2}(R)-1)/(1-3A^{-1/2}(R))$$
then $p_c \to \infty$ when $A(R) \to 9$. Then $A(R) = 9$ would give the maximum value of M/R (if $A'<0$ at every $R<r<\infty$).
b) Other interesting case is when the velocity of the sound v_s is constant, i.e. $p = \alpha\varepsilon$ with α = constant.

Let's propose the solution $p=p_o/r^2$ with p_o constant. Putting this in (2): $m(r) = Kp_o r/\alpha$. Replacint it in (6) $x/A(x)dA(x)/dx = 4/(\alpha(1+1/\alpha))$, with $x=M/R=Kp_o/\alpha$ from where we find p_o.

The limit $p_c \to \infty$ gives the maximum mass[6].

c) We shall now choose a more realistic equation of state and parametrize $A(r)$.

In the Schwarzschild metric $A(r)=(1-2M/r)^{-1}$. Then we study the following set of possible $A(r)$:

$A(r) = (1-2M/r+\mu M^2/r^2)^{-1}$

Replacing it in (6) we have:

$$dp/dr=-(\epsilon+p)(m(r)/r^2+Krp-\mu m^2(r)/r^3-K\mu p m(r))/$$
$$/(1-2m(r)/r+\mu m^2(r)/r^2)$$

Comparing with the results of P-P-N[3] we can associate the term $\mu\, m^2/r^3$ to the post-Newtonian order if $\mu = 5+3\gamma - 6\beta + \xi_2$, then under the light of the experimental results[7] $|\mu| < 10^{-2}$.

It is a very interesting result, because there is not experimental data on μ .

Furthermore, we take $\mu = 0$ and add an r^{-3} term to obtain $A(r) = (1-2M/r+\Omega M^3/r^3)$ and replacing it in (6):

$$dp/dr=-(\epsilon+p)(m(r)/r^2+Krp)(1-3/2\,\Omega(m(r)/r)^2)/$$
$$/(1-2m(r)/r+\Omega m^3(r)/r^3)$$

We have integrated this equation and (2) using the Baym et al. equation of state[8], for pressures lower than 6.10^{33} dyn/cm^2, and the one of Bethe & Johnson[9] for higher pressures. The results are plotted in Figure 1, where the full line represents the maximum mass M^{*n}_{max} as a function of the parameter Ω . It is also plotted the best determined neutron star mass, the binary pulsar PSR 1913+16[7] $M_p = 1.42 \pm 0.06\ M_\odot$.

The existence of a neutron star with this mass implies that Ω must be such that $M^{*n}_{max} > M_p$, then from figure 1 $\Omega > -6$.

This example show us how to use the observational data in restricting the values of a theoretical parameters as?

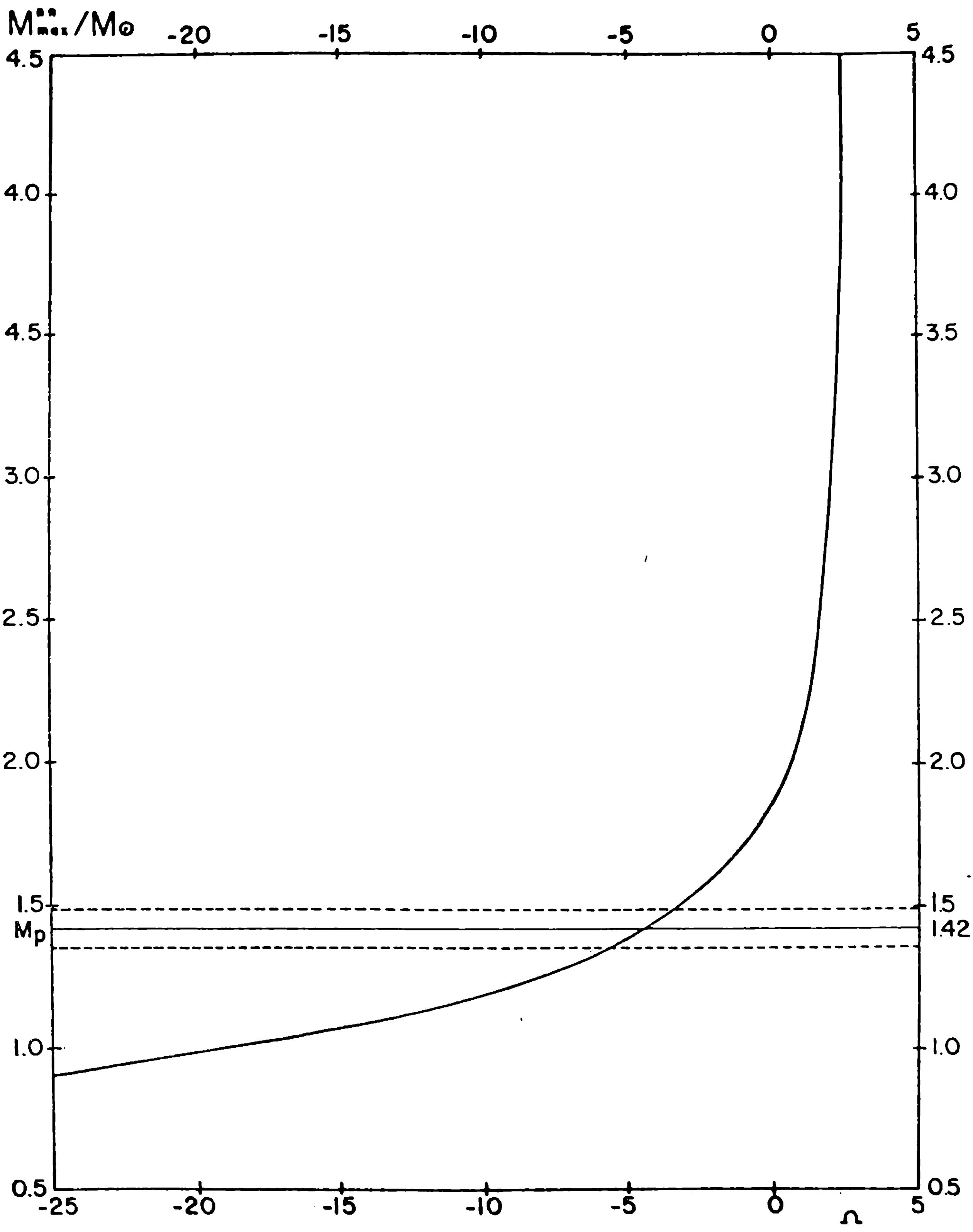
$M^{**}_{max}/M_\odot$
-20
-15
-10
-5
0
5
4.5
4.0
4.5
3.0
2.5
2.0
1.5
M_p
1.0
0.5
3.5
1.42
-25
Ω

Figure 1

4. CONCLUSIONS

Under some restrictions, we have found equations of equilibrium which only depend on the object exterior metric. This shows that one could find empirically the exterior metric and then, immediately know the interior structure of a neutron star. Besides, example c) displays the possibility to test gravitation in the stron field regime, where is not much information available.

Finally, this paper pretends to attract the attention on this kind of problems and go along a way which has not been very much travelled.

This work was supported by the Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina.

REFERENCES

1) Butterworth, E.M.; Ipser, J.R. 1975: Ap.J. Lett. 200, L103.
1976: Ap.J. 204, 200.
Shapiro, S.L.; Lightman, A.P. 1976: Ap.J. 207, 263.

2) Will, C.M. 1981, "Theory and experiment in gravitational Physics", Cambridge U.P., Cambridge.
Chang, D.B.; Johnson, H.H. 1980: Phys. Rev. D, 21, N^{o} 4, 874.

3) Wagoner, R.V.; Malone, R.C. 1974: Ap.J. Lett. 189, L75.
Ciufolini, I.; Ruffini, R. 1981: Astr. & Astrop. Lett. 97, 12.
1983: Ap.J. 275, 867.

4) Misner, C.W. et al. 1973: "Gravitation", Freeman & Co., San Francisco.

5) Weinsberg, S. 1972: "Gravitation and cosmology", Wiley, J. & Sons., New York.
6) Becher, K.; Caporaso, G. 1976: Nature **259**, 377.
7) Will, C.M. 1984: Phys. Rep. **113**, 345.
8) Baym, G. et al. 1971: Ap.J. **170**, 299.
9) Bethe, H.A.; Johnson, M.B. 1974: Nucl. Phys. A. **230**, 1.