LA PLATA-TH 98/18

# Finite density and temperature in hybrid bag models

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(May 4, 2019)

## Abstract

We introduce the chemical potential in a system of two-flavored massless fermions in a chiral bag by imposing boundary conditions in the Euclidean time direction. We express the fermionic mean number in terms of a functional trace involving the Green function of the boundary value problem, which is studied analytically. Numerical evaluations for the fermionic number are presented.

12.39.Ba, 11.10.Wx, 12.38.Mh

## I. INTRODUCTION

Functional determinants of elliptic differential operators are useful for describing one-loop effects in Quantum Field Theory and Statistical Mechanics.

When the fields are constrained to live inside a bounded manifold in the Euclidean space-time, boundary conditions become relevant in the calculation of physical quantities.

This is the case of Hybrid Chiral bags, which are effective models for confinement [1–3] (For a general review on bag models see [4]). They share the characteristics of two succesful approaches to describe baryons: bag models [5,6] and Skyrme model [7–9].

According to the Cheshire Cat Principle (CCP) [1,10], in these two-phase models, fermionic degrees of freedom could be replaced by bosonic ones in a region of space, physical quantities being insensitive to the position of the limit of separation of the two phases. In $1+1$-dimensions the CCP follows from the bosonization of fermionic fields [1]. In the $3+1$-case, topological quantities such as the baryonic number at $T = 0$ have a similar behavior [11], while for non-topological ones, as the total energy at $T = 0$, the CCP is approximately valid [12].

In this paper we will study the mean fermionic number for a two phase model at finite temperature and chemical potential. This represents an extension of previous calculations done in the frame of the MIT bag model [13].

In Section II we will set the problem, expressing the mean fermionic bag number as an adequate trace of the Green function of the system. In doing this, the chemical potential is introduced through the boundary conditions in the Euclidean time direction [14,13].

Section III is devoted to the calculation of the Green function for the Dirac operator obeying the above temporal boundary condition, and spatial (chiral bag) boundary conditions which relate the fermionic degrees of freedom in the inner phase with the Skyrme field in a hedgehog configuration outside of the bag.

In Section IV the expression of the mean fermionic number is built. By means of the asymptotic Debye expansion of Bessel functions we are able to split the mean fermionic number into two pieces: a regular one, to be evaluated numerically, and a possibly non-regular one, to be treated analytically following the techniques presented in Ref. [13].

In Section V we show our numerical results, for the total mean fermionic number including both the contributions of the bag sector and of the skyrmion sector as in Ref. [8].

Finally, in Section VI the conclusions are presented.

## II. THE HYBRID BAG MODEL

We will be interested in the study of the mean fermionic number for a two phase model: a two-flavored massless fermionic field confined inside a static sphere of radius $R$ and a hedgehog Skyrme field filling the exterior sector. These two phases are linked by the spatial chiral boundary conditions, which will be introduced in the following.

The fermionic field inside the bag $\Sigma \otimes [0,1]$, $0 \leq |\vec{x}|, t \leq 1$ is described by the Dirac operator

$$D(\beta, R) = \left[\frac{i}{\beta}\gamma^0\partial_t + \frac{i}{R}\vec{\gamma}\cdot\vec{\nabla}\right] \otimes \mathcal{I}_I \qquad (1)$$

Here $\mathcal{I}_I$ denotes the identity in the flavor (isospin) space, and $\beta$ stands for the inverse temperature.

The external phase is described by a Skyrme model [7,15], whose Lagrangian is given by

$$\mathcal{L} = \frac{1}{16} F_\pi^2 \mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{1}{32e^2} \mathrm{Tr}\left[\left(\partial_\mu U\right) U^\dagger, \left(\partial_\nu U\right) U^\dagger\right]^2, \tag{2}$$

where the scalar field $U(x)$ takes values in the $SU(2)$ group.

When the whole space is filled by $U(x)$, a topological stable classical solution is given by the hedgehog configuration

$$U_0 = e^{i\theta(r)(\vec{\tau}\cdot\vec{x})}, \tag{3}$$

where the profile of the skyrmion $\theta(r)$ is the chiral angle. In this pure skyrmionic model, the imposition of the boundary conditions

$$\theta(r=0) = \pi, \qquad \theta(r\to\infty) \to 0$$

leads to a topological charge (winding number) $\mathcal{B} = 1$, which is identified with the baryonic number of the skyrmion [8,9]. The profile of the chiral angle, $\theta(r)$, is numerically determined by minimizing the classical energy of the soliton [9,16].

When the skyrmion fills only the exterior of the bag, its contribution to the fermionic number, $N_{\mathrm{Sk}}$, is given by the integral of the topological charge density over that region, giving

$$N_{\mathrm{Sk}} = \frac{1}{\pi}\left(\theta - \sin\theta\cos\theta\right) \tag{4}$$

as discussed in Ref. [8].

In the hybrid bag model, fermionic and skyrmionic phases are connected through the spatial boundary conditions

$$B\psi(t,\vec{x}) = 0 \quad \text{for } |\vec{x}| = 1 \tag{5}$$

where

$$B = \mathcal{I}_4 \otimes \mathcal{I}_I + \left(i \not{n} \otimes \mathcal{I}_I\right) e^{-i\theta\gamma^5 \otimes (\vec{\tau}\cdot\vec{n})}, \tag{6}$$

being $\theta = \theta(R)$ the value of the skyrmion profile at the surface of the bag.

Since we are dealing with fermions, these fields must be antiperiodic in the Euclidean time direction,

$$\psi(1,\vec{x}) = -\psi(0,\vec{x}). \tag{7}$$

The Grand Canonical partition function is given by

$$\Xi(T,R,\mu) = e^{-\beta G(T,R,\mu)} \sim \mathrm{Det}\left[D(\beta,R) - i\mu\gamma^0\right]_{\mathrm{BC}}, \tag{8}$$

where “BC” means that the differential operator is defined on a space of functions satisfying both spatial and temporal boundary conditions. In the above expressions, $\mu$ stands for the chemical potential.

As discussed in Ref. [13] the mean fermionic number can be expressed as

$$\langle N_{\rm bag}\rangle(\beta,R,\mu;\theta) = -\frac{\partial G}{\partial \mu}(\beta,R,\mu;\theta) = \frac{1}{\beta} Tr\{-i\left(\gamma^0\otimes\mathcal{I}_I\right)k(t,x;t',x')\}, \tag{9}$$

where $k(t,\vec{x};t',\vec{x'})$ is the Green function of the problem

$$\begin{aligned}
D(\beta,R)k(t,\vec{x};t',\vec{x'}) &= \delta^{(3)}(\vec{x}-\vec{x'})\,\delta(t-t')\mathcal{I}_8, \qquad \text{for } \vec{x},\vec{x'}\in\Sigma, &(10)\\
Bk(t,\vec{x};t',\vec{x'}) &= 0, \qquad\qquad \text{for } \vec{x}\in\partial\Sigma, &(11)\\
k(1,\vec{x};t',\vec{x'}) + e^{-\mu\beta}k(0,\vec{x};t',\vec{x'}) &= 0. &(12)
\end{aligned}$$

The total mean fermionic number for the hybrid chiral bag model is defined as

$$\langle N\rangle(\beta,R,\mu;\theta) = \langle N_{\rm bag}\rangle(\beta,R,\mu;\theta) + N_{Sk}(\theta) \tag{13}$$

One can apply discrete transformations to these equations to obtain symmetries of $\langle N_{\rm bag}\rangle$. In fact,

$$k_1(t,\vec{x};t',\vec{x'}) = -\gamma^5 k(t,\vec{x};t',\vec{x'})\gamma^5 \tag{14}$$

is the Green function of the problem with $\theta\to\theta+\pi$ in the spatial boundary condition, equation (11). Then, from equation (9), it is easy to establish that

$$\langle N_{\rm bag}\rangle(\beta,R,\mu;\theta+\pi) = \langle N_{\rm bag}\rangle(\beta,R,\mu;\theta) \tag{15}$$

Taking into account (4) and (15), the total mean fermionic number satisfies

$$\langle N\rangle(\beta,R,\mu;\theta+\pi) = 1 + \langle N\rangle(\beta,R,\mu;\theta). \tag{16}$$

Moreover,

$$k_2(t,\vec{x};t',\vec{x'}) = \gamma^0\gamma^5 k(1-t,\vec{x};1-t',\vec{x'})\gamma^5\gamma^0 \tag{17}$$

is the Green function of the problem with $\theta\to-\theta$ in the spatial boundary condition and $\mu\to-\mu$ in the Euclidean time boundary condition (12). So, from (9) it can be seen that

$$\langle N_{\rm bag}\rangle(\beta,R,-\mu;-\theta) = -\langle N_{\rm bag}\rangle(\beta,R,\mu;\theta). \tag{18}$$

Notice that the total mean fermionic number has the same behavior, since $N_{Sk}(-\theta) = -N_{Sk}(\theta)$.

From the above relations, one can deduce that

$$\langle N\rangle(\beta,R,-\mu;\theta) = 1 - \langle N\rangle(\beta,R,\mu;\pi-\theta), \tag{19}$$

allowing to determine $\langle N\rangle$ for $\mu<0$ from its values for $\mu>0$ and $0\le\theta\le\pi$.

## III. GREEN FUNCTION

In this section we will obtain the Green function in equations (10),(11),(12). In order to satisfy the "temporal" boundary conditions, we propose for the Green function

$$k(t,\vec{x};t',\vec{x'}) = R\sum_{n=-\infty}^{\infty} k_n(\vec{x},\vec{x'})e^{-i\Omega_n\beta(t-t')}, \tag{20}$$

where

$$\Omega_n = \omega_n - i\mu, \qquad \omega_n = (2n+1)\frac{\pi}{\beta}.$$

We adopt the representation for the Euclidean Dirac matrices,

$$\gamma^0 = i\rho^3\otimes\mathcal{I}_2 \qquad \vec{\gamma} = i\rho^2\otimes\vec{\sigma} \qquad \gamma^5 = \rho^1\otimes\mathcal{I}_2, \tag{21}$$

where $\sigma^k$, $\rho^k$ are Pauli matrices. For each value of $n$, we obtain

$$\left[i\bar{\Omega}_n\rho^3\otimes\mathcal{I}_2\otimes\mathcal{I}_I - \rho^2\otimes\vec{\sigma}\cdot\vec{\nabla}\otimes\mathcal{I}_I\right]k_n(\vec{x},\vec{x'}) = \delta^{(3)}(\vec{x}-\vec{x'})\mathcal{I}_2\otimes\mathcal{I}_2\otimes\mathcal{I}_I, \tag{22}$$

with $\bar{\Omega}_n = \Omega_n R = (2n+1)\pi z - i\bar{\mu}$, being $z = RT$ and $\bar{\mu} = \mu R$ dimensionless variables to be used along the paper.

We write the Green function as the sum of a particular solution of the inhomogeneous differential equation plus a solution of the homogeneous equation,

$$k_n(\vec{x},\vec{x'}) = k_n^{(0)}(\vec{x},\vec{x'}) + \tilde{k}_n(\vec{x},\vec{x'}), \tag{23}$$

with

$$k_n^{(0)}(\vec{x},\vec{x'}) = \sum_{\ell=0}^{3}\rho^\ell\otimes A^{(\ell,n)}(\vec{x},\vec{x'}), \tag{24}$$

$$\tilde{k}_n(\vec{x},\vec{x'}) = \sum_{\ell=0}^{3}\rho^\ell\otimes a^{(\ell,n)}(\vec{x},\vec{x'}). \tag{25}$$

We choose $k_n^{(0)}(\vec{x},\vec{x'})$ so as to give the contribution of a free gas to the mean fermionic number. The second term (25) accounts for the correction, due to the boundary conditions, to the mean fermionic number.

The differential operator (1) and the boundary condition operator (6) are invariant under transformations in the diagonal subgroup of $SU(2)_{rotations}\otimes SU(2)_{isospin}$. So, the Green function is block-diagonal and can be expressed in each invariant subspace in terms of the eigenstates of $\{K^2, J^2, L^2, S^2, I^2, K_3\}$, where $\vec{K} = \vec{L}+\vec{S}+\vec{I}$. For this purpose, we choose the following basis in the $(k,m)$ subspace:

$$|1\rangle = |k, j=k+\tfrac{1}{2}, l=k, m\rangle \qquad |2\rangle = |k, j=k-\tfrac{1}{2}, l=k, m\rangle$$
$$|3\rangle = |k, j=k+\tfrac{1}{2}, l=k+1, m\rangle \quad |4\rangle = |k, j=k-\tfrac{1}{2}, l=k-1, m\rangle$$

for $k \neq 0$, and

$$|1\rangle_0 = |0, j = \tfrac{1}{2}, l = 0, 0\rangle$$
$$|3\rangle_0 = |0, j = \tfrac{1}{2}, l = 1, 0\rangle$$

for $k = 0$.

The free field contribution to the Green function in the invariant subspaces labeled by $n, k, m$ is given by

$$A^{(0)}_{(k,m)}(r, r') = A^{(1)}_{(k,m)}(r, r') = 0 \tag{26}$$

$$A^{(3)}_{(k,m)}(r, r') = iS_n\bar{\Omega}_n^2 J_{(k,m)}(r_<)H_{(k,m)}(r_>) \tag{27}$$

$$A^{(2)}_{(k,m)}(r, r') = -\bar{\Omega}_n^2 \begin{cases} J_{(k,m)}(r_<)M_{(k,m)}H_{(k,m)}(r_>) & r' > r \\ H_{(k,m)}(r_>)M_{(k,m)}J_{(k,m)}(r_<) & r' < r \end{cases}, \tag{28}$$

where the dependence on $n$ is implicit in the arguments of the spherical Bessel functions appearing in the expressions for the diagonal matrices

$$J_{(k,m)}(r) = \text{diag}\left(j_k(iS_n\bar{\Omega}_n r), j_k(iS_n\bar{\Omega}_n r), j_{k+1}(iS_n\bar{\Omega}_n r), j_{k-1}(iS_n\bar{\Omega}_n r)\right) \tag{29}$$

$$J_{(0,0)}(r) = \text{diag}\left(j_0(iS_n\bar{\Omega}_n r), j_1(iS_n\bar{\Omega}_n r)\right) \tag{30}$$

$$H_{(k,m)}(r) = \text{diag}\left(h^{(1)}_k(iS_n\bar{\Omega}_n r), h^{(1)}_k(iS_n\bar{\Omega}_n r), h^{(1)}_{k+1}(iS_n\bar{\Omega}_n r), h^{(1)}_{k-1}(iS_n\bar{\Omega}_n r)\right) \tag{31}$$

$$H_{(0,0)}(r) = \text{diag}\left(h^{(1)}_0(iS_n\bar{\Omega}_n r), h^{(1)}_1(iS_n\bar{\Omega}_n r)\right). \tag{32}$$

In previous equations, $S_n$ stands for the sign of $\omega_n$ and

$$M_{(k,m)} = \begin{pmatrix} 0 & \mathcal{I}_2 \\ \mathcal{I}_2 & 0 \end{pmatrix}, \qquad M_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We propose the following Ansatz for the coefficients introduced in (25) when considering the solution of the homogeneous differential equation,

$$\begin{aligned} a^{(0)}_{(k,m)}(r, r') &= iS_n\bar{\Omega}_n^2 J_{(k,m)}(r)c(0)_{(k,m)}J_{(k,m)}(r') \\ a^{(1)}_{(k,m)}(r, r') &= -i\bar{\Omega}_n^2 J_{(k,m)}(r)M_{(k,m)}c(0)_{(k,m)}J_{(k,m)}(r') \\ a^{(2)}_{(k,m)}(r, r') &= -\bar{\Omega}_n^2 J_{(k,m)}(r)M_{(k,m)}c(3)_{(k,m)}J_{(k,m)}(r') \\ a^{(3)}_{(k,m)}(r, r') &= iS_n\bar{\Omega}_n^2 J_{(k,m)}(r)c(3)_{(k,m)}J_{(k,m)}(r'). \end{aligned} \tag{33}$$

By imposing the boundary conditions, we can determine the unknown matrices $c(0)_{(k,m)}$ and $c(3)_{(k,m)}$ from the equations

$$\begin{aligned} A^{(2)}_{(k,m)}(\vec{x}, \vec{x'}) + a^{(2)}_{(k,m)}(\vec{x}, \vec{x'}) + i\cos\theta\ (i\vec{n}\cdot\vec{\sigma}\otimes\mathcal{I}_I)_{(k,m)}\, a^{(0)}_{(k,m)}(\vec{x}, \vec{x'}) \\ - i\sin\theta\ (i\vec{n}\cdot\vec{\sigma}\otimes i\vec{n}\cdot\vec{\tau})_{(k,m)}\, a^{(1)}_{(k,m)}(\vec{x}, \vec{x'}) = 0, \end{aligned} \tag{34}$$

$$A^{(3)}_{(k,m)}(\vec{x},\vec{x'}) + a^{(3)}_{(k,m)}(\vec{x},\vec{x'}) + \cos\theta\ (i\vec{n}\cdot\vec{\sigma}\otimes\mathcal{I}_I)_{(k,m)}\, a^{(1)}_{(k,m)}(\vec{x},\vec{x'})$$
$$- \sin\theta\ (i\vec{n}\cdot\vec{\sigma}\otimes i\vec{n}\cdot\vec{\tau})_{(k,m)}\, a^{(0)}_{(k,m)}(\vec{x},\vec{x'}) = 0, \tag{35}$$

for $|\vec{x}| = 1$. The matrices $(i\vec{n}\cdot\vec{\sigma}\otimes\mathcal{I}_I)_{(k,m)}$ and $(i\vec{n}\cdot\vec{\sigma}\otimes i\vec{n}\cdot\vec{\tau})_{(k,m)}$ are explicitly given in Ref. [17].

It is straightforward to see that, for $k \neq 0$,

$$c(0)_{(k,m)} = \begin{pmatrix} C_0^{(k,m)} & 0_{2\times2} \\ 0_{2\times2} & -C_0^{(k,m)} \end{pmatrix}, \qquad c(3)_{(k,m)} = \begin{pmatrix} C_{3,+}^{(k,m)} & 0_{2\times2} \\ 0_{2\times2} & C_{3,-}^{(k,m)} \end{pmatrix}, \tag{36}$$

where

$$\left[S_n\cos\theta\,\mathcal{I}_2 + \frac{\sin\theta}{\nu}E_k + \frac{\alpha}{2\nu}\sin\theta\, E_k A_k\right] C_0^{(k,m)} = D_k, \tag{37}$$

$$C_{3,\pm}^{(k,m)} = -J_\pm^{-1}H_\pm + \left[S_n\cos\theta\,\sigma_3 J_\pm^{-1}J_\mp \pm \frac{\sin\theta}{2\nu}\sigma_3 + \frac{\alpha}{2\nu}\sin\theta\, J_\pm^{-1}\sigma_1 J_\pm\right] C_0^{(k,m)}, \tag{38}$$

Here we have introduced $\alpha = \sqrt{4\nu^2-1}$, $\nu = k+1/2$, and

$$E_k = \begin{pmatrix} e_k & 0 \\ 0 & -e_{k-1} \end{pmatrix}, \qquad D_k = \begin{pmatrix} d_k & 0 \\ 0 & -d_{k-1} \end{pmatrix},$$

$$A_k = \begin{pmatrix} 0 & 1 - \frac{j_{k-1}(iS_n\bar{\Omega}_n)}{j_{k+1}(iS_n\bar{\Omega}_n)} \\ -\left(1 - \frac{j_{k+1}(iS_n\bar{\Omega}_n)}{j_{k-1}(iS_n\bar{\Omega}_n)}\right) & 0 \end{pmatrix},$$

with

$$e_k = \frac{j_{k+1}(iS_n\bar{\Omega}_n)j_k(iS_n\bar{\Omega}_n)}{j^2_{k+1}(iS_n\bar{\Omega}_n) - j^2_k(iS_n\bar{\Omega}_n)}, \qquad d_k = \frac{j_{k+1}(iS_n\bar{\Omega}_n)h^{(1)}_k(iS_n\bar{\Omega}_n) - j_k(iS_n\bar{\Omega}_n)h^{(1)}_{k+1}(iS_n\bar{\Omega}_n)}{j^2_{k+1}(iS_n\bar{\Omega}_n) - j^2_k(iS_n\bar{\Omega}_n)}.$$

Moreover, for $k = 0$,

$$c(0)_{(0,0)} = \begin{pmatrix} C_0^{(0,0)} & 0 \\ 0 & -C_0^{(0,0)} \end{pmatrix}, \qquad c(3)_{(0,0)} = \begin{pmatrix} C_{3,+}^{(0,0)} & 0 \\ 0 & C_{3,-}^{(0,0)} \end{pmatrix},$$

and the coefficients are determined by the equations

$$(S_n\cos\theta\ + 2\sin\theta e_0)\, C_0^{(0,0)} = d_0, \tag{39}$$

$$C_{3,+}^{(0,0)} = \left[S_n\cos\theta\,(j_0(iS_n\bar{\Omega}_n))^{-1}j_1(iS_n\bar{\Omega}_n) + \sin\theta\right] C_0^{(0,0)} - (j_0(iS_n\bar{\Omega}_n))^{-1}h^{(1)}_0(iS_n\bar{\Omega}_n), \tag{40}$$

$$C_{3,-}^{(0,0)} = \left[S_n\cos\theta\,(j_1(iS_n\bar{\Omega}_n))^{-1}j_0(iS_n\bar{\Omega}_n) - \sin\theta\right] C_0^{(0,0)} - (j_1(iS_n\bar{\Omega}_n))^{-1}h^{(1)}_1(iS_n\bar{\Omega}_n). \tag{41}$$

The above equations allow for the complete determination of the Green function. In the next section, we will give the explicit expressions for those coefficients needed for the evaluation of the mean fermionic number of the bag.

## IV. THE MEAN FERMIONIC NUMBER

The mean fermionic number for the bag in the hybrid model can be written as

$$\langle N_{\rm bag}\rangle = \langle N_0\rangle + \langle \tilde{N}\rangle. \tag{42}$$

In equation (42), $\langle N_0\rangle$ is the contribution corresponding to a free fermionic field. It has been calculated in [18], to give

$$\langle N_0\rangle = N_f \times \frac{4\pi}{9}\left(\bar{\mu} z^2 + \frac{\bar{\mu}^3}{\pi^2}\right), \tag{43}$$

with the number of flavors $N_f = 2$ in our case. See Ref. [13] for an evaluation of $\langle N_0\rangle$ employing techniques similar to the present paper.

Finally, we will evaluate the second term of (42) as

$$\langle \tilde{N}\rangle = \frac{1}{\beta}{\rm Tr}\left[-i\left(\gamma^0 \otimes \mathcal{I}_I\right)\tilde{k}(t,\vec{x};t',\vec{x'})\right]. \tag{44}$$

It is seen from the Dirac matrices representation (21) that only $a^{(3)}(t,\vec{x};t',\vec{x'})$ must be considered.

In taking the trace, the integration over angular variables can be done to obtain

$$\begin{aligned}\langle \tilde{N}\rangle &= 2z\int_0^1 r^2\,dr\int_\Omega d\Omega \sum_{n=-\infty}^{\infty} tr\{a^{(3,n)}(\vec{x},\vec{x'})\}\Big|_{\vec{x'}=\vec{x}}\\ &= 2z\sum_{n=-\infty}^{\infty} iS_n\bar{\Omega}_n^2\int_0^1 r^2\,dr\,\Bigg[{\rm tr}\left(J_{(0,0)}(r)c(3)_{(0,0)}J_{(0,0)}(r')\right)\\ &\quad+\sum_{k=1}^{\infty}\sum_{m=-k}^{k}{\rm tr}\left(J_{(k,m)}(r)c(3)_{(k,m)}J_{(k,m)}(r')\right)\Bigg]\Bigg|_{r'=r}.\end{aligned} \tag{45}$$

It is easily seen from Equation (45) and from the definitions (29,30) that only the diagonal elements of $c(3)_{(k,m)}$, $\left[C_{3,\pm}^{(k,m)}\right]_{11}$, $\left[C_{3,\pm}^{(k,m)}\right]_{22}$, $C_{3,\pm}^{(0,0)}$ are needed. Their explicit evaluation from equations (37–41) leads to the equalities

$$\left[C_{3,+}^{(k,m)}\right]_{11} = \left[C_{3,-}^{(k,m)}\right]_{11}, \qquad \left[C_{3,+}^{(k,m)}\right]_{22} = \left[C_{3,-}^{(k,m)}\right]_{22}, \qquad C_{3,+}^{(0,0)} = C_{3,-}^{(k,m)}, \tag{46}$$

allowing to write for $\langle \tilde{N}\rangle$

$$\begin{aligned}\langle \tilde{N}\rangle = 2z\sum_{n=-\infty}^{\infty} iS_n\bar{\Omega}_n^2\int_0^1 r^2\,dr\,&\Big\{C_{3,+}^{(0,0)}\left[j_0(iS_n\bar{\Omega}_n r)j_0(iS_n\bar{\Omega}_n r') + j_1(iS_n\bar{\Omega}_n r)j_1(iS_n\bar{\Omega}_n r')\right]\\ +\sum_{k=1}^{\infty}\sum_{m=-k}^{k}&\Big\{\left[C_{3,+}^{(k,m)}\right]_{11}\left[j_k(iS_n\bar{\Omega}_n r)j_k(iS_n\bar{\Omega}_n r') + j_{k+1}(iS_n\bar{\Omega}_n r)j_{k+1}(iS_n\bar{\Omega}_n r')\right]\\ +\left[C_{3,+}^{(k,m)}\right]_{22}&\left[j_k(iS_n\bar{\Omega}_n r)j_k(iS_n\bar{\Omega}_n r') + j_{k-1}(iS_n\bar{\Omega}_n r)j_{k-1}(iS_n\bar{\Omega}_n r')\right]\Big\}\Big\}\Big|_{r'=r}.\end{aligned} \tag{47}$$

Notice that we have changed the order in which the series and the integral are taken. To properly define the resulting series we will take $r' = r(1-\epsilon)$, keeping $\epsilon > 0$ up to the end of the calculation.

In equation (47), we can do the $m$-sum, to get a factor $2\nu$. We can also rearrange the $k$-sum of the first term to include the $k=0$ subspace contribution. In this way, we obtain

$$\begin{aligned} \tilde{N} &= \lim_{\epsilon\to 0} \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} N_{k,m}(z,\bar{\mu},\theta;\epsilon) \\ &= \lim_{\epsilon\to 0} \sum_{n=-\infty}^{\infty} \pi z \bar{\Omega}_n \sum_{k=0}^{\infty} \left[ 2\nu \left[ C_{3,+}^{(k,m)} \right]_{11} + 2(\nu+1) \left[ C_{3,+}^{(k+1,m)} \right]_{22} \right] (\mathrm{I}_\nu + \mathrm{I}_{\nu+1}) , \end{aligned} \tag{48}$$

where we have introduced

$$\mathrm{I}_\nu = (1-\epsilon)^{-1/2} \int_0^1 r\, dr\, J_\nu(iS_n\bar{\Omega}_n r) J_\nu(iS_n\bar{\Omega}_n r(1-\epsilon)),$$

with $J_\nu(x)$ the cilindrical Bessel functions.

One can see that

$$\mathrm{I}_\nu + \mathrm{I}_{\nu+1} = \frac{(1-\epsilon)^{-3/2}}{\epsilon} \frac{J_\nu(iS_n\bar{\Omega}_n(1-\epsilon)) J_\nu(iS_n\bar{\Omega}_n)}{(iS_n\bar{\Omega}_n)^2} \left[ d_j^\epsilon - d_j^{\epsilon=0} + \epsilon \left( d_j^{\epsilon=0} - \nu \right) \right] , \tag{49}$$

where $d_j^\epsilon = iS_n\bar{\Omega}_n(1-\epsilon) \left[ \frac{d}{dx} \log J_\nu(x) \right]_{x=iS_n\bar{\Omega}_n(1-\epsilon)}$.

The first factor in the $k$-series on the r.h.s. of (48) contains all the dependence of $\langle\tilde{N}\rangle$ on the chiral angle $\theta(R)$. Using the recurrence relations for the Bessel functions, we can express this factor in terms of modified Bessel functions,

$$2\nu \left[ C_{3,+}^{(k,m)} \right]_{11} + 2(\nu+1) \left[ C_{3,+}^{(k+1,m)} \right]_{22}$$

$$= \left( \frac{2i}{\pi} \right) \frac{x^2}{(J_\nu(iS_n\bar{\Omega}_n))^2} \left[ 2\nu \frac{M_\nu \bar{N}_\nu + P_\nu \bar{P}_\nu}{\bar{M}_\nu \bar{N}_\nu + \bar{P}_\nu^2} + \frac{x^2}{(d_\nu^{\epsilon=0} - \nu)^2} 2(\nu+1) \frac{\bar{M}_{\nu+1} N_{\nu+1} + P_{\nu+1} \bar{P}_{\nu+1}}{\bar{M}_{\nu+1} \bar{N}_{\nu+1} + \bar{P}_{\nu+1}^2} \right], \tag{50}$$

where

$$M_\nu = S_n \cos\theta \left[ u - v + w \right] - \frac{i\sin\theta}{2\nu} \frac{\sqrt{1-t^2}}{t} \left[ v - 2t^2 w \right], \tag{51}$$

$$N_\nu = S_n \cos\theta \left[ -u - v - w \right] - \frac{i\sin\theta}{2\nu} \frac{\sqrt{1-t^2}}{t} \left[ v + 2t^2 w \right], \tag{52}$$

$$P_\nu = -\frac{i\sin\theta}{2\nu} \frac{\sqrt{1-t^2}}{t} v \sqrt{4\nu^2 - 1}, \tag{53}$$

$$\bar{M}_\nu = S_n \cos\theta \left[ \left( d_\nu^{\epsilon=0} - \nu \right)^2 + \nu^2 \left( \frac{1-t^2}{t^2} \right) \right] - i\sin\theta \frac{\sqrt{1-t^2}}{t} \left( d_\nu^{\epsilon=0} - \nu \right), \tag{54}$$

$$\bar{N}_\nu = S_n \cos\theta \left[ -\left( d_\nu^{\epsilon=0} + \nu \right)^2 - \nu^2 \left( \frac{1-t^2}{t^2} \right) \right] - i\sin\theta \frac{\sqrt{1-t^2}}{t} \left( d_\nu^{\epsilon=0} + \nu \right), \tag{55}$$

$$\bar{P}_\nu = -i\sin\theta \frac{\sqrt{1-t^2}}{t} d_\nu^{\epsilon=0} \sqrt{4\nu^2 - 1}. \tag{56}$$

Here,

$$\rho^2 = \nu^2 + x^2, \qquad t = \frac{\nu}{\rho}, \qquad x = S_n \bar{\Omega}_n,$$

$$u = I'_\nu(S_n\bar{\Omega}_n)K'_\nu(S_n\bar{\Omega}_n), \qquad v = \frac{\nu}{x}\left(I_\nu(S_n\bar{\Omega}_n)K_\nu(S_n\bar{\Omega}_n)\right)', \qquad w = \frac{\rho^2}{x^2}I_\nu(S_n\bar{\Omega}_n)K_\nu(S_n\bar{\Omega}_n).$$

The double series (48) is not absolutely convergent for $\epsilon = 0$, so we must keep $\epsilon > 0$ in the general term up to the end of the calculations. A way to isolate this non regular behavior consists in the subtraction of the $M$-order asymptotic (Debye) expansion [19] of the general term $N_{k,m}(z, \bar{\mu}, \theta; \epsilon)$, $\Delta^M_{k,m}$, as discussed in Ref. [13].

So, we can write

$$\begin{aligned} \langle \tilde{N} \rangle &= \tilde{N}_1 + \tilde{N}_2 \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \Big[ N_{k,m}(z, \bar{\mu}, \theta; \epsilon = 0) - \Delta^6_{k,m}(z, \bar{\mu}, \theta; \epsilon = 0) \Big] \\ &\quad + \lim_{\epsilon \to 0} \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \Delta^6_{k,m}(z, \bar{\mu}, \theta; \epsilon) \end{aligned} \tag{57}$$

For $M \geq 3$ the $\epsilon \to 0$ limit can be taken inside the double sum for $\tilde{N}_1$ . In equation (57) we have used the expansion up to the order $M = 6$ to improve the numerical calculations to be done later. The non regular behavior of the series for $\langle \tilde{N} \rangle$ is isolated in the second term $\tilde{N}_2$, which can be studied analytically.

From equations (49,50), by defining

$$F_1(z, \bar{\mu}; \epsilon) = D.D.^{(6)} \left[ \frac{I_\nu(S_n\bar{\Omega}_n(1-\epsilon))}{I_\nu(S_n\bar{\Omega}_n)} \right] \tag{58}$$

and

$$\begin{aligned} F_2(z, \bar{\mu}, \theta; \epsilon) &= \\ &= D.D.^{(6)} \left[ (2iz)\bar{\Omega}_n \frac{(1-\epsilon)^{-3/2}}{\epsilon} \left[ d^\epsilon_j - d^{\epsilon=0}_j + \epsilon \left( d^{\epsilon=0}_j - \nu \right) \right] \right. \\ &\quad \left. \times \left[ 2\nu \frac{M_\nu \bar{N}_\nu + P_\nu \bar{P}_\nu}{\bar{M}_\nu \bar{N}_\nu + \bar{P}^2_\nu} + \frac{z^2}{(d^{\epsilon=0}_\nu - \nu)^2} 2\,(\nu+1)\, \frac{\bar{M}_{\nu+1} N_{\nu+1} + P_{\nu+1}\bar{P}_{\nu+1}}{\bar{M}_{\nu+1}\bar{N}_{\nu+1} + \bar{P}^2_{\nu+1}} \right] \right], \end{aligned} \tag{59}$$

where $D.D.^{(6)}$ stands for the asymptotic Debye expansion up to the sixth order, we can write

$$\Delta^6_{k,m}(z, \bar{\mu}, \theta; \epsilon) = F_1(z, \bar{\mu}; \epsilon) F_2(z, \bar{\mu}, \theta; \epsilon) \tag{60}$$

(consistently retaining in this product terms up to $\rho^{-6}$).

It is easy to see that [13]

$$F_1(z, \bar{\mu}; \epsilon) = \exp\left(-\epsilon \frac{\nu}{t}\right) [1 + O(\epsilon)] \tag{61}$$

So, we straightforwardly get

$$\tilde{N_1} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \left[N_{k,m}(z,\bar{\mu},\theta;\epsilon=0) - F_2(z,\bar{\mu},\theta;\epsilon=0)\right], \tag{62}$$

$$\tilde{N_2} = \lim_{\epsilon\to 0} \sum_{n=\infty}^{\infty} \sum_{k=0}^{\infty} \exp\left(-\epsilon\frac{\nu}{t}\right) F_2(z,\bar{\mu},\theta;\epsilon=0)(1+O(\epsilon)), \tag{63}$$

where $F_2(z,\bar{\mu},\theta;\epsilon=0)$. Employing the asymptotic Debye expansion for Bessel functions we obtain

$$\begin{aligned}
F_2(z,\bar{\mu},\theta;\epsilon=0) = (2iz\bar{\Omega}_n)\Bigg\{ & -\frac{t^5}{\nu^2} + \frac{t^5}{2\,\nu^3}\left[-3 - t + 5\,t^2 + 3\,t^3 - t\,\cos(2\,\theta)\right] \\
& + \frac{t^5}{8\,\nu^4}\Big[-6 - 6\,t + 25\,t^2 + 42\,t^3 + 35\,t^4 - 48\,t^5 - 71\,t^6 \\
& \quad + \left(-6\,t - 14\,t^2 + 12\,t^3 + 21\,t^4\right)\,\cos(2\,\theta)\Big] \\
& + \frac{t^5}{32\,\nu^5}\Big[-4 - 12\,t + 30\,t^2 + 212\,t^3 + 560\,t^4 - 253\,t^5 - 2044\,t^6 \\
& \quad -690\,t^7 + 1562\,t^8 + 852\,t^9 \\
& \quad + \left(-12\,t - 84\,t^2 - 76\,t^3 + 406\,t^4 + 456\,t^5 - 378\,t^6 - 450\,t^7\right)\,\cos(2\,\theta) \\
& \quad + \left(8\,t^3 - 3\,t^5\right)\,\cos(4\,\theta)\Big] \\
& + \frac{i}{4}\,\frac{t^8}{\nu^5\,(1-t^2)^{\frac{1}{2}}}\,\cos(\theta)\sin(\theta)^3\,\left[-8 + 7\,t^2\right] \\
& + \frac{t^6}{128\,\nu^6}\Big[-8 - 20\,t + 504\,t^2 + 2305\,t^3 + 414\,t^4 - 6696\,t^5 - 17400\,t^6 \\
& \quad -14025\,t^7 + 39576\,t^8 + 50414\,t^9 - 23856\,t^{10} - 32799\,t^{11} \\
& \quad + \Big(-8 - 168\,t - 744\,t^2 + 468\,t^3 + 7312\,t^4 + 4842\,t^5 - 16700\,t^6 \\
& \quad -15664\,t^7 + 10800\,t^8 + 11154\,t^9\Big)\,\cos(2\,\theta) \\
& \quad + \left(48\,t^2 + 136\,t^3 - 158\,t^4 - 270\,t^5 + 60\,t^6 + 99\,t^7\right)\,\cos(4\,\theta)\Big] \\
& - \frac{i}{8}\,\frac{t^8}{\nu^6\,(1-t^2)^{\frac{1}{2}}}\,\cos(\theta)\sin(\theta)^3\left[24 + 68\,t - 91\,t^2 - 169\,t^3 + 63\,t^4 + 100\,t^5\right]\Bigg\}
\end{aligned} \tag{64}$$

Now, equation (63) can be expressed in terms of the double series

$$s(p,q;\epsilon) = \sum_{k=0}^{\infty} \nu^q \mathcal{S}_p(\nu,\epsilon), \qquad t(p,q;\epsilon) = \sum_{k=0}^{\infty} \nu^q \mathcal{T}_p(\nu,\epsilon), \tag{65}$$

where

$$\mathcal{S}_p(\nu,\epsilon) = (-2z)\sum_{n=-\infty}^{\infty} \frac{i\bar{\Omega}_n \exp\left(-\epsilon\sqrt{\nu^2+\bar{\Omega}_n^2}\right)}{\left(\nu^2+\bar{\Omega}_n^2\right)^{p/2}} \tag{66}$$

and

$$\mathcal{T}_p(\nu,\epsilon) = (-2z) \sum_{n=-\infty}^{\infty} \frac{\exp\left(-\epsilon\sqrt{\nu^2+\bar{\Omega}_n^2}\right)}{\left(\nu^2+\bar{\Omega}_n^2\right)^{p/2}}. \tag{67}$$

The series $\mathcal{S}_p$ have been studied in Ref. [13], and the series $\mathcal{T}_p$ can be evaluated in a similar way. Both satisfy the same recursion relations [13]. So, they can be completely determined from the knowledge of $\mathcal{S}_2(\nu,\epsilon)$ and $\mathcal{T}_2(\nu,\epsilon)$. $\mathcal{S}_2(\nu,\epsilon)$ is evaluated in the Appendix II of Ref. [13], and $\mathcal{T}_2(\nu,\epsilon)$ can be evaluated following similar steps.

While $\mathcal{S}_p(\nu,\epsilon)$ is regular at $\epsilon = 0$ for all $p \geq 1$, $\mathcal{T}_p(\nu,\epsilon)$ is regular at $\epsilon = 0$ only for $p \geq 2$. In both cases they are exponentially decreasing with $\nu$, making the remaining $k$ series in equations (65) absolutely convergent, even for $\epsilon = 0$.

Consequently, it is sufficient to consider the $\epsilon = 0$ limit, thus getting the simplified expressions (see Ref. [13])

$$\mathcal{S}_{2\kappa+1}(\nu,0) = \frac{1}{[2\kappa-1]!!}\left(-\frac{1}{\nu}\frac{\partial}{\partial\nu}\right)^{\kappa} \mathcal{S}_1(\nu,0), \tag{68}$$

$$\mathcal{S}_{2\kappa}(\nu,0) = \frac{1}{[2(\kappa-1)]!!}\left(-\frac{1}{\nu}\frac{\partial}{\partial\nu}\right)^{\kappa-1} \mathcal{S}_2(\nu,0), \tag{69}$$

where

$$\mathcal{S}_1(\nu,0) = \frac{2\nu}{\pi}\int_0^\infty du \left[\frac{1}{1+e^{-\bar{\mu}/z}e^{\frac{\nu}{z}\sqrt{u^2+1}}} - \frac{1}{1+e^{+\bar{\mu}/z}e^{\frac{\nu}{z}\sqrt{u^2+1}}}\right], \tag{70}$$

$$\mathcal{S}_2(\nu,0) = \frac{1}{1+e^{-(\bar{\mu}-\nu)/z}} - \frac{1}{1+e^{(\bar{\mu}+\nu)/z}}, \tag{71}$$

and

$$\mathcal{T}_{2\kappa}(\nu,0) = \frac{1}{[2(\kappa-1)]!!}\left(-\frac{1}{\nu}\frac{\partial}{\partial\nu}\right)^{\kappa-1} \mathcal{T}_2(\nu,0), \tag{72}$$

$$\mathcal{T}_{2\kappa+1}(\nu,0) = \frac{1}{[2\kappa-1]!!}\left(-\frac{1}{\nu}\frac{\partial}{\partial\nu}\right)^{\kappa-1} \mathcal{T}_3(\nu,0), \tag{73}$$

with

$$\mathcal{T}_2(\nu,0) = \frac{1}{\nu}\left[\frac{1}{1+e^{-(\bar{\mu}-\nu)/z}} + \frac{1}{1+e^{(\bar{\mu}+\nu)/z}} - 1\right], \tag{74}$$

$$\mathcal{T}_3(\nu,0) = -\frac{2}{\pi}\int_0^\infty \frac{du}{\sqrt{u^2+1}}\left(\frac{1}{\nu}\frac{\partial}{\partial\nu}\right)\left[\frac{1}{1+e^{-\bar{\mu}/z}e^{\frac{\nu}{z}\sqrt{u^2+1}}} + \frac{1}{1+e^{+\bar{\mu}/z}e^{\frac{\nu}{z}\sqrt{u^2+1}}}\right] - \frac{2}{\pi\nu^2}. \tag{75}$$

In particular, by inspection of the powers of $\rho$ appearing in the expression of $F_2(z, \bar{\mu}, \theta; \epsilon = 0)$, equation (64), it can be seen that only $s(p, q, \epsilon = 0)$ for $p \geq 5$, and $t(p, q, \epsilon = 0)$ for $p \geq 8$ are present in $\tilde{N}_2$.

These results are suitable for performing the numerical evaluations of $\tilde{N}_2$, which will be described in the next section.

Numerical evaluations become more difficult when $z$ grows up. But, in the $z > 1$ region an asymptotic approximation valid for large $z$ applies.

In fact, the series in Eq. (65) for $p \geq 4$ can be expressed as

$$s(p, q; \epsilon = 0) = (-4z) \frac{1}{p-2} \frac{\partial}{\partial \bar{\mu}} \sigma_{p-2,q} \,, \tag{76}$$

$$t(p, q; \epsilon = 0) = (-4z)\, \sigma_{p,q} \,, \tag{77}$$

where

$$\sigma_{p,q} = \sum_{k=0}^{\infty} \nu^q \sum_{n=0}^{\infty} \rho^{-p} \,. \tag{78}$$

As discused in Ref. [13], $\sigma_{p,q}$ can be developed, for $z >> 1$ and $p \geq 2$, as

$$\sigma_{p,q} = \frac{1}{2} \Re \left[ \left( \frac{1}{2\pi z} \right)^{p-q-1} \frac{\Gamma\left(\frac{p-q-1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \zeta\left(p - q - 1, \frac{1}{2} - i \frac{\bar{\mu}}{2\pi z}\right) + \right.$$

$$\left. 2 \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \left( \frac{1}{2\pi z} \right)^{p+2\ell} \frac{\Gamma\left(\frac{p+2\ell}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \zeta\left(p + 2\ell, \frac{1}{2} - i \frac{\bar{\mu}}{2\pi z}\right) \zeta\left(-(q + 2\ell), \frac{1}{2}\right) \right] . \tag{79}$$

Notice that Eqs. (79,76,77) suggest that $\tilde{N}_1$ is $\mathcal{O}(z^{-3})$ (since the terms in the series in Eq. (62) are $\mathcal{O}(1/z)^{p-q-4}$ with $p - q = 7$, the next order in the Debye expansion). So, up to this order, we have

$$\langle N_{\rm bag} \rangle = \frac{8\pi}{9} \left( \bar{\mu} z^2 + \frac{\bar{\mu}^3}{\pi^2} \right) - \frac{4\,\bar{\mu}}{3\,\pi} + \frac{\bar{\mu}\,\zeta(3)}{90\,\pi^3\,z^2} \left(31 - 21\,\cos(2\,\theta)\right) + O(z^{-3}) \,. \tag{80}$$

The first term in the r.h.s. of this equation is the well known contribution from the free fermionic field. The remaining terms come from the large $z$ expansion of $\tilde{N}_2$. For $\theta = 0$ this reproduces the result found in Ref. [13] for the MIT-case.

## V. NUMERICAL EVALUATIONS

In this section we will show our numerical evaluation of $\langle N \rangle$, for a selected set of values of $z$ and $\theta$, as a function of $\bar{\mu}$.

For the double series in equation (62) we adopt a cutoff for $n$ and $k$, such that the tail of the series becomes negligible. Since we have taken the asymptotic expansion up to the

$M = 6$ order, the double series are strongly convergent, being the required maximum values, $n_0$ and $k_0$, small enough for shortening the numerical calculations.

On the other hand, since we have analytically solved the $n$-sum in equation (63), only the remaining $k$-sum in (65) must be numerically studied. We impose again a cutoff to these sums for large values of $k$, so as to also make negligible the tail of these series.

In Figure 1 we show the total mean fermionic number as function of $\bar{\mu}$, for a fixed value of $z = \frac{1}{8}$ and for different values of $\theta \leq \frac{\pi}{2}$. Similarly, Figure 2 shows the total fermionic number for $\theta \geq \frac{\pi}{2}$.

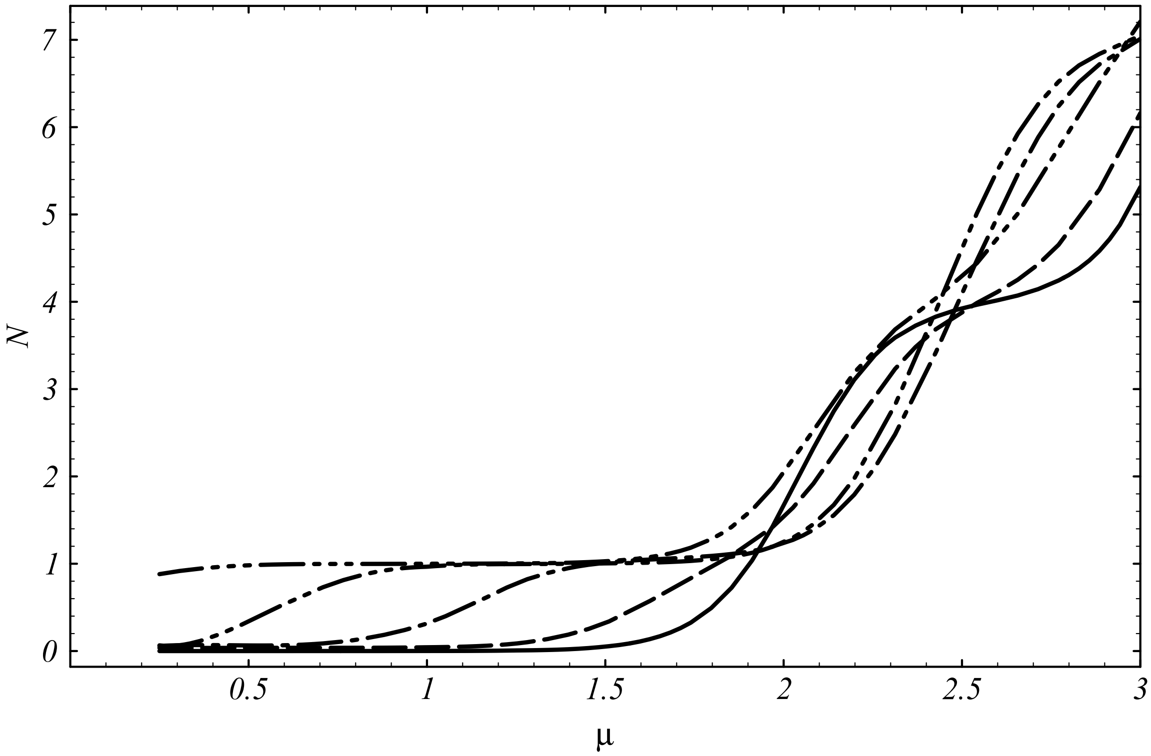


FIG. 1. Fermionic number for $z=\frac{1}{8}$,—: $\theta=0$, – –: $\theta=\frac{\pi}{8}$,–·–: $\theta=\frac{\pi}{4}$, –··–: $\theta=\frac{3\pi}{8}$,–· · ·–: $\theta=\frac{\pi}{2}$

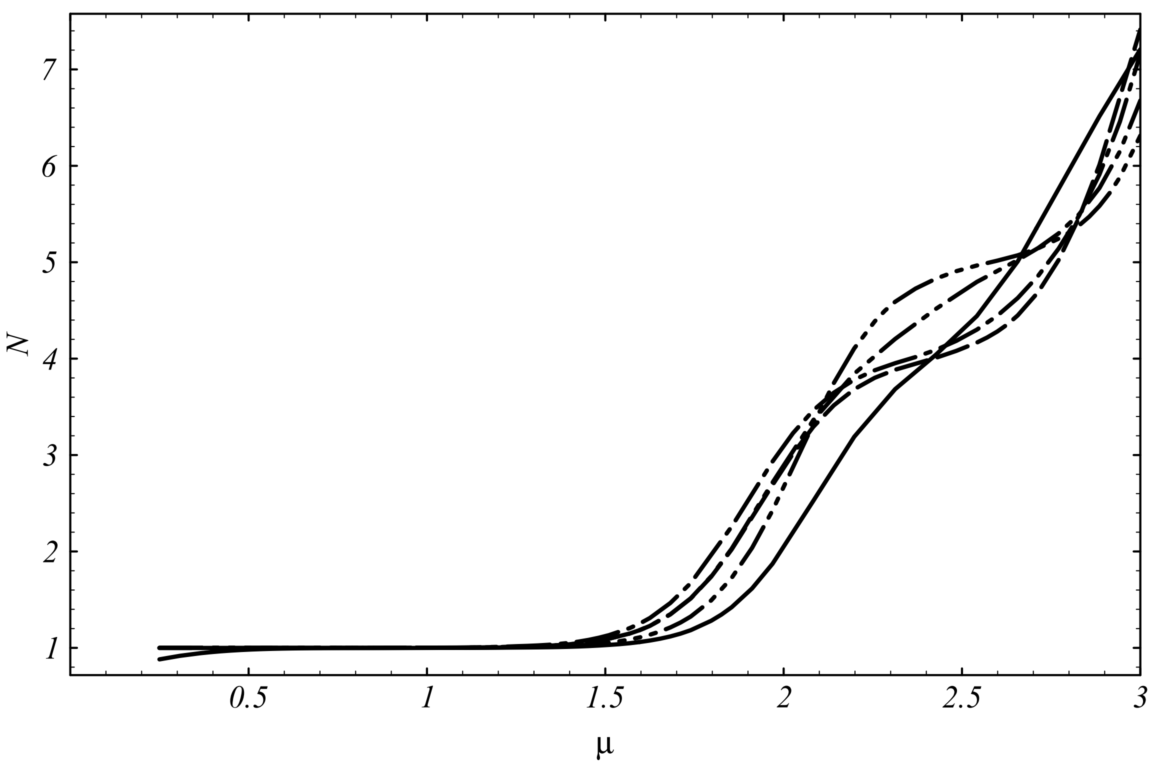

FIG. 2. Fermionic number for $z = \frac{1}{8}$,—: $\theta = \frac{\pi}{2}$, – –: $\theta = \frac{5\pi}{8}$,–·–: $\theta = \frac{3\pi}{4}$, –··–: $\theta = \frac{7\pi}{8}$,–· · ·–: $\theta = \pi$

The solid line in Fig. 1 corresponds to the fermionic number in the MIT bag model ($\theta = 0$) [13]. As expected, when $\bar{\mu}$ approaches the (adimensionalized) energy of the lowest state of the Dirac Hamiltonian ($RE_0 = 2.04$), the number jumps up to $\langle N \rangle = 4$, which is the corresponding degeneracy. When $\theta$ grows up, these four levels split into a singlet and a triplet state [17], the curve having a first step up to $\langle N \rangle = 1$ at a smaller value of $\bar{\mu}$. It is worthwhile to remark that $N_{Sk}$ produces a $\theta$-dependent (and $\bar{\mu}$-independent) shift in $\langle N \rangle$ (4), which is canceled by the contribution of the fermionic degrees of freedom, $\langle N_{\rm bag} \rangle$, giving the expected plateau at $\langle N \rangle = 1$.

Finally, we show the total fermionic number for different values of $z$, for $\theta = 0$ in Fig. 3 and $\theta = \pi/4$ in Fig. 4. These figures show how the way energy levels are populated smoothes when $z = RT$ grows up .

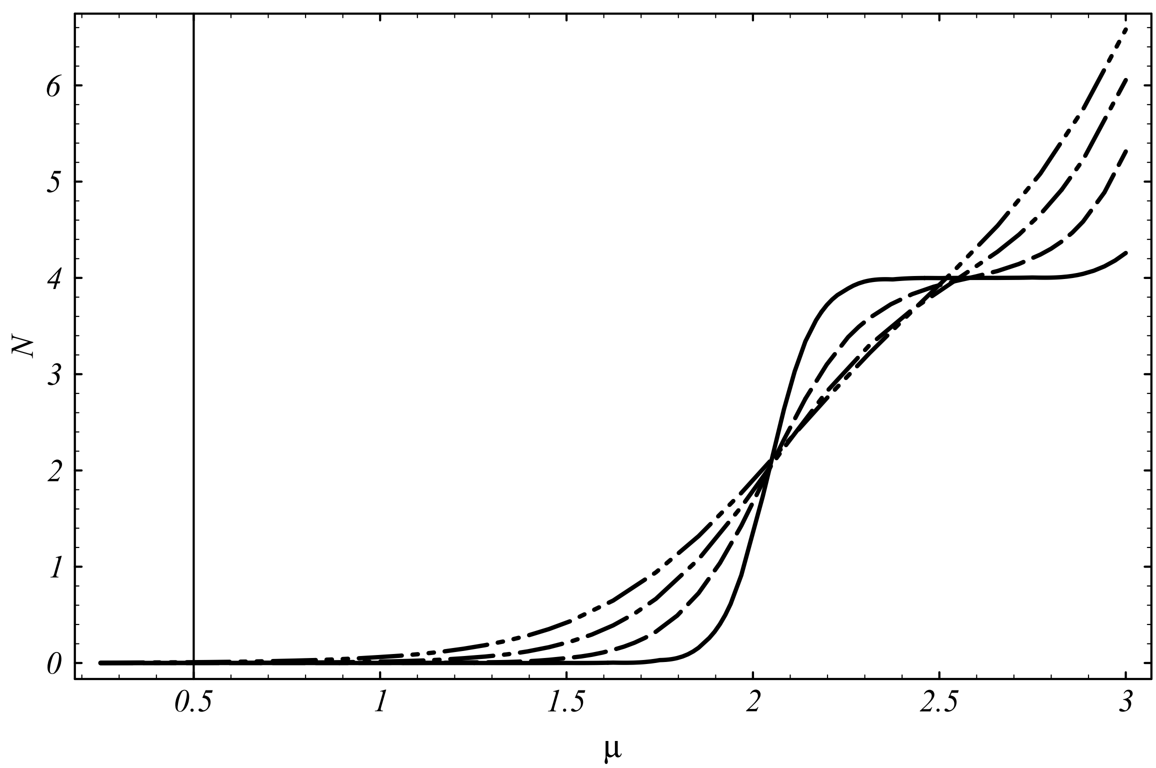


FIG. 3. Fermion number at $\theta = 0$; —:$z = \frac{3}{50}$, – –:$z = \frac{1}{8}$, –·–:$z = \frac{3}{16}$, –··–:$z = \frac{1}{4}$

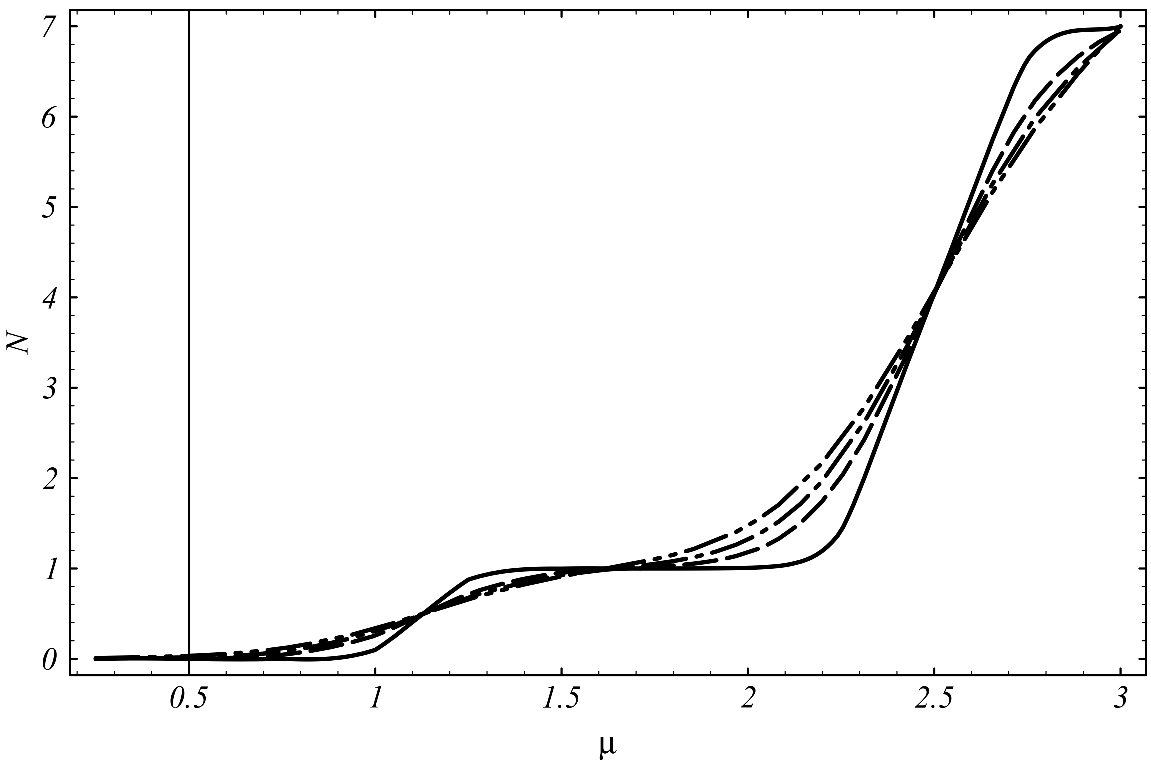


FIG. 4. Fermion number at $\theta = \frac{\pi}{4}$; —:$z = \frac{3}{50}$, – –:$z = \frac{1}{8}$, –·–:$z = \frac{3}{16}$, –··–:$z = \frac{1}{4}$

## VI. CONCLUSIONS

In previous sections we evaluated the mean fermionic number for a hybrid chiral model consisting in a static spherical bag containing a massless two flavored fermionic field and an exterior skyrmionic sector. Fermions are constrained to satisfy chiral boundary conditions depending on the Skyrmion profile $\theta(R)$ at the the bag surface.

Describing the one-loop Grand partition function through the functional determinant of the Dirac operator subject to (chiral bag) spatial and (antiperiodic) temporal boundary conditions, we were able to express the mean fermionic number in the bag, $\langle N_{\rm bag}\rangle = -\dfrac{\partial G}{\partial \mu}(\beta, R, \mu; \theta)$, as a trace involving the Green function of the boundary value problem. This trace has been expanded as a sum over finite dimensional invariant subspaces, and evaluated with help of the asymptotic (Debye) expansion, allowing to show the regularity of the final result.

The CCP requires to complement $\langle N_{\rm bag}\rangle$ with the contribution from the tail of the Skyrmion, $N_{Sk}(\theta)$. This introduces a $\theta$-dependent shift in the total mean fermionic number $\langle N\rangle$, which is exactly compensated by an opposite one coming from the fermionic field in the bag. Notice that this behavior of $\langle N_{\rm bag}\rangle$, which reflects the spectral asymmetry induced by a non vanishing $\theta$, can not be obtained by considering only a finite number of eigenvalues of the Dirac operator to approximate the Grand partition function.

We would like to remark that up to now we have considered the chiral angle $\theta(R)$ as a free parameter. In fact, the topological charge contained in the tail of a Skyrmion-like configuration, Eq. (4), depends only on $\theta(R)$ at the surface, and on the boundary condition $\lim_{r\to\infty}\theta(r)=0$ [8,7,11].

In the description of a "baryon" ($\langle N\rangle = 1$, since we are considering only one color) through this two phase hybrid model, the following picture emerges: for large bag radius ($\theta(R)\to 0$, i.e. $N_{Sk}\to 0$), the chemical potential $\mu$ must be fixed so that $\langle N_{\rm bag}\rangle = 1$. When $R$ becomes smaller, the chiral angle, and consequently $N_{Sk}$, rise up. The $\mu$ necessary to get $\langle N_{\rm bag}\rangle(\beta,R,\mu;\theta)+N_{Sk}(\theta)=1$ diminishes. When the radius is such that $\theta(R)=\frac{\pi}{2}$ the lowest state of the Dirac Hamiltonian dives into the Dirac sea. At this point $N_{Sk}=\frac{1}{2}$, so $\mu$ must be chosen to give $\langle N_{\rm bag}\rangle = \frac{1}{2}$. When $R\to 0$ ($\theta(R)\to\pi$), the topological charge reaches one. Then, $\langle N_{\rm bag}\rangle = 0$, which is compatible with $\mu=0$, as can be seen from (15,18,19).

The behavior of hadronic matter at finite temperature and density has received considerable attention during the last years (See for example [20]). The main motivation behind this effort is an attempt to understand properly the generally accepted possibility of a deconfining phase transition from hadronic matter to the quark-gluon plasma [21], with applications to relativistic heavy ion collisions and to the early Universe. In this framework, the study of the thermodynamical stability of the hybrid model considered above, through the analysis of the free energy, may be relevant.

The Gibbs free energy can be obtained by integrating $\langle N_{\rm bag}\rangle$,

$$G(\beta,R,\mu;\theta)=F(\beta,R;\theta)-\int_0^{\mu}\langle N_{\rm bag}\rangle(\beta,R,\mu';\theta)d\mu', \tag{81}$$

where $F(\beta,R;\theta)$ is the ($\mu=0$) Helmholtz free energy of the chiral bag, evaluated in [22].

When studying the system for a fixed "baryonic" number the relation between $\mu$ and $\langle N\rangle$ must be inverted, which requires more detailed numerical information.

Moreover, the study of the free energy as function of $\beta$ and $R$ needs the knowledge of the detailed profile $\theta(\beta,R)$. Even though one can rely on approximated explicit versions as in Ref. [23,24], this also would require a greater numerical computational effort. We will report on that subject elsewhere.

## ACKNOWLEDGMENTS

The authors thank E. M. Santangelo for many useful discussions and comments.

This work was supported in part by Fundación Andes – Antorchas under Contract No. C-13398/6, ANPCyT (PICT-0421 and PICT-00039) and CONICET (Argentina), FONDECyT (Chile) under Grant No. 1980577 and Commission of European Communities under contract C11*-CT93-0315.

# REFERENCES


[1] M. Rho, Physics Reports **240**, 1 (1994).

[2] G. E. Brown, A. D. Jackson, M. Rho, and V. Vento, Physics Letters **140B**, 285 (1984).

[3] I. Zahed, A. Wirzba, and U.-G. Meissner, Annals of Physics **165**, 406 (1985).

[4] R. F. Alvarez-Estrada, F. Fernandez, J. L. Sánchez-Gomez, and V. Vento, *Models of Hadron Structure Based on Quantum Chromodynamics*, Vol. 259 of *Lecture Notes in Physics* (Springer–Verlag, Germany, 1986).

[5] A. Chodos, R. L. Jaffe, C. B. Thorn, and V. Weisskopf, Physical Review **9D**, 3471 (1974).

[6] A. Chodos and C. B. Thorn, Physical Review **12D**, 2733 (1975).

[7] T. H. R. Skyrme, Nuclear Physics **31**, 556 (1962).

[8] E. Witten, Nuclear Physics **223B**, 433 (1983).

[9] G. S. Adkins, C. R. Nappi, and E. Witten, Nuclear Physics **228B**, 552 (1983).

[10] S. Nadkarni, H. B. Nielsen, and I. Zahed, Nuclear Physics **B253**, 308 (1985).

[11] J. Goldstone and R. L. Jaffe, Physical Review Letters **51**, 1518 (1983).

[12] M. D. Francia, H. Falomir, and E. M. Santangelo, Physics Letters **B 371**, 285 (1996).

[13] M. D. Francia, H. Falomir, and M. Loewe, Phys. Rev. **D55**, 2477 (1997).

[14] M. D. Francia, H. Falomir, M. Loewe, and E. M. Santangelo, Physical Review **D51**, 7132 (1995).

[15] T. H. R. Skyrme, Proceedings of the Royal Society **260**, 127 (1961).

[16] M. Rho, A. S. Goldhaber, and G. E. Brown, Physical Review Letters **51**, 747 (1983).

[17] P. J. Mulders, Physical Review **30D**, 1073 (1984).

[18] G. W. Gibbons, Physics Letters **60A**, 385 (1977).

[19] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1970).

[20] E. V. Shuryak, Physics Reports **61**, 73 (1980).

[21] A. Smilga, Surveys High Energ.Phys. **10**, 233 (1997).

[22] M. D. Francia, H. Falomir, and E. M. Santangelo, Physical Review **D45**, 2129 (1992).

[23] M. F. Atiyah and N. S. Manton, Physics Letters **222**, 438 (1989).

[24] K. J. E. y K. Kajantie, Zeitschrift für Physik **C44**, 347 (1989).