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Statistical mechanics and the description of the early universe. (II). Principle of detailed balance and primordial ^4He formation

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Abstract

If the universe is slightly non-extensive, and the distribution functions are not exactly given by those of Boltzmann–Gibbs, the primordial production of light elements will be non-trivially modified. In particular, the principle of detailed balance (PDB), of fundamental importance in the standard analytical analysis, is no longer valid, and a non-extensive correction appears. This correction is computed and its influence is studied and compared with previous works, where, even when the universe was considered as an slightly non-extensive system, the PDB was assumed valid. We analytically track the formation of helium and deuterium, and study the kind of deviation one could expect from the standard regime. The correction to the capture time, the moment in which deuterium can no longer be substantially photo-disintegrated, is also presented. This allows us to take into account the process of the free decay of neutrons, which was absent in all previous treatments of the topic. We show that even when considering a first (linear) order correction in the quantum distribution functions, the final output on the primordial nucleosynthesis yields cannot be reduced to a linear correction in the abundances. We finally obtain new bounds upon the non-extensive parameter, both comparing the range of physical viability of the theory, and using the latest observational data. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

Primordial nucleosynthesis provides an interesting testing arena where to analyze the viability of physical theories, particularly, of the statistical description. It is in

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this epoch where the earliest bounds upon a given theory with cosmological influence can be imposed. Thermal processes (see Ref. [1], hereafter referred as Paper I) are non-trivially modified by a non-extensive correction to quantum distribution functions. Then, different abundances of light elements are a possible outcome.

Some of the predictions for primordial nucleosynthesis in a non-extensive setting have been analyzed before by some of us, using the asymptotic approach of the quantum distribution functions, see Refs. [2–4]. Here, instead, we shall consistently continue within the formalism given in Paper I. Since this approach is simpler, we shall obtain analytical results far beyond the point where previous works have reached, see for instance Ref. [4,5]. Together with Paper I, we shall then provide a complete history of the early universe, accordingly modified taking into account a non-extensive setting. In this paper, we shall focus on the study of the changes that non-extensive statistics introduces in the principle of detailed balance, for which we provide a detailed analysis, both numerical and analytical. We shall then enhance the study presented in Ref. [4], by framing it in a larger picture which encompasses an smaller number of approximations and a larger number of predictions.

Primordial nucleosynthesis was recently used as well to demonstrate that macroscopic samples of neutrinos in thermal equilibrium are indeed distributed according to Fermi–Dirac statistics [6]. These latter authors considered that neutrinos were distributed by a Bose–Einstein statistics, and established bounds (not very restrictive though) upon this unexpected possibility. It is interesting to compare with our case: we assume that neutrinos are indeed $1/2$ particles, as a large amount of data coming from particles accelerators show, but that even when continuing being fermions, and fulfilling the Pauli’s exclusion principle, their distribution functions could slightly deviate from an exact Fermi–Dirac one.

Since we have provided a detailed account of non-extensive statistics, and the reasons why we choose the analytical form of the quantum distribution functions we adopted (together with its derivation) in Paper I, we shall skip such topics here. We have also considered in Paper I some physical reasons why to expect that Boltzmann–Gibbs distribution functions could be considered as an approximation. The same is valid for citations to previous works, for which we adopted here the criterion to quote just those needed for the explicit computations we are carrying on. This does not mean that our references are the only works in cosmological applications of non-extensivity, but only that for proper citation of some of the others, we refer the reader to Paper I.

The layout of this work is as follows. Section 2 presents the basics of the neutron to proton ratio in an evolving universe. This section does not use much the fact that we are immersed in a non-extensive setting, but just presents general results which are valid disregarding the statistical mechanics used. Indeed, the derivation being presented in Section 2 was already given by others [4,7], and we provide it here just for the ease of discussion. In Sections 3–7 we give the details of the analysis of the principle of detailed balance, and show how to obtain a priori results on the possible range of physically admitted values of $(q - 1)$ without the need to compare with experiments. Much of it is done in an analytical form, some is solved numerically. In Section 8,

we present a detailed comparison between the two situations (full and approximate cases) that we found possible for the principle of detailed balance. Towards the end of this latter section we provide a comparison with the latest data available. In Section 9 we compute, for the first time in a non-extensive framework, which is the modified capture time, the time in which neutrons are captured into deuterons. Using this result we are able to compute the primordial abundance of ${}^4\text{He}$ with a greater degree of precision than that obtained in all previous works. We show that there are non-linear effects introduced by the appearance of a slight non-extensivity. Finally, we give some general discussion in our concluding remarks.

2. The neutron to proton ratio

We begin by turning again to the issue of the evolution of the neutron abundance as the universe evolves. We shall base this discussion in the work by, Bernstein et al. [7]. As we have done before, we shall denote by $\lambda_{pn}(T(t))$ the rate for the weak processes to convert protons into neutrons and by $\lambda_{np}(T(t))$ the rate for the reverse ones [4]. $X(T(t))$ will be, as usual, the number of neutrons to the total number of baryons. For it, a valid kinetic equation is

$$\frac{dX(t)}{dt} = \lambda_{pn}(T)(1 - X(t)) - \lambda_{np}(T)X(t). \tag{1}$$

The solution to it is given by

$$X(T) = \int_{t_0}^T dt' I(t, t') \lambda_{pn}(t') + X(t_0) I(t, t_0). \tag{2}$$

Here, $I(t, t')$ is

$$I(t, t') = \exp\left(-\int_{t'}^t d\hat{t} A(\hat{t})\right) \tag{3}$$

with

$$A(t) = \lambda_{pn}(t) + \lambda_{np}(t). \tag{4}$$

Note that this solution is completely general, and does not depend on the statistical mechanics used, except by the implicit changes introduced in the new reaction rates. As explained in Ref. [4], we simplify by taking $t_0 = 0$ and omitting the $X(t_0)I(t, t_0)$ term. These approximations yield

$$X(t) = \int_0^t dt' I(t, t') \lambda_{pn}(t'). \tag{5}$$

Finally, we note that

$$I(t, t') = \frac{1}{A(t')} \frac{d}{dt'} I(t, t') \tag{6}$$

or, equivalently

$$X(t) = \frac{\lambda_{pn}(t)}{A(t)} - \int_0^t dt' I(t, t') \frac{d}{dt'} \left(\frac{\lambda_{pn}(t')}{A(t')} \right). \tag{7}$$

To compute Eq. (7), we need to know the reaction rates. Let us consider $\lambda_{np}(t)$:

$$\lambda_{np} = \lambda_{\nu+n \rightarrow p+e^-} + \lambda_{e^+n \rightarrow p+\bar{\nu}} + \lambda_{n \rightarrow p+e^-+\bar{\nu}} \tag{8}$$

that are individually given in Ref. [8]

$$\lambda_{\nu+n \rightarrow p+e^-} = A \int_0^\infty dp_\nu p_\nu^2 p_e E_e (1 - f^e) f^\nu, \tag{9}$$

$$\lambda_{e^+n \rightarrow p+\bar{\nu}} = A \int_0^\infty dp_e p_e^2 p_\nu E_\nu (1 - f^\nu) f^e, \tag{10}$$

$$\lambda_{n \rightarrow p+e^-+\bar{\nu}} = A \int_0^{p_0} dp_e p_e^2 p_\nu E_\nu (1 - f^\nu)(1 - f^e) \tag{11}$$

with A a constant, fixed by the experimental value of $\lambda_{n \rightarrow p+e^-+\bar{\nu}}$, $p_{\nu,e}$ are the neutrino and electron momenta, and $E_{\nu,e}$ their energies. In the energy domain we are interested, some approximations are in order, see Refs. [4,7] for discussion: (1) energy conservation is $E_\nu + m_n = E_e + m_p$, to be used in Eq. (9) and $E_\nu + m_p = E_e + m_n$, to be used in Eq. (10). (2) In Eq. (11), $E_\nu = \Delta m - E_e > 0$, with $\Delta m = m_n - m_p = 1.29$ MeV, from here comes the upper limit of the integration range. (3) Pauli blocking factors are assumed equal to 1.

Since computations will involve the inverse processes of Eqs. (9) and (10), we shall quote their form below

$$\lambda_{e^-+p \rightarrow n+\nu} = A \int_{p_e^0}^\infty dp_e p_e^2 p_\nu E_\nu (1 - f^\nu) f^e, \tag{12}$$

$$\lambda_{\bar{\nu}+p \rightarrow n+e^+} = A \int_{\Delta m}^\infty dp_\nu p_\nu^2 p_e E_e (1 - f^e) f^\nu, \tag{13}$$

where $E_\nu + m_n = E_e + m_p$ for Eq. (12) and $E_\nu + m_p = E_e + m_n$ for Eq. (13). The lower limit of the integral (12) is given by the minimum momentum that electrons must have in order to result $E_\nu > 0$, $p_e^0 = (\Delta m^2 - m_e^2)^{1/2}$. On the other hand, the lower limit in the integral (13) comes from the constraint $E_\nu = E_e + \Delta m > \Delta m$, since we suppose that final states are unbounded.

As remarked in Ref. [4], in general, the electron and neutrino temperatures, T_e and T_ν , may differ because at the end of the freezing out period, electrons and positrons annihilate, heating only the photons and maintaining with them thermal equilibrium. Indeed, in Paper I we have computed the amount of this deviation for our non-extensive setting. This difference is, however, small, and we shall follow Bernstein et al., and set all temperatures equal, $T = T_e = T_\nu = T_\gamma$. In the standard case this assumption ensure

that the rates for reverse reactions, such as $e^- + p \rightarrow n + \nu$, obey the principle of detailed balance (PDB).

3. Principle of detailed balance

The PDB states that if we know the expression for a reaction rate, say λ_{np} , then this is related with the reaction rate for the inverse reaction by an exponential factor

$$\lambda_{pn} = e^{-\Delta m/T} \lambda_{np} . \tag{14}$$

We shall study what happen in the non-extensive case, by analyzing each reaction appearing in Eq. (8) in a separate way.

3.1. $\nu + n \leftrightarrow p + e^-$

We would like to see what is the relationship between $\lambda_{e^-+p \rightarrow n+\nu}^q$ and $\lambda_{\nu+n \rightarrow p+e^-}^q$, with

$$\lambda_{e^-+p \rightarrow n+\nu}^q = A \int_{p_e^0}^{\infty} dp_e p_e^2 p_\nu E_\nu (1 - f_q^\nu) f_q^e \tag{15}$$

and

$$\lambda_{\nu+n \rightarrow p+e^-}^q = A \int_0^{\infty} dp_\nu p_\nu^2 p_e E_e (1 - f_q^e) f_q^\nu . \tag{16}$$

Starting from $\lambda_{e^-+p \rightarrow n+\nu}^q$, and taking into account that $p_e dp_e = E_e dE_e$, $E_\nu = E_e - \Delta m$ and $E_\nu = p_\nu$, it is possible to show that, using changes of variable

$$\lambda_{e^-+p \rightarrow n+\nu}^q = A \int_0^{\infty} dp_\nu p_\nu^2 p_e E_e (1 - f_q^\nu) f_q^e . \tag{17}$$

In the regime $x = E/T \gg 1$, f_q^i can be approximated by

$$f_q^i = e^{-x_i} + \frac{q-1}{2} x_i^2 e^{-x_i} \quad \text{with } x_i = E_i/T . \tag{18}$$

Since during the period of freezing, the temperature T is low compared with the energies appearing in the reaction rates, it is a good approximation to neglect Pauli factors. Then, to analyze the relationship between

$$\lambda_{e^-+p \rightarrow n+\nu}^q = A \int_0^{\infty} dp_\nu p_\nu^2 p_e E_e f_q^e \tag{19}$$

and

$$\lambda_{\nu+n \rightarrow p+e^-}^q = A \int_0^{\infty} dp_\nu p_\nu^2 p_e E_e f_q^\nu \tag{20}$$

it is enough to see how f_q^e and f_q^ν relate themselves. From Eq. (18), and recalling that in the reactions $\nu + n \leftrightarrow p + e^-$, $E_\nu = E_e - \Delta m$, we can directly write f_q^e in terms

of f_q^v . This yields

$$f_q^e = e^{-\Delta m/T} f_q^v + e^{-\Delta m/T} \frac{q-1}{2} (2E_v \Delta m + \Delta m^2) \frac{e^{-E_v/T}}{T^2} . \tag{21}$$

Then, the PDB is no longer valid, since the reaction rates in this case are related by

$$\begin{aligned} \lambda_{e^-+p \rightarrow n+v}^q &= e^{-\Delta m/T} \lambda_{v+n \rightarrow p+e^-}^q + \frac{q-1}{2} e^{-\Delta m/T} \frac{A}{T^2} \\ &\times \int_0^\infty dp_v p_v^2 p_e E_e (2E_v \Delta m + \Delta m^2) e^{-E_v/T} \end{aligned} \tag{22}$$

with $E_v = E_e - \Delta m$. We shall say that we are working within a detailed balance with non-extensive corrections (BNE).

It is worth noting that we can obtain an approximate expression among the rates, similar to what one obtains for the standard situation. Indeed, using $E_v/T \gg 1$ in Eq. (21), we get

$$f_q^e = e^{-\Delta m/T} f_q^v \tag{23}$$

what immediately yields

$$\lambda_{e^-+p \rightarrow n+v}^q = e^{-\Delta m/T} \lambda_{v+n \rightarrow p+e^-}^q . \tag{24}$$

We can recover the standard PDB, invoking this additional approximation, what we shall refer as to standard PDB (BST). In what follows, we shall show results both in the full BNE as in the BST approximation, and we shall discuss the range of validity of Eq. (23).

3.2. $e^+ + n \leftrightarrow p + \bar{\nu}$

A similar analysis yields

$$\begin{aligned} \lambda_{e^++n \rightarrow p+\bar{\nu}}^q &= e^{-\Delta m/T} \lambda_{\bar{\nu}+p \rightarrow n+e^+}^q + \frac{q-1}{2} e^{-\Delta m/T} \frac{A}{T^2} \\ &\times \int_0^\infty dp_e p_e^2 p_v E_v (2E_v \Delta m + \Delta m^2) e^{-E_e/T} , \end{aligned} \tag{25}$$

where $E_v = E_e + \Delta m$ and we neglected the electron mass with respect to the energies. In fact, a careful look at Eqs. (22) and (25) show that both integrals are identical.

Within BST, the relationship between the rates again simplifies, to give

$$\lambda_{e^++n \rightarrow p+\bar{\nu}}^q = e^{-\Delta m/T} \lambda_{\bar{\nu}+p \rightarrow n+e^+}^q . \tag{26}$$

3.3. $n \rightarrow p + e^- + \bar{\nu}$

Because of the fact that we have taken $1 - f_q^i$ as 1 in Eq. (11), no modification arises in this case

$$\frac{1}{\tau} = \lambda_{n \rightarrow p+e^-+\bar{\nu}} = 0.0157 A \Delta m^5 . \tag{27}$$

This allow us to determine A in terms of a measurable quantity, the neutron mean life τ ,

$$A = \frac{a}{\tau} \frac{1}{4} \Delta m^5, \quad a = 255. \tag{28}$$

We shall neglect the free decay of the neutron in this section, particularly when computing λ_{np} . This makes Eq. (8) to assume the form $\lambda_{np}^q = \lambda_{\nu+n \rightarrow p+e^-}^q + \lambda_{e^+n \rightarrow p+\bar{\nu}}^q$. Neglecting the blocking factors in Eqs. (9) and (10) it is evident that this two terms are identical. The same happens with Eqs. (12) and (13), but in this case we have to neglect, in addition, the electron mass in order to have p_0 equal to Δm . Due to this, $\lambda_{np}^q = 2\lambda_{\nu+n \rightarrow p+e^-}^q$ and $\lambda_{pn}^q = 2\lambda_{e^-+p \rightarrow n+\nu}^q$.

Using all previous results, the valid relationship between λ_{np}^q and λ_{pn}^q , both in the BNE as in the BST approximation are

$$\lambda_{pn}^q = e^{-\Delta m/T} \lambda_{np}^q + (q - 1)e^{-\Delta m/T} I, \tag{29}$$

where we have defined

$$I = \frac{A}{T^2} \int_0^\infty dp_\nu p_\nu^2 p_e E_e (2E_\nu \Delta m + \Delta m^2) e^{-E_\nu/T} \tag{30}$$

with $E_e = E_\nu + \Delta m$, for BNE, and

$$\lambda_{pn}^q = e^{-\Delta m/T} \lambda_{np}^q \tag{31}$$

for the BST.

4. Explicit form for the rates

To compute λ_{np}^q and λ_{pn}^q , it is clear that we have to know just one reaction rate, for instance $\lambda_{\nu+n \rightarrow p+e^-}^q$, and the value of I defined in Eq. (30). We can solve for $\lambda_{\nu+n \rightarrow p+e^-}^q$ using Eq. (20). With the distribution functions being as in Paper I we obtain

$$\lambda_{\nu+n \rightarrow p+e^-}^q = A \int_0^\infty dE_\nu E_\nu^2 (E_\nu + \Delta m)^2 f_q^\nu. \tag{32}$$

These integrals are easily computed changing variables to $x=E_\nu/T$ and using the integral form of the function $\Gamma(n)$. Noting that $\Gamma(n + 1) = n!$ for $n \in \mathbf{Z}$, defining $y = \Delta m/T$, and using (27), we obtain

$$\lambda_{\nu+n \rightarrow p+e^-}^q = \frac{1}{2} \frac{a}{\tau y^5} [12 + 6y + y^2] + \frac{a}{\tau y^5} (q - 1) [180 + 60y + 6y^2]. \tag{33}$$

These same considerations are valuable to compute I . It is given as

$$I(y) = \frac{1}{2} \frac{a}{\tau y^5} [120y + 60y^2 + 12y^3 + y^4]. \tag{34}$$

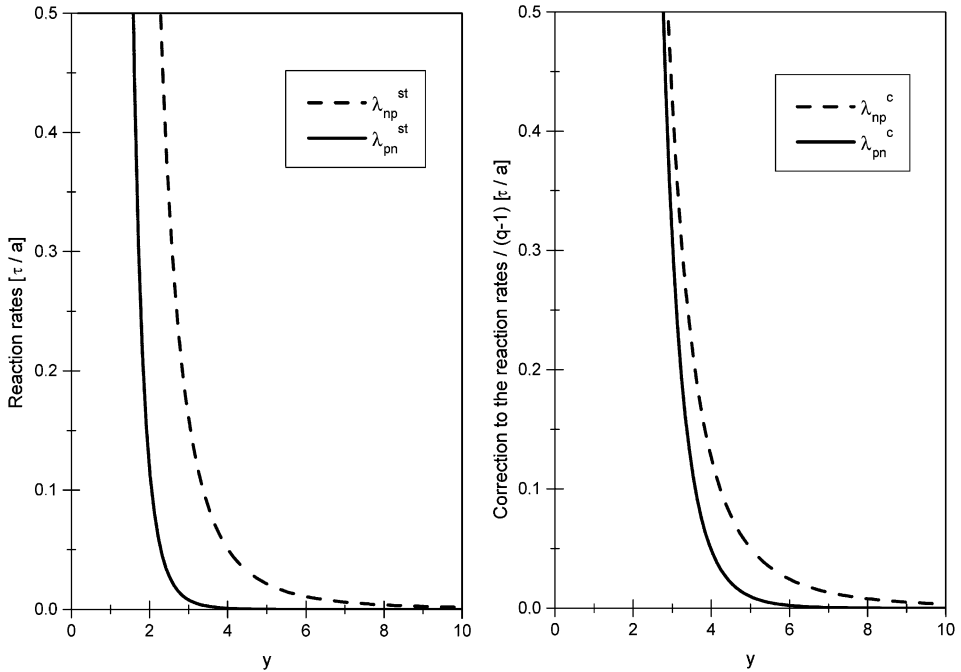


Fig. 1. Left Standard reaction rates $\lambda_{pn}^{st}(y)$ and $\lambda_{np}^{st}(y)$. Right: Corrections produced by non-extensive statistics, normalized by $(q - 1)$, using the full BNE.

4.1. λ_{np}^q

We have already mentioned that $\lambda_{np}^q = 2\lambda_{v+n \rightarrow p+e}^q$, i.e.,

$$\lambda_{np}^q(y) = \frac{a}{\tau y^5} [12 + 6y + y^2] + \frac{a}{\tau y^5} (q - 1) [180 + 60y + 6y^2]. \tag{35}$$

This equation is valid in both, the full BNE and the BST approximation.

4.2. λ_{pn}^q

We use Eqs. (29) and (34) to get, within the full BNE

$$\lambda_{pn}^q(y) = e^{-y} \lambda_{np}^q(y) + \frac{q - 1}{2} \frac{a}{\tau y^5} e^{-y} [120y + 60y^2 + 12y^3 + y^4]. \tag{36}$$

It is also immediate to obtain this rate in the BST, i.e.,

$$\lambda_{pn}^q(y) = e^{-y} \lambda_{np}^q(y). \tag{37}$$

This last equation was used by Torres and Vucetich [4], where the BST was assumed from the beginning.

It is worth noticing that since $\lambda_{pn}^{st}(y)$ and $\lambda_{np}^{st}(y)$ have an asymptotic behavior analogous to that of the standard rates, see Fig. 1, also in this case we expect that in the

limit of low temperatures (i.e., $y \rightarrow \infty$) the neutron abundance behaves as a constant. That is, for low temperatures, the reaction rates are practically zero and then, processes interchanging neutrons into protons and viceversa are negligible. In this sense, $X^q(y \rightarrow \infty)$ stabilizes towards a constant.

5. Evolution of the neutron abundance

The formal solution for the evolution of the neutron abundance is given by Eq. (7). We shall use the variable $y = \Delta m/T$, so that

$$X^q(y) = \frac{\lambda_{pn}^q(y)}{A^q(y)} - \int_0^y dy' I^q(y, y') \frac{d}{dy'} \left(\frac{\lambda_{pn}^q(y')}{A^q(y')} \right) \tag{38}$$

and the factor I becomes

$$I^q(y, y') = \exp \left(- \int_{y'}^y d\hat{y} \left(\frac{d\hat{t}}{d\hat{y}} \right) A^q(\hat{y}) \right). \tag{39}$$

To evaluate the Jacobian $d\hat{t}/d\hat{y}$, we recall that the scale factor, R , in a Friedmann–Robertson–Walker metric, goes as $R \simeq 1/T$, independently of the statistics [9]. Then, $\dot{T}/T = -\dot{R}/R$, and the rhs is given by Einstein equations

$$\frac{\dot{R}}{R} = \left(\frac{8\pi G}{3} \rho_q \right)^{1/2}. \tag{40}$$

Here, ρ_q is the energy density of relativistic species, given in Paper I. When the universe is dominated by e^-, e^+, ν and γ 's, and then $g_b = 2, g_f = 2 + 2 + 2 \times 3 = 10$ and $g = \sum_b g^b + \frac{7}{8} \sum_f g^f = 43/4$, we obtain

$$\rho_q = \frac{\pi^2}{30} [g + 35.85(q - 1)] T^4. \tag{41}$$

We can now compute dt/dy ,

$$\frac{dt}{dy} = \frac{dt}{dT} \frac{dT}{dy} = - \frac{1}{\dot{T}} \frac{\Delta m}{y^2}. \tag{42}$$

Since, $\dot{T}/T = -\dot{R}/R$,

$$\dot{T} = -T \left(\frac{8\pi G}{3} \rho_q \right)^{1/2} \tag{43}$$

and to first order in $(q - 1)$,

$$\frac{1}{\dot{T}} = - \frac{1}{T^3} \left(\frac{45}{4\pi^3 G g} \right)^{1/2} \left[1 - \frac{15}{g\pi^2} 35.85(q - 1) \right]. \tag{44}$$

In this sense, we see that

$$\frac{dt}{dy} = \frac{\tau}{a} b y [1 - c(q - 1)], \tag{45}$$

where b and c are given

$$b = \left(\frac{45}{4\pi^3 Gg} \right)^{1/2} \frac{a}{\tau \Delta m^2}, \quad c = \frac{15}{g\pi^2} 35.85. \tag{46}$$

Numerical values for these constants can be obtained taking into account that: $a = 255$, $g = 43/4$, $\Delta m = 1.29$ MeV, $\tau = 887 \pm 2$ s [10] and $G = m_{pl}^{-2}$ and are $b = 0.25$ and $c = 5.07$. This follows the treatment given in Ref. [4].

Using all previous remarks

$$I^q(y, y') = \exp(K^q(y) - K^q(y')) \tag{47}$$

with $K^q(y)$ given by

$$K^q(y) = - \int d\hat{y} \left(\frac{d\hat{t}}{d\hat{y}} \right) A^q(\hat{y}) \tag{48}$$

or, equivalently, using Eq. (45)

$$K^q(y) = - \frac{\tau b}{a} (1 - c(q - 1)) \int d\hat{y} \hat{y} A^q(\hat{y}). \tag{49}$$

We then see that the needed steps to solve for the evolution of X^q as a function of y are: (1) find the function $K^q(y)$ as in Eq. (49), (2) substitute this result in the expression (47) for $I^q(y, y')$, and (3) finally solve the integral in (38). The complexity of all functions involved requires a numerical procedure. However, we can do further before going to numerical work, since our aim is to primarily see how the asymptotic value of X , $X^q(y \rightarrow \infty)$, is affected as a function of the parameter $(q - 1)$, and ultimately, how this asymptotic value changes if one assume the full BNE as compared with the approximate BST approach.

6. Standard balance: BST

In this case, $\lambda^q(y)/A^q(y)$ coincides with its standard analogous $\lambda^{st}(y)/A^{st}(y)$. Indeed, by definition, $A^q(y) = \lambda_{pm}^q(y) + \lambda_{np}^q(y)$, and since BST establishes $\lambda_{pn}^q(y) = e^{-y} \lambda_{np}^q(y)$, we obtain $\lambda^q(y)/A^q(y) = (1 + e^y)^{-1} = \lambda^{st}(y)/A^{st}$.

We can get $K^q(y)$ taking into account that, using Eqs. (35) and (36)

$$A^q(y) = \frac{a}{\tau y^5} (1 + e^y) [12 + 6y + y^2] + (q - 1) [180 + 60y + 6y^2]. \tag{50}$$

Computing the integral in Eq. (49)

$$K^q(y) = b(1 - c(q - 1)) \left[K_1(y) + \frac{q - 1}{2} K_2(y) \right]. \tag{51}$$

with

$$K_1(y) = \left[\left(\frac{4}{y^3} + \frac{3}{y^2} + \frac{1}{y} \right) + \left(\frac{4}{y^3} + \frac{1}{y^2} \right) e^{-y} \right] \tag{52}$$

and

$$K_2(y) = \left[\left(\frac{30}{y^3} + \frac{15}{y^2} + \frac{3}{y} \right) + \left(\frac{30}{y^3} + \frac{3}{y} \right) e^{-y} - 3Ei(1, y) \right] \tag{53}$$

and where the functions $Ei(1, y)$ are

$$Ei(1, y) = \int_1^\infty \frac{e^{-yt}}{t} dt. \tag{54}$$

We recognize in Eq. (51) the standard term $K^{st}(y) = bK_1(y)$. The presence of the factor $b(1 - c(q - 1))$ in the Jacobian dt/dy produces the appearance in $K_2(y)$ of a second order term in $(q - 1)$. It is important to note that because this function diverges as y^{-3} at the origin, it is not a priori obvious that we can retain (as was done in Ref. [4]), only the term linear in $(q - 1)$. This is a key remark, and we shall come back on this later.

We shall then try to obtain the whole solution, without linearizing any of the functions involving $(q - 1)$. Having shown that $\lambda^q(y)/A^q(y) = (1 + e^y)^{-1}$, we can write, using Eq. (38), the asymptotic value $X^q(\infty) \equiv \lim_{y \rightarrow \infty} X^q(y)$, as

$$X^q(\infty) = \lim_{y \rightarrow \infty} e^{K^q(y)} \int_0^y e^{-K^q(y')} \frac{e^{y'}}{(1 + e^{y'})^2} dy'. \tag{55}$$

This expression allows for an analytical study. Based on the divergent behavior of K^q at the origin and because of the good behavior of

$$\frac{d}{dy'} \left(\frac{\lambda_{pn}^q(y')}{A^q(y')} \right) = \frac{e^{y'}}{(1 + e^{y'})^2} \tag{56}$$

near zero, it is possible to discard a priori, i.e., without having the need to compare with any observation, the range of $(q - 1)$ -values such that the non-extensive theory, considered within the BST, loses its sense (i.e., the cases in which the asymptotic values predicted for X^q are out of the interval $[0, 1/2]$). Indeed, in order to avoid problems at the origin in Eq. (55), $K^q(y)$ has to be positive when $y \rightarrow 0$. Using Eqs. (51)–(53), this condition translates immediately into an allowed interval for $(q - 1)$, since

$$\lim_{y \rightarrow 0} K^q(y) \simeq \lim_{y \rightarrow 0} b(1 - c(q - 1)) \left[\frac{4}{y^3} + \frac{4}{y^3} e^{-y} + \frac{q - 1}{2} \left(\frac{30}{y^3} + \frac{30}{y} e^{-y} \right) \right], \tag{57}$$

i.e.,

$$\lim_{y \rightarrow 0} K^q(y) \simeq b(1 - c(q - 1)) \left[\frac{8}{y^3} + (q - 1) \frac{30}{y^3} \right]. \tag{58}$$

We see that, to have $K^q(y) > 0$, $(q - 1)$ must be such that the inequality $-8/30 < (q - 1) < 1/c$ is fulfilled. Recalling that $c = 5.07$ we see that values that do not fulfill the condition

$$-0.27 < (q - 1) < 0.2 \tag{59}$$

must be automatically discarded. We mention in addition that no matter the sign of K^q , the factor $e^{K(y)}$ multiplying the integral in Eq. (55) tends to 1 when $y \rightarrow \infty$.

Fig. 2 shows the behavior of the integrand of Eq. (55) for different $(q - 1)$ -values. Note the abrupt change of the plots when $(q - 1)$ is near the extremes of the range given by Eq. (59). The analytical conclusion is then reinforced by these plots and is completely confirmed when numerical computations are made. These latter show that the integral (55) grows without limit when $(q - 1)$ assumes values out of the range given by Eq. (59). We have then showed that there exist a range of $(q - 1)$ such that the asymptotic value of $X^q(\infty)$ has physical sense. We can now ask if there also exist a range of $(q - 1) \in [-0.27, 0.2]$ such that the $X^q(\infty)$ obtained to first order in $(q - 1)$ is consistent with the whole computation. This would be so if we can prove that the first order result do not differ much from the real result that we have already got.

6.1. First order computation

To compute $X^q(\infty)$ to first order in $(q - 1)$, we need to write Eq. (55) to first order and neglect quadratic terms in Eq. (51). This gives

$$K^q(y) = K^{st}(y) + (q - 1)K^c(y) \tag{60}$$

with

$$K^{st}(y) = b \left[\left(\frac{4}{y^3} + \frac{3}{y^2} + \frac{1}{y} \right) + \left(\frac{4}{y^3} + \frac{1}{y^2} \right) e^{-y} \right] \tag{61}$$

and

$$K^c(y) = \frac{b}{2} \left[\left(\frac{30}{y^3} + \frac{15}{y^2} + \frac{3}{y} \right) + \left(\frac{30}{y^3} + \frac{3}{y} \right) e^{-y} - 3Ei(1, y) \right] - cK^{st}. \tag{62}$$

We write e^{K^q} to first order as

$$e^{K^q} = e^{K^{st}} e^{(q-1)K^c} \simeq e^{K^{st}} [1 + (q - 1)K^c]. \tag{63}$$

Substituting this expansion into (55) and retaining only the linear term we get

$$X^q(\infty) = X^{st}(\infty) + (q - 1)X^c(\infty) \tag{64}$$

with

$$X^{st}(\infty) = \lim_{y \rightarrow \infty} e^{K^{st}(y)} \int_0^y e^{-K^{st}(y')} \frac{e^{y'}}{(1 + e^{y'})^2} dy' \tag{65}$$

and

$$X^c(\infty) = \lim_{y \rightarrow \infty} e^{K^{st}(y)} \int_0^y e^{-K^{st}(y')} [K^c(y) - K^c(y')] \frac{e^{y'}}{(1 + e^{y'})^2} dy'. \tag{66}$$

This integrals can be computed numerically, and the result is

$$X^q(\infty) = 0.15 + (q - 1)0.18, \tag{67}$$

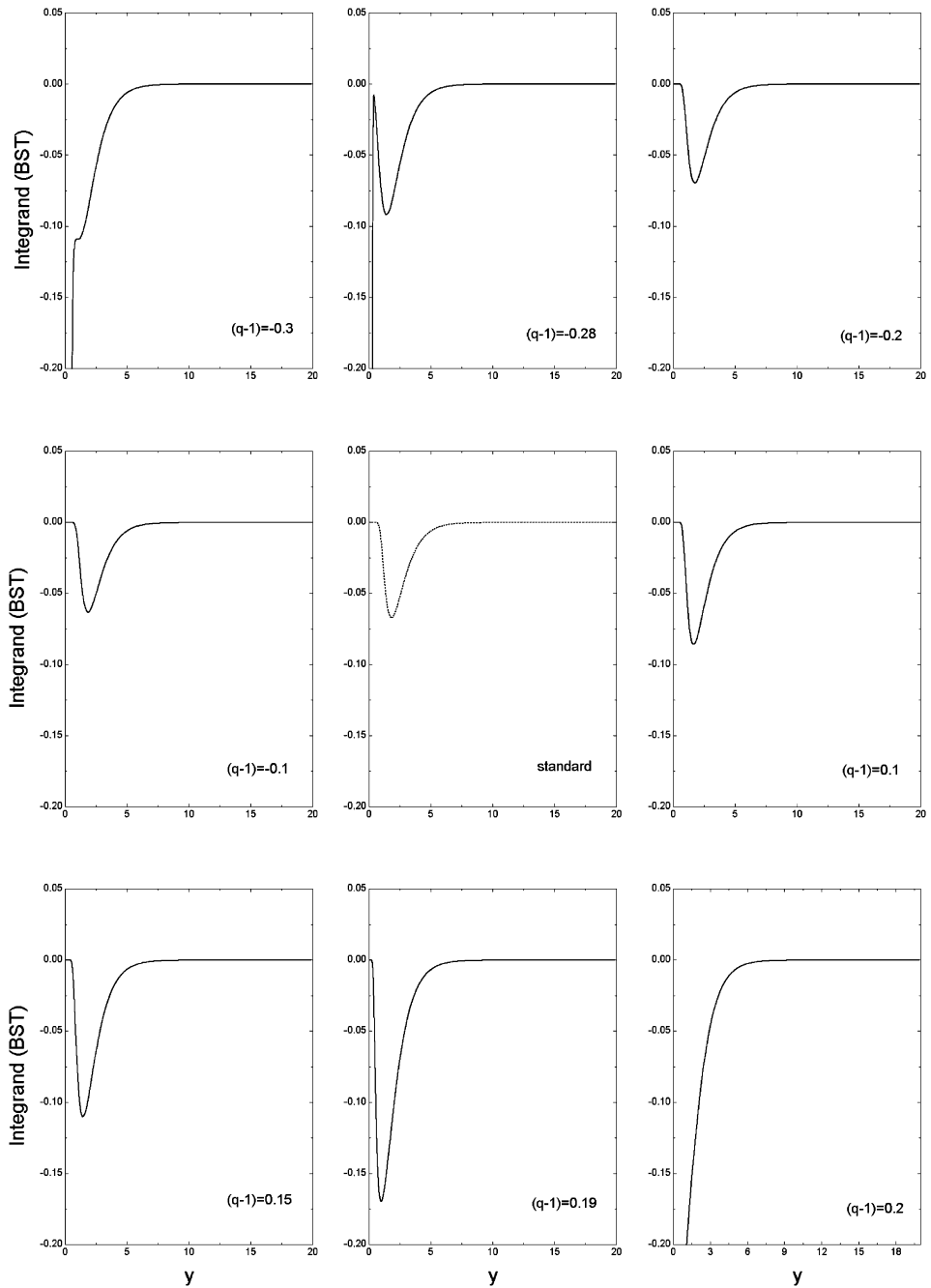


Fig. 2. Behavior of the integrand needed in the computation of $X^q(\infty)$ for different values of $(q - 1)$ within the BST approach.

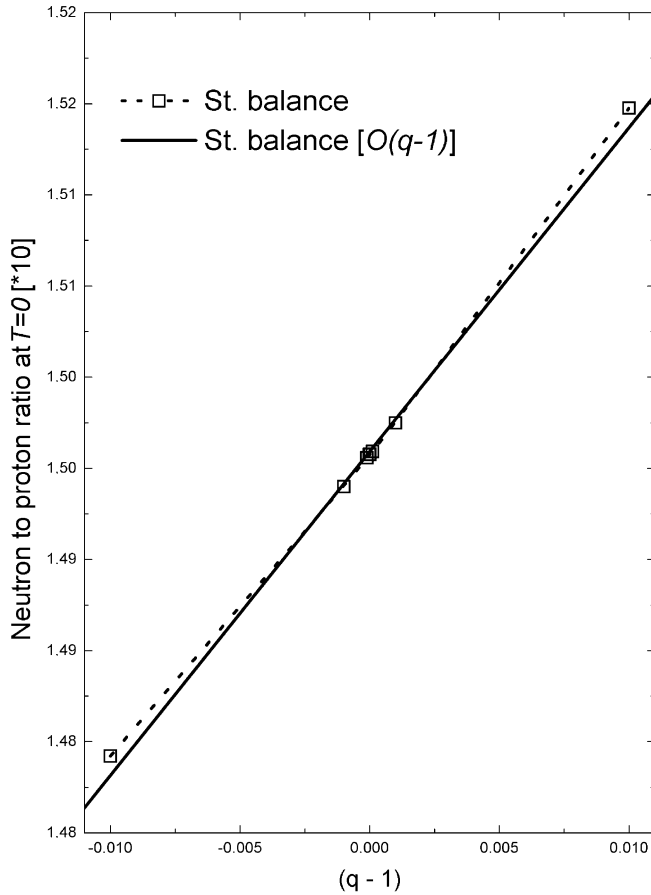


Fig. 3. Comparison between $X^q(\infty)$ as a function of $(q - 1)$, to first order, and using the full numerical solution, both obtained within the BST approach.

where the value $X^{st}(\infty)=0.15$ is the standard one. Fig. 3 shows the linear dependence of Eq. (67), together with the full solution given by Eq. (55). Although within the framework of the asymptotic approach to quantum distribution functions, an analogous equation to (67) was presented already by some of us [4], we have now shown, for the first time, what is behind that approach, what sustains it, and the range of its validity.

7. Non-extensive balance: BNE

In this section we shall solve Eq. (38) without approximations. That is, we shall take for λ_{pn}^q and λ_{np}^q , the expressions given by (35) and (36), respectively. Our aim will be to quantify the amount of the deviation that it is produced when the BST, instead of the full BNE, is considered.

By definition, $A^q(y) = \lambda_{pn}^q(y) + \lambda_{np}^q(y)$, adding (35) and(36) we see that $A^q(y)$ is given by the sum of two terms

$$A^{st}(y) = \frac{a}{\tau y^5} (12 + 6y + y^2)[1 + e^{-y}], \tag{68}$$

$$A^c(y) = \frac{1}{2} \frac{a}{\tau y^5} [90[1 + e^{-y}] + 30y[1 + 5e^{-y}] + y^2[1 + 21e^{-y}] + 12y^3 + y^4]. \tag{69}$$

Because of the linearity of the integral operator, the same happens to $K^q(y)$ in Eq. (49). Using the previous expressions, we get

$$K^q(y) = b(1 - c(q - 1)) \left[\hat{K}_1(y) + \frac{q - 1}{2} \hat{K}_2(y) \right] \tag{70}$$

with

$$\hat{K}_1(y) = \left[\left(\frac{4}{y^3} + \frac{3}{y^2} + \frac{1}{y} \right) + \left(\frac{4}{y^3} + \frac{1}{y^2} \right) e^{-y} \right] \tag{71}$$

and

$$\hat{K}_2(y) = \left[\left(\frac{30}{y^3} + \frac{15}{y^2} + \frac{3}{y} \right) + \left(\frac{30}{y^3} + \frac{60}{y^2} + \frac{3}{y} \right) e^{-y} - y - 12 \ln y - 3Ei(1, y) \right], \tag{72}$$

where $Ei(1, y)$ were given in (54). As before, we do recognize the standard term $K^{st}(y) = b\hat{K}_1(y)$. Also here, within the BNE, it happens that

$$\lim_{y \rightarrow \infty} \frac{\lambda_{pn}^q(y)}{A^q(y)} = 0. \tag{73}$$

Indeed, since within the BNE, in the limit $y \rightarrow \infty$ we have $\lambda_{pn}^q(y) \propto y^{-3}$ and $\lambda_{np}^q(y) \propto y^{-1}e^{-y}$, we obtain

$$\lim_{y \rightarrow \infty} \frac{\lambda_{pn}^q(y)}{A^q(y)} = \lim_{y \rightarrow \infty} \frac{1}{1 + \lambda_{pn}^q(y)/A^q(y)} = \lim_{y \rightarrow \infty} \frac{1}{1 + e^y} = 0. \tag{74}$$

We then obtain an analogous to Eq. (55)

$$X^q(\infty) = - \lim_{y \rightarrow \infty} e^{K^q(y)} \int_0^y e^{-K^q(y')} \frac{d}{dy'} \left(\frac{\lambda_{np}^q(y)}{\lambda_{pn}^q(y)} \right) dy'. \tag{75}$$

It is important to note that $K^q(y)$ diverges as y^{-3} at the origin, and as $-y$ at infinity. Then, first order developments will not do. The complex dependencies of the integrand of Eq. (38) with $(q - 1)$ makes much harder to a priori analyze the validity range, as done within the BST. We can, however, make a detailed analysis of the behavior of the derivative of the function $\lambda_{pn}^q(y)/A^q(y)$ and of the integrand of Eq. (38), for different $(q - 1)$ -values. We show this in Figs. 4 and 5. From the analysis of these figures we see that the BST and the BNE differ in a fundamental way, which translates into the value of $X^q(\infty)$. Within the BNE, and because of the behavior of the derivative of the function $\lambda_{pn}^q(y)/A^q(y)$ for negative values of $(q - 1)$, it is not guaranteed that the

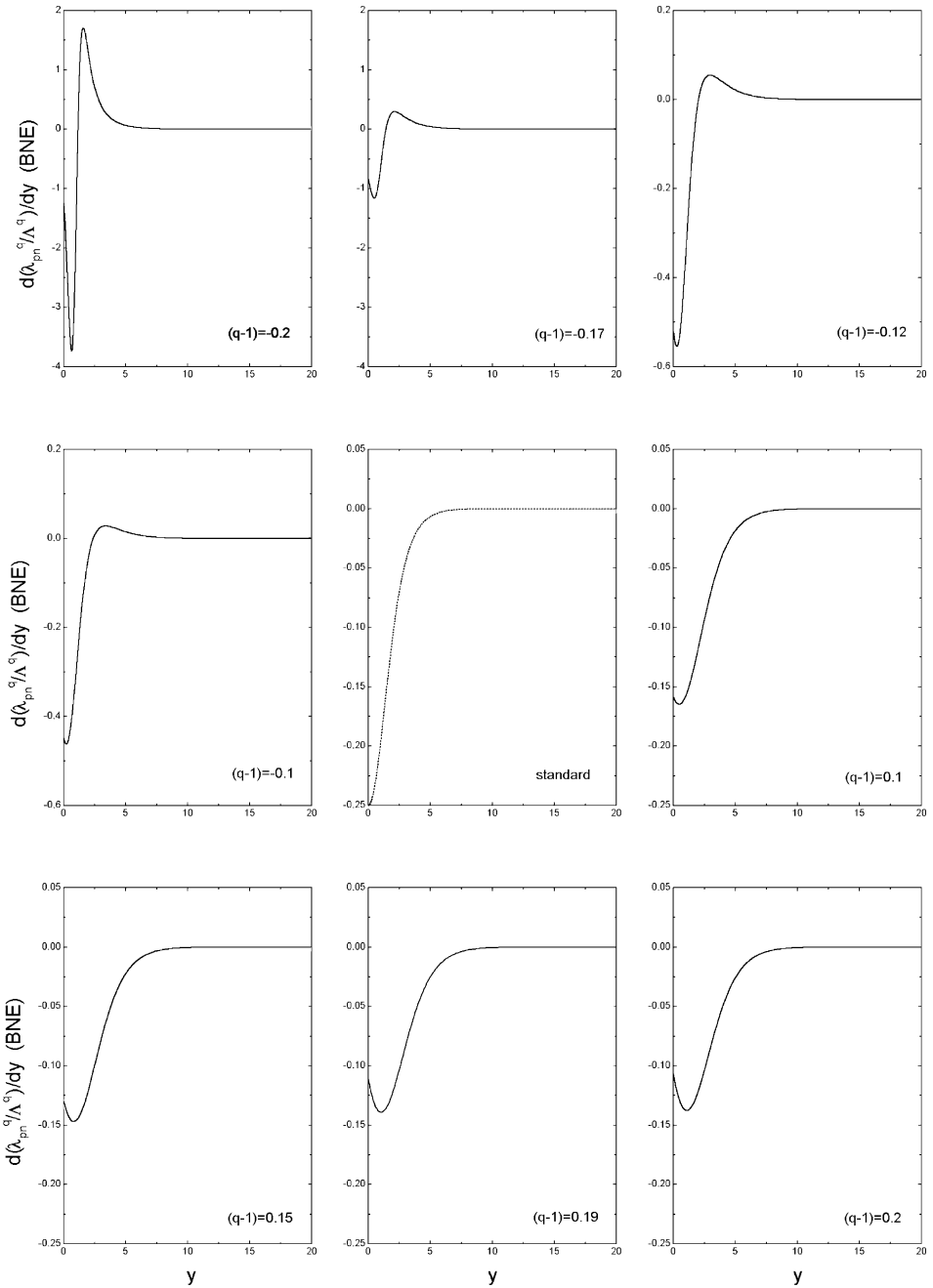


Fig. 4. Behavior of the derivative of $\lambda_{pn}^q(y)/A^q(y)$ for different $(q - 1)$ -values within the BNE.

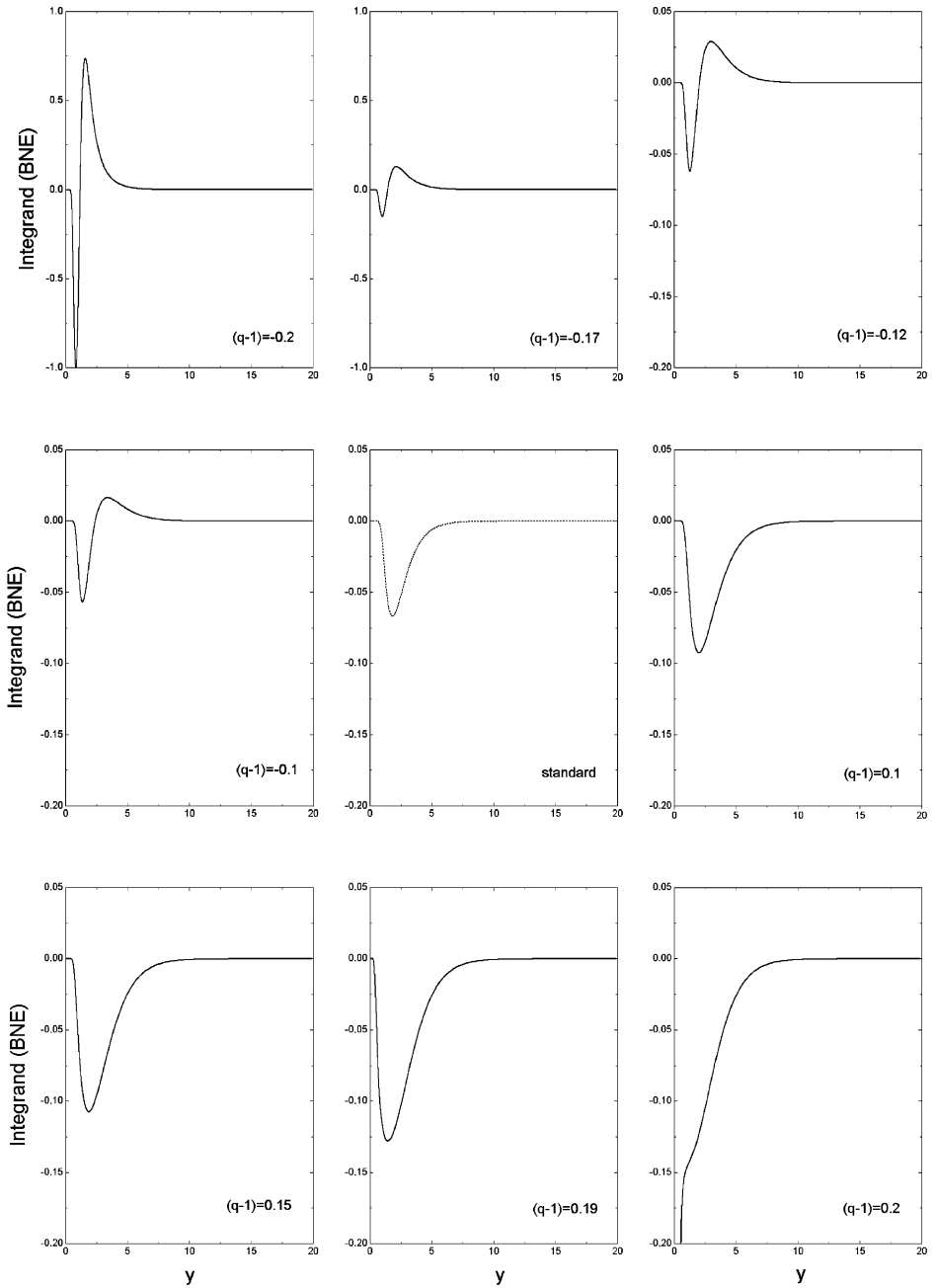


Fig. 5. Behavior of the integrand involved in the computation of $X^q(\infty)$ for different $(q - 1)$ -values, within the BNE.

integrand in Eq. (38) is negative. For $(q - 1) < -0.1$ the integral of (38) is positive (i.e., $X^q(\infty) < 0$), and this discards values of q such that $(q - 1) < -0.1$. On the other hand, looking at Fig. 4 for positive values of $(q - 1)$, we see that the upper bound is between 0.19 and 0.2, in this case similarly to the BST. Then, for the BNE, the values for which the theory has physical sense a priori of any experimental comparison are

$$-0.1 < (q - 1) < 0.2 . \tag{76}$$

The BNE is more restrictive than the BST. Out of this range, the physical sense of the description is lost.

8. Comparing both approaches, and bounding the value of q

We now compare the results obtained in Sections 6 and 7. Fig. 6 shows $X^q(\infty)$ for different values of $(q - 1)$, both in the full BNE as well as in the BST approximation. Differences are notable, and they grow with increasing values of $|q - 1|$, as it tends to its limiting values. Table 1 gives the explicit values of $X^q(\infty)$ obtained using a numerical integration scheme for the BST and for the full BNE, and the analytical results of the first order approximation for the BST. For values of $(q - 1)$ near 0, the BST numerical computation is perfectly described by the linear approximation. The deviation from the standard case is greater for the BNE than for the BST, for equal values of $(q - 1)$, i.e., the BST underestimates the effect of non-extensivity. This can be seen directly from Fig. 6, or equivalently from Table 2, where we show the coefficients of the linear fit of $X^q(\infty)$ near 0, both for the full BNE and the BST. For the sake of completeness, we also show the first order result for the BST (this is not a fit, but comes directly from the analytical computations shown before).

Table 1
Comparison of $X^q(\infty)$ -values as a function of $(q - 1)$ for the different approaches

$(q - 1)$	X^q (BST)	X^q (BST) ($O(q - 1)$)	X^q (BNE)
-0.150	0.1472	0.1230	-0.6571
-0.120	0.1441	0.1284	-0.1061
-0.100	0.1431	0.1320	0.0261
-0.050	0.1440	0.1410	0.1299
-0.010	0.1484	0.1482	0.1476
-0.001	0.1499	0.1498	0.1498
0.0	0.1500	0.1500	0.1500
0.001	0.1502	0.1502	0.1503
0.010	0.1520	0.1518	0.1523
0.050	0.1625	0.1590	0.1635
0.100	0.1851	0.1680	0.1915
0.120	0.1991	0.1716	0.2109
0.150	0.2303	0.1770	0.2561

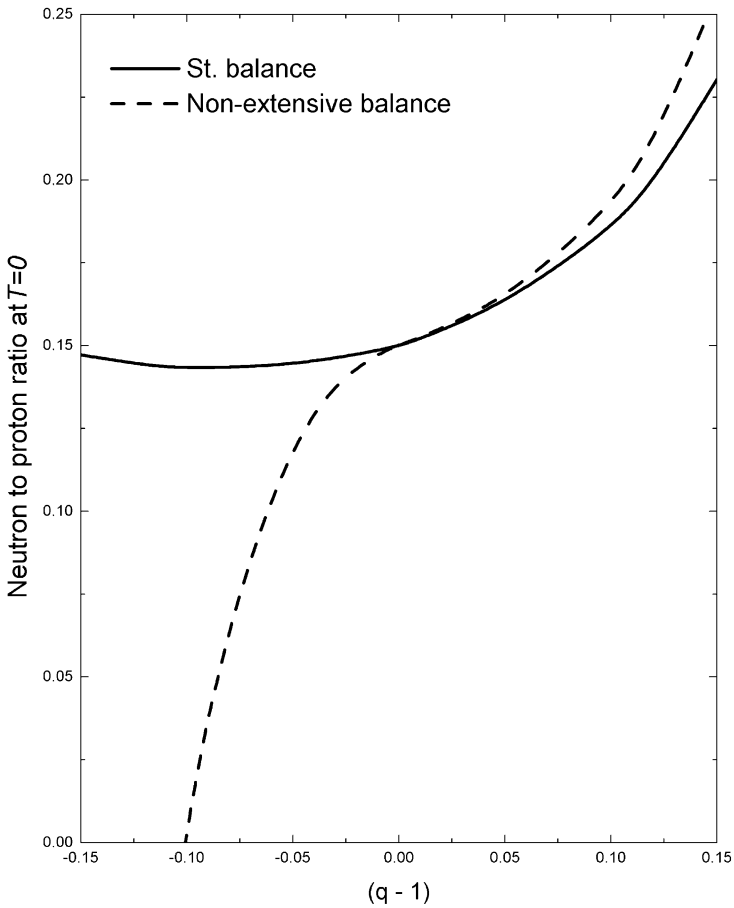


Fig. 6. $X^q(\infty)$ as a function of $(q - 1)$, both for the full BNE and the BST.

Table 2
Coefficients of the linear fits of $X^q(\infty)$ near $(q-1)=0$ for the full BNE, the BST approximation, and for the first order computation of the BST

Case	X^{st}	X^c
BST approximation	0.15	0.18
BST $O(q - 1)$	0.15	0.18
BNE Full	0.15	0.23

Until now, we have discussed the physical viability of the statistical description, establishing a range of a priori discarded values of $(q - 1)$ both for the full BNE case, the real situation, and for the BST approximation. These values are those for which $X^q(\infty)$ is < 0 or > 0.5 . This can be done before than the comparison with

any given observational data or experiment. Recall that $X(\infty)$ is the neutron-to-proton ratio, for which clearly, a negative value has no sense. In addition, $X(\infty) > 0.5$ is in conflict with the reaction rates dependence with temperature, since $X = 0.5$ is the initial condition for the kinetic equation (1) when $T \rightarrow \infty$ (i.e., $y \rightarrow 0$). Within the range of physical viability we now look for consistency with observations. This will further restrict the range of admitted values of $(q - 1)$. To obtain a direct bound upon $(q - 1)$ using the primordial abundance of ${}^4\text{He}$ it is necessary to study in detail the free decay of neutrons, happening between the moment of freezing out of the weak interactions ($t \simeq 1$ s) and the moment in which the temperature of the universe is similar to the binding energy of D ($t \simeq 3$ min). We shall provide, for the first time in a non-extensive setting, a detailed account of this in the following section, but nevertheless, let us give here some preliminary considerations in the sense of Ref. [2,3].

Including the effects of the neutron decay in the equation for the evolution of $X^q(t)$ we have

$$X^q(t) = \exp(-t/\tau)\bar{X}^q(t), \quad (77)$$

where \bar{X}^q is the already obtained ratio and τ is the neutron mean life. In the capture time, $t = t_c$, when the temperature is below the D binding energy (2.23 MeV), neutrons are captured into D. Then, almost all neutrons present at $t = t_c$ are converted into ${}^4\text{He}$. Substituting the value of t_c into Eq. (77) and using the value of $\bar{X}^q(\infty)$ previously found, we shall obtain half of the mass fraction of all ${}^4\text{He}$ primordially produced. For the time being, we shall adopt the standard value for $\exp(t_c/\tau)$, which is $\simeq 0.8$ [7]. With this we obtain

$$\begin{aligned} Y_p^q &\equiv 2X_4^q \simeq e^{-t_c/\tau} 2\bar{X}^q(\infty) \\ &= 0.8 \times 2 \times [0.15 + (q - 1)0.23] \\ &= 0.24 + (q - 1)0.37. \end{aligned} \quad (78)$$

There is no absolute consensus about the observational value of ${}^4\text{He}$. The two greatest compilations give [11,12],

$$Y_p = 0.244 \pm 0.004 \quad \text{and} \quad Y_p = 0.234 \pm 0.004. \quad (79)$$

To consider a typical case, we average over the two mean values and twice the error bar

$$Y_p = 0.239 \pm 0.008. \quad (80)$$

If at the same time we neglect the difference between the standard theoretical and the observational values, which is given by 0.001, in order to obtain a first bound, we can get, using Eq. (78) and asking for $|q - 1|0.37 < 0.008$,

$$|q - 1| < 0.021. \quad (81)$$

Within the most complete treatment of the principle of detailed balance (BNE), which accounts for the more in depth analytical study on the influence of a slight non-extensivity in primordial nucleosynthesis, the value of q could differ from 1 at the level of a few percent and still be in agreement with current constraints. See below for a more detailed treatment.

9. The capture time

The aim of this section is to show how the capture time t_c is modified in the non-extensive framework. Recalling that at early times $\dot{R}/R = (8\pi\rho_q/3m_{pl})^2$ and $R(t)T_\nu(t) = \text{constant}$ after the e^+e^- annihilation, it is straightforward to write

$$t = \left(\frac{3}{8\pi G}\right)^{1/2} \int_{T_\nu}^\infty \rho_q^{-1/2} \frac{dT'_\nu}{T'_\nu} + t_0, \tag{82}$$

where t_0 is a constant whose standard value is of the order 2 s, see Ref. [7]. In Paper I, it was shown that the energy density could be written as $\rho_R = (\pi^2/30)g_*^q T^4$. Taking also into account that the photon and neutrino temperatures are related by $T_\gamma = (11/4)^{1/3}[1 + 0.109(q - 1)]T_\nu$, we can solve the definite integral to obtain

$$t = \left(\frac{45}{16\pi^3 g_{eff}}\right)^{1/2} \left(\frac{11}{4}\right)^{2/3} \frac{m_{pl}}{T_\gamma^2} [1 - 5.06(q - 1)] + t_0, \tag{83}$$

where $g_{eff} = 2(11/4)^{4/3} + 21/4 \simeq 12.95$.

In equilibrium, neutrons, protons and deuterons behave as free non-relativistic gases, with number densities given by

$$n_q^i = g_i \left(\frac{m_i T_\gamma}{2\pi}\right)^{3/2} e^{-(m_i - \mu_i)/T_\gamma} \times \left[1 + \frac{q - 1}{2} \left(\frac{15}{4} + 3\frac{m_i - \mu_i}{T_\gamma} + \left(\frac{m_i - \mu_i}{T_\gamma}\right)^2\right)\right]. \tag{84}$$

Here, the subscript i stands for n, p and D , for neutrons, protons and deuterons, respectively, $g_n = g_p = 2, g_D = 3$ and μ_i, m_i are the chemical potentials and masses of the particles. At early times, these gases are in chemical equilibrium, so that $\mu_D = \mu_n + \mu_p$. Therefore

$$\frac{n_q^n n_q^p}{n_q^D} = \frac{g_n g_p}{g_D} \left(\frac{m_n m_p}{m_D}\right)^{3/2} \left(\frac{T_\gamma}{2\pi}\right)^{3/2} e^{-\varepsilon_D/T_\gamma} \left[\frac{\bar{u}((m_n - \mu_n)/T)\bar{u}((m_p - \mu_p)/T)}{\bar{u}((m_D - \mu_D)/T)}\right] \tag{85}$$

with $\varepsilon_D = m_n + m_p - m_D$, and where we have made use of the definition

$$\bar{u}((m_i - \mu_i)/T) \equiv 1 + \frac{q - 1}{2} \left(\frac{15}{4} + 3\frac{m_i - \mu_i}{T} + \left(\frac{m_i - \mu_i}{T}\right)^2\right). \tag{86}$$

Eq. (85) is the generalized Saha equation that describes the deuterium formation (see Paper I for a general derivation). Helium appearance is inhibited by the “deuteron bottleneck”, represented by the reactions



Eq. (85) can be given in terms of the normalized abundance fractions $X_q^i = n_q^i/n_q^B$. To this purpose we introduce the baryon to photon ratio $\eta_q = n_q^B/n_q^\gamma$ and recall that $n_q^\gamma = [2\zeta(3) + 3!\zeta(4)(q - 1)](T_\gamma^3/\pi^2)$. We get for the quantity $G_{np}^q = X_q^n X_q^p / X_q^D$, to first order in $(q - 1)$, the following result

$$G_{np}^q = \frac{\pi^{1/2}}{\zeta(3)} \frac{2}{3\eta_q} \left(\frac{m_p m_n}{2m_D T_\gamma} \right)^{3/2} e^{-\varepsilon_D/T_\gamma} \left[1 + \frac{q - 1}{2}(u_n + u_p - u_D) \right]. \tag{90}$$

Here, as in Paper I, $u_i = 15/4 + 3(m_i - \mu_i)/T + ((m_i - \mu_i)/T)^2$. As in the standard case analyzed in Ref. [7], to determine the neutron capture time t_c we need to examine the sequence of reactions (87)–(89). It is convenient to do so in terms of an scaled temperature $z = \varepsilon_D/T_\gamma$, so that they involve rate parameters of the form

$$R^q = \frac{dt}{dz} \langle \sigma v \rangle_T n_q^B, \tag{91}$$

where $\langle \sigma v \rangle_T$ denotes the thermal average of the relevant cross-section times the relative velocity [7]. Using Eq. (83) and writing the baryon number density in terms of the baryon to photon number density ratio, η_q , we get

$$R^q = \frac{\eta_q}{z^2} \left(\frac{45}{\pi^7 g_{eff}} \right)^{1/2} \left(\frac{11}{4} \right)^{2/3} \zeta(3) \varepsilon_D m_{pl} \langle \sigma v \rangle_T [1 + 0.34(q - 1)]. \tag{92}$$

Taking $\eta_0 = 3.57 \times 10^{10}$ [10] as a nominal value for η_q and considering $\langle \sigma_{np} v \rangle_T = 4.6 \times 10^{-20}$ [13] we obtain

$$R_{np}^q = 3.63 \left(\frac{29}{z^2} \right)^2 \frac{\eta_q}{\eta_0} [1 + 0.34(q - 1)]. \tag{93}$$

Neutron and proton populations are mainly determined by the kinetic equations

$$\frac{dX_q^n}{dz} = -R_{np}^q [X_q^n X_q^p - G_{np}^q X_q^D], \quad \frac{dX_q^p}{dz} = -R_{np}^q [X_q^n X_q^p - G_{np}^q X_q^D]. \tag{94}$$

As in the usual case, if the neutron population is not depleted by other reactions such as Eqs. (88)–(89), protons, neutrons and deuterons are kept in equilibrium, relationships $X_p^q + X_n^q + 2X_D^q = 1$ and $X_D^q = (G_{np}^q)^{-1} X_p^q X_n^q$ being valid. Since $(G_{np}^q)^{-1}$ is very small, the

deuterium number density will be also small, and we can write a first approximation as

$$X_q^{D,(1)} = (G_{np}^q)^{-1} X_q^{n,(0)} X_q^{p,(0)}, \tag{95}$$

where $X_q^{n,(0)}$ and $X_q^{p,(0)}$ are the unperturbed populations, that obey $X_q^{n,(0)} + X_q^{p,(0)} = 1$. Using this we now have

$$X_p^q + X_n^q \simeq 1 - 2(G_{np}^q)^{-1} X_q^{n,(0)} X_q^{p,(0)}. \tag{96}$$

Recalling Eq. (90), one sees that for the z -values we are interested ($z \simeq 30$), the major z dependence of $(G_{np}^q)^{-1}$ goes as e^z . Hence, to first order

$$\frac{d}{dz}(X_p^q + X_n^q) \simeq -2(G_{np}^q)^{-1} X_q^{n,(0)} X_q^{p,(0)}. \tag{97}$$

Adding Eqs. (94) and using Eq. (95) we find that

$$X_q^{D,(1)} = R_{np}^q [X_q^n X_q^p - G_{np}^q X_q^D]. \tag{98}$$

When deuterium number density is depleted, its number density will change according to

$$\frac{dX_q^D}{dz} = R_{np}^q [X_q^n X_q^p - G_{np}^q X_q^D] - R_{DD}^q [2(X_q^D)^2 - G_{DD}^q X_q^T X_q^q] + \dots, \tag{99}$$

where R_{DD} is the scaled rate for the reaction (88) and G_{DD}^q is the Saha factor which gives the equilibrium value for the ratio $(X_q^D)^2 / X_q^T X_q^q$. It can be shown that, in contrast to G_{np} , G_{DD} is always a small number (see Section III of Ref. [7]). In fact, in the standard framework G_{DD} is of order e^{-60} . We shall obviously be safe in considering $G_{DD}^q = G_{DD}$, and therefore in neglecting it. We can obtain R_{DD}^q inserting the value of $\langle \sigma_{DD} v \rangle_T$ in Eq. (92). The calculation of $\langle \sigma_{DD} v \rangle_T$ is a non-trivial task, which involves the use of a phenomenological fit to the cross-section plus a thermal average carried out using Boltzmann velocity distributions for the deuterons. For the sake of simplicity, we shall adopt the value of the thermal averaged cross-section used by Bernstein et al. This can be written as

$$\langle \sigma_{DD} v \rangle_T = 0.016 \frac{z^{2/3}}{\epsilon_D^{7/2} m_{pl}^{5/6}} e^{-1.44z^{1/3}}. \tag{100}$$

Introducing the nominal value of η_0 as we did it when we wrote R_{np}^q , we can now write for R_{DD}^q

$$R_{DD}^q = 1.76 \times 10^7 \left(\frac{\eta_q}{\eta_0} \right) z^{-4/3} e^{-1.44z^{1/3}} [1 + 0.34(q - 1)]. \tag{101}$$

When the deuterium number density decreases enough, the chain of reactions that converts almost all neutrons into helium is initiated. We may therefore identify the temperature $T_{\gamma,c}$, at which the neutrons are captured, or equivalently, the value of

$z = z_c = \varepsilon_D/T_{\gamma,c}$, as in the standard case, by the condition that

$$\left(\frac{dX_q^D}{dz} \right)_{z=z_c} \simeq 0. \tag{102}$$

Once we have neglected the factor G_{DD}^q , the condition stated above, together with Eqs. (95) and (98) and the approximation $X_q^D = X_q^{D,(1)}$, gives

$$2X_q^{D,(1)}R_{DD}^q \simeq 1. \tag{103}$$

In order to solve this equation we proceed as follows. Let us first rewrite the previous equation as

$$G_{np}^q = 2X_q^{n,(0)}X_q^{p,(0)}R_{DD}^q. \tag{104}$$

Note that we can put the rhs in terms of z . In Section 8 we have already given the asymptotic value of $X_q^{n,(0)}$ as $0.15 + 0.23(q - 1)$, and therefore the value of $X_q^{p,(0)}$ can be found through the relation $X_q^{n,(0)} + X_q^{p,(0)} = 1$. Now, using Eq. (101) leads, to first order in $(q - 1)$, to

$$G_{np}^q = 4.5 \times 10^5 \left(\frac{\eta_q}{\eta_0} \right) [1 + (q - 1)1.6]z^{-4/3}e^{-1.44z^{1/3}}. \tag{105}$$

On the other hand, we also have an expression for G_{np}^q : it is given by Eq. (90). Therefore, if we were able to write Eq. (90) in terms of z we can equal both expressions and solve for the unknown z_c , which will satisfy the equality for a given value of $(q - 1)$. Unfortunately, this is not easy. The difficulty is due to the presence of the chemical potentials in G_{np}^q in combinations other than the relation $\mu_D = \mu_n + \mu_p$. In this sense, we confront here a problem similar to that found in the recombination study of Paper I. We shall just mention that it is possible to show that the function $u_n + u_p - u_D$ has 10^3 as an upper bound, see the appendix for details. Therefore, in the range of z -values that concerns us, the corrections in G_{np}^q due to non-extensivity will be at most of order $[(q - 1)/2]10^3$ and we can then write for $G_{np}^q = G_{np}^q(z)$,

$$G_{np}^q = 2.98 \times 10^{12} \left(\frac{\eta_0}{\eta_q} \right) [1 + 5 \times 10^2(q - 1)]z^{3/2}e^{-z}. \tag{106}$$

Thus, equating expressions (105) and (106), and taking $\eta_q = \eta_0$, we finally found that z_c must satisfy the following relation,

$$1 = 1.5 \times 10^{-6} \frac{1 + 1.6(q - 1)}{1 + 5 \times 10^2(q - 1)} z_c^{-17/6} e^{-1.44z_c^{1/3}} e^{z_c}. \tag{107}$$

The standard result (i.e., $(q - 1) = 0$) gives $z_c = 27.08$. The numerical solution of this equation for different values of the parameter $(q - 1)$, and the corresponding capture time, are shown in Fig. 7. It is interesting to notice that while values of $(q - 1) > 0$ can be used within the analytical study presented in this section without any limit,

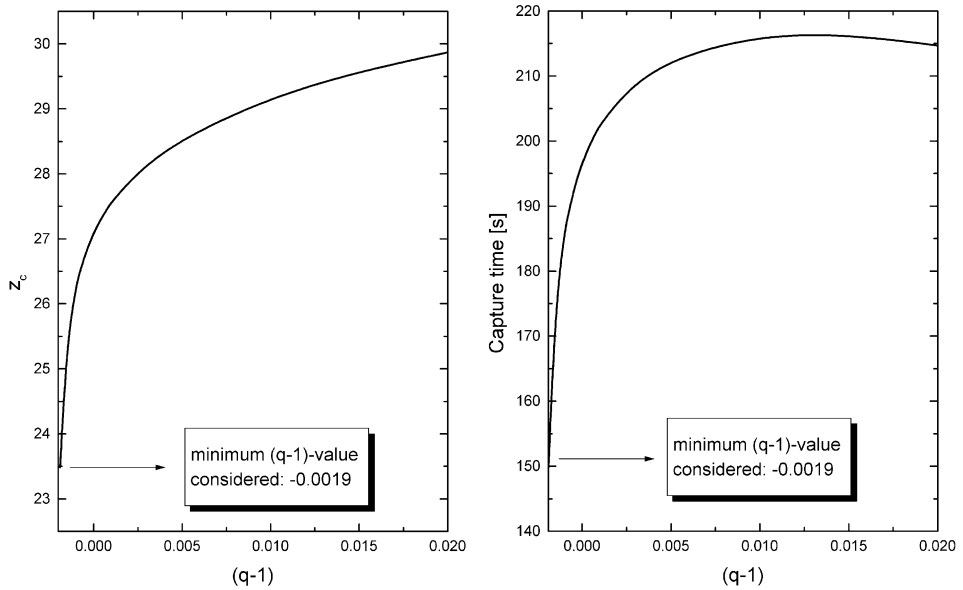


Fig. 7. Left z_c as a function of $(q - 1)$. Right: Capture time as a function of $(q - 1)$. Both plots are constructed for a nominal value of $\eta_q = 3.57 \times 10^{-10}$. Note the asymmetric range of the $(q - 1)$ -values considered. See text for explanation.

this is not the case for values of the parameter such that $(q - 1) < 0$. The problem arises in that in these cases, G_{np}^q can become negative, what clearly has no sense. In particular, for $(q - 1) = -0.002$, the denominator in Eq. (107) is identically zero, while for values of $(q - 1) < -0.002$ it is negative. Although this mines the physical viability of the description for these values of q , in the same sense that what was referred before, other assumptions are taken in this section, and only a numerical code can give a more precise answer in either way. Note that to obtain Fig. 7 we have made use of the non-extensive time–temperature correction. In Fig. 8 (left panel), we show the dependence of the ^4He abundance as a function of $(q - 1)$, taking into the free neutron decay correction.

Some comments about Figs. 7 and 8 are worth doing. Firstly, note that the dependence of the ^4He abundance, when neutron free decay is taken into account, is no longer linear as it was before, see for instance Eq. (78). This makes the study of the nucleosynthesis process much more interesting, and the use of the full numerical code a worth doing task. Now we know that even when considering a first (linear) order correction in the quantum distribution functions, the final output in the primordial nucleosynthesis cannot be reduced to a linear correction in the abundances. Secondly, it appears that for a given value of Y_p (and η) there are two values of $(q - 1)$ that could fit well the observations. Finally note that the y -axis of Fig. 8 (left panel) is showing variations at the level of less than 1%, and in principle, all values of $(q - 1)$ shown there are admitted by the current constraints on the primordial abundance of ^4He .

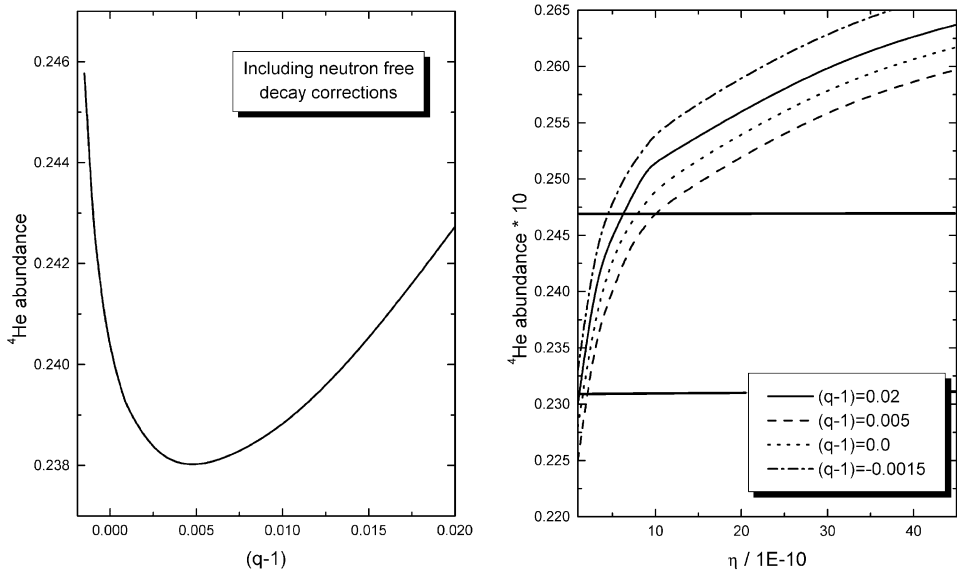


Fig. 8. Left: ${}^4\text{He}$ abundance as a function of $(q-1)$, taking into account neutron decay corrections. This plot is constructed for a nominal value of $\eta_q = 3.57 \times 10^{-10}$. Right: Primordial abundance of ${}^4\text{He}$ as a function of η_q . Several $(q-1)$ -values are shown. Two horizontal lines stand for a conservative current constraint on Y_p .

In Fig. 8 (right panel) we show for the first time the dependence of the ${}^4\text{He}$ abundance as a function of η_q , for different $(q-1)$ -values. The general trend of an increasing abundance as a function of η is maintained, but for different values of q , the curves cross the current constraints in different positions. This plot actually represents a 3D problem, with a two parameter space. The influence of η appears when numerically solving the equation for z_c . Indeed, the computation of the correction to the value of Y_p in the full analytical treatment presented here has to take into account different contributions. A direct linear dependence on $(q-1)$ is within the time–temperature relationship (see the first equations in this section, especially Eqs. (83)) and (78). In addition, we can see that if z_c is smaller than in the standard case, then T_c will be bigger, and t_c will be smaller than their respective standard counterparts. Thus, $e^{-t/\tau}$ will in turn be bigger, what happens for instance in the case of a negative value of $(q-1)$, and this would provide a bigger Y_p . However, the linear correction in Eq. (78) is competing with that positive deviation; this is what finally makes the curve to deviate from a straight line.

10. Conclusions

In this paper, based on previous works (Paper I), we have revisited the problem of the primordial genesis of light elements within the framework of non-extensive

statistics. We have particularly paid attention to the form of the principle of detailed balance that is valid in this new setting. We have shown that its usual form is no longer valid, but instead, that it is just an approximation, dubbed here as the BST. The full, new form of the principle of detailed balance was derived and named as BNE, and we have studied it in detail too. By doing so, we were able to disguise the range of validity of previous approaches to the topic, and to formally see the origin and range of the deviations between the standard and non-extensive scenarios for the primordial production of elements. We have also analyzed the neutron free decay correction to the capture time, and found that even when considering a first (linear) order correction in the quantum distribution functions, the final output cannot be reduced to a linear correction in the abundances, as Fig. 8 explicitly show. By comparing with the latest observational data, we have obtained a new bound on the upper limit of the parameter $|q - 1|$. We think that the bound being presented here (referring also to Figs. 7 and 8) is obtained within a much more detailed basis than the previous ones, see for instance [2–5], and then that this bound should be considered as the more reliable of them all. Indeed, we believe that the only way to improve it is to go to a direct implementation of the primordial nucleosynthesis code. By doing so we would have the possibility of studying the modifications to the abundances of all light elements in a simultaneous way. This further extension is currently under analysis.

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Appendix A. $u_n + u_p - u_D$

To have a rough idea of the order of magnitude of the function $u_n + u_p - u_D$, we need to compute $(m_i - \mu_i)/T$ in the range of temperatures we are interested in, with $i = n, p$, and D . Firstly, we note that for this purpose it is enough to consider the standard values of all these quantities. This is because non-extensive effects here would introduce only second order corrections in $(q - 1)$, that we are disregarding in our computation. Recalling the relation $X_i = (g_i/n_B)(m_i T/2\pi)^{3/2} \exp((m_i - \mu_i)/T)$, we see that in order to estimate the values of $(m_i - \mu_i)/T$ we need to know the abundances X_i . Then, to get $u_n + u_p - u_D$, we have made the following steps:

1. Using the time–temperature relation we can put $X_n(t) = \exp(-\tau/t)X_n(t_{freeze})$ as an explicit function of T . Thus $X_n(T)$ follows, as well as $(m_n - \mu_n)/T$.

2. Considering $X_n(T)$ together with the obtainable function $X_n X_p / X_D$ (remember that $\mu_n + \mu_p - \mu_D = 0$), and imposing the (approximate) constrain $X_n + X_p + 2X_D = 1$, we get the values of X_p and X_D . Then, the corresponding values of $(m_p - \mu_p)/T$ and $(m_D - \mu_D)/T$ can be obtained.

These results allow us to construct $u_n + u_p - u_D$ and thus to see that 10^3 constitutes an upper bound in the range $0.07 < T < 0.1$ MeV (i.e., $22 < z < 32$).

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