



Original articles

A new approach to deal with C^2 cubic splines and its application to super-convergent quasi-interpolation

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Abstract

In this paper, we construct a novel normalized B-spline-like representation for C^2 -continuous cubic spline space defined on an initial partition refined by inserting two new points inside each sub-interval. The basis functions are compactly supported non-negative functions that are geometrically constructed and form a convex partition of unity. With the help of the control polynomial theory introduced herein, a Marsden identity is derived, from which several families of super-convergent quasi-interpolation operators are defined.

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1. Introduction

Nowadays, high smoothness splines have widespread applications in many fields, including approximation theory, computer-aided geometric design, the entertainment industry, etc. The highest regularity with the lowest degree is a commonly used option. In particular, C^2 cubic splines are very attractive since they fulfill this feature and allow to deal efficiently with many different problems.

As shown in [8], C^2 cubic splines on a partition endowed with a specific refinement are obtained if all values and derivative values up to order two at the break-points of the initial partition are given. More specifically, to get global C^2 cubic splines, the initial partition should be refined by inserting two new knots inside each sub-interval induced by the primary partition (for the general case, see [5]).

The idea of introducing a split knot was introduced for the first time by L. L. Schumaker in [16] to address the case of quadratic splines. Adopting the same procedure, C. Manni in [10] has investigated interpolation by means of C^1 quadratic and C^2 cubic many-knot splines with shape parameters. More recently, the same idea has been used in [8,12] when addressing the problem of Hermite interpolation with C^2 cubic splines with the aid of

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blossoming. Unfortunately, the strategies outlined in these papers have some drawbacks. In fact, the B-spline-like bases constructed in [8] are non-positive, while the strategy developed in [12] is somewhat complicated. The latter may be seen as a special case of the approach proposed here.

The strategy of refining a given partition is commonly used in multivariate approximation by splines [7]. In fact, in order to construct smooth splines of low degree, the given partition must be refined to a number of smaller simplices. The construction of bases to properly represent the spline functions defined on the refined triangulation is essential (see for instance [4,11] for Clough–Tocher and Powell–Sabin refinements, respectively). The methods developed to deal with these issues in the bivariate case (see [6]) have inspired the construction in [1], where splines of any degree and appropriate smoothness defined on an arbitrary partition refined by adding a knot in each sub-interval are constructed and used to define quasi-interpolants. Super-convergent quasi-interpolants defined on Powell–Sabin triangulations have also been discussed in the literature (see [14]).

As mentioned above, in this paper we consider a refinement of the initial partition by inserting two split knots inside each initial sub-interval and define a space of C^2 cubic splines. Each spline in this space is uniquely determined by its value and that of those of its derivatives up to order 2 at each knot of the initial partition. Since the C^2 cubic spline space is characterized by an interpolation problem, a B-spline-like basis is constructed by defining its basis functions as duals of the interpolation functionals. This will be done in a completely geometric form in order to get compactly supported non-negative B-spline-like functions forming a convex partition of unity.

The solution of a Hermite interpolation problem in this space gives rise to a many-knot spline, which can be considered as a differential quasi-interpolant. Therefore, the notion of control polynomial allows us to obtain a Marsden identity from which we define quasi-interpolants that reproduce the cubic polynomials.

Super-convergence is a phenomenon that appears when the order of convergence at some particular points is higher than the order of convergence over the whole domain of definition [2,3,15]. It is an advantageous theoretical property that can be exploited successfully in practice. The theory of control polynomials used here, allows to define a family of super-convergent quasi-interpolation operators.

The rest of the paper is organized as follows: In Section 2, we review some notions related to Bernstein–Bézier representation and polar forms. In Section 3, we introduce the space of many-knot C^2 cubic splines, and we describe the geometric approach used to construct B-spline-like bases with interesting properties. In Section 4, we develop a general theory of super-convergent quasi-interpolants based on control polynomials. In Section 5, the construction of several families of quasi-interpolants is addressed. Section 6 is devoted to illustrate the theoretical results obtained by some numerical tests. A section of conclusions is also included.

2. Preliminaries

Bernstein–Bézier representation and blossoming are the basic tools to deal with the construction addressed in this paper, so we recall some results about them and establish some others.

Each $x \in \mathbb{R}$ can be written with respect to a given interval $I := [a, b]$ as $x = (1 - t)a + tb$, being $(1 - t, t)$ the corresponding barycentric coordinates. They are non-negative when x belongs to I .

Polynomials of degree less than or equal to d can be conveniently represented on the interval I from the Bernstein polynomials $\mathfrak{B}_{\beta, I}^d$ of degree d , defined as

$$\mathfrak{B}_{\beta, I}^d(t) := \frac{d!}{\beta!} t^{\beta_1} (1 - t)^{\beta_2},$$

where $t := \frac{x-a}{b-a}$, $\beta := (\beta_1, \beta_2)$, $\beta! := \beta_1! \beta_2!$ and $|\beta| := \beta_1 + \beta_2 = d$. They form a convex partition of unity on I , and provide a basis for the space $\mathbb{P}_d(I)$ of polynomials of degree less than or equal to d defined over I . Each $p_d \in \mathbb{P}_d(I)$ can be uniquely expressed as

$$p_d(x) = \sum_{|\beta|=d} b_\beta \mathfrak{B}_{\beta, I}^d(t) =: b_d(t), \quad x \in I, \tag{1}$$

for some values $\{b_\beta\}_{|\beta|=d}$ which will be referred to as Bernstein–Bézier (BB-) coefficients or Bézier (B-) ordinates of p_d with respect to I . Equality (1) is said to be the Bernstein–Bézier (BB-) representation of p_d .

The B-ordinates $\{b_\beta\}_{|\beta|=d}$ of p_d can be neatly expressed in terms of blossoms or polar forms. The blossom or polar form $\mathbf{B}[p_d]$ of a given polynomial p_d of degree less than or equal to d is the unique, symmetric, multi-affine map from \mathbb{R}^d to \mathbb{R} fulfilling the following property [13]:

$$\mathbf{B}[p_d](A[d]) = p_d(A),$$

where $A[d]$ marks that the point A is repeated d times as a blossom argument. When $d = 1$, A will be used instead of $A[1]$.

The blossom $\mathbf{B}[p_d]$ of $p_d \in \mathbb{P}_d$ allows to write

$$p_d(x) = \sum_{|\beta|=d} \mathbf{B}[p](a[\beta_1], b[\beta_2]) \mathfrak{B}_{\beta, I}^d(\tau), \quad x \in I. \tag{2}$$

It also leads to simple expressions for directional derivatives. The derivative of $p_d \in \mathbb{P}_d$ at the point v in the directions u_1, u_2, \dots, u_q is given as,

$$\mathbf{D}_{u_1, \dots, u_q}^q p_d(v) := \frac{d!}{(d-q)!} \mathbf{B}[p_d](v[d-q], u_1, \dots, u_q). \tag{3}$$

Regularity of a piecewise polynomial function can be easily described in terms of B-ordinates with respect to the intervals. Let $I_1 = [a, c]$ and $I_2 = [c, b]$ be two adjacent intervals, and let p and \tilde{p} be two polynomials of total degree d defined on I_1 and I_2 , with B-ordinates $b_{1,\beta}$ and $b_{2,\beta}$, respectively. Assume that $\hat{\tau} := (\hat{\tau}_1, \hat{\tau}_2)$ are the barycentric coordinates of b with respect to I_1 . Then, the piecewise function defined as p on I_1 and \tilde{p} on I_2 is of class C^r at c if, for $\beta_1 = 0, \dots, r$, and $\beta_2 = d - r$, it holds

$$b_{2,\beta} = \sum_{|\alpha|=\beta_1} b_{1,\alpha+\beta_2 e_2} \mathfrak{B}_{\alpha, I_1}^r(\hat{\tau}),$$

where $e_2 := (0, 1)$.

In order to express the blossom of a product in terms of blossoms of its factors, we use the following result introduced in [9].

Proposition 1. *Let ℓ_1, \dots, ℓ_m be m polynomials in \mathbb{P}_1 . If $p = \prod_{i=1}^m \ell_i$, then, we have*

$$\mathbf{B}[p](u_1, \dots, u_m) = \frac{1}{m!} \sum_{\pi \in \sigma_m} \prod_{i=1}^m \ell_i(u_{\pi(i)}),$$

where σ_m stands for the symmetric group of all permutations of the set $\{1, \dots, m\}$.

Now, by using the relationship between polynomials and their blossoms, we will obtain a result that will allow to define the so-called control polynomials, which will be the main tool for establishing a Marsden identity which is the key for building super-convergent quasi-interpolation schemes based on C^2 cubic splines space.

Let us recall the following result [9].

Lemma 2. *Let d_1 and d_2 be two positive integers, with $d_2 \leq d_1$. Then, for any polynomial $p \in \mathbb{P}_{d_1}$ and any set of values $x_1, \dots, x_{d_1-d_2}$ in \mathbb{R} , the function*

$$q(x) := \mathbf{B}[p](x_1, \dots, x_{d_1-d_2}, x[d_2])$$

is a polynomial of degree less than or equal to d_2 . Moreover, for any set of values y_1, \dots, y_{d_2} in \mathbb{R} , it holds

$$\mathbf{B}[q](y_1, \dots, y_{d_2}) = \mathbf{B}[p](x_1, \dots, x_{d_1-d_2}, y_1, \dots, y_{d_2}).$$

The behavior of the controlled spline function at any knot can be detected from the behavior of control polynomials at the same knot. The following result, that defines the control polynomial of degree d_2 at the knot x_1 of a polynomial p of degree d_1 , is an alternative way to establish Marsden’s identity.

Proposition 3. *Let d_1 and d_2 be two positive integers, with $d_2 \leq d_1$. Let $p \in \mathbb{P}_{d_1}$ and $x_1 \in \mathbb{R}$. For any real number θ , the polynomial q of degree d_2 defined by*

$$q(x) := \mathbf{B}[p](x_1[d_1 - d_2], (\theta x + (1 - \theta)x_1)[d_2]), \tag{4}$$

satisfies

$$D^j p(x_1) = \frac{1}{\theta^j} \binom{d_1}{j} D^j q(x_1)$$

for all $0 \leq j \leq d_2$.

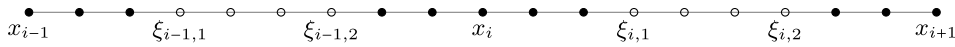


Fig. 1. Schematic representation of domain points corresponding to the BB-representation of a C^2 cubic spline. The points depicted by (•) represent the degree of freedom, while, the points represented by (○) mark the B-ordinates computed from imposed C^2 smoothness at the inserted split points.

Proof. We prove the result by induction on d_2 . As polar forms are multi-affine, then q can also be written as

$$q(x) = \sum_{i=0}^{d_2} \binom{d_2}{i} \theta^i (1-\theta)^{d_2-i} \mathbf{B}[p](x_1[d_1-i], x[i]).$$

From Lemma 2, q is a polynomial of degree less than or equal to d_2 . Define the polynomial q_i of degree i as

$$q_i(X) := \mathbf{B}[p](x_1[d_1-i], x[i]),$$

and let $\xi = \vec{e}$, i.e. a unit vector in \mathbb{R} .

Since $q_i \in \mathbb{P}_i$, we consider only the case when $a \leq i$ to derive the equality

$$D^j q_i(x_1) = \frac{i!}{(i-j)!} \mathbf{B}[q_i](x_1[i-j], \xi[a]) = \frac{i!}{(i-j)!} \mathbf{B}[p](x_1[i-j], \xi[j]).$$

Then,

$$\begin{aligned} D^j q(x_1) &= \sum_{i=j}^{d_2} \frac{d_2!}{(d_2-i)!(i-j)!} \theta^i (1-\theta)^{d_2-i} \mathbf{B}[p](x_1[i-j], \xi[j]) \\ &= \sum_{\ell=0}^{d_2-j} \frac{d_2!}{(d_2-j)!\ell!} \theta^{\ell+j} (1-\theta)^{d_2-j-\ell} \mathbf{B}[p](x_1[d_1-j], \xi[j]) \\ &= \theta^j \frac{d_2!}{(d_2-j)!} \mathbf{B}[p](x_1[d_1-j], \xi[j]), \end{aligned}$$

and the proof is complete. \square

When $\theta := \frac{d_1}{d_2}$, q is said to be control polynomial of degree d_2 at x_1 of the polynomial p .

3. A space of C^2 many-knot splines

For a given $n \geq 2$, let $X_n := \{x_0 < x_1 < \dots < x_n\}$ be a subset of knots providing a partition of I into sub-intervals $I_i = [x_i, x_{i+1}]$, $0 \leq i \leq n-1$. A refinement X_n^{ref} of the initial partition X_n is defined by inserting two split points $\xi_{i,1} := \frac{1}{3}(2x_i + x_{i+1})$ and $\xi_{i,2} := \frac{1}{3}(x_i + 2x_{i+1})$ in each sub-interval I_i that defines the micro-intervals $I_{i,1} := [x_i, \xi_{i,1}]$, $I_{i,2} := [\xi_{i,1}, \xi_{i,2}]$ and $I_{i,3} := [\xi_{i,2}, x_{i+1}]$.

In this work, we focus on the spline space

$$S_3^2(X_n^{\text{ref}}) := \{s \in C^2(I) : s|_{I_{i,j}} \in \mathbb{P}_3, j = 1, 2, 3, 0 \leq i \leq n-1\}.$$

A spline $s \in S_3^2(X_n^{\text{ref}})$ can be uniquely characterized by three specific values at each knot x_i [5].

Theorem 4. Given values $f_{i,0}, f_{i,1}, f_{i,2}$, $0 \leq i \leq n$, there exists a unique spline $s \in S_3^2(X_n^{\text{ref}})$ such that

$$s(x_i) = f_{i,0}, \quad s'(x_i) = f_{i,1}, \quad s''(x_i) = f_{i,2}, \tag{5}$$

Fig. 1 shows a graphical representation relative to Theorem 4. The B-ordinates of s corresponding to x_i and its neighboring domain points depicted by dark bullets (•) are computed from interpolation conditions (5). The remaining B-ordinates are determined from the C^2 smoothness conditions at the inserted split points.

In what follows, we will look for a normalized representation of the spline $s \in S_3^2(X_n^{\text{ref}})$ of the form

$$s = \sum_{i=0}^n \sum_{|\alpha|=2} c_{i,\alpha} \mathcal{B}_{i,\alpha}, \tag{6}$$

in which the basis functions $\mathcal{B}_{i,\alpha}$ are non-negative, have local supports and form partition of unity.

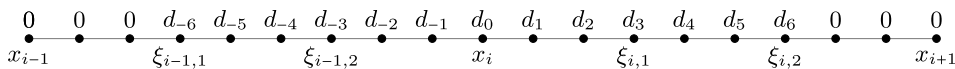


Fig. 2. B-ordinates of the B-spline-like $B_{i,\alpha}$ associated with the break-point x_i .

3.1. Setting up a normalized B-spline-like representation

This subsection is devoted to construct suitable B-spline-like functions $B_{i,\alpha}$, $i = 0, \dots, n$, $|\alpha| = 2$, such that (6) holds for any spline $s \in S_3^2(X_n^{\text{ref}})$.

The construction used herein is entirely geometric. For every knot x_i , $1 \leq i \leq n - 1$, define

$$W_{i,1} := \frac{4}{3}\xi_{i-1,2} - \frac{1}{3}x_i, \quad W_{i,2} := \frac{4}{3}\xi_{i,1} - \frac{1}{3}x_i, \tag{7}$$

and the interval $W_i := [W_{i,1}, W_{i,2}]$. From W_i we introduce nine parameters relative to x_i . Let $\mathfrak{B}_{W_i,\alpha}^2$, $|\alpha| = 2$, denote the Bernstein polynomials of degree 2 w.r.t. W_i , and define, for $0 \leq j \leq 2$, $|\alpha| = 2$ and a given integer $m \geq 3$, the values

$$\gamma_{i,\alpha}^j := \frac{\binom{j}{m}}{\binom{j}{2}} \left(\frac{2}{m}\right)^j D^j \mathfrak{B}_{W_i,\alpha}^2(x_i). \tag{8}$$

The B-spline-like functions for $S_3^2(X_n^{\text{ref}})$ are defined in terms of conditions (5) provided in Theorem 4. The definition of those associated with x_i , i.e. $B_{i,\alpha}$, $|\alpha| = 2$, is based entirely on parameters $\gamma_{i,\alpha}^j$. Indeed, $B_{i,\alpha}$ is the unique function in $S_3^2(X_n^{\text{ref}})$ such that

$$B_{i,\alpha}(x_i) = \gamma_{i,\alpha}^0, \quad B'_{i,\alpha}(x_i) = \gamma_{i,\alpha}^1, \quad B''_{i,\alpha}(x_i) = \gamma_{i,\alpha}^2,$$

and $B_{i,\alpha}(x_\ell) = B'_{i,\alpha}(x_\ell) = B''_{i,\alpha}(x_\ell) = 0$ at any knot x_ℓ different from x_i .

A schematic representation of the B-ordinates corresponding to the B-spline-like $B_{i,\alpha}$ associated with the break-point x_i of X_n is depicted in Fig. 2. By definition, the B-ordinates at the domain points in a neighborhood of x_{i-1} and x_{i+1} are equal to zero. Because of C^2 smoothness at x_i , B-ordinates d_{-2}, d_{-1}, d_0, d_1 and d_2 are completely determined by the value $\gamma_{i,\alpha}^j$. They are given explicitly as follows:

$$\begin{aligned} d_0 &= \gamma_{i,\alpha}^0, \\ d_1 &= \gamma_{i,\alpha}^0 + \gamma_{i,\alpha}^1 \frac{\xi_{i,1} - x_i}{3}, \quad d_2 = \gamma_{i,\alpha}^0 + 2\gamma_{i,\alpha}^1 \frac{\xi_{i,1} - x_i}{3} + \gamma_{i,\alpha}^2 \frac{(\xi_{i,1} - x_i)^2}{6}, \\ d_{-1} &= \gamma_{i,\alpha}^0 + \gamma_{i,\alpha}^1 \frac{\xi_{i-1,2} - x_i}{3}, \quad d_{-2} = \gamma_{i,\alpha}^0 + 2\gamma_{i,\alpha}^1 \frac{\xi_{i-1,2} - x_i}{3} + \gamma_{i,\alpha}^2 \frac{(\xi_{i-1,2} - x_i)^2}{6}. \end{aligned}$$

Since $B_{i,\alpha}$ is C^2 continuous at $\xi_{i-1,1}, \xi_{i-1,2}, \xi_{i,1}$ and $\xi_{i,2}$, then

$$\begin{aligned} d_3 &= \frac{1}{6}(7d_2 - 2d_1), & d_4 &= \frac{1}{3}(4d_2 - 2d_1), & d_5 &= \frac{1}{3}(2d_2 - d_1), & d_6 &= \frac{1}{6}(2d_2 - d_1), \\ d_{-3} &= \frac{1}{6}(7d_{-2} - 2d_{-1}), & d_{-4} &= \frac{1}{3}(4d_{-2} - 2d_{-1}), & d_{-5} &= \frac{1}{3}(2d_{-2} - d_{-1}), & d_{-6} &= \frac{1}{6}(2d_{-2} - d_{-1}). \end{aligned}$$

Remark 1. Boundary B-spline-like functions for $S_3^2(X_n^{\text{ref}})$ are constructed according to the same procedure highlighted in Section 3.1, with a particular choice of points in (7), namely $W_{0,1} := x_0$ and $W_{n,2} := x_n$.

Fig. 3 shows the graphs of the vertex B-spline-like functions for interior and boundaries vertices.

3.2. Properties of B-spline-like basis functions

In many practical applications, especially in the area of computer aided geometric design, bases with compactly supported non-negative functions that form a partition of unity are desired. In what follows, we are going to prove that the B-spline-like basis functions constructed here meet these properties.

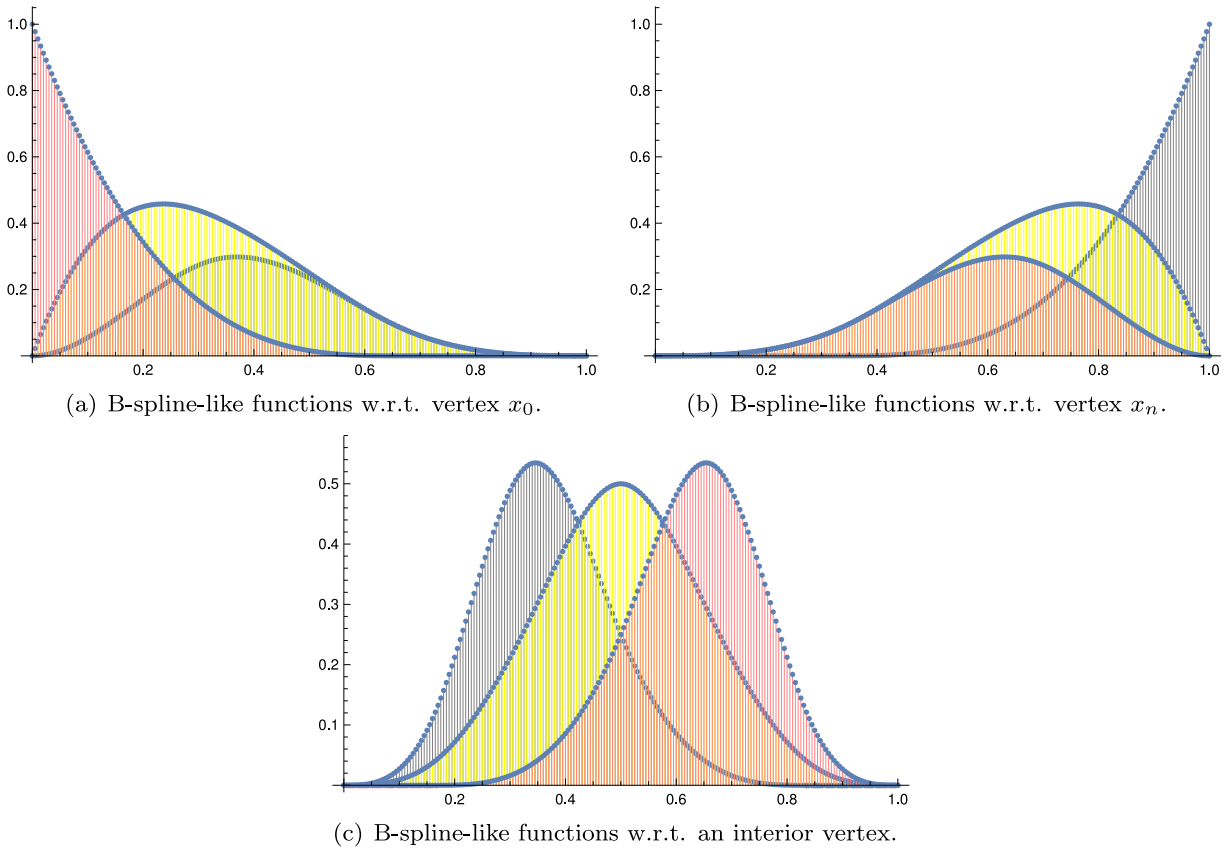


Fig. 3. B-spline-like functions for interior and boundary knots.

Property 5. The B-spline-like functions $\mathcal{B}_{i,\alpha}$, $i = 0, \dots, n$, $|\alpha| = 2$, form a partition of unity, i.e.

$$1 = \sum_{i=0}^n \sum_{|\alpha|=2} \mathcal{B}_{i,\alpha}.$$

Proof. It follows from the definition of $\mathcal{B}_{i,\alpha}$ that only three basis functions have non-zero function and derivative values at x_i . Moreover, the Bernstein polynomials in (8) form a partition of unity on W_i . Then, it holds

$$\sum_{|\alpha|=2} \gamma_{i,\alpha}^0 = 1, \quad \sum_{|\alpha|=2} \gamma_{i,\alpha}^1 = \sum_{|\alpha|=2} \gamma_{i,\alpha}^2 = 0, \tag{9}$$

and the claim follows by considering interpolation problem (5) and (9). \square

Property 6. The B-splines $\mathcal{B}_{i,\alpha}$, $i = 0, \dots, n$, $|\alpha| = 2$, are non-negative.

Proof. It suffices to prove that the B-ordinates of $\mathcal{B}_{i,\alpha}$ are all non-negative. Let

$$u := \frac{\xi_{i,1} - x_i}{|\xi_{i,1} - x_i|}.$$

A quadratic polynomial p defined on the interval $[P_1, P_2]$ with $P_1 = x_i$ and $P_2 = \frac{1}{3}x_i + \frac{2}{3}\xi_{i,1}$ has B-ordinates d_0 , d_1 and d_2 if and only if

$$p(x_i) = \mathcal{B}_{i,\alpha}(x_i) = d_0,$$

$$\frac{1}{2} \frac{2}{3} D_u p(x_i) = \frac{1}{3} D_u \mathcal{B}_{i,\alpha}(x_i) = \frac{d_1 - d_0}{|\xi_{i,1} - x_i|},$$

$$\frac{1}{2} \left(\frac{2}{3}\right)^2 D_u^2 p(x_i) = \frac{1}{6} D_u^2 \mathcal{B}_{i,\alpha}(x_i) = \frac{d_0 - 2d_1 + d_2}{|\xi_{i,1} - x_i|}.$$

From (8), it follows that p must be equal to a certain Bernstein polynomial of degree 2 w.r.t. W_i .

Since P_1, P_2 can be written as

$$P_1 = x_i, \quad P_2 = \frac{1}{2}x_i + \frac{1}{2}W_{i,2},$$

them P_1 and P_2 lie in W_i , which means that the barycentric coordinates of P_1 and P_2 w.r.t. W_i are non-negative. Suppose they are $\sigma^1 = (\sigma_1^1, \sigma_2^1)$ and $\sigma^2 = (\sigma_1^2, \sigma_2^2)$, respectively. Then, we get,

$$d_0 = \mathbf{B}[p](\sigma^1, \sigma^1), \quad d_1 = \mathbf{B}[p](\sigma^1, \sigma^2), \quad d_2 = \mathbf{B}[p](\sigma^2, \sigma^2).$$

Since polar forms are multi-affine, we get

$$d_6 = \frac{1}{6}(2d_2 - d_1) = \frac{1}{6}\mathbf{B}[p](2\sigma^2 - \sigma^1, \sigma^2),$$

and the barycentric coordinates $2\sigma^2 - \sigma^1$ corresponding to $2P_2 - P_1$ w.r.t W_i , i.e.,

$$2P_2 - P_1 = (2\sigma_1^2 - \sigma_1^1) W_{i,1} + (2\sigma_2^2 - \sigma_2^1) W_{i,2}.$$

Since $W_{i,2} = \frac{4}{3}\xi_{i,1} - \frac{1}{3}x_i = 2P_2 - P_1$, then, $2\sigma^2 - \sigma^1$ are the barycentric coordinates corresponding to the point $W_{i,2} \in W_i$, which means that $2\sigma^2 - \sigma^1$ are non negative.

Then, it follows that $2d_2 - d_1 \geq 0$, and therefore, $d_3 = \frac{1}{6}(3d_2 + 2(2d_2 - d_1))$, $d_4 = \frac{1}{3}(2(2d_2 - d_1))$ and $d_5 = \frac{1}{3}(2d_2 - d_1)$ are all non-negative. Following the same strategy, we can prove also that $d_{-3}, d_{-4}, d_{-5}, d_{-6} \geq 0$. \square

Any B-spline-like $\mathcal{B}_{i,\alpha}$ is related to a quadratic Bernstein polynomial, and the coefficients $c_{i,\alpha}$, $|\alpha| = 2$, corresponding to $\mathcal{B}_{i,\alpha}$ are B-ordinates of a quadratic polynomial defined on the interval W_i called control polynomial w.r.t. the knot x_i and defined as

$$T_i(x) := \sum_{|\alpha|=2} c_{i,\alpha} \mathfrak{B}_{W_i,\alpha}^2(x), \quad x \in W_i.$$

The following result justifies the name.

Property 7. *The polynomial T_i is tangent to the spline $s \in S_3^2(X_n^{ref})$ at x_i .*

Proof. For $s \in S_3^2(X_n^{ref})$ and $a = 0, 1$, it holds

$$s^{(j)}(x_i) = \sum_{|\alpha|=2} c_{i,\alpha} \gamma_{i,\alpha}^j = \sum_{|\alpha|=2} c_{i,\alpha} D^j \mathfrak{B}_{W_i,\alpha}^2(x) = T_i^{(j)}(x_i),$$

and the claim follows. \square

3.3. B-spline-like representation

This subsection aims to derive the coefficients of (6) for an interpolation spline.

Suppose that $s \in S_3^2(X_n^{ref})$ is determined by the Hermite interpolation problem (5). The evaluation of $s^{(j)}$, $0 \leq j \leq 2$, at x_i yields the linear system

$$\begin{pmatrix} \mathcal{Y}_{i,(2,0)}^0 & \mathcal{Y}_{i,(1,1)}^0 & \mathcal{Y}_{i,(0,2)}^0 \\ \mathcal{Y}_{i,(2,0)}^1 & \mathcal{Y}_{i,(1,1)}^1 & \mathcal{Y}_{i,(0,2)}^1 \\ \mathcal{Y}_{i,(2,0)}^2 & \mathcal{Y}_{i,(1,1)}^2 & \mathcal{Y}_{i,(0,2)}^2 \end{pmatrix} \begin{pmatrix} c_{i,(2,0)} \\ c_{i,(1,1)} \\ c_{i,(0,2)} \end{pmatrix} = \begin{pmatrix} f_{i,0} \\ f_{i,1} \\ f_{i,2} \end{pmatrix}.$$

The definition of the parameters $\gamma_{i,\alpha}^j$ in (8) involves the values of Bernstein polynomials and those derivatives. Since they are linear independent, the solution of the linear system is then unique, and given by

$$\begin{aligned} c_{i,(2,0)} &= f_{i,0} + f_{i,1} (W_{i,1} - x_i) + \frac{m}{4(m-1)} f_{i,2} (W_{i,1} - x_i)^2, \\ c_{i,(1,1)} &= f_{i,0} + \frac{1}{2} f_{i,1} (W_{i,1} + W_{i,2} - 2x_i) + \frac{m}{4(m-1)} f_{i,2} (W_{i,1} - x_i) (W_{i,2} - x_i), \\ c_{i,(0,2)} &= f_{i,0} + f_{i,1} (W_{i,2} - x_i) + \frac{m}{4(m-1)} f_{i,2} (W_{i,2} - x_i)^2. \end{aligned} \tag{10}$$

With $h_{i-1} := x_i - x_{i-1}$ and $h_i := x_{i+1} - x_i$, they can be simplified to get

$$\begin{aligned} c_{i,(2,0)} &= f_{i,0} + \frac{4}{81} h_{i-1} \left(-9f_{i,1} + \frac{m}{m-1} h_{i-1} f_{i,2} \right), \\ c_{i,(1,1)} &= f_{i,0} + \frac{2}{9} f_{i,1} (h_i - h_{i-1}) + \frac{-4m}{81(m-1)} h_{i-1} h_i f_{i,2}, \\ c_{i,(0,2)} &= f_{i,0} + \frac{4}{81} h_i \left(9f_{i,1} + \frac{m}{m-1} h_i f_{i,2} \right). \end{aligned}$$

Any cubic spline $s \in S_3^2(X_n^{ref})$ can be uniquely expressed in the form (6). Thus, in the BB-representation of a polynomial p , the coefficients $c_{i,\alpha}$ of s can be expressed in terms of polar form values of a polynomial obtained by restricting s to a specific sub-interval.

Proposition 8. For $m = 3$, let $s \in S_3^2(X_n^{ref})$. Denote by $s_{|[x_i, \xi_{i,1}]}$ the restriction of s to the interval $[x_i, \xi_{i,1}]$. Then, the coefficients $c_{i,\alpha}$ in the B-splines representation (6) of s can be expressed as

$$\begin{aligned} c_{i,(2,0)} &= \mathbf{B} [s_{|[x_i, \xi_{i,1}]}] (x_i, \tilde{W}_{i,1}, \tilde{W}_{i,1}), \\ c_{i,(1,1)} &= \mathbf{B} [s_{|[x_i, \xi_{i,1}]}] (x_i, \tilde{W}_{i,1}, \tilde{W}_{i,2}), \\ c_{i,(0,2)} &= \mathbf{B} [s_{|[x_i, \xi_{i,1}]}] (x_i, \tilde{W}_{i,2}, \tilde{W}_{i,2}), \end{aligned}$$

where $\tilde{W}_{i,1} := \frac{3}{2}W_{i,1} - \frac{1}{2}x_i$ and $\tilde{W}_{i,2} := \frac{3}{2}W_{i,2} - \frac{1}{2}x_i$.

Proof. From (3), the values of the above blossoms are expressed in terms of function and derivative values up to order 2 of s at x_i as

$$\begin{aligned} &\mathbf{B} [s_{|[x_i, \xi_{i,1}]}] (x_i, \tilde{W}_{i,1}, \tilde{W}_{i,1}) \\ &= \mathbf{B} [s_{|[x_i, \xi_{i,1}]}] \left(x_i, \frac{3}{2}W_{i,1} - \frac{1}{2}x_i, \frac{3}{2}W_{i,1} - \frac{1}{2}x_i \right) \\ &= \mathbf{B} [s_{|[x_i, \xi_{i,1}]}] \left(x_i, \frac{3}{2}(W_{i,1} - x_i) + x_i, \frac{3}{2}(W_{i,1} - x_i) + x_i \right) \\ &= \frac{9}{4} \mathbf{B} [s_{|[x_i, \xi_{i,1}]}] (x_i, W_{i,1} - x_i, W_{i,1} - x_i) + 3 \mathbf{B} [s_{|[x_i, \xi_{i,1}]}] (x_i, W_{i,1} - x_i, x_i) \\ &+ \mathbf{B} [s_{|[x_i, \xi_{i,1}]}] (x_i, x_i, x_i) \\ &= \frac{9}{4} \frac{1}{6} D_{W_{i,1} - x_i}^2 s(x_i) + 3 \frac{1}{3} D_{W_{i,1} - x_i} s(x_i) + s(x_i) \\ &= \frac{3}{8} s''(x_i) (W_{i,1} - x_i)^2 + s'(x_i) (W_{i,1} - x_i) + s(x_i). \end{aligned}$$

By the same technique, we can get

$$\mathbf{B} [s_{|[x_i, \xi_{i,1}]}] (x_i, \tilde{W}_{i,2}, \tilde{W}_{i,2}) = \frac{3}{8} s''(x_i) (W_{i,2} - x_i)^2 + s'(x_i) (W_{i,2} - x_i) + s(x_i),$$

and,

$$\mathbf{B}[s_{[x_i, \xi_{i,1}]}] \left(x_i, \tilde{W}_{i,1}, \tilde{W}_{i,2} \right) = \frac{3}{8} s''(x_i) (W_{i,1} - x_i) (W_{i,2} - x_i) + \frac{1}{2} s'(x_i) (W_{i,1} + W_{i,2} - 2x_i) + s(x_i).$$

The expressions obtained are the same as the ones in (10) for $m = 3$, which completes the proof. \square

After this result, a Marsden identity is obtained: any spline $s \in S_3^2(X_n^{\text{ref}})$ can be expressed compactly as

$$s = \sum_{i=0}^n \sum_{|\alpha|=2} \mathbf{B}[s_{[x_i, \xi_{i,1}]}] \left(x_i, \tilde{W}_{i,1}[\alpha_1], \tilde{W}_{i,2}[\alpha_2] \right) \mathcal{B}_{i,\alpha}. \tag{11}$$

4. Super-convergent quasi-interpolants

In what follows, we aim to construct some super-convergent quasi-interpolation operators that map an element of the linear space of polynomials of degree less than or equal to $m \geq 3$ to an element of $S_3^2(X_n^{\text{ref}})$.

Define

$$Q_{i,\ell} := \frac{m}{2} W_{i,\ell} + \left(1 - \frac{m}{2}\right) x_i, \quad \ell = 1, 2,$$

Then, we have the following result.

Theorem 9. For any integer $m \geq 3$ and $p \in \mathbb{P}_m$, let $Q_m p$ be defined as

$$Q_m p = \sum_{i=0}^n \sum_{|\alpha|=2} \mathbf{B}[p] \left(x_i[m-2], Q_{i,1}[\alpha_1], Q_{i,2}[\alpha_2] \right) \mathcal{B}_{i,\alpha}. \tag{12}$$

Then, $Q_m p \in S_3^2(X_n^{\text{ref}})$ and $Q_m p = p$ for all $p \in \mathbb{P}_3$.

Proof. We will prove that

$$D^j Q_m p(x_i) = D^j p(x_i), \quad i = 0, \dots, n, \quad 0 \leq j \leq 2, \quad \text{for all } p \in \mathbb{P}_m.$$

We have

$$Q_m p(x_i) = \sum_{|\alpha|=2} \mathbf{B}[p] \left(x_i[m-2], Q_{i,1}[\alpha_1], Q_{i,2}[\alpha_2] \right) \mathcal{B}_{i,\alpha}(x_i).$$

Define

$$q_{x_i}(x) := \sum_{|\alpha|=2} \mathbf{B}[p] \left(x_i[m-2], Q_{i,1}[\alpha_1], Q_{i,2}[\alpha_2] \right) \mathcal{B}_{i,\alpha}(x).$$

Then,

$$D^j q_{x_i}(x) = \left(\frac{2}{m}\right)^j \frac{\binom{j}{m}}{\binom{j}{2}} \sum_{|\alpha|=2} \mathbf{B}[p] \left(x_i[m-2], Q_{i,1}[\alpha_1], Q_{i,2}[\alpha_2] \right) \mathfrak{B}_{W_i,\alpha}^2(x).$$

Using Proposition 3, we define

$$\tilde{q}(x) := \mathbf{B}[p] \left(x_i[m-2], \left(\frac{m}{2}x + \left(1 - \frac{m}{2}\right)x_i\right) [2] \right),$$

Function \tilde{q} can be written on W_i as

$$\begin{aligned} \tilde{q}(x) &= \sum_{|\alpha|=2} \mathbf{B}[\tilde{q}] \left(W_{i,1}[\alpha_1], W_{i,2}[\alpha_2] \right) \mathfrak{B}_{W_i,\alpha}^2(x) \\ &= \sum_{|\alpha|=2} \mathbf{B}[p] \left(x_i[m-2], Q_{i,1}[\alpha_1], Q_{i,2}[\alpha_2] \right) \mathfrak{B}_{W_i,\alpha}^2(x). \end{aligned}$$

Therefore,

$$D^j p(x_i) = \left(\frac{2}{m}\right)^j \frac{\binom{j}{m}}{\binom{j}{2}} D^j \tilde{q}(x_i) = D^j q_{x_i}(x_i) = D^j Q_m p(x_i),$$

and the proof is complete. \square

Remark 2. To get the expression of $\tilde{W}_{i,\ell}$, $\ell = 1, 2$, it suffices to choose $m = 3$.

Next, we define from (12) several quasi-interpolation operators $Q_m : \mathcal{F}(I) \rightarrow S_3^2(X_n^{\text{ref}})$ giving quasi-interpolants $Q_m(f) = Q_m f$ having the form

$$Q_m f := \sum_{i=0}^n \sum_{|\alpha|=2} \mu_{i,\alpha}^m(f) \mathcal{B}_{i,\alpha}, \tag{13}$$

where $\mu_{i,\alpha}^m$ are linear functionals such that

$$Q_m p = p \quad \text{for all } p \in \mathbb{P}_3. \tag{14}$$

Here $\mathcal{F}(I)$ stands for an appropriate space of functions defined on I that includes polynomials of any degree.

4.1. Differential quasi-interpolation operator

Let u, v, w be three points in \mathbb{R} . Consider a polynomial $p \in \mathbb{P}_m$, $m \geq 3$. By using (3), we have

$$\mathbf{B}[p](u[m-2], v, w) := p(u) + \frac{1}{m} (D_{v-u} p(u) + D_{w-u} p(u)) + \frac{1}{m(m-1)} D_{(v-u)(w-u)}^2 p(u).$$

Then, we extend these results to the space $\mathcal{F}(I) = C^2(I)$ to get

$$\mathbf{N}[f](u[m-2], v, w) := f(u) + \frac{1}{m} (D_{v-u} f(u) + D_{w-u} f(u)) + \frac{1}{m(m-1)} D_{(v-u)(w-u)}^2 f(u),$$

from which we define linear functionals providing differential quasi-interpolation operators.

Theorem 10. Define

$$\mu_{i,\alpha}^m(f) := \mathbf{N}[f](x_i[m-2], Q_{i,1}[\alpha_1], Q_{i,2}[\alpha_2]). \tag{15}$$

Then, the operator Q_m defined by (13) satisfies (14).

Proof. It is enough to notice that

$$\mathbf{N}[p](x_i[m-2], Q_{i,1}[\alpha_1], Q_{i,2}[\alpha_2]) = \mathbf{B}[p](x_i[m-2], Q_{i,1}[\alpha_1], Q_{i,2}[\alpha_2])$$

for all $p \in \mathbb{P}_m$, $m \geq 3$. \square

4.2. Quasi-interpolation based on point values

In order to construct a super-convergent discrete quasi-interpolation operator based on point values, let $t_{i,\alpha,k}^m$, $k = 0, \dots, m$, be $m + 1$ distinct points in the support of each $\mathcal{B}_{i,\alpha}$, $i = 0, \dots, n$, and let $L_{i,\alpha,k}^m \in \mathbb{P}_m$, $k = 0, \dots, m$, the associated Lagrange polynomials, i.e. they satisfy the conditions $L_{i,\alpha,k}^m(t_{i,\alpha,j}^m) = \delta_{k,j}$, $j, k = 0, \dots, m$, $\delta_{k,j}$ being the Kronecker’s delta. The polynomial

$$\mathcal{I}_m(f) := \sum_{k=0}^m f(t_{i,\alpha,k}^m) L_{i,\alpha,k}^m$$

interpolates f at those points. In the following theorem, we give an explicit formula of the coefficients $\mu_{i,\alpha}^m(f)$ in terms of $f(t_{i,\alpha,k}^m)$.

Theorem 11. For $i = 1, \dots, n, k = 0, \dots, m$, define

$$\begin{aligned} t_{i,(2,0),k}^m &:= \beta_{i,(2,0),k}^m Q_{i,1} + (1 - \beta_{i,(2,0),k}^m) x_i, \\ t_{i,(1,1),k}^m &:= \beta_{i,(1,1),k}^m Q_{i,1} + (1 - \beta_{i,(1,1),k}^m) Q_{i,2}, \\ t_{i,(0,2),k}^m &:= \beta_{i,(0,2),k}^m Q_{i,2} + (1 - \beta_{i,(0,2),k}^m) x_i, \end{aligned}$$

where $x_i := \bar{\beta}_i Q_{i,1} + (1 - \bar{\beta}_i) Q_{i,2}$. Then, the quasi-interpolation operator Q_m defined by (13) with

$$\mu_{i,\alpha}^m(f) := \sum_{k=0}^m q_{i,\alpha,k}^m f(t_{i,\alpha,k}^m) \tag{16}$$

satisfies (14), if and only if

$$\begin{aligned} q_{i,(2,0),k}^m &= \frac{1}{m} \frac{\sum_{\substack{s_1, s_2=0 \\ s_1 \neq s_2 \neq k}}^m (1 - \beta_{i,(2,0),s_1}^m) (1 - \beta_{i,(2,0),s_2}^m) \prod_{\substack{n=0 \\ n \neq s_1, s_2, k}}^m -\beta_{i,(2,0),n}^m}{\prod_{j \neq k}^m (\beta_{i,(2,0),k}^m - \beta_{i,(2,0),j}^m)}, \\ q_{i,(1,1),k}^m &= \frac{1}{m(m-1)} \frac{\sum_{\substack{s_1, s_2=0 \\ s_1 \neq s_2 \neq k}}^m (1 - \beta_{i,(1,1),s_1}^m) - \beta_{i,(1,1),s_2}^m \prod_{\substack{n=0 \\ n \neq s_1, s_2, k}}^m (\bar{\beta}_i - \beta_{i,(1,1),n}^m)}{\prod_{j \neq k}^m (\beta_{i,(1,1),k}^m - \beta_{i,(1,1),j}^m)}, \\ q_{i,(0,2),k}^m &= \frac{1}{m} \frac{\sum_{\substack{s_1, s_2=0 \\ s_1 \neq s_2 \neq k}}^m (1 - \beta_{i,(0,2),s_1}^m) (1 - \beta_{i,(0,2),s_2}^m) \prod_{\substack{n=0 \\ n \neq s_1, s_2, k}}^m -\beta_{i,(0,2),n}^m}{\prod_{j \neq k}^m (\beta_{i,(0,2),k}^m - \beta_{i,(0,2),j}^m)}. \end{aligned}$$

Proof. According to (12), we have

$$\begin{aligned} \mu_{i,\alpha}^m(f) &= \mathbf{B}[\mathcal{I}_m(f)](x_i[m-2], Q_{i,1}[\alpha_1], Q_{i,2}[\alpha_2]), \\ &= \sum_{k=0}^m f(t_{i,\alpha,k}^m) \mathbf{B}[L_{i,\alpha,k}^m](x_i[m-2], Q_{i,1}[\alpha_1], Q_{i,2}[\alpha_2]). \end{aligned}$$

Then, $q_{i,\alpha,k}^m = \mathbf{B}[L_{i,\alpha,k}^m](x_i[m-2], Q_{i,1}[\alpha_1], Q_{i,2}[\alpha_2])$.

The values to $q_{i,\alpha,k}^m, k = 0, \dots, m$, follow from Proposition 1, and the proof is complete. \square

Next, we provide an example of discrete quasi-interpolation operator based on point values for a uniform partition. Associated with the knot x_0 ,

$$\begin{aligned} \mu_{0,(2,0)}^3(f) &= f(x_0), \\ \mu_{0,(1,1)}^3(f) &= -\frac{2}{9}f(x_0) + \frac{2}{9}f(x_0 + h_0) + 2f\left(x_0 + \frac{1}{3}h_0\right) - f\left(x_0 + \frac{2}{3}h_0\right), \\ \mu_{0,(0,2)}^3(f) &= -\frac{1}{9}f(x_0) - \frac{2}{9}f(x_0 + h_0) + \frac{2}{3}f\left(x_0 + \frac{1}{3}h_0\right) + \frac{2}{3}f\left(x_0 + \frac{2}{3}h_0\right). \end{aligned}$$

For the interior knots,

$$\begin{aligned} \mu_{i,(2,0)}^3(f) &= \frac{4}{27}f(x_{i-1}) + \frac{47}{27}f(x_i) - \frac{32}{27}f\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{8}{27}f(x_{i+1}), \\ \mu_{i,(1,1)}^3(f) &= \frac{-2}{27}f(x_{i-1}) + \frac{31}{27}f(x_i) - \frac{2}{27}f(x_{i+1}), \\ \mu_{i,(0,2)}^3(f) &= -\frac{1}{27}f(x_i) + \frac{32}{27}f\left(\frac{x_i + x_{i+1}}{2}\right) - \frac{4}{27}f(x_{i+1}). \end{aligned}$$

Finally, the coefficients of the functionals $\mu_{n,\alpha}^m$ associated with the boundary knot x_n are symmetric to those associated with x_0 .

4.3. Discrete quasi-interpolation operator based on polarization

Polarization with constant coefficients can be used to obtain functions in the form of combination of discrete values. The polarization formula is given by

$$\mathbf{B}[p](u_1, \dots, u_m) = \frac{1}{m!} \sum_{\substack{S \subset \{1, \dots, m\} \\ k=|S|}} (-1)^{m-k} k^m p \left(\frac{1}{k} \sum_{i \in S} u_i \right).$$

We extend it to $\mathcal{F}(I) = C(I)$ in order to define

$$\mathbf{M}[f](u_1, \dots, u_m) := \frac{1}{m!} \sum_{\substack{S \subset \{1, \dots, m\} \\ k=|S|}} (-1)^{m-k} k^m f \left(\frac{1}{k} \sum_{i \in S} u_i \right).$$

From Marsden’s identity, we have the following result.

Theorem 12. *The quasi-interpolation operator \mathcal{Q}_m defined by (13) with*

$$\mu_{i,\alpha}^m(f) = \mathbf{M}[f](x_i[m-2], \mathcal{Q}_{i,1}[\alpha_1], \mathcal{Q}_{i,1}[\alpha_1]) \tag{17}$$

satisfies (14).

4.4. Error estimate of super-convergent quasi-interpolation operators

The exactness on \mathbb{P}_3 of the differential and discrete operators \mathcal{Q}_m , $m \geq 3$, defined above ensures for functions f in $C^4(I)$ the existence of constants C_k independent of m and f such that

$$\|\mathcal{Q}_m^{(k)} f - f^{(k)}\|_{\infty, I} \leq C_k \bar{h}^{4-k} \|f^{(4-k)}\|_{\infty, I},$$

where, $\|\cdot\|_{\infty, I}$ stands for the infinity norm on the interval I and $\bar{h} := \max_i h_i$.

The following result claims the super-convergence of \mathcal{Q}_m at the break-points of X_n .

Proposition 13. *For all $i = 0, \dots, n$, and for any function f in $C^{m+1}(I)$ it holds*

$$|\mathcal{Q}_m^{(k)} f(x_i) - f^{(k)}(x_i)| = \mathcal{O}(\bar{h}^{m+1-k}), \quad k = 0, 1, 2.$$

Proof. Let m be an integer greater than or equal to 3. Let f be a function in $C^{m+1}(I)$. The Taylor expansion of f at a point $x_{i,0}$ in the neighborhood of x_i , $i = 0, \dots, n$, is given by

$$f(x) = \sum_{j=0}^m \frac{f^{(j)}(x_{i,0})}{j!} (x - x_{i,0})^j + \mathcal{O}((x - x_{i,0})^{m+1}).$$

The operator \mathcal{Q}_m is exact on the space of cubic polynomials, then,

$$\mathcal{Q}_m f(x) = \sum_{j=0}^3 \frac{f^{(j)}(x_{i,0})}{j!} (x - x_{i,0})^j + \sum_{j=4}^m \frac{f^{(j)}(x_{i,0})}{j!} \mathcal{Q}_m((x - x_{i,0})^j) + \mathcal{O}((x - x_{i,0})^{m+1}).$$

Therefore,

$$\mathcal{Q}_m f(x) - f(x) = \sum_{j=4}^m \frac{f^{(j)}(x_{i,0})}{j!} \left(\mathcal{Q}_m((x - x_{i,0})^j) - (x - x_{i,0})^j \right) + \mathcal{O}((x - x_{i,0})^{m+1}).$$

If we denote $g_j = (x - x_{i,0})^j$, $j = 4, \dots, m$, and using [Theorem 9](#), we get

$$\mathcal{Q}_m^{(k)} g_j(x_i) = g_j^{(k)}(x_i), \quad k = 0, 1, 2.$$

which concludes the proof. \square

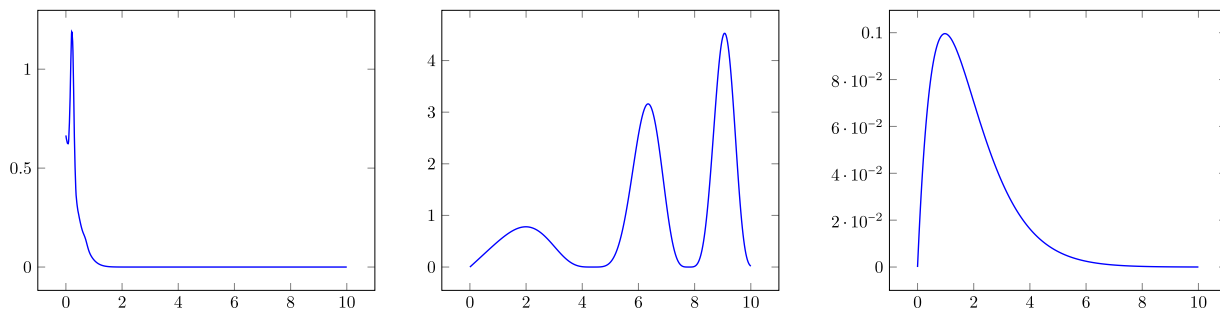


Fig. 4. Plots of the tests functions: f_1 (left), f_2 (middle) and f_3 (right).

Table 1

Estimated errors of the differential QI (15) for functions f_1, f_2 and f_3 and NCOs with $n = 10\ell, \ell = 1, \dots, 9$.

n	$\mathcal{E}_{df,3,n}(f_1)$	NCO	$\mathcal{E}_{df,3,n}(f_2)$	NCO	$\mathcal{E}_{df,3,n}(f_3)$	NCO
10	3.5239×10^{-3}	–	1.2861×10^{-3}	–	5.5739×10^{-3}	–
20	2.8618×10^{-4}	3.6222	9.5743×10^{-5}	3.7477	4.0099×10^{-4}	3.7970
30	7.1933×10^{-5}	3.4056	1.9565×10^{-5}	3.9163	8.9648×10^{-5}	3.6946
40	2.3741×10^{-5}	3.8533	6.2143×10^{-6}	3.9866	3.2991×10^{-5}	3.4748
50	9.7510×10^{-6}	3.9876	2.5611×10^{-6}	3.9723	1.3853×10^{-5}	3.8884
60	5.0067×10^{-6}	3.6560	1.2503×10^{-6}	3.9325	6.6813×10^{-6}	3.9998
70	2.7104×10^{-6}	3.9809	6.6895×10^{-7}	4.05763	3.7777×10^{-6}	3.6989
80	1.6092×10^{-6}	3.9044	3.9255×10^{-7}	3.9919	2.1791×10^{-6}	4.1202
90	9.0493×10^{-7}	4.8874	2.4106×10^{-7}	4.1396	1.3219×10^{-6}	4.2438

5. Numerical tests

This section provides some numerical results to illustrate the performance of the above quasi-interpolation operators. To this end, we will use the test functions

$$f_1(x) = \frac{3}{4}e^{-2(9x-2)^2} - \frac{1}{5}e^{-(9x-7)^2-(9x-4)^2} + \frac{1}{2}e^{-(9x-7)^2-\frac{1}{4}(9x-3)^2} + \frac{3}{4}e^{\frac{1}{10}(-9x-1)-\frac{1}{39}(9x+1)^2},$$

$$f_2(x) = e^{-x} \sin(5\pi x),$$

and,

$$f_3(x) = \frac{1}{2}x \cos^4(4(x^2 + x - 1))$$

whose plots appear in Fig. 4.

Let us consider the interval $I = [0, 1]$. The tests are carried out for a sequence of uniform mesh \mathcal{J}_n associated with the break-points $ih, i = 0, \dots, n$, where $h = \frac{1}{n}$.

The quasi-interpolation error is estimated as

$$\mathcal{E}_{m,n} := \max_{0 \leq \ell \leq 200} |\mathcal{Q}_m f(x_\ell) - f(x_\ell)|, \quad m = 3, 4, 5.$$

where $x_\ell, \ell = 0, \dots, 200$, are equally spaced points in I . $\mathcal{E}_{df,m,n}, \mathcal{E}_{di,m,n}, \mathcal{E}_{dp,m,n}$ represent the estimated error $\mathcal{E}_{m,n}$ for the differential quasi-interpolant (15), the discrete quasi-interpolant (16) and the discrete one based on polarization (17), respectively. The estimated numerical convergence order (NCO) is given by the rate

$$NCO := \frac{\log\left(\frac{\mathcal{E}_{m,n_1}}{\mathcal{E}_{m,n_2}}\right)}{\log\left(\frac{n_2}{n_1}\right)}.$$

The estimated errors relative to the differential quasi-interpolant (15) and the NCOs for f_1, f_2 and f_3 are shown in Table 1. They confirm the theoretical results.

Table 2

Estimated errors of the discrete QI (16) for functions f_1 , f_2 and f_3 and NCOs with $n = 10\ell$, $\ell = 1, \dots, 9$.

n	$\mathcal{E}_{di,3,n}(f_1)$	NCO	$\mathcal{E}_{di,3,n}(f_3)$	NCO	$\mathcal{E}_{di,3,n}(f_2)$	NCO
10	1.8646×10^{-2}	–	2.9778×10^{-2}	–	5.2251×10^{-3}	–
20	1.1630×10^{-3}	4.0030	1.9875×10^{-3}	3.9052	3.2841×10^{-4}	3.9918
30	2.3050×10^{-4}	3.9916	3.9273×10^{-4}	3.9991	7.0841×10^{-5}	3.7828
40	8.5906×10^{-5}	3.4308	1.2000×10^{-4}	4.1211	2.0878×10^{-5}	4.2468
50	3.3864×10^{-5}	4.1717	4.5507×10^{-5}	4.3454	8.1204×10^{-6}	4.2319
60	1.7205×10^{-5}	3.7140	2.4659×10^{-5}	3.3605	4.1315×10^{-6}	3.7062
70	9.5539×10^{-6}	3.8160	1.3593×10^{-5}	3.8634	2.2698×10^{-6}	3.8856
80	5.3656×10^{-6}	4.3206	7.3688×10^{-6}	4.5859	1.3000×10^{-6}	4.1737
90	3.1927×10^{-6}	4.4074	4.9837×10^{-6}	3.3203	8.6994×10^{-7}	3.4104

Table 3

Estimated errors of the discrete QI (17) for function f_1 and NCOs with $n = 10\ell$, $\ell = 1, \dots, 9$, and $m = 3, 4, 5$.

n	$\mathcal{E}_{dp,3,n}(f_1)$	NCO	$\mathcal{E}_{dp,4,n}(f_1)$	NCO	$\mathcal{E}_{dp,5,n}(f_1)$	NCO
10	4.2245×10^{-3}	–	6.3612×10^{-3}	–	1.9123×10^{-3}	–
20	3.6695×10^{-4}	3.2213	1.4630×10^{-4}	5.4423	5.1646×10^{-5}	5.2105
30	7.9143×10^{-5}	3.7832	1.3716×10^{-5}	5.8379	6.0072×10^{-6}	5.3061
40	2.5806×10^{-5}	3.8953	2.4968×10^{-6}	5.9216	1.1658×10^{-6}	5.6989
50	1.0717×10^{-5}	3.9380	6.6135×10^{-7}	5.9535	3.1758×10^{-7}	5.8280
60	5.2073×10^{-6}	3.9589	2.2273×10^{-7}	5.9692	1.0855×10^{-7}	5.8880
70	2.8234×10^{-6}	3.9708	8.8626×10^{-8}	5.9781	4.3575×10^{-8}	5.9210
80	1.6598×10^{-6}	3.9781	3.9862×10^{-8}	5.9836	1.9710×10^{-8}	5.9412
90	1.0383×10^{-6}	3.9830	1.9692×10^{-8}	5.9872	9.7747×10^{-9}	5.9545

Table 4

Estimated errors of the discrete QI (17) for function f_2 and NCOs with $n = 10\ell$, $\ell = 1, \dots, 9$, and $m = 3, 4, 5$.

n	$\mathcal{E}_{dp,3,n}(f_2)$	NCO	$\mathcal{E}_{dp,4,n}(f_2)$	NCO	$\mathcal{E}_{dp,5,n}(f_2)$	NCO
10	2.1886×10^{-3}	–	5.6590×10^{-4}	–	2.3397×10^{-4}	–
20	1.4931×10^{-4}	3.8735	9.3265×10^{-6}	5.9230	4.4871×10^{-6}	5.7043
30	2.9963×10^{-5}	3.9610	8.2685×10^{-7}	5.9758	4.0812×10^{-7}	5.9127
40	9.5327×10^{-6}	3.9809	1.4766×10^{-7}	5.9880	7.3529×10^{-8}	5.9575
50	3.9145×10^{-6}	3.9886	3.8771×10^{-8}	5.9929	1.9384×10^{-8}	5.9747
60	1.8904×10^{-6}	3.9924	1.2995×10^{-8}	5.9952	6.5114×10^{-9}	5.9832
70	1.0212×10^{-6}	3.9946	5.1563×10^{-9}	5.9966	2.5870×10^{-9}	5.9881
80	5.9895×10^{-7}	3.9959	2.3149×10^{-9}	5.9974	1.1624×10^{-9}	5.9911
90	3.7406×10^{-7}	3.9968	1.1421×10^{-9}	5.9980	5.7384×10^{-10}	5.9931

In Table 2, the estimated errors of discrete quasi-interpolants (16) for the functions f_1 , f_2 and f_3 are shown, as well as the corresponding NCOs.

In Tables 3–5, the errors and NCOs relative to f_1 , f_2 and f_3 , respectively, given by the polarization-based discrete spline quasi-interpolant (17) for different values of m are shown. The results are in good agreement with the theoretical ones.

6. Conclusion

In this paper, we dealt with the space of C^2 -continuous cubic splines defined on a partition endowed with a specific refinement. We have constructed a B-spline basis, having the usual properties required for its use in CAGD and developed a theory of control polynomials which is used to establish a Marsden identity, from which various families of super-convergent quasi-interpolation operators have been defined. The numerical tests show the good performance of the defined quasi-interpolants.

Table 5Estimated errors of the discrete QI (17) for function f_3 and NCOs with $n = 10\ell$, $\ell = 1, \dots, 9$, and $m = 3, 4, 5$.

n	$\mathcal{E}_{dp,3,n}(f_3)$	NCO	$\mathcal{E}_{dp,4,n}(f_3)$	NCO	$\mathcal{E}_{dp,5,n}(f_3)$	NCO
10	2.8071×10^{-3}	–	5.8048×10^{-3}	–	2.3020×10^{-3}	–
20	2.0400×10^{-4}	3.7824	1.5420×10^{-4}	5.2343	4.6572×10^{-5}	5.6272
30	4.4729×10^{-5}	3.7426	1.4793×10^{-5}	5.7812	6.1440×10^{-6}	4.9955
40	1.5111×10^{-5}	3.7722	2.7143×10^{-6}	5.8940	1.2327×10^{-6}	5.5833
50	6.3742×10^{-6}	3.8682	7.2158×10^{-7}	5.9372	3.4059×10^{-7}	5.7645
60	3.1227×10^{-6}	3.9136	2.4349×10^{-7}	5.9584	1.1728×10^{-7}	5.8473
70	1.7015×10^{-6}	3.9388	9.7005×10^{-8}	5.9703	4.7288×10^{-8}	5.8925
80	1.0035×10^{-6}	3.9544	4.3664×10^{-8}	5.9778	2.1450×10^{-8}	5.9201
90	6.2909×10^{-7}	3.9647	2.1582×10^{-8}	5.9827	1.0658×10^{-8}	5.9383

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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