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Research Article

Mining Temporal Association Rules with Temporal Soft Sets

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Traditional association rule extraction may run into some difficulties due to ignoring the temporal aspect of the collected data. Particularly, it happens in many cases that some item sets are frequent during specific time periods, although they are not frequent in the whole data set. In this study, we make an effort to enhance conventional rule mining by introducing temporal soft sets. We define temporal granulation mappings to induce granular structures for temporal transaction data. Using this notion, we define temporal soft sets and their Q-clip soft sets to establish a novel framework for mining temporal association rules. A number of useful characterizations and results are obtained, including a necessary and sufficient condition for fast identification of strong temporal association rules. By combining temporal soft sets with NegNodeset-based frequent item set mining techniques, we develop the negFIN-based soft temporal association rule mining (negFIN-STARM) method to extract strong temporal association rules. Numerical experiments are conducted on commonly used data sets to show the feasibility of our approach. Moreover, comparative analysis demonstrates that the newly proposed method achieves higher execution efficiency than three well-known approaches in the literature.

1. Introduction

In modern society, vast amounts of data are produced and collected daily by all walks of life. With an increasing amount of data, there has been an urgent need for developing powerful models, methods and apparatuses to facilitate data analysis. In response to this demand, data mining has emerged and become a fast-growing research field with various fascinating topics and practical applications. Data mining is a multidisciplinary field, which involves applied mathematics, computer science, information science, statistics, and other disciplines. In the process of knowledge discovery in databases (KDD), data mining is viewed as the most essential step in which sophisticated methods are applied to extract knowledge or patterns from data. As shown in Figure 1, six fundamental tasks in data mining are association rule mining, clustering, classification, regression, summarization, and sequence analysis. In a more general perspective, some researchers treat data mining as a synonym for KDD. Data mining has proven to be useful in a myriad of areas including biological statistics [1], case-based reasoning [2], factor analysis of heart disease [3], pattern classification [4], and group role assignment [5].

Association rule mining, such as association analysis and association rule learning, is of great importance in the realm of knowledge discovery and data mining. It was originally proposed in [6] with the aim to find frequent patterns in transactional databases and potential association rules between different item sets. Most rule extraction algorithms belong to one of the following two categories. The first one is known as the class of "candidate generation" methods with the Apriori [7] algorithm as its typical representative. The

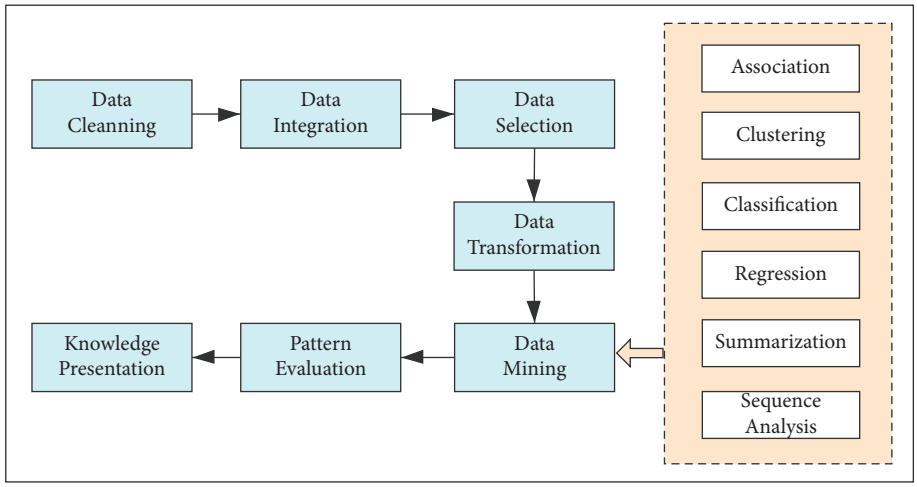

Figure 1: The generic framework of KDD.

main drawback of these methods is that all of them require multiple database scans. The second category consists of "pattern growth" methods such as the FP-growth algorithm [8], which relies on the tree-based data structure (like FP-trees) to store basic information about frequent item sets. More specifically, it does not generate candidate sets of items, and it does not require multiple scans of the database by saving basic information about frequent sets of items into a custom-built data structure. In addition, Zaki [9] proposed another lower I/O costs vertical mining algorithm, called the equivalence class transformation (ECLAT). However, the performance of ECLAT can be affected in dense databases. By using the bitmap representation of sets, Aryabarzan et al. [10] presented a crucial data structure named NegNodeset and developed the NegNodeset-based Frequent Itemset Mining (negFIN) algorithm. The prominent features of the negFIN algorithm are three-fold. Firstly, it makes use of bitwise operators in order to extract NegNodesets of item sets. Secondly, it significantly reduces the complexity of computing supports. Lastly, it generates frequent item sets by using the structure called set-enumeration tree, and meanwhile, it efficiently prunes the search space with the promotion method. Djenouri et al. [11] developed an efficient parallel genetic algorithm for extracting diversified association rules in big data sets. To further improve pattern mining in big data, Luna et al. [12] designed several sophisticated algorithms which rely on a novel paradigm called MapReduce and related implementation named Hadoop. Nevertheless, it should be noticed that the abovementioned rule extraction methods may sometimes produce redundant or incoherent association rules. In view of this, Feldman et al. [13] proposed the maximal association rule, which is a novel complementary apparatus to extract interesting association rules that are frequently lost when using regular association rules. Amir et al. [14] contributed to additional developments regarding exact conceptualization and efficient identification of maximal association rules. In addition to objective measures such as support, confidence, and correlation, some researchers have been interested in considering subjective measures such as risk, interest, and utility to discover useful item sets and association rules. In particular, a new research direction named utility pattern mining [15–17] has received considerable attention in recent years.

Temporal association rule mining (TARM) is one of the most fascinating topics in the field of association rule mining. It has been successfully applied to a wide range of domains such as cancer treatment [18], gene analysis [19], and web mining [20]. Depending on whether the time variable is considered as an implied or integral component, Segura-Delgado et al. [21] systematically classified the existing TARM approaches into two main categories. Agrawal and Srikant [22] coined the terminology of sequential pattern to facilitate the analysis of a transaction database. Inspired by this seminal idea, many scholars have conducted in-depth research with regard to sequential rule mining. Zhai et al. [23] designed a time constraint-based rule mining algorithm, called T-Apriori, to analyse the sequence of ecological events. Gan et al. [24] presented a projection-based utility mining method which is useful for mining high-utility sequential patterns from sequence data. Hong et al. [25] constructed a hierarchical granular framework to enhance TARM by considering different levels of time granules. Song et al. [26] detected changes of customer behavior by using temporal association rules mining from customer profiles and sales data at different time snapshots. Yun et al. [27] designed an efficient algorithm to discover high-utility patterns from incremental databases by constructing a global data structure through a single scan.

Molodtsov's soft set theory [28] provides a formal framework for coping with uncertainty. Its basic principle relies on the perspective of parameterization, suggesting that one should recognize uncertainly defined objects from various facets, and every solo feature yields an approximate description of this object. Maji et al. [29] soon presented several operations of soft sets to complement [28]. Ali et al. [30] introduced several new operations to consolidate the basis of soft set theory. Babitha and Sunil [31] extended the ideas of functions and relations by virtue of soft set theory. Feng and Li [32] clarified the relations among several kinds

of soft subsets and discovered that soft sets satisfy new algebraic properties. By the combination of soft sets and fuzzy sets, Maji et al. [33] proposed a hybrid concept named as fuzzy soft sets. Later on, several more complicated extensions of soft sets have been developed and investigated [34–38]. Ali and Shabir [39] developed some logic connectives in (fuzzy) soft set theory. In [40], a distance-based algorithm was designed for fuzzy soft set parameter reduction. Several works pointed out that rough sets, soft sets, and fuzzy sets are closely connected models [41–43]. They model uncertainty from independent perspectives, namely, gradualness, granularity, and parameterization. Feng et al. initiated several hybrid structures combining rough sets, soft sets, and fuzzy sets [44]. Taking a soft set as the underlying granulation structure, Feng et al. [45] proposed soft rough sets. Soft sets and related extensions have been widely used in many distinct domains, such as decision-making [46–51], valuation of assets [52], clustering [53], medical diagnosis [54], parameter reduction [55], feature selection [56], data analysis [57], BCK/BCI-algebras [58–60], graph theory [61], and computational biology [62]. The reader is referred to John's latest monograph [63] for more details regarding soft set theory and its applications.

With the assistance of soft set theory, Herawan and Deris [64] made an innovative proposal of identifying association rules from transaction data sets. Their pioneering work opened up a new research direction, aiming at developing soft set-based approach to rule extraction. Some concepts were first introduced in [65] to study the approximate reasoning theory based on soft sets, inclusive of logical formulas over soft sets, and basic soft truth degree of formulas. Feng et al. [66] revisited Herawan and Deris's initial idea and refined several important notions to promote (maximal) association rule mining by virtue of soft set theory. Two important observations motivate us to continue this line of exploration:

(1) The ignorance of the temporal aspect of data in the abovementioned association rule extraction approaches [64, 66] may cause some limitations. For instance, some item sets are indeed frequent within certain time periods, even if they are not frequent in the whole data set and the entire time-span. Nonetheless, it is meaningful to discover such item sets since a commodity may sell exceptionally well in a specific season but not during the rest of the year.

(2) The identification of temporal frequent item sets, as an essential step in TARM process, can be facilitated by integrating time as a new component into soft set theory. In fact, the BitMap Coding (BMC) tree [10] must be built to generate node sets corresponding to frequent 1-item sets in the NegNodeset-based frequent item set mining process. The bit value at the index of each temporal frequent 1-item set can be combined to form the bitmap code of a temporal frequent item set. This indicates that temporal soft sets and Q-clip soft sets to be introduced in current work will provide a helpful apparatus for the construction of BMC trees.

To address these issues, the current study focuses on enhancing association rule extraction with the aid of temporal soft sets. The main contributions of this study are summarized as follows:

(1) We define some new concepts such as temporal granulation mappings, temporal soft sets and Q-clip soft sets in order to establish a conceptual framework for extracting temporal association rules

(2) We present a number of useful characterizations and results within the established framework, including a necessary and sufficient condition for fast identification of strong temporal association rules

(3) We develop an effective approach, called negFIN-STARM, to extract strong temporal association rules by virtue of temporal soft sets and NegNodeset-based frequent item set mining

The rest of this paper is arranged in the following way: Section 2 provides the rudiments with regard to TARM. Section 3 proposes several fundamental notions such as temporal soft sets and Q-clip soft sets. Section 4 focuses on soft temporal association rule mining to develop the negFIN-STARM approach. Section 5 is devoted to numerical experiments and comparative analysis of four different methods for extracting temporal association rules. Section 6 concludes this research and points out future research directions.

2. Temporal Association Rules

This section focuses on temporal association rules. First, we quote some fundamental definitions from [19].

Definition 1 (see [19]). An item endowed with a time-stamp is called a temporal item. A temporal item set means a nonempty set $\dot{I}$ of temporal items.

Definition 2 (see [19]). Assume that T is a set of transactions on a temporal item set $\dot{I}$ and the positive integer α is the selected support threshold. Then, we say that $\dot{I}$ is a temporal frequent item set with respect to T and α, when **support** $T(\dot{I}) \geq \alpha$.

Definition 3 (see [19]). A pair of disjoint temporal item sets is called a temporal association rule (TAR). Let RHS and LHS, respectively, represent the right and left temporal item sets. Then, we denote a TAR by LHS $>$ (Δ)RHS, where the time-stamp of every temporal item in LHS precedes that, of any temporal item in RHS, Δ is the interval of two different time-stamps.

Since temporal items in a transaction are associated with respective time-stamps, TARs can be generated by finding temporal frequent item sets in the temporal transaction set with interval Δ. TARs defined in [19] are therefore useful for capturing temporal dependence among items within different time spans.

Nevertheless, it is also interesting to see that some item sets are indeed frequent within certain period of time, even if they are not frequent in the whole data set during the entire time-span. In order to better describe such cases, we revisit some basic concepts in TARM and refine them in what follows.

Suppose that $I = \{i_1, \ldots, i_{|I|}\}$ is an item domain. Any subset t of I is a transaction, and a transaction data set consists of a set $D = \{t_1, \ldots, t_{|D|}\}$ formed by all transactions under inspection. Each transaction in D has a unique transaction identifier (TID).

In classical association rule extraction, an item set X is a subset of I. When it is formed by k distinct items, we call it a k-item set. To simplify notation, the item set $\{i_s\}$ is denoted by i_s. An item set X appears in t (alternatively, t supports X, when $X \subseteq t$).

Now, let $P = \{p_1, p_2, \ldots, p_{|P|}\}$ be a collection of pairwise disjoint periods of time. If $t \in D$ is related to a unique period $p_t = \tau(t) \in P$ (indicating that t occurs during the period p_t), then p_t is called the period marker of t. In fact, this defines a mapping τ from D to P such that $\tau(t) = p_t$. In what follows, $T = (D, \tau, P)$ is called a temporal transaction data set.

Definition 4. Assume that $T = (D, \tau, P)$ is a temporal transaction data set, $t \in D$, $Q \subseteq P$, and X is an item set. Then, t supports X in T during a period in Q if $X \subseteq t$ and $\tau(t) \in Q$.

Definition 5. Assume that $T = (D, \tau, P)$ is a temporal transaction data set, $Q \subseteq P$ and Z is an item set. The set

$$\Delta_T^Q(Z) = \{t \in D: \tau(t) \in Q \wedge Z \subseteq t\}, \tag{1}$$

is the temporal realization of Z in T during a period in Q.

The set $\Delta_T^Q(Z)$ consists of all the transactions in D which contain all the items in Z and occur during a period in Q. The cardinality of this set is written as $S_T^Q(Z)$, called the temporal support of Z in T during a period in Q. For simplicity, $\Delta_T^{\{p_k\}}(Z)$ and $S_T^{\{p_k\}}(Z)$ are written as $\Delta_T^{p_k}(Z)$ and $S_T^{p_k}(Z)$, respectively.

Definition 6. Let $T = (D, \tau, P)$ be a temporal transaction data set and $Q \subseteq P$. Given two disjoint nonempty item sets $G, Z \subseteq I$, an expression $G \Longrightarrow^Q Z$ is called a temporal association rule (TAR).

We refer to Z and G as the consequent and antecedent of the rule $G \Longrightarrow^Q Z$. The rule $G \Longrightarrow^{\{p_k\}} Z$ is simply written as $G \Longrightarrow^{p_k} Z$.

Definition 7. Let $T = (D, \tau, P)$ be a temporal transaction data set, $Q \subseteq P$ and $G \Longrightarrow^Q Z$ be a TAR. Then, the temporal realization of $G \Longrightarrow^Q Z$ in T is given by

$$\Delta_T^Q\left(G \Longrightarrow^Q Z\right) = \Delta_T^Q(G \cup Z). \tag{2}$$

The cardinality of the set $\Delta_T^Q(G \cup Z)$, denoted by $S_T^Q(G \Longrightarrow^Q Z)$, is called the temporal support of $G \Longrightarrow^Q Z$.

Definition 8. The temporal confidence of a TAR $G \Longrightarrow^Q Z$ is given by

$$C_T^Q\left(G \Longrightarrow^Q Z\right) = \frac{S_T^Q\left(G \Longrightarrow^Q Z\right)}{S_T^Q(G)} = \frac{S_T^Q(G \cup Z)}{S_T^Q(G)}. \tag{3}$$

In particular, $C_T^Q(G \Longrightarrow^Q Z) = 0$ if $S_T^Q(G) = 0$.

The temporal confidence serves as an essential measure in the evaluation of temporal association rules. It reflects the strength of the association between antecedent and the consequent of a TAR during concerned periods.

For simplicity, $\Delta_T^{\{p_k\}}(G \Longrightarrow^{\{p_k\}} Z)$, $S_T^{\{p_k\}}(G \Longrightarrow^{\{p_k\}} Z)$, and $C_T^{\{p_k\}}(G \Longrightarrow^{\{p_k\}} Z)$ are written as $\Delta_T^{p_k}(G \Longrightarrow^{p_k} Z)$, $S_T^{p_k}(G \Longrightarrow^{p_k} Z)$, and $C_T^{p_k}(G \Longrightarrow^{p_k} Z)$, respectively. Let $\mathbb{N}^*$ stand for the set of all positive integers. To find significant and interesting TARs from a temporal transaction data set $T = (D, \tau, P)$, the users or experts should specify the minimum temporal support (min-TS) $\alpha_Q \in \mathbb{N}^*$ and the minimum temporal confidence (min-TC) $\beta_Q \in (0, 1]$ for a given subset Q of P. An item set Z is temporal frequent during a period in Q if $S_T^Q(Z) \geq \alpha_Q$. A TAR $G \Longrightarrow^Q Z$ is frequent during a period in Q if $S_T^Q(G \Longrightarrow^Q Z) \geq \alpha_Q$. If $C_T^Q(G \Longrightarrow^Q Z) \geq \beta_Q$, $G \Longrightarrow^Q Z$ is a confident TAR during a period in Q. A TAR $G \Longrightarrow^Q Z$ is strong during a period in Q if it is both frequent and confident.

The next example illustrates some concepts mentioned above.

Example 1. Consider a temporal transaction data set adapted from [25]. Let us assume that $T = (D, \tau, P)$ be a sample temporal transaction data set, where $D = \{t_1, t_2, \ldots, t_{16}\}$ consisting of all the transactions. Assume that every $t \in D$ is related to a unique period $p_t = \tau(t) \in P$, where $P = \{p_1, p_2, p_3, p_4\}$. From Table 1, it can be seen that T is divided into four parts by P. For example, the item set $\{\gamma\}$ appears in the transaction t_2 and t_3 during the period p_1.

Now, let us consider the subset $Q = \{p_1\}$ of P. By Definition 5, we have $\Delta_T^{p_1}(\{\gamma\}) = \{t_2, t_3\}$ and $S_T^{p_1}(\{\gamma\}) = 2$. In a similar fashion, $\Delta_T^{p_1}(\{\delta\}) = \{t_1, t_2, t_4\}$ and $S_T^{p_1}(\{\delta\}) = 3$. In addition, the 2-item set $\{\gamma, \delta\}$ appears in the transaction t_2, and transaction t_2 occurs during the period p_1. Thus by Definition 4, we can say that t_2 supports $\{\gamma, \delta\}$ in T during the period p_1. Also, it is clear that $\Delta_T^{p_1}(\{\gamma, \delta\}) = \{t_2\}$ and $S_T^{p_1}(\{\gamma, \delta\}) = 1$.

Next, we consider the TAR $\{\gamma\} \Longrightarrow^{p_1} \{\delta\}$. By Definition 7, its temporal realization in T is

$$\Delta_T^{p_1}(\{\gamma\} \Longrightarrow^{p_1} \{\delta\}) = \Delta_T^{p_1}(\{\gamma\} \cup \{\delta\}) = \{t_2\}, \tag{4}$$

and the temporal support of this rule is

$$S_T^{p_1}(\{\gamma\} \Longrightarrow^{p_1} \{\delta\}) = S_T^{p_1}(\{\gamma\} \cup \{\delta\}) = 1. \tag{5}$$

Table 1: An example of a temporal transaction data set.

Period	TID	Items
p_1	t_1	δ
p_1	t_2	γ, δ
p_1	t_3	γ
p_1	t_4	δ
p_2	t_5	α, γ, δ
p_2	t_6	$\alpha, \beta, \gamma, \delta$
p_2	t_7	β, γ, δ
p_2	t_8	α, δ
p_3	t_9	β
p_3	t_{10}	α, γ
p_3	t_{11}	α, β, γ
p_3	t_{12}	β, γ
p_4	t_{13}	β, δ
p_4	t_{14}	β, γ, δ
p_4	t_{15}	β
p_4	t_{16}	β, γ, δ

By Definition 8, the temporal confidence of this rule is

$$C_T^{p_1}\left(\{\gamma\} \Longrightarrow^{p_1} \{\delta\}\right) = \frac{S_T^{p_1}\left(\{\gamma\} \Longrightarrow^{p_1} \{\delta\}\right)}{S_T^{p_1}\left(\{\gamma\}\right)} = \frac{S_T^{p_1}\left(\{\gamma\} \cup \{\delta\}\right)}{S_T^{p_1}\left(\{\gamma\}\right)} = 50\%. \tag{6}$$

Finally, assume that $\alpha_{p_1} = 1$ and $\beta_{p_1} = 35\%$. We conclude that $\{\gamma\} \Longrightarrow^{p_1} \{\delta\}$ is a strong TAR during the period p_1.

3. Temporal Soft Sets

In this section, we define some new concepts such as temporal granulation mappings, temporal soft sets, and Q-clip soft sets which will play a role of fundamental importance in this study. In the following, $\mho$ represents a universal set of objects and E stands for the parameter space consisting of all parameters associated with objects in $\mho$. The power set of $\mho$ is written as $2^{\mho}$.

Definition 9 (see [28]). A soft set $\mathfrak{G} = (\widehat{F}, K)$ over $\mho$ is an ordered pair, in which $K \subseteq E$ and $\widehat{F}\colon K \longrightarrow 2^{\mho}$ is called the approximate function of $\mathfrak{G}$.

Definition 10 (see [67]). Assume that $\mho$ and K are nonempty finite sets of alternatives and attributes, respectively. The pair $\mathcal{I} = (\mho, K)$ is called an information system (IS), when every attribute $k \in K$ can be identified with an information function $k\colon \mho \longrightarrow V_k$ and V_k is the value set of k.

When $\mathfrak{G} = (\widehat{F}, K)$ is a soft set over $\mho$, it naturally induces an IS $\mathcal{I}_{\mathfrak{G}} = (\mho, K)$ in the following fashion. Given every $k \in K$ and $v \in \mho$, associate the corresponding information function $k\colon \mho \longrightarrow V_k = \{0, 1\}$ as follows:

$$k(v) = \begin{cases} 0, & \text{when } v \notin \widehat{F}(k), \\ 1, & \text{otherwise.} \end{cases} \tag{7}$$

Definition 11. Let P be a set of pairwise disjoint periods of time. Then, $\tau\colon \mho \longrightarrow P$ is called a temporal granulation mapping.

Definition 12. A temporal soft set (TSS) over $\mho$ is a quadruple $\mathfrak{T} = (\widehat{F}, K, \tau, P)$ such that

(1) $(\widehat{F}, K)$ is a soft set over $\mho$

(2) $\tau\colon \mho \longrightarrow P$ is a temporal granulation mapping

The soft set $(\widehat{F}, K)$ is said to be the underlying soft set (USS) of the TSS $\mathfrak{T}$. We also refer to $\wp = (P, \tau)$ as a temporal granulation of $\mho$. The TSS $\mathfrak{T}$, as an abstract representation of data, can additionally capture temporal information, which is unable to be expressed by its underlying soft set.

Definition 13. Assume that $\mathfrak{T} = (\widehat{F}, K, \tau, P)$ is a TSS over $\mho$ and $Q \subseteq P$. Then the Q-clip of $\mathfrak{T}$ is a soft set $\mathfrak{T}_Q = (\widehat{G}, K)$ over

$$\tau^{-1}(Q) = \{u \in \mho \mid \tau(u) \in Q\}, \tag{8}$$

where $\widehat{G}(k) = \widehat{F}(k) \cap \tau^{-1}(Q)$ for all $k \in K$.

Note that $\{p\}$-clip soft set is simply called p-clip soft set. Next, we consider an example that illustrates the above-mentioned notions.

Example 2. The Nobel Prizes are awarded annually to individuals and organizations in recognition of outstanding contributions in several categories: literature, chemistry, physics, physiology or medicine, and peace. In the following, we focus on three types of prizes, which are the Nobel Prizes in Physics (NPP), Physiology or Medicine (NPPM), and Chemistry (NPC).

We consider

$$\mho = \{v_1, v_2, \ldots, v_9\}, \tag{9}$$

as a universal set that consists of all Nobel Prizes in scientific categories, namely, NPP, NPPM, and NPC awarded between 1901 and 1903. Detailed information regarding these prizes can be found in Table 2. Suppose that $C = \{c_1, c_2, \ldots, c_6\}$ is a set of parameters, containing all the affiliation countries associated with the prizes in $\mho$. More specifically, let c_i $(i = 1, 2, \ldots, 6)$ stand for "Denmark," "France," "Germany," "The Netherlands," "Sweden," and "United Kingdom," respectively. Based on the information in Table 2, we can construct a soft set $(\widehat{F}, C)$ over $\mho$, with its approximate function defined as $\widehat{F}(c_1) = \{v_9\}$, $\widehat{F}(c_2) = \{v_8\}$, $\widehat{F}(c_3) = \{v_1, v_2, v_3, v_4\}$, $\widehat{F}(c_4) = \{v_5\}$, $\widehat{F}(c_5) = \{v_7\}$, and $\widehat{F}(c_6) = \{v_6\}$.

In addition, a temporal granulation $\wp = (Y, \tau)$ of $\mho$ can be derived from Table 2 in a natural way. In fact, let $Y = \{y_1, y_2, y_3\}$ with $y_i = 1900 + i$ for $i = 1, 2, 3$. Then, the temporal granulation mapping $\tau\colon \mho \longrightarrow Y$ is given by

$$\tau(v_3) = \tau(v_2) = \tau(v_1) = y_1 = 1901, \tag{10}$$

$$\tau(v_6) = \tau(v_5) = \tau(v_4) y_2 = 1902, \tag{11}$$

Table 2: Information on NPCs, NPPs, and NPPMs (1901–1903).

Award	Year of award	Prize category	Affiliation country
v_1	1901	NPC	Germany
v_2	1901	NPP	Germany
v_3	1901	NPPM	Germany
v_4	1902	NPC	Germany
v_5	1902	NPP	The Netherlands
v_6	1902	NPPM	United Kingdom
v_7	1903	NPC	Sweden
v_8	1903	NPP	France
v_9	1903	NPPM	Denmark

$$\tau(v_9) = \tau(v_8) = \tau(v_7) = y_3 = 1903. \tag{12}$$

The intuitive meaning of τ is apparent. For instance, equation (10) says that the prizes v_1, v_2, and v_3 were bestowed in 1901. With this mapping, we can construct a TSS $\mathfrak{T} = (\widehat{F}, C, \tau, Y)$ over $\mathcal{O}$, as shown in Table 3. As seen from the equations (10)–(12), the temporal granulation mapping τ: $\mathcal{O} \longrightarrow Y$ induces a partition of $\mathcal{O}$ as follows:

$$\left\{\tau^{-1}(y_i) | y_i \in Y\right\} = \{\{v_4, v_5, v_6\}, \{v_7, v_8, v_9\}, \{v_1, v_2, v_3\}\}. \tag{13}$$

Finally, by Definition 13, the y_i-clip soft set $\mathfrak{T}_{y_i} = (\widehat{G}_i, C)$ of the TSS $\mathfrak{T}$ for $i = 1, 2, 3$ are as follows:

(1) The y_1-clip of $\mathfrak{T}$ is a soft set $\mathfrak{T}_{y_1} = (\widehat{G}_1, C)$ over $\tau^{-1}(y_1)$, where $\widehat{G}_1(c_3) = \tau^{-1}(y_1) = \{v_1, v_2, v_3\}$ and $\widehat{G}_1(c_j) = \varnothing$ for all $c_j \in C$ with $j \neq 3$

(2) The y_2-clip of $\mathfrak{T}$ is a soft set $\mathfrak{T}_{y_2} = (\widehat{G}_2, C)$ over $\tau^{-1}(y_2)$, where $\widehat{G}_2(c_3) = \{v_4\}$, $\widehat{G}_2(c_4) = \{v_5\}$, $\widehat{G}_2(c_6) = \{v_6\}$, and $\widehat{G}_2(c_j) = \varnothing$ for all $c_j \in C$ with $j \neq 3, 4, 6$

(3) The y_3-clip of $\mathfrak{T}$ is a soft set $\mathfrak{T}_{y_3} = (\widehat{G}_3, C)$ over $\tau^{-1}(y_3)$, where $\widehat{G}_3(c_1) = \{v_9\}$, $\widehat{G}_3(c_2) = \{v_8\}$, $\widehat{G}_3(c_5) = \{v_7\}$, and $\widehat{G}_3(c_j) = \varnothing$ for all $c_j \in C$ with $j \neq 1, 2, 5$

4. Soft Temporal Association Rule Mining

This section aims to establish a formal framework for mining TARs by means of TSSs. Let P be a set of pairwise disjoint periods of time and $Q \subseteq P$ throughout this section.

Definition 14. (see [66]). Assume that $\mathfrak{G} = (\widehat{F}, K)$ is a soft set over $\mathcal{O}$ and $v \in \mathcal{O}$. Then, we call

$$\mathrm{Co}_{\mathfrak{G}}(v) = \left\{k \in K: v \in \widehat{F}(k)\right\}, \tag{14}$$

as the parameter coset of the alternative v in $\mathfrak{G}$.

It can be seen that $\mathrm{Co}_{\mathfrak{G}}(v)$ contains all the parameters that the alternative v meets, according to the information contained in $\mathfrak{G}$.

Definition 15. Assume that $\mathfrak{T} = (\widehat{F}, K, \tau, P)$ is a TSS over $\mathcal{O}$ with its USS $\mathfrak{G} = (\widehat{F}, K)$. For any $\varnothing \neq B \subseteq K$,

Table 3: Tabular representation of the TSS $\mathfrak{T} = (\widehat{F}, C, \tau, Y)$.

$\mathcal{O}$	c_1	c_2	c_3	c_4	c_5	c_6	$\tau(\cdot)$
v_1	0	0	1	0	0	0	y_1
v_2	0	0	1	0	0	0	y_1
v_3	0	0	1	0	0	0	y_1
v_4	0	0	1	0	0	0	y_2
v_5	0	0	0	1	0	0	y_2
v_6	0	0	0	0	0	1	y_2
v_7	0	0	0	0	1	0	y_3
v_8	0	1	0	0	0	0	y_3
v_9	1	0	0	0	0	0	y_3

$$\Delta_{\mathfrak{T}}^{Q}(B) = \left\{v \in \tau^{-1}(Q): B \subseteq \mathrm{Co}_{\mathfrak{G}}(v)\right\}, \tag{15}$$

is called the Q-realization of B in the TSS $\mathfrak{T}$.

When $v_0 \in \Delta_{\mathfrak{T}}^{Q}(B)$, it is said that B is Q-supported by the alternative $v_0 \in \mathcal{O}$. The Q-support of B in $\mathfrak{T}$ is the cardinality of $\Delta_{\mathfrak{T}}^{Q}(B)$, represented by $\sup p_{\mathfrak{T}}^{Q}(B)$. Note that $\Delta_{\mathfrak{T}}^{\{p\}}(B)$ and $\sup p_{\mathfrak{T}}^{\{p\}}(B)$ are respectively written as $\Delta_{\mathfrak{T}}^{p}(B)$ and $\sup p_{\mathfrak{T}}^{p}(B)$.

Definition 16. Assume that $\mathfrak{T} = (\widehat{F}, K, \tau, P)$ is a TSS over $\mathcal{O}$ and G, Z are two disjoint non-empty subsets of K. We call the expression $G \Longrightarrow^{Q} Z$ as a temporal association rule (TAR) in the TSS $\mathfrak{T}$. The non-empty parameter sets Z and G are respectively called consequent and antecedent of the TAR $G \Longrightarrow^{Q} Z$.

Definition 17. Suppose that $\mathfrak{T} = (\widehat{F}, K, \tau, P)$ is a TSS over $\mathcal{O}$ and $G \Longrightarrow^{Q} Z$ is a TAR in $\mathfrak{T}$. We refer to

$$\Delta_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) = \Delta_{\mathfrak{T}}^{Q}(G \cup Z), \tag{16}$$

as the Q-realization of $G \Longrightarrow^{Q} Z$ in the TSS $\mathfrak{T}$.

The Q-support of $G \Longrightarrow^{Q} Z$, written as $\sup p_{\mathfrak{T}}^{Q}(G \Longrightarrow^{Q} Z)$, is the cardinality of $\Delta_{\mathfrak{T}}^{Q}(G \Longrightarrow^{Q} Z)$. For convenience, $G \Longrightarrow^{\{p\}} Z$, $\Delta_{\mathfrak{T}}^{\{p\}}(G \Longrightarrow^{\{p\}} Z)$ and $\sup p_{\mathfrak{T}}^{\{p\}}(G \Longrightarrow^{\{p\}} Z)$ are simply written as $G \Longrightarrow^{p} Z$, $\Delta_{\mathfrak{T}}^{p}(G \Longrightarrow^{p} Z)$ and $\sup p_{\mathfrak{T}}^{p}(G \Longrightarrow^{p} Z)$, respectively.

Proposition 1. *Assume that $\mathfrak{T} = (\widehat{F}, K, \tau, P)$ is a TSS over $\mathcal{O}$ and fix $\varnothing \neq B \subseteq K$. Then*

$$\Delta_{\mathfrak{T}}^{Q}(B) = \left(\bigcap_{k \in B} \widehat{F}(k)\right) \cap \left(\tau^{-1}(Q)\right). \tag{17}$$

Proof. We denote by $\mathfrak{G}$ the USS of the TSS $\mathfrak{T} = (\widehat{F}, K, \tau, P)$. Let $x \in \Delta_{\mathfrak{T}}^{Q}(B)$. Equation (15) assures $B \subseteq \mathrm{Co}_{\mathfrak{G}}(x)$ and $x \in \tau^{-1}(Q)$. By Definition 14, $x \in \widehat{F}(k)$ when $k \in B$. Thus we have

$$x \in \left(\bigcap_{k \in B} \widehat{F}(k)\right) \cap \left(\tau^{-1}(Q)\right). \tag{18}$$

This proves

$$\Delta_{\mathfrak{T}}^{Q}(B) \subseteq \left(\bigcap_{k\in B}\widehat{F}(k)\right)\cap\left(\tau^{-1}(Q)\right). \tag{19}$$

Now, suppose that $y \in (\cap_{k\in B}\widehat{F}(k))\cap(\tau^{-1}(Q))$. Then $y \in \tau^{-1}(Q)$ and $y \in \widehat{F}(k)$ for any $k \in B$. From the definition of the parameter coset $\mathrm{Co}_{\mathfrak{G}}(y)$, it follows that $B \subseteq \mathrm{Co}_{\mathfrak{G}}(y)$ and $y \in \tau^{-1}(Q)$. Hence $y \in \Delta_{\mathfrak{T}}^{Q}(B)$, which also shows that

$$\left(\bigcap_{k\in B}\widehat{F}(k)\right)\cap\left(\tau^{-1}(Q)\right) \subseteq \Delta_{\mathfrak{T}}^{Q}(B). \tag{20}$$

Therefore we derive that

$$\Delta_{\mathfrak{T}}^{Q}(B) = \left(\bigcap_{k\in B}\widehat{F}(k)\right)\cap\left(\tau^{-1}(Q)\right). \tag{21}$$

This ends the proof.

By Proposition 1, the following results can be deduced. □

Corollary 1. *Assume that $\mathfrak{T} = (\widehat{F}, K, \tau, P)$ is a TSS over $\mathfrak{O}$ and $\mathfrak{T}_Q = (\widehat{G}, K)$ is its Q-clip soft set. Then we have*

$$\Delta_{\mathfrak{T}}^{Q}(B) = \bigcap_{k\in B}\widehat{G}(k), \tag{22}$$

for all non-empty subset B of K.

Corollary 2. *Assume that $\mathfrak{T} = (\widehat{F}, K, \tau, P)$ is a TSS over $\mathfrak{O}$ and K_1, K_2 are subsets of K. Then we have*

$$K_1 \subseteq K_2 \Longrightarrow \Delta_{\mathfrak{T}}^{Q}(K_2) \subseteq \Delta_{\mathfrak{T}}^{Q}(K_1). \tag{23}$$

Proposition 2. *Assume that $\mathfrak{T} = (\widehat{F}, K, \tau, P)$ is a TSS over $\mathfrak{O}$ and $G \Longrightarrow^{Q} Z$ is a TAR in $\mathfrak{T}$. Then,*

$$\begin{aligned}\Delta_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) &= \Delta_{\mathfrak{T}}^{Q}(G)\cap\Delta_{\mathfrak{T}}^{Q}(Z)\\ &= \left(\bigcap_{k\in G\cup Z}\widehat{F}(k)\right)\cap\left(\tau^{-1}(Q)\right).\end{aligned} \tag{24}$$

Proof:. For simplicity, let J_1 and J_2 stand for $\cap_{k\in G}\widehat{F}(k)$ and $\cap_{k\in Z}\widehat{F}(k)$, respectively. According to Definition 17,

$$\Delta_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) = \Delta_{\mathfrak{T}}^{Q}(G\cup Z). \tag{25}$$

By Proposition 1, we have

$$\begin{aligned}\Delta_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) &= \left(\bigcap_{k\in G\cup Z}\widehat{F}(k)\right)\cap\left(\tau^{-1}(Q)\right)\\ &= (J_1\cap J_2)\cap\left(\tau^{-1}(Q)\right)\\ &= \left(J_1\cap\tau^{-1}(Q)\right)\cap\left(J_2\cap\tau^{-1}(Q)\right)\\ &= \Delta_{\mathfrak{T}}^{Q}(G)\cap\Delta_{\mathfrak{T}}^{Q}(Z).\end{aligned} \tag{26}$$

This ends the proof. □

Remark 1. The above assertion reveals that the Q-realization of a TAR $G \Longrightarrow^{Q} Z$ in a TSS coincides with the intersection of the Q-realizations of the consequent and antecedent of $G \Longrightarrow^{Q} Z$.

By Proposition 2, the following results can be deduced.

Corollary 3. *Assume that $\mathfrak{T} = (\widehat{F}, K, \tau, P)$ is a TSS over $\mathfrak{O}$ and $\mathfrak{T}_Q = (\widehat{G}, K)$ is its Q-clip soft set. Then,*

$$\Delta_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) = \bigcap_{k\in G\cup Z}\widehat{G}(k), \tag{27}$$

where $G \Longrightarrow^{Q} Z$ is a TAR in $\mathfrak{T}$.

Corollary 4. *Assume that $\mathfrak{T} = (\widehat{F}, K, \tau, P)$ is a TSS over $\mathfrak{O}$ and $G \Longrightarrow^{Q} Z$ is a TAR in $\mathfrak{T}$. Then,*

$$\min\left\{\sup p_{\mathfrak{T}}^{Q}(G), \sup p_{\mathfrak{T}}^{Q}(Z)\right\} \geq \sup p_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right). \tag{28}$$

Definition 18. Assume that $\mathfrak{T} = (\widehat{F}, K, \tau, P)$ is a TSS over $\mathfrak{O}$ and $G \Longrightarrow^{Q} Z$ is a TAR in $\mathfrak{T}$. The Q-confidence of $G \Longrightarrow^{Q} Z$ is given by

$$\mathrm{conf}_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) = \begin{cases}\dfrac{\sup p_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right)}{\sup p_{\mathfrak{T}}^{Q}(G)}, & \text{if } \sup p_{\mathfrak{T}}^{Q}(G) \neq 0,\\ 0, & \text{if } \sup p_{\mathfrak{T}}^{Q}(G) = 0.\end{cases} \tag{29}$$

For convenience, $\mathrm{conf}_{\mathfrak{T}}^{\{p\}}(G \Longrightarrow^{\{p\}} Z)$ is simply written as $\mathrm{conf}_{\mathfrak{T}}^{p}(G \Longrightarrow^{p} Z)$.

Theorem 1. *Assume that $\mathfrak{T} = (\widehat{F}, K, \tau, P)$ is a TSS over $\mathfrak{O}$ and $G \Longrightarrow^{Q} Z$ is a TAR in $\mathfrak{T}$. Then, $G \Longrightarrow^{Q} Z$ is strong during a period in Q if and only if*

$$\sup p_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) \geq \max\left\{\alpha_Q, \beta_Q \cdot \sup p_{\mathfrak{T}}^{Q}(G)\right\}, \tag{30}$$

where $\alpha_Q \in \mathbb{N}^*$ is the min-TS and $\beta_Q \in (0, 1]$ is the min-TC.

Proof. Suppose that $G \Longrightarrow^{Q} Z$ is strong in $\mathfrak{T}$ during a period in Q. Then, we have

$$\begin{aligned}&\sup p_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) \geq \alpha_Q,\\ &\mathrm{conf}_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) = \frac{\sup p_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right)}{\sup p_{\mathfrak{T}}^{Q}(G)} \geq \beta_Q.\end{aligned} \tag{31}$$

It follows that

$$\sup p_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) \geq \beta_Q \cdot \sup p_{\mathfrak{T}}^{Q}(G). \tag{32}$$

Thus, we have

$$\sup p_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) \geq \max\left\{\alpha_Q, \beta_Q \cdot \sup p_{\mathfrak{T}}^{Q}(G)\right\}. \tag{33}$$

Conversely, let $G \Longrightarrow^{Q} Z$ be a temporal association rule in $\mathfrak{T}$ such that

$$\sup p_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) \geq \max\left\{\alpha_Q, \beta_Q \cdot \sup p_{\mathfrak{T}}^{Q}(G)\right\}. \tag{34}$$

It follows that

$$\begin{aligned}&\sup p_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) \geq \alpha_Q,\\ &\sup p_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) \geq \beta_Q \cdot \sup p_{\mathfrak{T}}^{Q}(G).\end{aligned} \tag{35}$$

Hence, we deduce that

$$\frac{\sup p_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right)}{\sup p_{\mathfrak{T}}^{Q}(G)} = \operatorname{conf}_{\mathfrak{T}}^{Q}\left(G \Longrightarrow^{Q} Z\right) \geq \beta_{Q}. \tag{36}$$

Thus, $G \Longrightarrow^{Q} Z$ is strong in $\mathfrak{T}$ during a period in Q, completing the proof.

Using the aforementioned concepts and results, we can obtain the following result. □

Proposition 3. *Suppose that $\mathfrak{T} = (\widehat{F}, K, \tau, P)$ is a TSS over $\mathcal{O}$ and $G \Longrightarrow^{Q} Z$ is a TAR in $\mathfrak{T}$ with $\sup p_{\mathfrak{T}}^{Q}(G \Longrightarrow^{Q} Z) = \sup p_{\mathfrak{T}}^{Q}(G)$. Then, the following are equivalent:*

(1) G is temporal frequent during a period in Q

(2) $G \Longrightarrow^{Q} Z$ is frequent during a period in Q

(3) $G \Longrightarrow^{Q} Z$ is confident during a period in Q

(4) $G \Longrightarrow^{Q} Z$ is strong during a period in Q

To illustrate the new notions above, we consider the following example, which is a continuation of Example 2.

Example 3. Assume that $A = \{a_1, a_2, a_3\}$ is a set of parameters, consisting of the three types of prizes under consideration, i.e., a_1 is NPC, a_2 is NPP, and a_3 is NPPM. Before using the proposed concepts regarding soft temporal association rule mining for mathematical modeling and analysis, we now first establish another TSS based on the information in Table 2. The TSS $\mathfrak{T} = (\widehat{F}, B, \tau, Y)$ over $\mathcal{O}$ is shown in Table 4, where the parameter set $B = C \cup A$ and the temporal granulation mapping τ: $\mathcal{O} \longrightarrow Y$ is identical with what is defined in Example 2. In what follows, let us consider three different cases in which $Q_1 = \{y_1, y_2\}$, $Q_2 = \{y_2\}$, and $Q_3 = Y = \{y_1, y_2, y_3\}$, respectively. Suppose that the min-TS $\alpha_{Q_1} = \alpha_{Q_2} = 1$ and $\alpha_{Q_3} = 2$. The min-TC $\beta_{Q_i} = 75\%$ for $i = 1, 2, 3$.

Let us first focus on the case when $Q_1 = \{y_1, y_2\}$. Recall first that $\tau^{-1}(Q_1) = \{v_1, v_2, \ldots, v_6\}$. By Definition 13, the Q_1-clip of the TSS $\mathfrak{T}$ is a soft set $\mathfrak{T}_{Q_1} = (\widehat{G_1}, B)$ over $\tau^{-1}(Q_1)$, where $\widehat{G_1}(c_3) = \{v_1, v_2, v_3, v_4\}$, $\widehat{G_1}(c_4) = \{v_5\}$, $\widehat{G_1}(c_6) = \{v_6\}$, $\widehat{G_1}(a_1) = \{v_1, v_4\}$, $\widehat{G_1}(a_2) = \{v_2, v_5\}$, $\widehat{G_1}(a_3) = \{v_3, v_6\}$ and $\widehat{G_1}(c_j) = \varnothing$ for all $c_j \in C$ with $j = 1, 2, 5$. By Proposition 1 and Corollary 1, we can easily get

$$\begin{aligned} \Delta_{\mathfrak{T}}^{Q_1}(\{a_1\}) &= \widehat{F}(a_1) \cap \tau^{-1}(Q_1) = \widehat{G_1}(a_1) = \{v_1, v_4\}, \\ \Delta_{\mathfrak{T}}^{Q_1}(\{c_3\}) &= \widehat{F}(c_3) \cap \tau^{-1}(Q_1) = \widehat{G_1}(c_3) = \{v_1, v_2, v_3, v_4\}. \end{aligned} \tag{37}$$

Next, we consider the TAR $\{a_1\} \Longrightarrow^{Q_1} \{c_3\}$. By Proposition 2 and Corollary 3, its Q_1-realization in $\mathfrak{T}$ can be calculated as follows:

$$\begin{aligned} \Delta_{\mathfrak{T}}^{Q_1}\left(\{a_1\} \Longrightarrow^{Q_1} \{c_3\}\right) &= \Delta_{\mathfrak{T}}^{Q_1}(\{a_1\}) \cap \Delta_{\mathfrak{T}}^{Q_1}(\{c_3\}) \\ &= \widehat{F}(a_1) \cap \widehat{F}(c_3) \cap \tau^{-1}(Q_1) \\ &= \widehat{G_1}(a_1) \cap \widehat{G_1}(c_3) \\ &= \{v_1, v_4\}. \end{aligned} \tag{38}$$

In fact, as indicated by Corollary 3, the Q_1-realization of the TAR $\{a_1\} \Longrightarrow^{Q_1} \{c_3\}$ in the TSS $\mathfrak{T}$ is completely determined by the approximate function of the corresponding Q_1-clip $\mathfrak{T}_{Q_1} = (\widehat{G_1}, B)$. It is clear that the Q_1-support of this rule is

$$\sup p_{\mathfrak{T}}^{Q_1}\left(\{a_1\} \Longrightarrow^{Q_1} \{c_3\}\right) = 2. \tag{39}$$

By Definition 18, the Q_1-confidence of this rule is

$$\begin{aligned} \operatorname{conf}_{\mathfrak{T}}^{Q_1}\left(\{a_1\} \Longrightarrow^{Q_1} \{c_3\}\right) &= \frac{\sup p_{\mathfrak{T}}^{Q_1}\left(\{a_1\} \Longrightarrow^{Q_1} \{c_3\}\right)}{\sup p_{\mathfrak{T}}^{Q_1}(\{a_1\})} \\ &= 100\%. \end{aligned} \tag{40}$$

Hence, by definition, $\{a_1\} \Longrightarrow^{Q_1} \{c_3\}$ is strong during a period in Q_1. On the other hand, we can draw the same conclusion from Theorem 1 since

$$\begin{aligned} \sup p_{\mathfrak{T}}^{Q_1}\left(\{a_1\} \Longrightarrow^{Q_1} \{c_3\}\right) &> \max\left\{\alpha_{Q_1}, \beta_{Q_1} \cdot \sup p_{\mathfrak{T}}^{Q_1}(\{a_1\})\right\} \\ &= \max\{1, 1.5\} = 1.5. \end{aligned} \tag{41}$$

Note also that

$$\begin{aligned} \sup p_{\mathfrak{T}}^{Q_1}\left(\{a_1\} \Longrightarrow^{Q_1} \{c_3\}\right) &= \sup p_{\mathfrak{T}}^{Q_1}(\{a_1\}) = 2, \\ \sup p_{\mathfrak{T}}^{Q_1}(\{a_1\}) &> \alpha_{Q_1} = 1. \end{aligned} \tag{42}$$

Thus, we can also conclude that $\{a_1\} \Longrightarrow^{Q_1} \{c_3\}$ is strong during a period in Q_1 by Proposition 3. This rule indicates that "From 1901 to 1902, all the Nobel Prizes in Chemistry were awarded to Germany." Conversely, we can consider the TAR $\{c_3\} \Longrightarrow^{Q_1} \{a_1\}$. Its Q_1-support is

$$\sup p_{\mathfrak{T}}^{Q_1}\left(\{c_3\} \Longrightarrow^{Q_1} \{a_1\}\right) = 2 > \alpha_{Q_1}, \tag{43}$$

but its Q_1-confidence is

$$\operatorname{conf}_{\mathfrak{T}}^{Q_1}\left(\{c_3\} \Longrightarrow^{Q_1} \{a_1\}\right) = 50\% < \beta_{Q_1} = 75\%. \tag{44}$$

Hence, this rule is frequent but not confident during a period in Q_1. It reveals that "From 1901 to 1902, 50% of the Nobel Prizes awarded to Germany pertain to the category of chemistry."

Now, let us consider the second case when $Q_2 = \{y_2\}$. Similarly, we can get

$$\begin{aligned} \sup p_{\mathfrak{T}}^{y_2}(\{c_3\} \Longrightarrow^{y_2} \{a_1\}) &= \sup p_{\mathfrak{T}}^{y_2}(\{c_3\}) = \alpha_{y_2} = 1, \\ \operatorname{conf}_{\mathfrak{T}}^{y_2}(\{c_3\} \Longrightarrow^{y_2} \{a_1\}) &= 100\% > \beta_{y_2} = 75\%. \end{aligned} \tag{45}$$

Hence, we conclude that $\{c_3\} \Longrightarrow^{y_2} \{a_1\}$ is strong during the period y_2. This rule says that "In 1902, Germany was only awarded the NPC, instead of the NPP or NPPM."

Finally, we consider the third case when $Q_3 = Y = \{y_1, y_2, y_3\}$. Clearly, $\tau^{-1}(Q_3) = \{v_1, v_2, \ldots, v_9\} = \mathcal{O}$ in this case. It follows that the Q_3-clip of the TSS $\mathfrak{T}$ is a soft set $\mathfrak{T}_{Q_3} = (\widehat{G_3}, B)$ over $\mathcal{O}$, which coincide with the USS $(\widehat{F}, B)$ of the TSS $\mathfrak{T}$. That is, $\widehat{G_3}(e) = \widehat{F}(e)$ for all $e \in B$. By Proposition 1 and Corollary 1, we have

Table 4: Tabular representation of the TSS $\mathfrak{T} = (\widehat{F}, B, \tau, Y)$.

$\mathcal{O}$	c_1	c_2	c_3	c_4	c_5	c_6	a_1	a_2	a_3	$\tau(\cdot)$
v_1	0	0	1	0	0	0	1	0	0	y_1
v_2	0	0	1	0	0	0	0	1	0	y_1
v_3	0	0	1	0	0	0	0	0	1	y_1
v_4	0	0	1	0	0	0	1	0	0	y_2
v_5	0	0	0	1	0	0	0	1	0	y_2
v_6	0	0	0	0	0	1	0	0	1	y_2
v_7	0	0	0	0	1	0	1	0	0	y_3
v_8	0	1	0	0	0	0	0	1	0	y_3
v_9	1	0	0	0	0	0	0	0	1	y_3

$$\begin{aligned}\Delta_{\mathfrak{T}}^{Q_3}(\{a_3\}) &= \widehat{F}(a_3) \cap \mathcal{O} = \widehat{G_3}(a_3) = \{v_3, v_6, v_9\},\\ \Delta_{\mathfrak{T}}^{Q_3}(\{c_6\}) &= \widehat{F}(c_6) \cap \mathcal{O} = \widehat{G_3}(c_6) = \{v_6\}.\end{aligned} \tag{46}$$

Next, let us consider the TAR $\{a_3\} \Longrightarrow^{Q_3} \{c_6\}$, which can also be seen as an association rule $\{a_3\} \Longrightarrow \{c_6\}$ in conventional sense. By Proposition 2 and Corollary 3, its Q_3-realization in $\mathfrak{T}$ can be calculated as follows:

$$\begin{aligned}\Delta_{\mathfrak{T}}^{Q_3}\left(\{a_3\} \Longrightarrow^{Q_3} \{c_6\}\right) &= \Delta_{\mathfrak{T}}^{Q_3}(\{a_3\}) \cap \Delta_{\mathfrak{T}}^{Q_3}(\{c_6\})\\ &= \widehat{F}(a_3) \cap \widehat{F}(c_6) \cap \mathcal{O}\\ &= \widehat{G_3}(a_3) \cap \widehat{G_3}(c_6)\\ &= \{v_6\}.\end{aligned} \tag{47}$$

Obviously, $\sup p_{\mathfrak{T}}^{Q_3}(\{a_3\} \Longrightarrow^{Q_3} \{c_6\}) = 1$. By Definition 18, we also have

$$\begin{aligned}\mathrm{conf}_{\mathfrak{T}}^{Q_3}\left(\{a_3\} \Longrightarrow^{Q_3} \{c_6\}\right) &= \frac{\sup p_{\mathfrak{T}}^{Q_3}\left(\{a_3\} \Longrightarrow^{Q_3} \{c_6\}\right)}{\sup p_{\mathfrak{T}}^{Q_3}(\{a_3\})}\\ &= \frac{1}{3}.\end{aligned} \tag{48}$$

It is clear that

$$\begin{aligned}\sup p_{\mathfrak{T}}^{Q_3}\left(\{a_3\} \Longrightarrow^{Q_3} \{c_6\}\right) &= 1 < \alpha_{Q_3} = 2,\\ \mathrm{conf}_{\mathfrak{T}}^{Q_3}\left(\{a_3\} \Longrightarrow^{Q_3} \{c_6\}\right) &= \frac{1}{3} < \beta_{Q_3} = 0.75.\end{aligned} \tag{49}$$

Thus, we deduce that $\{a_3\} \Longrightarrow^{Q_3} \{c_6\}$ is neither frequent nor confident during a period in Q_3. This rule reveals that "From 1901 to 1903, only one Nobel Prize in Physiology or Medicine was awarded to the United Kingdom." In addition, it can be seen that the rule $\{c_3\} \Longrightarrow^{Q_3} \{a_3\}$ is neither frequent nor confident during a period in Q_3 since

$$\begin{aligned}\sup p_{\mathfrak{T}}^{Q_3}\left(\{c_3\} \Longrightarrow^{Q_3} \{a_3\}\right) &= 1 < \alpha_{Q_3} = 2,\\ \mathrm{conf}_{\mathfrak{T}}^{Q_3}\left(\{c_3\} \Longrightarrow^{Q_3} \{a_3\}\right) &= 0.25 < \beta_{Q_3} = 0.75.\end{aligned} \tag{50}$$

This rule says that "From 1901 to 1903, only a quarter of the Nobel Prizes awarded to Germany pertain to the category of physiology or medicine."

Compared with the case of Q_3 consisting of all time periods, we see that some rules such as $\{a_1\} \Longrightarrow^{Q_1} \{c_3\}$ and $\{c_3\} \Longrightarrow^{y_2} \{a_1\}$ can only be identified as strong TARs when we restrict to the cases of Q_1 or Q_2 consisting of fewer time periods. This is mainly due to the fact that some item sets can be frequent during certain time periods rather than all of them. In a nutshell, we conclude that the TARM based on TSSs can help find some strong TARs which might be ignored in conventional rule extraction process.

Based on the results obtained in this section and the concepts such as TSSs and Q-clip soft sets proposed in Section 3, we present a novel TARM method by combining NegNodeset-based frequent item set mining with TSS-based rule mining. Our method will be abbreviated as negFIN-STARM in the sequel. The pseudocode description of the negFIN-STARM method is given in Algorithm 1. This algorithm takes a temporal transaction data set T, a set $Q \subseteq P$, the min-TS α_Q, and the min-TC β_Q as the input. The output of Algorithm 1 is the class $\mathrm{SRTAR}(T, Q, \alpha_Q, \beta_Q)$, which contains all strong TARs during a period in Q. The main procedure of the negFIN-STARM method can be divided into three stages:

(1) In the first stage, we construct a TSS $\mathfrak{T} = (\widehat{F}, I, \tau, P)$ over D from the provided temporal transaction data set T. Then, according to Definition 13, we determine $\tau^{-1}(Q)$ and construct the Q-clip soft set $\mathfrak{T}_Q = (\widehat{H}, I)$ of the TSS $\mathfrak{T}$. Next, we derive the IS $\mathcal{I}_{\mathfrak{T}_Q} = (\tau^{-1}(Q), I)$ from the Q-clip soft set $\mathfrak{T}_Q = (\widehat{H}, I)$ of $\mathfrak{T}$.

(2) In the second stage, NegNodeset-based frequent item set mining technique and temporal soft sets are combined for generating all temporal frequent item sets. More specifically, we first employ the information function of $\mathcal{I}_{\mathfrak{T}_Q} = (\tau^{-1}(Q), I)$ to construct the BMC tree. Then, the Nodesets of all frequent 1-item sets are generated by traversing the BMC tree. Furthermore, we identify the NegNodesets of all frequent k-item sets ($k \geq 2$). Eventually, the set-enumeration tree is built to generate the class $\mathrm{TFIS}(\mathfrak{T}_Q, \alpha_Q)$, which consists of all temporal frequent item sets. These item sets will function as potential consequents and antecedents for finding strong TARs. Here, we would like to emphasize a crucial issue. To apply the NegNodeset-based frequent item set mining, the BMC tree must be built to generate the node set related to every frequent 1-item set. Each frequent item set is represented by a bitmap code, and every frequent 1-item set is mapped to one

Input: a temporal transaction data set $T = (D, \tau, P)$, a set $Q \subseteq P$, the min-TS $\alpha_Q \in \mathbb{N}^*$, and the min-TC $\beta_Q \in (0, 1]$.
Output: the class $\mathrm{SRTAR}(T, Q, \alpha_Q, \beta_Q)$ that contains all strong TARs during a period in Q.
(1) Construct a TSS $\mathfrak{T} = (\widehat{F}, I, \tau, P)$ over D from the temporal transaction data set T with the item domain I
(2) Calculate $\tau^{-1}(Q)$ and construct the Q-clip soft set $\mathfrak{T}_Q = (\widehat{H}, I)$ of $\mathfrak{T}$
(3) Construct the IS $\mathscr{I}_{\mathfrak{T}_Q} = (\tau^{-1}(Q), I)$ induced by $\mathfrak{T}_Q = (\widehat{H}, I)$
(4) Construct the BMC tree by $\mathscr{I}_{\mathfrak{T}_Q} = (\tau^{-1}(Q), I)$
(5) Traverse the BMC tree to get the Nodesets of all frequent 1-item sets
(6) Identify the NegNodesets of all frequent k-item sets $(k \geq 2)$
(7) Build the set-enumeration tree to generate the class $\mathrm{TFIS}(\mathfrak{T}_Q, \alpha_Q)$, which consists of all temporal frequent item sets
(8) **for** $G \in \mathrm{TFIS}(\mathfrak{T}_Q, \alpha_Q)$ **do**
(9) **for** $Z \in \mathrm{TFIS}(\mathfrak{T}_Q, \alpha_Q)$ **do**
(10) **if** $G \cap Z = \varnothing$ **then**
(11) Calculate $\Delta^Q_{\mathfrak{T}}(G \Longrightarrow {}^Q Z) = \cap_{a \in G \cup Z} \widehat{H}(a)$;
(12) **end if**
(13) **if** $\sup p^Q_{\mathfrak{T}}(G \Longrightarrow {}^Q Z) = |\Delta^Q_{\mathfrak{T}}(G \Longrightarrow {}^Q Z)| \geq \max\{\alpha_Q, \beta_Q \cdot \sup p^Q_{\mathfrak{T}}(G)\}$ **then**
(14) Put $G \Longrightarrow {}^Q Z$ into $\mathrm{SRTAR}(T, Q, \alpha_Q, \beta_Q)$;
(15) **end if**
(16) **end for**
(17) **end for**
(18) **return** $\mathrm{SRTAR}(T, Q, \alpha_Q, \beta_Q)$;

Algorithm 1: The negFIN-STARM method.

of its bits. In other words, the bit value at the corresponding index of each temporal frequent 1-item set can be combined to form the bitmap code of the temporal frequent item set. It is worth noting that the use of TSSs and Q-clip soft sets can facilitate the calculation of bitmap codes and the construction of BMC trees in this important stage.

(3) In the last stage, by Corollary 3, we can calculate the Q-realization of $G \Longrightarrow {}^Q Z$ using the Q-clip soft set $\mathfrak{T}_Q$ for all $G, Z \in \mathrm{TFIS}(\mathfrak{T}_Q, \alpha_Q)$ which are disjoint. Next, by Theorem 1, it is easy to check whether or not the $G \Longrightarrow {}^Q Z$ is strong during a period in Q. If this is true, then we put $G \Longrightarrow {}^Q Z$ into the class $\mathrm{SRTAR}(T, Q, \alpha_Q, \beta_Q)$.

5. Numerical Experiments

In this section, we conduct numerical experiments on the commonly used chess and mushroom data sets to compare the performance of our newly presented negFIN-STARM approach with three well-known approaches in the literature, namely, the T-Apriori [23], T-FPGrowth [8], and T-ECLAT methods [9]. Hereinafter, the abbreviations such as T-Apriori, T-FPGrowth, and T-ECLAT stand for Temporal Apriori, Temporal FPGrowth, and Temporal Eclat, respectively.

5.1. Running Environment. The numerical experiment was conducted on a laptop computer equipped with a 2.00 GHz Intel Core i7 processor and 8 GB of RAM running the 64-bit Microsoft Windows 10 operating system. The algorithms are coded in Java 13.0.1 using IntelliJ IDEA 2019.2.2. The performance of the selected methods is evaluated by the runtime over the aforementioned data sets. For higher accuracy, the codes corresponding to the four methods were executed 5 times under the same conditions. The comparison is made in terms of the average values of the runtime.

5.2. Description of Data Sets. Two commonly used data sets are employed for comparing our method with existing methods mentioned above. These data sets are available from the open-source data mining library SPMF (The SPMF library at http://www.philippe-fournier-viger.com/spmf/.) founded by Philippe Fournier-Viger. The first data set is the chess data set adapted based on the UCI chess data set. The second one is the mushroom data drawn from The Audubon Society Field Guide to North American Mushrooms. Table 5 gives a basic description of these data sets.

5.3. Results and Comparative Analysis. At first, we conduct numerical experiments and comparative analysis of four different methods using the chess data set. This data set contains 3196 transactions, each uniquely related to a period in $P = \{p_1, p_2, p_3, p_4, p_5\}$. There are 75 different items in the item domain of this data set. We consider the cases when $Q = \{p_1\}$ and $Q = \{p_5\}$. The min-TS and min-TC are simply denoted by α_{p_i} and β_{p_i} $(i = 1, 5)$, respectively. The runtime comparison based on the chess data set of four methods under different thresholds is shown in Figure 2. More details regarding the average runtime (in milliseconds) of four methods on the chess data set are listed in Tables 6 and 7.

As shown in Figure 2(a), the negFIN-STARM method is faster than the T-Apriori, T-ECLAT, and T-FPGrowth methods when $Q = \{p_1\}$, the min-TS $\alpha_{p_1} = 500$, and the min-TC β_{p_1} is 75%, 85%, and 95%, respectively. In addition, Figure 2(b) illustrates that the negFIN-STARM method runs faster than the other methods when $\beta_{p_1} = 85\%$ and α_{p_1} is designated as 400, 450, and 500, respectively.

Table 5: Basic description of the data sets.

Data set	Type	Items	Transactions	Periods
Chess	Real	75	3196	5
Mushroom	Real	119	8124	17

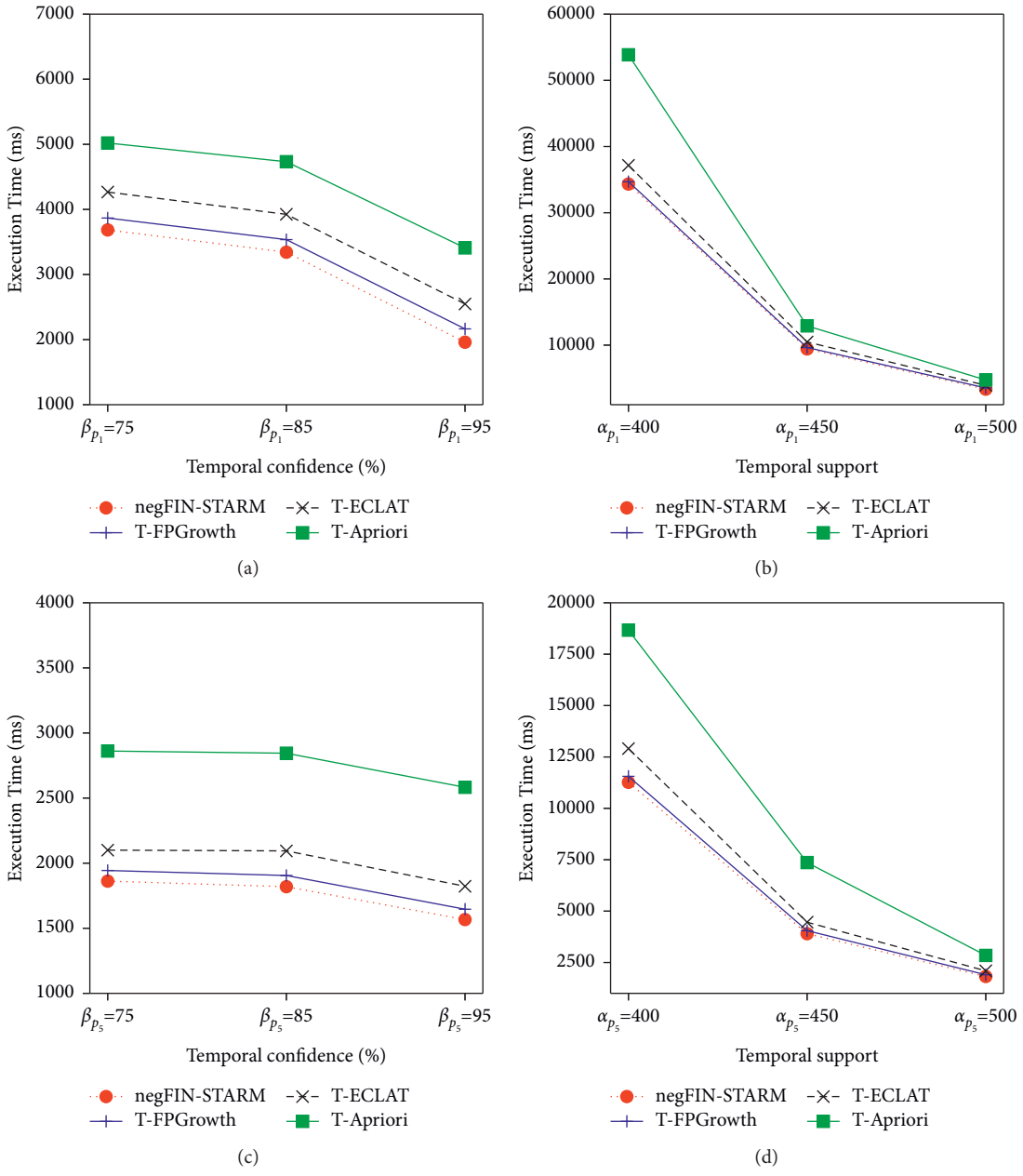

Figure 2: The runtime comparison of four methods based on the chess data set. (a) $Q = \{p_1\}$ and $\alpha_{p_1} = 500$. (b) $Q = \{p_1\}$ and $\beta_{p_1} = 85\%$. (c) $Q = \{p_5\}$ and $\alpha_{p_5} = 500$. (d) $Q = \{p_5\}$ and $\beta_{p_5} = 85\%$.

From Figure 2(c), we see that the negFIN-STARM method is faster than the T-Apriori, T-ECLAT, and T-FPGrowth methods when $Q = \{p_5\}$, $\alpha_{p_5} = 500$, and β_{p_5} is designated as 75%, 85%, and 95%, respectively. In addition, Figure 2(d) illustrates that our new method performs better than those existing methods when $\beta_{p_5} = 85\%$ and α_{p_5} is set as 400, 450, and 500, respectively.

Furthermore, the quantity comparison of the obtained TARs based on the chess data set under different thresholds is demonstrated in Figure 3. In brief, we can find that the rule number decreases when the threshold increases.

More specifically, Figure 3(a) shows that if $Q = \{p_1\}$, the min-TS $\alpha_{p_1} = 500$, and the min-TC β_{p_1} is specified as 75%, 85%, and 95%, the number of TARs is 1481 178, 1290 849,

Table 6: Execution time (ms) of four methods on the chess data set when $Q = \{p_1\}$.

Method	$\alpha_{p_1} = 500$			$\beta_{p_1} = 85\%$		
	$\beta_{p_1} = 75\%$	$\beta_{p_1} = 85\%$	$\beta_{p_1} = 95\%$	$\alpha_{p_1} = 400$	$\alpha_{p_1} = 450$	$\alpha_{p_1} = 500$
negFIN-STARM	**3684.786**	**3342.824**	**1960.214**	**34 331.857**	**9424.531**	**3342.824**
T-FPGrowth	3867.453	3535.803	2164.636	34 692.863	9634.365	3535.803
T-ECLAT	4266.350	3924.420	2547.601	37 159.247	10 479.533	3924.420
T-apriori	5020.186	4732.639	3408.667	53 857.785	12 905.558	4732.639

Table 7: Execution time (ms) of four methods on the chess data set when $Q = \{p_5\}$.

Method	$\alpha_{p_5} = 500$			$\beta_{p_5} = 85\%$		
	$\beta_{p_5} = 75\%$	$\beta_{p_5} = 85\%$	$\beta_{p_5} = 95\%$	$\alpha_{p_5} = 400$	$\alpha_{p_5} = 450$	$\alpha_{p_5} = 500$
negFIN-STARM	**1861.970**	**1819.180**	**1566.782**	**11 266.145**	**3901.763**	**1819.180**
T-FPGrowth	1943.752	1905.275	1646.243	11 556.546	4049.539	1905.275
T-ECLAT	2100.131	2094.229	1822.637	12 908.227	4460.329	2094.229
T-apriori	2861.286	2844.442	2582.787	18 667.517	7362.813	2844.442

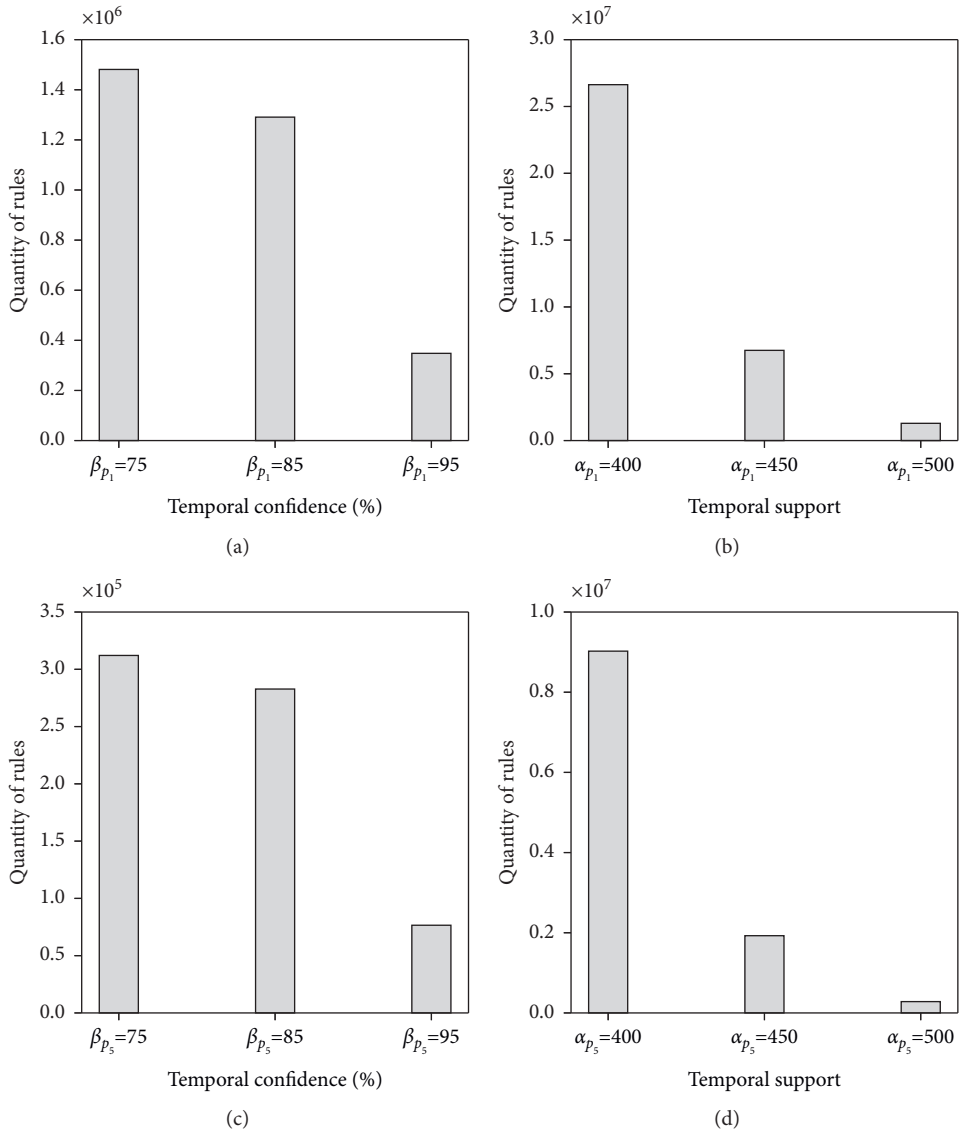

Figure 3: The quantity comparison of TARs based on the chess data set. (a) $Q = \{p_1\}$ and $\alpha_{p_1} = 500$. (b) $Q = \{p_1\}$ and $\beta_{p_1} = 85\%$. (c) $Q = \{p_5\}$ and $\alpha_{p_5} = 500$. (d) $Q = \{p_5\}$ and $\beta_{p_5} = 85\%$.

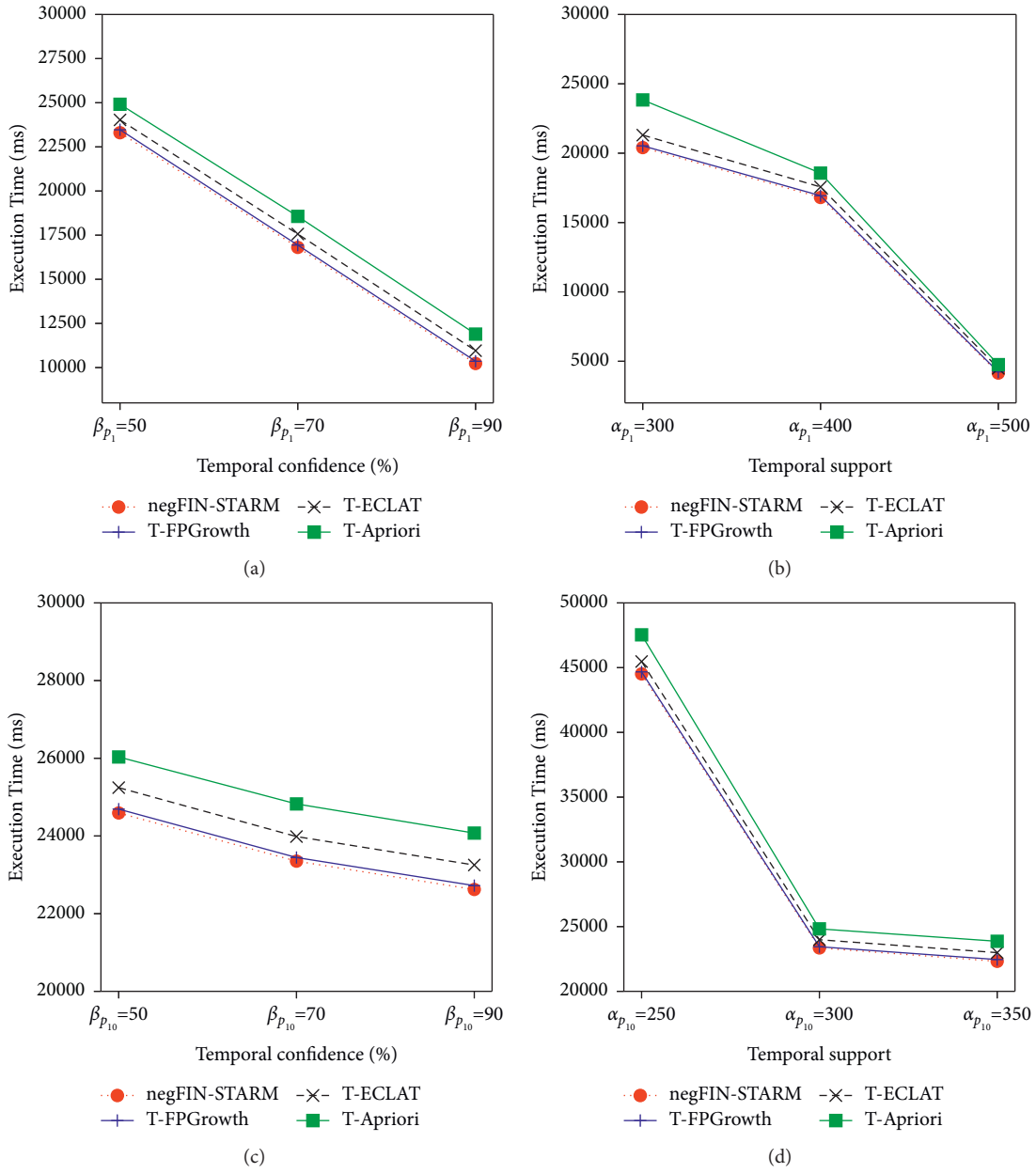

Figure 4: The runtime comparison of four methods based on the mushroom data set. (a) $Q = \{p_1\}$ and $\alpha_{p_1} = 400$. (b) $Q = \{p_1\}$ and $\beta_{p_1} = 70\%$. (c) $Q = \{p_{10}\}$ and $\alpha_{p_{10}} = 300$. (d) $Q = \{p_{10}\}$ and $\beta_{p_{10}} = 70\%$.

and 348 100, respectively. In addition, as illustrated in Figure 3(b), when the min-TC $\beta_{p_1} = 85\%$, and α_{p_1} is specified as 400, 450, and 500, the rule number is 26 632 597, 6750 662, and 1290 849, respectively.

Figure 3(c) shows that when $Q = \{p_5\}$, $\alpha_{p_5} = 500$, and β_{p_5} is specified as 75%, 85%, and 95%, the rule number is 312 092, 282 769, and 76 564, respectively. In addition, as shown in Figure 3(d), when $\beta_{p_5} = 85\%$ and α_{p_5} is specified as 400, 450, and 500, the rule number is 9024 350, 1926 626, and 282 769, respectively.

Similarly, we also conduct numerical experiments and comparative analysis of four methods using the mushroom data set. This data set contains 8124 transactions, each uniquely related to a period in $P = \{p_1, p_2, \ldots, p_{17}\}$. There are 119 different items in the item domain of this data set. We consider the cases when $Q = \{p_1\}$ and $Q = \{p_{10}\}$. The min-TS and min-TC are simply denoted by α_{p_i} and β_{p_i} $(i = 1, 10)$, respectively. The runtime comparison with respect to the mushroom data set of four methods under different thresholds is shown in Figure 4. The quantity comparison of the obtained TARs based on the mushroom data set under different thresholds is illustrated in Figure 5. The average runtime (in milliseconds) of four methods on the mushroom data set is listed in Tables 8 and 9.

The above numerical experiments demonstrate that our newly proposed method is a helpful apparatus for mining TARs. The comparative analysis illustrates that the negFIN-STARM method performs better than three well-known

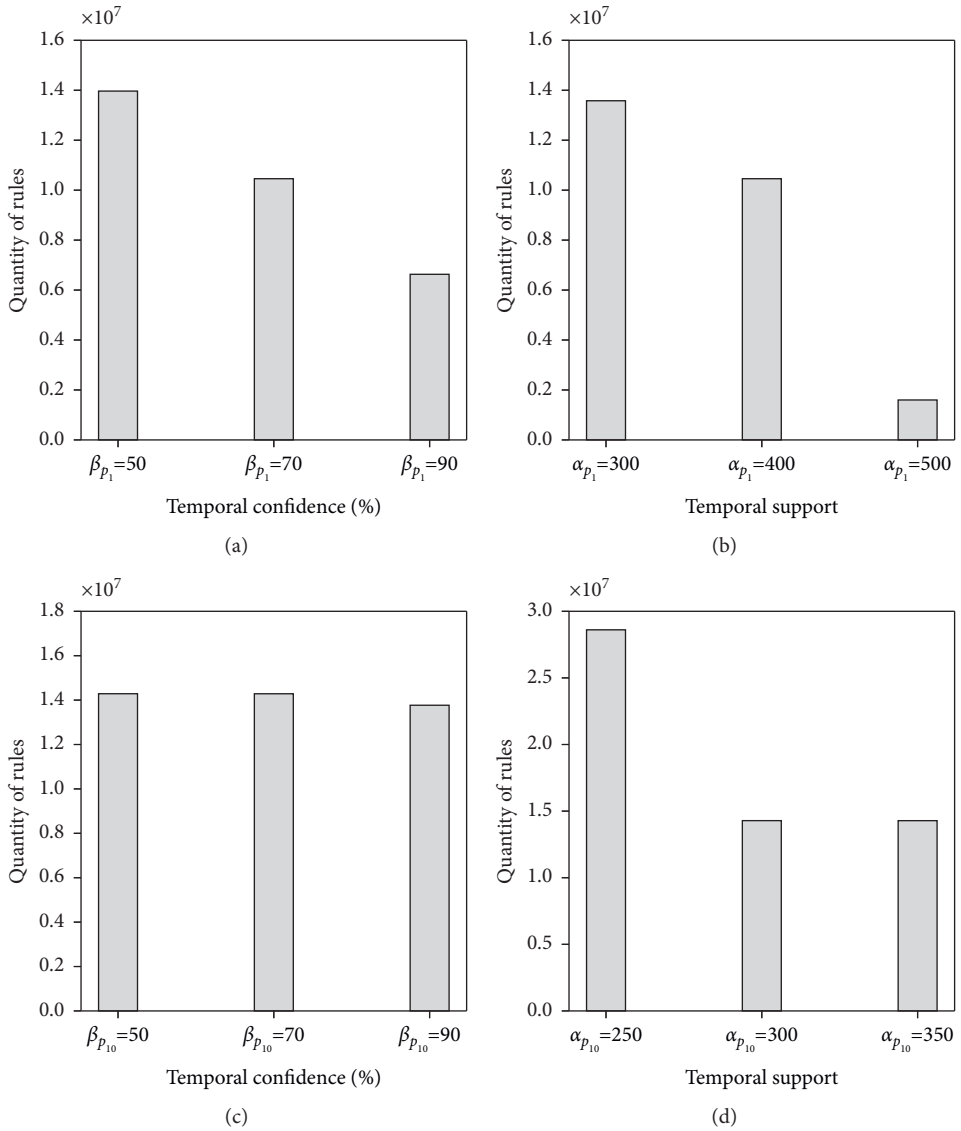

Figure 5: The quantity comparison of TARs based on the mushroom data set. (a) $Q = \{p_1\}$ and $\alpha_{p_1} = 400$. (b) $Q = \{p_1\}$ and $\beta_{p_1} = 70\%$. (c) $Q = \{p_{10}\}$ and $\alpha_{p_{10}} = 300$. (d) $Q = \{p_{10}\}$ and $\beta_{p_{10}} = 70\%$.

Table 8: Execution time (ms) of four methods on the mushroom data set when $Q = \{p_1\}$.

Method	$\alpha_{p_1} = 400$			$\beta_{p_1} = 70\%$		
	$\beta_{p_1} = 50\%$	$\beta_{p_1} = 70\%$	$\beta_{p_1} = 90\%$	$\alpha_{p_1} = 300$	$\alpha_{p_1} = 400$	$\alpha_{p_1} = 500$
negFIN-STARM	**23 301.314**	**16 805.882**	**10 229.619**	**20 402.433**	**16 805.882**	**4147.764**
T-FPGrowth	23 471.043	16 936.191	10 358.534	20 530.658	16 936.191	4228.420
T-ECLAT	24 027.201	17 564.730	10 956.091	21 303.879	17 564.730	4496.445
T-apriori	24 912.996	18 563.465	11 896.175	23 838.725	18 563.465	4757.468

Table 9: Execution time (ms) of four methods on the mushroom data set when $Q = \{p_{10}\}$.

Method	$\alpha_{p_{10}} = 300$			$\beta_{p_{10}} = 70\%$		
	$\beta_{p_{10}} = 50\%$	$\beta_{p_{10}} = 70\%$	$\beta_{p_{10}} = 90\%$	$\alpha_{p_{10}} = 250$	$\alpha_{p_{10}} = 300$	$\alpha_{p_{10}} = 350$
negFIN-STARM	**24 591.737**	**23 351.536**	**22 622.411**	**44 517.337**	**23 351.536**	**22 317.219**
T-FPGrowth	24 691.053	23 444.934	22 721.846	44 671.873	23 444.934	22 453.963
T-ECLAT	25 243.845	23 984.707	23 251.537	45 474.467	23 984.707	22 990.058
T-apriori	26 033.490	24 824.828	24 074.110	47 534.273	24 824.828	23 866.514

existing methods, which are the T-Apriori, T-FPGrowth, and T-ECLAT methods.

6. Conclusions

This paper is devoted to enhancing association rule mining by virtue of temporal soft sets. The notion of temporal granulation mappings was defined to induce the granular structure of a given temporal transaction data set. With the help of temporal granulation mappings, we introduced temporal soft sets and their Q-clip soft sets, which enable us to establish a conceptual framework for extracting TARs. Specially, we presented a number of useful characterizations and related results within this framework, including a necessary and sufficient condition for fast identification of strong TARs. An illustrative example regarding the Nobel Prizes was presented to show how these concepts and results can help facilitate TARM. We also developed a novel method, named negFIN-STARM, for extracting strong TARs by taking advantage of both temporal soft sets and NegNodeset-based frequent item set mining techniques. In addition, two commonly used data sets were employed to verify the feasibility of the negFIN-STARM method. Numerical results have shown that the negFIN-STARM method has better performance than existing approaches such as T-Apriori, T-ECLAT, and T-FPGrowth. It is robust with respect to the selection of different min-TS and min-TC thresholds as well. In future, it will be interesting to investigate the mining of maximal TARs using TSSs and consider its potential applications to dynamic detection, fault diagnosis, and optimal control in industrial processes.

Data Availability

The data used to support the findings of this study are available from the SPMF library at http://www.philippe-fournier-viger.com/spmf/ founded by Dr. Philippe Fournier-Viger.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

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