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Abstract

In this paper we build a Continuous Wavelet Transform (CWT) on the upper sheet of the 2-hyperboloid H^2_+. First, we define a class of suitable dilations on the hyperboloid through conic projection. Then, incorporating hyperbolic motions belonging to $SO_0(1,2)$, we define a family of hyperbolic wavelets. The continuous wavelet transform $W_f(a,x)$ is obtained by convolution of the scaled wavelets with the signal. The wavelet transform is proved to be invertible whenever wavelets satisfy a particular admissibility condition, which turns out to be a zero-mean condition. We then provide some basic examples and discuss the limit at null curvature.

Keywords

Non-commutative harmonic analysis, Wavelets, Fourier-Helgason transform.

1 Introduction

The continuous wavelet transform is already a well established procedure for analysing data. Its main advantages over the classical Fourier transform are its local and multiresolution nature, which provide the interesting properties of a mathematical microscope. Its theory is well known in the case of the line or other higher dimensional Euclidean spaces [Antoine et al., 2005, Mallat, 1998] and the wavelet transform has certainly become a standard in data analysis. However, data analysis has undergone deep changes recently and the field faces new exciting challenges. On one hand the *volume* of data is exploding due to the ubiquity of digital sensors (just think of digital cameras). The first challenge resides in extracting information from very high dimensional data. On the other hand, the *type* of data has also evolved tremendously over the past few decade, from images or volumetric

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data to non-scalar valued signals. One can cite for example tensor diffusion imaging, a new modality in medical imaging [Hagmann et al., 2003], or multimodal signals, i.e signals obtained when the same physical scene is observed through different sensors. Finally, there are instances where data are collected on a surface, or more generally a manifold, or through a complicated interface (think of the human eye for example). This is often the case in astrophysics and cosmology [Martinez-Gonzalez et al., 2002, Cayon et al., 2003], geophysics but also in neurosciences [Angenent et al., 1999], computational chemistry [Max and Getzoff, 1988] ... The list is truly endless. The second challenge thus resides in the complexity of data sources and a field one could call Complex Data Processing might be emerging.

A given feature shared by all these problems is the importance of geometry : either important information is localized around highly structured submanifolds, or data types obey intrinsic nonlinear constraints. As a consequence of the challenges in complex data processing, the representation and analysis of signals in non-Euclidean geometry is now a recurrent problem in many scientific domains. Because of these demands, spherical wavelets [Antoine and Vandergheynst, 1999] were recently developed and applied in various fields, from Cosmology [McEwen et al., 2004] to Computer Vision [Tosic et al., 2005].

Although the sphere is a manifold most desirable for applications, the mathematical analysis made so far invites us to consider other manifolds with similar geometrical properties, and first of all, other Riemannian symmetric spaces of constant curvature. Among them, the two-sheeted hyperboloid H^2 stands as a very interesting case. For instance, such a manifold may be viewed as the phase space for the motion of a free particle in $1+1$-anti de Sitter spacetime [Gazeau et al., 1989, Gazeau and Hussin, 1992]. Other examples come from physical systems constrained on a hyperbolic manifold, for instance, an open expanding model of the universe. A completely different example of application is provided by the emerging field of catadioptric image processing [Makadia and Daniilidis, 2003, Daniilidis et al., 2002]. In this case, a normal (flat) sensor is overlooking a curved mirror in order to obtain an omnidirectional picture of the physical scene. An efficient system is obtained using a hyperbolic mirror, since it has a single effective viewpoint. Finally, from a purely conceptual point of view, having already built the CWT for data analysis in Euclidean spaces and on the sphere, it is natural to raise the question of its existence and form on the dual manifold.

In general, for constructing a CWT on H^2, few basic requirements should be satisfied

- wavelets and signals must "live" on the hyperboloid;
- the transform must involve dilations of some kind; and
- the CWT on H^2 should reduce locally to the usual CWT on the plane.

The paper is organized as follows. In Section 2 we sketch the geometry of the two-sheeted hyperboloid H^2. In Section 3 we define affine transformations on the upper sheet

H_+^2 of H^2. There are two fundamental operations : dilations and hyperbolic motions represented by the group $SO_0(1,2)$. Then, the action of the dilation on the hyperboloid is derived in Section 4. In Section 5, harmonic analysis on the hyperboloid is introduced by means of the Fourier-Helgason transform : this is a central tool for constructing and studying the wavelet transform. Section 6 really constitutes the core of this paper. First we define the CWT on H_+^2 through a hyperbolic convolution. Then we prove a hyperbolic version of the Fourier convolution theorem which allows us to work conveniently in the Fourier-Helgason domain. Theorems 2 and 3 are our main results. We would like to state them roughly here in order to wet our readers' appetite since these results are reminiscent of their Euclidean counterparts. The first one states a generic *admissibility condition* for the existence of hyperbolic wavelets :

Theorem 1 (Admissible wavelets) *Let ψ be a compactly supported, square integrable, continuous function on H_+^2 whose Fourier-Helgason coefficients satisfy :*

$$0 < \mathcal{A}_\psi(\nu) = \int_0^\infty |\widehat{\psi_a}(\nu)|^2 \alpha(a)\,\mathrm{d}a < +\infty,$$

where $a \mapsto \alpha(a)$ is a positive continuous function on $\mathbb{R}_*^+$. Then the hyperbolic wavelet transform is a bounded operator from $L^2(H_+^2)$ to a subset of $L^2(\mathbb{R}_*^+ \times SO_0(1,2))$ that is invertible on its range.

Our second featured theorem shows that the admissibility condition simplifies to a zero-mean condition and really motivates the wavelet terminology.

Theorem 2 (Zero-Mean Condition) *Moreover, if $\alpha(a)\mathrm{d}a$ is a homogeneous measure of the form $a^{-\beta}\mathrm{d}a$, $\beta > 2$, then the following zero-mean condition has to be satisfied :*

A square integrable function on H_+^2 with bounded support is a wavelet if its integral vanishes when it is conveniently weighted, that is

$$\int_{H_+^2} \mathrm{d}\mu(\chi,\varphi) \left[\frac{\sinh 2p\chi}{\sinh\chi}\right]^{\frac{1}{2}} \psi(\chi,\varphi) = 0,$$

for $p > 0$.

Finally we conclude this paper with illustrating examples of hyperbolic wavelets and wavelet transforms and give directions for future work.

2 Geometry of the two-sheeted hyperboloid. Projective structures.

We start by recalling basic facts about the upper sheet of the two-sheeted hyperboloid of radius ρ, $H_{+\rho}^2$. Let χ, φ be a system of polar coordinates for $H_{+\rho}^2$. To each point $\theta = (\chi, \varphi)$

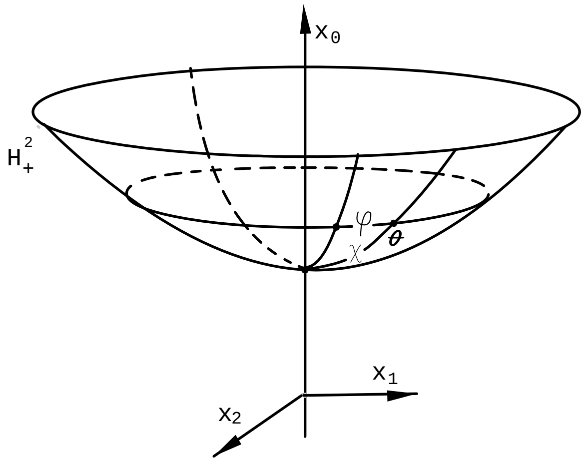

Figure 1: Geometry of the 2-hyperboloid.

we shall associate the vector $x = (x_0, x_1, x_2)$ of $\mathbb{R}^3$ given by

$$\begin{aligned} x_0 &= \rho \cosh \chi, \\ x_1 &= \rho \sinh \chi \cos \varphi, \quad \rho > 0, \quad \chi \geqslant 0, \quad 0 \leq \varphi < 2\pi, \\ x_2 &= \rho \sinh \chi \sin \varphi, \end{aligned}$$

where $\chi \geqslant 0$ is the arc length from the pole to the given point on the hyperboloid, while φ is the arc length over the equator, as shown in Figure 1. The meridians ($\varphi = const$) are geodesics.

The squared metric element in hyperbolic coordinates is:

$$(\mathrm{ds})^2 = -\rho^2 \Big((\mathrm{d}\chi)^2 + \sinh^2 \chi (\mathrm{d}\varphi)^2 \Big), \tag{1}$$

called *Lobachevskian metric*, whereas the measure element on the hyperboloid is

$$\mathrm{d}\mu = \rho^2 \sinh \chi \mathrm{d}\chi \mathrm{d}\varphi. \tag{2}$$

In the sequel, we shall put $\rho = 1$ for convenience and designate the unit hyperboloid $H^2_{+\rho=1}$ by H^2_+.

Various projections can be used to endow H^2_+ with a local Euclidean structure. One of them is immediate : it suffices to *flatten* the hyperboloid onto $\mathbb{R}^2 \simeq \mathbb{C}$. Another possibility is to project the hyperboloid onto a cone. Let us consider a half null cone $C^2_+ \in \mathbb{R}^3$ of equation $(x_0)^2 - \frac{1}{\tan \psi_0}((x_1)^2 + (x_2)^2) = 0, x_0 \geqslant 0$. This cone C^2_+ has Euclidean nature. The cone surface unrolled is a circular sector and all points of H^2_+ will be mapped onto C^2_+ using a specific conic projection. The characteristic parameter of a conic projection is

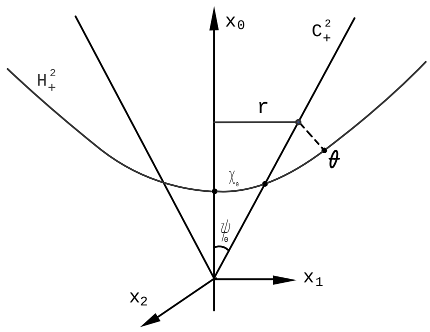

Figure 2: Cross-section of a conic projection

the constant of the cone $m = \cos\psi_0$, where ψ_0 is the Euclidean angle of inclination of the generatrix of the cone as shown in Figure 2.

Considering a radial conic projection, it is more convenient to use a radius r defined by the Euclidean distance between a point on the cone, conic projection of the point $(\chi, \varphi) \in H^2_+$, and the x_0-axis:

$$r = f(\chi), \quad \mathrm{d}r = f'(\chi)\mathrm{d}\chi \quad \text{with} \quad \left.\frac{\mathrm{d}r}{\mathrm{d}\chi}\right|_{\chi=0} = 1. \tag{3}$$

Each suitable projection is determined by a specific choice of $f(\chi)$. It is clear that dilation of the cone $C^2_+ \mapsto aC^2_+ = C^2_+$ entails $r \mapsto ar$. Consequently, the resulting map $\chi \mapsto \chi_a$ is determined by $f(\chi_a) = af(\chi)$. This is precisely the point at the heart of our approach and we shall discuss this more precisely in Section 4.

3 Affine transformations on the 2-hyperboloid

We recall that our purpose is to build a total family of functions in $L^2(H^2_+, \mathrm{d}\mu)$ by picking a wavelet or probe $\psi(\chi)$ with suitable localization properties and applying on it hyperbolic motions, belonging to the group $SO_0(1,2)$, supplemented by appropriate dilations

$$\psi(x) \to \lambda(a,x)\psi(d_{1/a}g^{-1}x) \equiv \psi_{a,g}(x), \quad g \in SO_0(1,2),\ a \in \mathbb{R}^+_*. \tag{4}$$

Dilations d_a will be studied below. Hyperbolic rotations and motions, $g \in SO_0(1,2)$, act on x in the following way.

A motion $g \in SO_0(1,2)$ can be factorized as $g = k_1 h k_2$, where $k_1, k_2 \in SO(2)$, $h \in$

$SO_0(1,1)$, and the respective action of k and h are the following

$$k(\varphi_0).x(\chi,\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_0 & -\sin\varphi_0 \\ 0 & \sin\varphi_0 & \cos\varphi_0 \end{pmatrix} \begin{pmatrix} \cosh\chi \\ \sinh\chi\cos\varphi \\ \sinh\chi\sin\varphi \end{pmatrix} \tag{5}$$

$$= x(\chi,\varphi+\varphi_0), \tag{6}$$

$$h(\chi_0).x(\chi,\varphi) = \begin{pmatrix} \cosh\chi_0 & \sinh\chi_0 & 0 \\ \sinh\chi_0 & \cosh\chi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\chi \\ \sinh\chi\cos\varphi \\ \sinh\chi\sin\varphi \end{pmatrix} \tag{7}$$

$$= x(\chi+\chi_0,\varphi)\,. \tag{8}$$

On the other hand, the dilation is a homeomorphism $d_a : H^2_+ \to H^2_+$ and we require that d_a fulfills the two conditions:

(i) it monotonically dilates the azimuthal distance between two points on H^2_+:

$$\mathrm{dist(d_a(x), d_a(x'))}, \tag{9}$$

where $\mathrm{dist(x,x')}$ is defined by

$$\mathrm{dist(x,x')} = \cosh^{-1}(\mathrm{x}\cdot\mathrm{x}'), \tag{10}$$

and the dot product is the Minkowski product in $\mathbb{R}^3$; note that $\mathrm{dist(x,x')}$ reduces to $|\chi-\chi'|$ when $\varphi=\varphi'$;

(ii) it is homomorphic to the group $\mathbb{R}^+_*$;

$$\mathbb{R}^+_* \ni a \to d_a, \quad d_{ab} = d_a d_b, \quad d_{a^{-1}} = d_a^{-1}, \quad d_1 = \mathbb{I}_d.$$

The action of a motion on a point $x \in H^2_+$ is trivial: it displaces (rotates) by a hyperbolic angle $\chi \in \mathbb{R}_+$ (respectively by an angle φ). It has to be noted that, as opposed to the case of the sphere, attempting to use the conformal group $SO_0(1,3)$ for describing dilation, our requirements are not satisfied. In particular, conformal dilations do not preserve the upper sheet H^2_+ of the hyperboloid. In this paper we adopt an alternative procedure that describes different maps for dilating the hyperboloid.

4 Dilations on hyperboloid

Considering the half null-cone of equation

$$C^2_+ = \left\{\xi \in \mathbb{R}^3 : \xi\cdot\xi = \xi_0^2 - \xi_1^2 - \xi_2^2 = 0, \quad \xi_0 > 0\right\}, \tag{11}$$

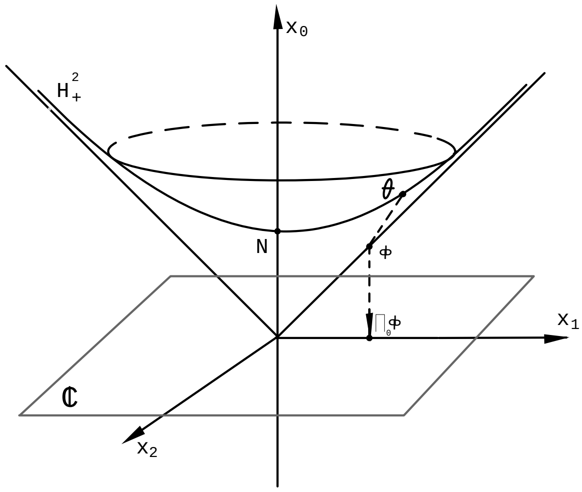

Figure 3: Conic projection and flattening.

there exist the $SO_0(1,2)$-motions and the obvious Euclidean dilations

$$\xi \in C_+^2 \to a\xi \in C_+^2 \equiv d_a^{C_+^2}(\xi), \tag{12}$$

which form a multiplicative one-parameter group isomorphic to $\mathbb{R}_*^+$.

In order to lift dilation (12) back to H_+^2, it is natural to use possible *conic projections* of H_+^2 onto C_+^2, as defined in Section 2.

$$H_+^2 \ni x \to \Phi(x) \in C_+^2 \to \Pi_0\Phi(x) \in \mathbb{R}^2 \simeq \mathbb{C}, \tag{13}$$

where Π_0 stands for *flattening*, defined by

$$\Pi_0\Phi(x) : x(r,\varphi) \in C_+^2 \mapsto re^{i\varphi} \in \mathbb{C}. \tag{14}$$

Flattening reveals the Euclidean nature of the conic projection and the full action of (14) is depicted on Figure 3. Then, we might wish to find a form of $\Pi_0\Phi$ such that, expressed in polar coordinates, the measure is

$$\mathrm{d}\mu = r\mathrm{d}r\mathrm{d}\varphi. \tag{15}$$

In this case, dilating r will quadratically dilate the measure $\mathrm{d}\mu$ as well. By expressing the measure (15) with the radius defined in (3) we obtain

$$f(\chi)f'(\chi) = \sinh\chi \quad \Longrightarrow \quad f(\chi) = 2\sinh\frac{\chi}{2}. \tag{16}$$

Consequently, the radius of this particular conic projection is $r = 2\sinh\frac{\chi}{2}$.

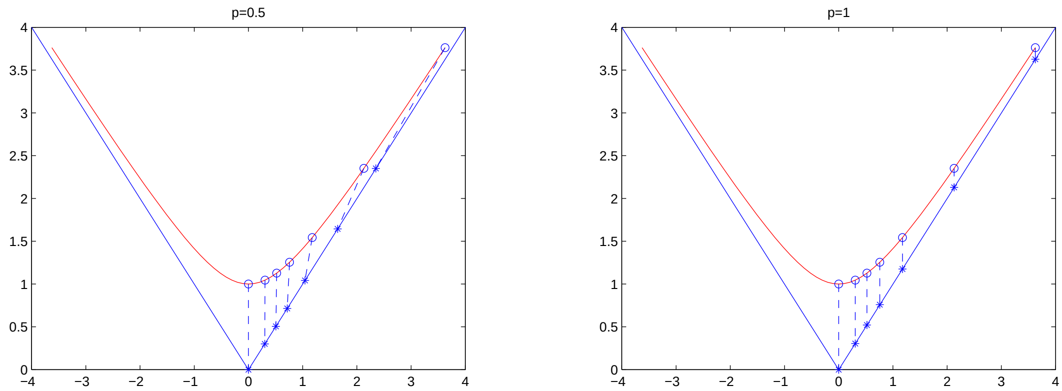

Figure 4: Cross-section of conic projections for different values of parameter p.

Thus, this conic projection $\Phi : H_+^2 \to C_+^2$ is a bijection given, after flattening, by

$$\Pi_0 \Phi(x) = 2 \sinh \frac{\chi}{2} e^{i\varphi},$$

with $x \equiv (\chi, \varphi), \quad \chi \in \mathbb{R}_+, \quad 0 \leq \varphi < 2\pi.$

Then, the lifted dilation is of the form

$$\sinh \frac{\chi_a}{2} = a \sinh \frac{\chi}{2}. \tag{17}$$

This particular example leads us to consider the following family of conic projections and flattening indexed by a positive parameter p :

$$H_+^2 \ni x = x(\chi, \varphi) \to \Pi_0 \Phi(x) = \frac{1}{p} \sinh p\chi e^{i\varphi} = re^{i\varphi} \in \mathbb{C}. \tag{18}$$

The action of Φ for different values of the conic projection parameter p is shown on Figure 4.

The invariant metric and measure on H_+^2, respectively (1) and (2), are then transformed into

$$(\mathrm{d}s)^2 \quad \to \quad -\left(\frac{1}{1+p^2r^2}(\mathrm{d}r)^2 + \frac{1}{4}\left(\varpi(r)^2 + (\varpi(r))^{-2} - 2\right)(\mathrm{d}\varphi)^2\right), \tag{19}$$

$$\mathrm{d}\mu(\chi, \varphi) \quad \to \quad \frac{1}{2}\frac{\varpi(r) - (\varpi(r))^{-1}}{\sqrt{1+p^2r^2}}\mathrm{d}r\mathrm{d}\varphi, \tag{20}$$

where $\varpi(r) = \sqrt[1/p]{pr + \sqrt{1+p^2r^2}}$. This also shows that this class of dilations is not conformal.

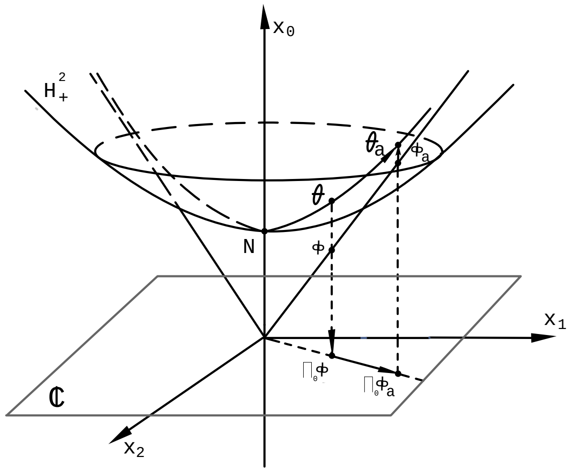

Figure 5: Action of a dilation a on the hyperboloid H^2_+ by conic projection with parameter $p = 1$.

The action of dilation by conic projection is given by

$$\sinh p\chi_a = a \sinh p\chi \tag{21}$$

The particular case $p = 1$ is depicted in Figure 5. The dilated point $x_a \in H^2_+$ is

$$x_a = (\cosh \chi_a, \sinh \chi_a \cos \varphi, \sinh \chi_a \sin \varphi), \tag{22}$$

with polar coordinates $\theta_a = (\chi_a, \varphi)$. The behaviour of $\mathrm{dist(x_N, x_a)}$, with x_N being the North Pole, is shown in Figure 6 in the case $p = 0.1$, $p = 0.5$ and $p = 1$. We can see that this is an increasing function with respect to the dilation a.

It is also interesting to compute the action of dilations in the bounded version of H^2_+. The latter is obtained by applying the stereographic projection from the South Pole of H^2 and it maps the upper sheet H^2_+ onto the open unit disc in the equatorial plane:

$$x = x(\chi, \varphi) \to \Phi(x) = \tanh \frac{\chi}{2} e^{i\varphi}. \tag{23}$$

In the case $p = \frac{1}{2}$ by using (17) and basic trigonometric relations, we obtain

$$\tanh \frac{\chi_a}{2} = \sqrt{\frac{a^2 \tanh^2 \frac{\chi}{2}}{1 + (a^2 - 1) \tanh^2 \frac{\chi}{2}}} \equiv \zeta. \tag{24}$$

In this case, the dilation leaves invariant both $\zeta = 0$ and $\zeta = 1$, the center and the border of the disc, respectively. Figure 7 depicts the action of this transformation on a point $x \in H^2_+$. A dilation from the North Pole (D_N) is considered as a dilation in the unit disc in

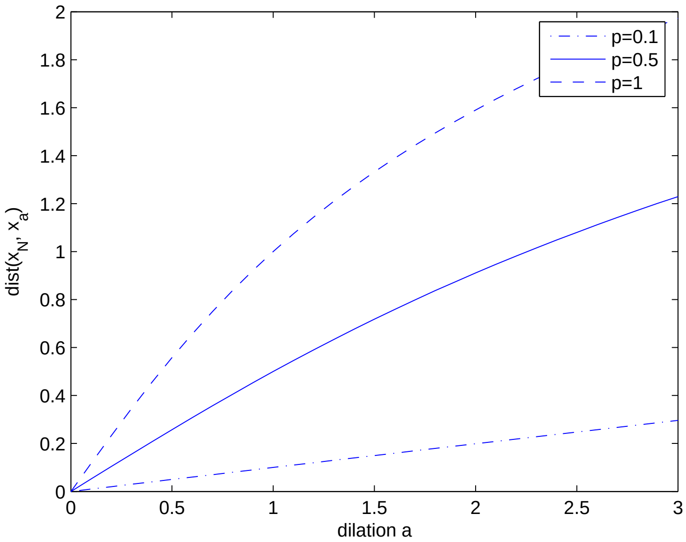

Figure 6: Analysis of the distance (9) as a function of dilation a, with x_N being the North Pole and using conic projection for different parameter p.

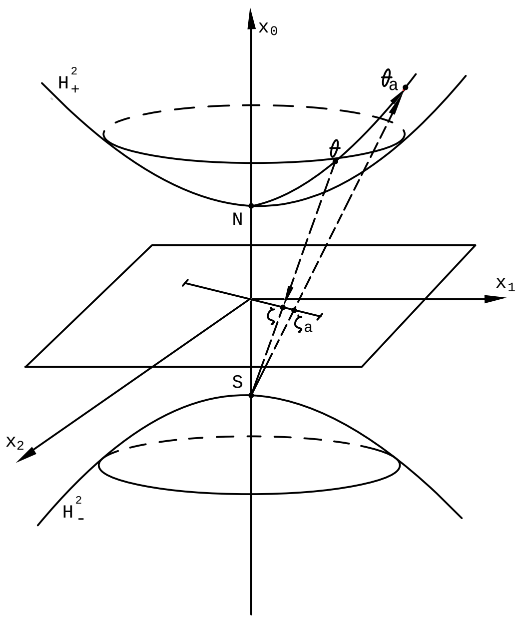

Figure 7: Action of a dilation a on the hyperboloid H^2_+ through a stereographic projection (case $p = \frac{1}{2}$).

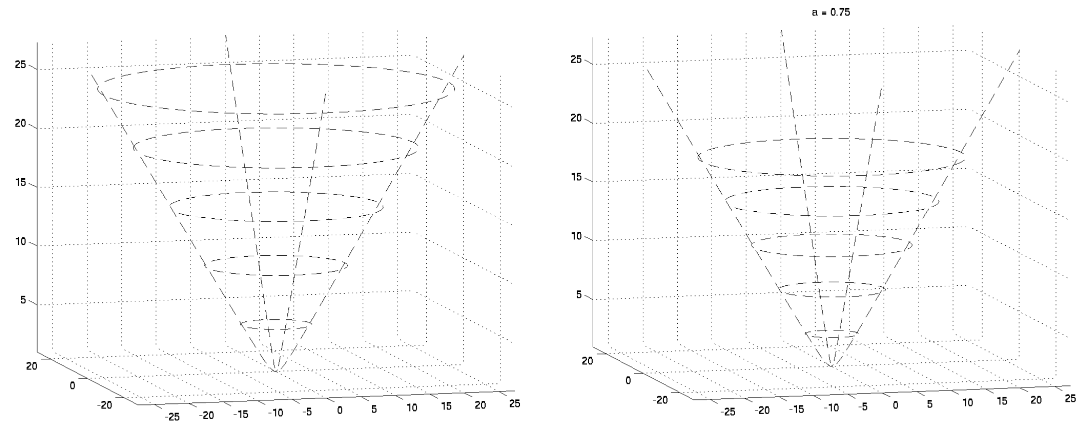

Figure 8: Visualization of the dilation on the hyperboloid H^2_+ (case $p = \frac{1}{2}$).

equatorial plane and lifted back to H^2_+ by inverse stereographic projection from the South Pole. A dilation from any other point $x \in H^2_+$ is obtained by moving x to the North Pole by a rotation $g \in SO_0(1,2)$, performing dilation D_N and going back by inverse rotation:

$$D_x = g^{-1} D_N g.$$

The visualization of the dilation on the hyperboloid H^2_+, with $p = 0.5$, is provided in Figure 8. There, each circle represents points on the hyperboloid at constant χ and is dilated by the scale factor $a = 0.75$.

5 Harmonic analysis on the 2-hyperboloid

5.1 Fourier-Helgason Transform

This integral transform is the precise analog of the Fourier-Plancherel transform on $\mathbb{R}^n$. It consists of an isometry between two Hilbert spaces

$$\mathcal{FH} : L^2(H^2_+, \mathrm{d}\mu) \longrightarrow L^2(\mathcal{L}, \mathrm{d}\eta), \tag{25}$$

where the measure $\mathrm{d}\mu$ is the $SO_0(1,2)$-invariant measure on H^2_+ and $L^2(\mathcal{L}, \mathrm{d}\eta)$ denotes the Hilbert space of sections of a line-bundle $\mathcal{L}$ over another suitably defined manifold, the so-called Helgason-dual of H^2_+ and denoted by Ξ. We note here that $\mathbb{R}^n$ is its own Helgason-dual.

Let us see what is the concrete realization of the dual space Ξ. Most of the following discussion can be found in [Ali and Bertola, 2002], and we summarize it here for convenience. In fact Ξ can be realized as the *projective half null-cone*, as defined in (11) and asymptotic to H^2_+, times the positive real line. The space Ξ is given by

$$\Xi = \mathbb{R}_+ \times \mathbb{P}C_+ \equiv \{k = (\nu, \vec{\xi})\}, \tag{26}$$

where $\mathbb{P}C_+$ denotes the projective forward cone $\{\xi \in C^2_+ \mid \lambda\xi \equiv \xi,\ \lambda > 0,\ \xi_0 > 0\}$ (the set of "rays" on the cone). A convenient realization of $\mathbb{P}C_+$ makes it diffeomorphic to the 1-sphere S^1 as follows

$$\mathbb{P}C_+ \simeq \{\vec{\xi} \in \mathbb{R}^2 : \|\vec{\xi}\| = 1\} \sim S^1 \tag{27}$$

$$\xi \equiv (\xi_0, \vec{\xi}) = (\xi_0, \xi_1, \xi_2) \mapsto \frac{1}{\xi_0}\vec{\xi}. \tag{28}$$

The Fourier - Helgason transform is defined in a way similar to the ordinary Fourier transform by using the eigenfunctions of the invariant differential operator of second order, i.e. the Laplacian on H^2_+. In our case, the functions of the (unique) invariant differential operator are named *hyperbolic plane waves* [Bros et al., 1994]

$$\mathcal{E}_{\nu,\xi}(x) = (\xi \cdot x)^{-\frac{1}{2}-i\nu},\ \nu \in \mathbb{R}_+,\ \xi \in C^2_+. \tag{29}$$

These waves are not parametrized by points of $\mathbb{R}_+ \times \mathbb{P}C_+$ but rather by points of $\mathbb{R}_+ \times C^2_+$; however the action of $\mathbb{R}_+$ on C^2_+ just rescales them by a factor which is constant in $x \in H^2_+$. In other words, they are sections of an appropriate line-bundle over Ξ, which we denote by $\mathcal{L}$, and C^2_+ is thought of as total space of $\mathbb{R}_+$ over $\mathbb{P}C_+$. As well, we note that the inner product $\xi \cdot x$ is positive on the product space $C^2_+ \times H^2_+$, so that the complex exponential is uniquely defined.

Let us express the plane waves in polar coordinates for a point $x \equiv (x_0, \vec{x}) \in H^2_+$

$$\begin{aligned}
\mathcal{E}_{\nu,\xi}(x) &= (\xi \cdot x)^{-\frac{1}{2}-i\nu} && (30)\\
&\equiv \left(\cosh\chi - \frac{\vec{\xi}\cdot\vec{x}}{\xi_0}\right)^{-\frac{1}{2}-i\nu} && (31)\\
&= (\cosh\chi - (\hat{n}\cdot\hat{x})\sinh\chi)^{-\frac{1}{2}-i\nu}, && (32)
\end{aligned}$$

where $\hat{n} \in S^1$ is a unit vector in the direction of $\vec{\xi}$ and $\hat{x} \in S^1$ is the unit vector in the direction of $\vec{x}$. Applying any rotation $\varrho \in SO(2) \subset SO_0(1,2)$ on this wave, it immediately follows

$$R(\varrho) : \mathcal{E}_{\nu,\xi}(x) \to \mathcal{E}_{\nu,\xi}(\varrho^{-1}\cdot x) = \mathcal{E}_{\nu,\varrho\cdot\xi}(x). \tag{33}$$

Finally, the Fourier - Helgason transform $\mathcal{FH}$ and its inverse $\mathcal{FH}^{-1}$ are defined as

$$\begin{aligned}
\hat{f}(\nu,\xi) \equiv \mathcal{FH}[f](\nu,\xi) &= \int_{H^2_+} f(x)(x\cdot\xi)^{-\frac{1}{2}+i\nu}\mathrm{d}\mu(x), \quad \forall f \in \mathcal{C}^\infty_0(H^2_+), && (34)\\
\mathcal{FH}^{-1}[g](x) &= \int_{\mathbf{j}\Xi} g(\nu,\xi)(x\cdot\xi)^{-\frac{1}{2}-i\nu}\mathrm{d}\eta(\nu,\xi), \quad \forall g \in \mathcal{C}^\infty_0(\mathcal{L}), && (35)
\end{aligned}$$

where $\mathcal{C}_0^\infty(\mathcal{L})$ denotes the space of compactly supported smooth sections of the line-bundle $\mathcal{L}$. The integration in (35) is performed along any smooth embedding $\mathbf{j}\Xi$ into the total space of the line-bundle $\mathcal{L}$ and the measure $\mathrm{d}\eta$ is given by

$$\mathrm{d}\eta(\nu,\xi) = \frac{\mathrm{d}\nu}{|\mathbf{c}(\nu)|^2}\,\mathrm{d}\sigma_0, \tag{36}$$

with $\mathbf{c}(\nu)$ being the Harish-Chandra $\mathbf{c}$-function [Helgason, 1994]

$$\mathbf{c}(\nu) = \frac{2^{i\nu}\Gamma(i\nu)}{\sqrt{\pi}\Gamma(\frac{1}{2}+i\nu)}. \tag{37}$$

The factor $|\mathbf{c}(\nu)|^{-2}$ can be simplified to

$$|\mathbf{c}(\nu)|^{-2} = \nu\sinh(\pi\nu)|\Gamma(\frac{1}{2}+i\nu)|^2. \tag{38}$$

The 1-form $\mathrm{d}\sigma_0$ in the measure (36) is defined on the null cone C_+^2, it is closed on it and hence the integration is independent of the particular embedding of Ξ. Thus, such an embedding can be the following

$$\mathbf{j}:\Xi \longrightarrow \mathbb{R}_+\times C_+^2, \tag{39}$$

$$(\nu,\xi) \mapsto (\nu,(1,\frac{\xi_1}{\xi_0},\frac{\xi_2}{\xi_0})) = (\nu,(1,\hat{\xi})). \tag{40}$$

Note that the transform $\mathcal{FH}$ maps functions on H_+^2 to sections of $\mathcal{L}$ and the inverse transform maps sections to functions. Thus, we have

Proposition 1 *[Helgason, 1994] The Fourier-Helgason transform defined in equations (34, 35) extends to an isometry of $L^2(H_+^2,\mathrm{d}\mu)$ onto $L^2(\mathcal{L},\mathrm{d}\eta)$ so that we have*

$$\int_{H_+^2}|f(x)|^2\mathrm{d}\mu(x) = \int_{\mathbf{j}\Xi}|\hat{f}(\xi,\nu)|^2\mathrm{d}\eta(\xi,\nu). \tag{41}$$

6 Continuous Wavelet Transform on the Hyperboloid

One way of constructing the CWT on the hyperboloid H_+^2 would be to find a suitable group containing both $SO_0(1,2)$ and the group of dilations, and then find its square-integrable representations in the Hilbert space $\psi\in L^2(H_+^2,\mathrm{d}\mu)$, where $\mathrm{d}\mu$ is the normalized $SO_0(1,2)$-invariant measure on H_+^2. We will take another approach by directly studying the following wavelet transform

$$\int\overline{\psi_{a,g}(x)}f(x)\mathrm{d}\mu(x) = \langle\psi_{a,g},f\rangle,$$

where the notation $\psi_{a,g}$ has been introduced in (4) and will be now made more precise in terms of group representation. Looking at pseudo-rotations (motions) only, we have the unitary action :

$$[\mathcal{U}_g\psi](x) = f(g^{-1}x), \quad g \in SO_0(1,2), \quad \psi \in L^2(H_+^2, \mathrm{d}\mu). \tag{42}$$

Clearly, $\mathcal{U}_g$ is a quasi-regular representation of $SO_0(1,2)$ on $L^2(H_+^2)$.

We now have to incorporate the dilation. However, the measure $d\mu$ is not dilation invariant, so that a Radon-Nikodym derivative $\lambda(a,x)$ must be inserted, namely:

$$\lambda(a,x) = \frac{\mathrm{d}\mu(a^{-1}x)}{\mathrm{d}\mu(x)}, \quad a \in \mathbb{R}_*^+. \tag{43}$$

The function λ is a 1-cocycle and satisfies the equation

$$\lambda(a_1a_2, x) = \lambda(a_1, x)\lambda(a_2, a_1^{-1}x). \tag{44}$$

In the case of dilating the hyperboloid through conic dilation with parameter $p > 0$, we have

$$\lambda(a,\chi) = \frac{\mathrm{d}\cosh\chi_{1/a}}{\mathrm{d}\cosh\chi} = \frac{1}{a}\,\frac{\sinh\chi_{1/a}}{\sinh\chi}\,\frac{\cosh p\chi}{\cosh p\chi_{1/a}}, \tag{45}$$

with $\sinh p\chi_{1/a} = \frac{1}{a}\sinh p\chi$. Note here that the case $p = \dfrac{1}{2}$ is unique in the sense that $\lambda(a,\chi)$ does not depend on χ : $\lambda(a,\chi) = a^{-2}$. In the case $p = 1$, we get the more elaborate expression

$$\lambda(a,\chi) = \frac{\mathrm{d}\cosh\chi_{1/a}}{\mathrm{d}\cosh\chi} = \frac{\cosh\chi}{a^2\sqrt{1 + a^{-2}\sinh^2\chi}}. \tag{46}$$

Thus, the action of the dilation operator on the function is

$$D_a\psi(x) \equiv \psi_a(x) = \lambda^{\frac{1}{2}}(a,\chi)\psi(d_a^{-1}x) = \lambda^{\frac{1}{2}}(a,\chi)\psi(x_{\frac{1}{a}}) \tag{47}$$

with $x_a \equiv (\chi_a, \varphi) \in H_+^2$ and it reads

$$\psi_a(x) = \sqrt{\frac{1}{a}\,\frac{\sinh\chi_{1/a}}{\sinh\chi}\,\frac{\cosh p\chi}{\cosh p\chi_{1/a}}}\,\psi(x_{\frac{1}{a}}).$$

One easily checks using (45) that D_a is unitary in $L^2(H_+^2)$.

Finally, the *hyperbolic wavelet function* can be written as

$$\psi_{a,g}(x) = \mathcal{U}_g D_a\psi(x) = \mathcal{U}_g\psi_a(x).$$

Accordingly, the *hyperbolic continuous wavelet transform* of a signal (function) $f \in L^2(H^2_+)$ is defined as:

$$\begin{aligned} W_f(a,g) &= \langle \psi_{a,g}|f\rangle & (48)\\ &= \int_{H^2_+} \overline{[\mathcal{U}_g D_a \psi](x)} f(x)\mathrm{d}\mu(x) & (49)\\ &= \int_{H^2_+} \overline{\psi_a(g^{-1}x)} f(x)\mathrm{d}\mu(x) & (50)\end{aligned}$$

where $x \equiv (\chi,\varphi) \in H^2_+$ and $g \in SO_0(1,2)$.

In the next section, we show how this expression can be conveniently interpreted and studied as a hyperbolic convolution.

6.1 Convolutions on H^2

Since H^2_+ is a homogeneous space of $SO_0(1,2)$, one can easily define a convolution. Indeed, let $f \in L^2(H^2_+)$ and $s \in L^1(H^2_+)$, their *hyperbolic convolution* is the function of $g \in SO_0(1,2)$ defined as

$$(f * s)(g) = \int_{H^2_+} f(g^{-1}x)s(x)\mathrm{d}\mu(x). \qquad (51)$$

Then $f * s \in L^2(SO_0(1,2),\mathrm{d}g)$, where $\mathrm{d}g$ stands for the Haar measure on the group and

$$\|f * s\|_2 \leq \|f\|_2\|s\|_1, \qquad (52)$$

by the Young convolution inequality.

In this paper however, we will deal with a simpler definition where the convolution is a function defined on H^2_+. Let $[\cdot] : H^2_+ \longrightarrow SO_0(1,2)$ be a section in the fiber bundle defined by the group and its homogeneous space. In the following we will make use of the *Euler section*, whose construction is now presented. Recall from Section 3 that any $g \in SO_0(1,2)$ can be uniquely decomposed as a product of three elements $g = k(\varphi)h(\chi)k(\psi)$. Using this parametrization, we thus define :

$$\begin{aligned} [\cdot] &: H^2_+ \longrightarrow SO_0(1,2)\\ [\cdot] &: x(\chi,\varphi) \mapsto g = k(\varphi)h(\chi) = [x]\ .\end{aligned}$$

The hyperbolic convolution, restricted to H^2_+, thus takes the following form:

$$(f * s)(y) = \int_{H^2_+} f([y]^{-1}x)s(x)\mathrm{d}\mu(x),\ y \in H^2_+\,.$$

We will mostly deal with convolution kernels that are axisymmetric (or rotation invariant) functions on H^2_+ (i.e. functions of the variable χ alone). The Fourier-Helgason transform of such an element has a simpler form as shown by the following proposition.

Proposition 2 *If f is a rotation invariant function, i.e. $f(\varrho^{-1}x) = f(x)$, $\forall \rho \in SO(2)$, its Fourier-Helgason transform $\hat{f}(\xi, \nu)$ is a function of ν alone, i.e. $\hat{f}(\nu)$.*

Proof : Applying the Fourier-Helgason transform on a rotation-invariant function we write:

$$\begin{aligned}
\hat{f}(\xi, \nu) &= \int_{H_+^2} f(x)\, \mathcal{E}_{\xi,\nu}(x) \mathrm{d}\mu(x) && (53)\\
&= \int_{H_+^2} f(\varrho^{-1}x)(\xi \cdot x)^{-\frac{1}{2}-i\nu} \mathrm{d}\mu(x), \quad \xi \in \mathbb{P}C_+,\ \rho \in SO(2) && (54)\\
&= \int_{H_+^2} f(x')(\xi \cdot \varrho x')^{-\frac{1}{2}-i\nu} \mathrm{d}\mu(x') && (55)\\
&= \hat{f}(\varrho^{-1}\xi, \nu), && (56)
\end{aligned}$$

and so $\hat{f}(\xi, \nu)$ does not depend on ξ. ■

We now have all the basic ingredients for formulating a useful convolution theorem in the Fourier-Helgason domain. As we will now see the FH transform of a convolution takes a simple form, provided one of the kernels is rotation invariant.

Theorem 1 (Convolution) *Let f and s be two measurable functions, with $f, s \in L^2(H_+^2)$ and let s be rotation invariant. The convolution $(s * f)(y)$ is in $L^1(H_+^2)$ and its Fourier-Helgason transform satisfies*

$$\widehat{(s * f)}(\nu, \xi) = \hat{f}(\nu, \xi)\, \hat{s}(\nu). \tag{57}$$

Proof : The convolution of s and f is given by:

$$(s * f)(y) = \int_{H_+^2} s([y]^{-1}x) f(x) \mathrm{d}\mu(x).$$

Since s is $SO(2)$-invariant, we write its argument in this equation in the following way :

$$\begin{pmatrix} \cosh\chi & -\sinh\chi & 0 \\ -\sinh\chi & \cosh\chi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_0 \cosh\chi - x_1 \sinh\chi \\ -x_0 \sinh\chi + x_1 \cosh\chi \\ 0 \end{pmatrix}, \tag{58}$$

where $x = (x_0, x_1, x_2)$ and we have used polar coordinates for $y = y(\chi, \varphi)$. On the other hand we can alternatively write this equation in the form :

$$\begin{pmatrix} x_0 \cosh\chi - x_1 \sinh\chi \\ -x_0 \sinh\chi + x_1 \cosh\chi \\ 0 \end{pmatrix} = \begin{pmatrix} x_0 & -x_1 & 0 \\ -x_1 & x_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh\chi \\ \sinh\chi \\ 0 \end{pmatrix}. \tag{59}$$

Thus we have

$$s([y]^{-1}x) = s([x]^{-1}y). \tag{60}$$

Therefore, the convolution with a rotation invariant function is given by

$$\begin{aligned}(s*f)(y) &= \int_{H^2_+} f(x)s([y]^{-1}x)\,\mathrm{d}\mu(x) &(61)\\ &= \int_{H^2_+} f(x)s([x]^{-1}y)\,\mathrm{d}\mu(x). &(62)\end{aligned}$$

On the other hand, applying the Fourier-Helgason transform on $s*f$ we get

$$\begin{aligned}\widehat{(s*f)}(\nu,\xi) &= \int_{H^2_+} (s*f)(y)(y\cdot\xi)^{-\frac{1}{2}-i\nu}\mathrm{d}\mu(y)\\ &= \int_{H^2_+} \mathrm{d}\mu(y)\int_{H^2_+} \mathrm{d}\mu(x)s([y]^{-1}x)f(x)(y\cdot\xi)^{-\frac{1}{2}-i\nu}\\ &= \int_{H^2_+} \mathrm{d}\mu(x)f(x)\int_{H^2_+} \mathrm{d}\mu(y)s([y]^{-1}x)(y\cdot\xi)^{-\frac{1}{2}-i\nu}\\ &= \int_{H^2_+} \mathrm{d}\mu(x)f(x)\int_{H^2_+} \mathrm{d}\mu(y)s([x]^{-1}y)(y\cdot\xi)^{-\frac{1}{2}-i\nu}\\ &= \int_{H^2_+} \mathrm{d}\mu(x)f(x)\int_{H^2_+} \mathrm{d}\mu(y)s(y)([x]y\cdot\xi)^{-\frac{1}{2}-i\nu}\\ &= \int_{H^2_+} \mathrm{d}\mu(x)f(x)\int_{H^2_+} \mathrm{d}\mu(y)s(y)(y\cdot[x]^{-1}\xi)^{-\frac{1}{2}-i\nu}.\end{aligned}$$

Since ξ belongs to the projective null cone, we can write

$$(y\cdot[x]^{-1}\xi) = ([x]^{-1}\xi)_0\left(y\cdot\frac{[x]^{-1}\xi}{([x]^{-1}\xi)_0}\right), \tag{63}$$

and using $([x]^{-1}\xi)_0 = (x\cdot\xi)$, we finally obtain

$$\begin{aligned}\widehat{(s*f)}(\nu,\xi) &= \int_{H^2_+} \mathrm{d}\mu(x)f(x)(x\cdot\xi)^{-\frac{1}{2}+i\nu}\int_{H^2_+} \mathrm{d}\mu(y)s(y)\left(y\cdot\frac{[x]^{-1}\xi}{([x]^{-1}\xi)_0}\right)^{-\frac{1}{2}-i\nu}\\ &= \hat{f}(\nu,\xi)\hat{s}(\nu)\end{aligned}$$

where we used the rotation invariance of s. ■

Based on Theorem 1, we can write the hyperbolic continuous wavelet transform of a function f with respect to an axisymmetric wavelet ψ as

$$W_f(a,g) \equiv W_f(a,[x]) = \big(\bar{\psi}_a * f\big)(x). \tag{64}$$

6.2 Wavelets on the hyperboloid

We now come to the heart of this paper : we prove that the hyperbolic wavelet transform is a well-defined invertible map, provided the wavelet satisfies an admissibility condition.

Theorem 2 (Admissibility condition) *Let $\psi \in L^1(H^2_+)$ be an axisymmetric function, $a \mapsto \alpha(a)$ a positive function on $\mathbb{R}^+_*$ and m, M two constants such that*

$$0 < m \leq \mathcal{A}_\psi(\nu) = \int_0^\infty |\hat{\psi}_a(\nu)|^2 \, \alpha(a) \mathrm{d}a \leq M < +\infty. \tag{65}$$

Then the linear operator A_ψ defined by :

$$A_\psi f(x') = \int_{\mathbb{R}^*_+} \int_{H^2_+} W_f(a,x) \psi_{a,x}(x') \mathrm{d}x \, \alpha(a) \mathrm{d}a, \tag{66}$$

where $\psi_{a,x} \equiv \psi_{a,[x]}$, is bounded and with bounded inverse. More precisely A_ψ is univocally characterized by the following Fourier-Helgason multiplier :

$$\widehat{A_\psi} \hat{f}(\nu,\varphi) \equiv \widehat{A_\psi f}(\nu,\varphi) = \hat{f}(\nu,\varphi) \int_0^\infty |\hat{\psi}_a(\nu)|^2 \, \alpha(a) \mathrm{d}a = \mathcal{A}_\psi(\nu) \hat{f}(\nu,\varphi).$$

Proof : Let the wavelet transform W_f be defined as in equation (64) and consider the following quantity :

$$\Delta_a(x') = \int_{H^2_+} W_f(a,x) \psi_{a,x}(x') \mathrm{d}x. \tag{67}$$

A close inspection reveals that $\Delta_a(x')$ is itself a convolution. Taking the Fourier-Helgason transform on both sides of (67) and applying Theorem 1 twice, we thus obtain:

$$\widehat{\Delta_a}(\nu,\varphi) = |\hat{\psi}_a(\nu)|^2 \hat{f}(\nu,\varphi) \, .$$

Finally, integrating over all scales we obtain :

$$\int_{\mathbb{R}^*_+} \alpha(a) \mathrm{d}a \, \widehat{\Delta_a}(\nu,\varphi) = \hat{f}(\nu,\varphi) \int_{\mathbb{R}^*_+} \alpha(a) \mathrm{d}a \, |\hat{\psi}_a(\nu)|^2 \tag{68}$$

which is the expected result. ■

There are three important remarks concerning this result. First, Theorem 2 shows that the wavelet family $\{\psi_{a,x}, a \in \mathbb{R}^+_*, \, x \in H^2_+\}$ forms a continuous frame [Ali et al., 2000] provided the admissibility condition (65) is satisfied. In this case, the wavelet transform W_f of any f can be inverted in the following way. Let $\widetilde{\psi_{a,x}}$ be a *reconstruction wavelet* defined by :

$$\widehat{\widetilde{\psi_{a,x}}}(\nu) = \mathcal{A}_\psi^{-1}(\nu) \widehat{\psi_{a,x}}(\nu).$$

As a direct consequence of Theorem 2, the inversion formula, to be understood in the strong sense in $L^2(H^2_+)$, reads :

$$f(x') = \int_{\mathbb{R}^*_+} \int_{H^2_+} W_f(a,x)\widetilde{\psi_{a,x}}(x')\mathrm{d}x\, \alpha(a)\mathrm{d}a\,. \tag{69}$$

As a second remark, the reader can check that Theorem 2 does not depend on choice of dilation! This is not exactly true, actually. The *architecture* of the proof does not depend on the explicit form of the dilation operator, but the admissibility condition explicitly depends on it. As we shall see later, it will be of crucial importance when trying to construct admissible wavelets. Finally the third remark concerns the somewhat arbitrary choice of measure $\alpha(a)$ in the formulas. The reader may easily check that the usual 1-D wavelet theory can be formulated along the same lines, keeping an arbitrary scale measure. In that case though, the choice $\alpha = a^{-2}$ leads to a tight continuous frame, i.e. the frame operator A_ψ is a constant. The situation here is more complicated in the sense that no choice of measure would yield to a tight frame, a particularity shared by the continuous wavelet transform on the sphere [Antoine and Vandergheynst, 1999]. Some choices of measure though lead to simplified admissibility conditions as we will now discuss.

Theorem 3 *Let $a \mapsto \alpha(a)$ be a positive continuous function on $\mathbb{R}^+_*$ which for large a behaves like $a^{-\beta}$, $\beta > 0$. If D_a is the conic dilation with parameter p defined by equations* (18)*,* (45) *and* (47)*, then an axisymmetric, compactly supported, continuous function $\psi \in L^2(H^2_+, \mathrm{d}\mu(\chi,\varphi))$ is admissible for all $p > 0$ and $\beta > \frac{2}{p} + 1$. Moreover, if $\alpha(a)\mathrm{d}a$ is a homogeneous measure of the form $a^{-\beta}\mathrm{d}a$, then the following zero-mean condition has to be satisfied :*

$$\int_{H^2_+} \psi(\chi,\varphi) \left[\frac{\sinh 2p\chi}{\sinh \chi}\right]^{\frac{1}{2}} \mathrm{d}\mu(\chi,\varphi) = 0. \tag{70}$$

Proof : Let us assume $\psi(x)$ belongs to $\mathcal{C}_0(H^2_+)$, i.e. it is continuous and compactly supported

$$\psi(x) = 0 \quad \text{if} \quad \chi > \tilde{\chi}, \quad \tilde{\chi} < const.$$

We wish to prove that

$$\int_0^\infty |\langle \mathcal{E}_{\xi,\nu}|D_a\psi\rangle|^2\, \alpha(a)\mathrm{d}a < \infty. \tag{71}$$

First, we compute the Fourier-Helgason coefficients of the dilated function ψ:

$$\begin{aligned} \langle \mathcal{E}_{\xi,\nu}|D_a\psi\rangle &= \int_{H^2_+} D_a\psi(\chi,\varphi)\, \overline{\mathcal{E}_{\xi,\nu}(\chi,\varphi)}\, \mathrm{d}\mu(\chi,\varphi) \\ &= \int_0^{2\pi}\!\!\int_0^{\tilde{\chi}_{1/a}} \lambda^{\frac{1}{2}}(a,\chi)\psi(\chi_{\frac{1}{a}},\varphi)\overline{\mathcal{E}_{\xi,\nu}(\chi,\varphi)}\sinh\chi \mathrm{d}\chi\mathrm{d}\varphi. \end{aligned}$$

By performing the change of variable $\chi' = \chi_{\frac{1}{a}}$, we get $\chi = \chi'_a$ and $\mathrm{d}\cosh\chi = \mathrm{d}\cosh\chi'_a = \lambda(a^{-1},\chi')\mathrm{d}\cosh\chi'$. The Fourier-Helgason coefficients become

$$\langle \mathcal{E}_{\xi,\nu}|D_a\psi\rangle = \int_0^{2\pi}\!\!\int_0^{\tilde{\chi}} \lambda^{\frac{1}{2}}(a,\chi'_a)\psi(\chi',\varphi)\overline{\mathcal{E}_{\xi,\nu}(\chi'_a,\varphi)}\lambda(a^{-1},\chi')\sinh\chi'\mathrm{d}\chi'\mathrm{d}\varphi. \tag{72}$$

From the cocycle property

$$\lambda^{\frac{1}{2}}(a^{-1},\chi') = \frac{1}{\lambda^{\frac{1}{2}}(a,\chi'_a)} = \left[a\,\frac{\sinh\chi_a}{\sinh\chi}\,\frac{\cosh p\chi}{\cosh p\chi_a}\right]^{\frac{1}{2}}, \tag{73}$$

we get

$$\langle \mathcal{E}_{\xi,\nu}|D_a\psi\rangle = \int_0^{2\pi}\!\!\int_0^{\tilde{\chi}} \lambda^{\frac{1}{2}}(a^{-1},\chi')\,\psi(\chi',\varphi)\,\overline{\mathcal{E}_{\xi,\nu}(\chi'_a,\varphi)}\sinh\chi'\,\mathrm{d}\chi'\mathrm{d}\varphi. \tag{74}$$

Then, we split (71) in three parts:

$$\int_0^\infty (.)\alpha(a)\mathrm{d}a = \underbrace{\int_0^\sigma (.)\alpha(a)\mathrm{d}a}_{I_1} + \underbrace{\int_\sigma^{\frac{1}{\sigma}} (.)\alpha(a)\mathrm{d}a}_{I_2} + \underbrace{\int_{\frac{1}{\sigma}}^\infty (.)\alpha(a)\mathrm{d}a}_{I_3}. \tag{75}$$

Let us focus on the first integral. Developing the Fourier-Helgason kernel $\mathcal{E}_{\xi,\nu}$ in (74), we obtain :

$$\begin{aligned} I_1 = &\int_0^\sigma \alpha(a)\,\mathrm{d}a\times \\ &\times \Big|\int_0^{\tilde{\chi}}\int_0^{2\pi} \mathrm{d}\mu(\chi',\varphi)\,\lambda^{\frac{1}{2}}(a^{-1},\chi')\,\psi(\chi')\,(\cosh\chi'_a - \sinh\chi'_a\cos\varphi)^{-\frac{1}{2}+i\nu}\Big|^2. \end{aligned}$$

Using the explicit form of χ'_a, we have for various involved quantities the following asymptotic behaviors at small scale $a \approx 0$:

$$\begin{aligned} \cosh p\chi_a &\sim 1 + o(a), \\ \cosh\chi_a &\sim 1 + o(a), \\ \sinh\chi_a &\sim \frac{a}{p}\sinh p\chi + o(a), \\ (\cosh\chi'_a - \sinh\chi'_a\cos\varphi)^{-\frac{1}{2}+i\nu} &\sim 1 - (-\frac{1}{2}+i\nu)\frac{a}{p}\cos\varphi. \end{aligned}$$

So we have the approximation

$$\begin{aligned} &\int_0^{\tilde{\chi}}\int_0^{2\pi} \mathrm{d}\mu(\chi',\varphi)\,\lambda^{\frac{1}{2}}(a^{-1},\chi')\,\psi(\chi')\,(\cosh\chi'_a - \sinh\chi'_a\cos\varphi)^{-\frac{1}{2}+i\nu} \\ \sim &\frac{a}{\sqrt{2p}}\int_0^{\tilde{\chi}}\int_0^{2\pi} \mathrm{d}\mu(\chi',\varphi)\,\left[\frac{\sinh 2p\chi}{\sinh\chi}\right]^{\frac{1}{2}}\left(1 - (-\frac{1}{2}+i\nu)\frac{a}{p}\cos\varphi\right). \end{aligned}$$

Integrating over φ and using the rotation invariance of ψ, we obtain the following approximation for I_1 :

$$I_1 \sim \int_0^{\sigma} \alpha(a) a^2 \mathrm{d}a \Big| \int_0^{\tilde{\chi}} \sinh\chi' \mathrm{d}\chi' \left[\frac{\sinh 2p\chi'}{\sinh\chi'}\right]^{\frac{1}{2}} \psi(\chi') \Big|^2. \tag{76}$$

The second subintegral (I_2) is straightforward, since the operator D_a is strongly continuous and thus the integrand is bounded on $[\sigma, \frac{1}{\sigma}]$.

Consider now the inequality :

$$I_3 \leq \int_{\frac{1}{\sigma}}^{+\infty} \alpha(a) \mathrm{d}a \times$$
$$\times \left(\int_0^{\tilde{\chi}} \int_0^{2\pi} \mathrm{d}\mu(\chi', \varphi)\, \lambda^{\frac{1}{2}}(a^{-1}, \chi')\, |\psi(\chi')| |\cosh\chi'_a - \sinh\chi'_a \cos\varphi|^{-1/2} \right)^2 .$$

The term $|\cosh\chi'_a - \sinh\chi'_a \cos\varphi|^{-1/2}$ is bounded from above and from below by :

$$e^{-\frac{\chi'_a}{2}} \leq |\cosh\chi'_a - \sinh\chi'_a \cos\varphi|^{-1/2} \leq e^{\frac{\chi'_a}{2}}. \tag{77}$$

Now, we have

$$e^{\frac{\chi'_a}{2}} = \left(e^{p\chi'_a}\right)^{\frac{1}{2p}} = \left[\sqrt{1 + a^2 \sinh^2 p\chi'} + a \sinh p\chi'\right]^{\frac{1}{2p}},$$

and so we get the asymptotic behavior of this upper bound at large scale:

$$e^{\frac{\chi'_a}{2}} \sim a^{\frac{1}{2p}} (\sinh p\chi')^{\frac{1}{2p}}.$$

Again using the explicit form of χ'_a and the following asymptotic behaviors at large scale $a \to \infty$:

$$\cosh p\chi_a \sim a \sinh p\chi,$$
$$\sinh \chi_a \sim a^{\frac{1}{p}} (\sinh p\chi)^{\frac{1}{p}},$$

we reach the following majoration of I_3 :

$$I_3 \leq \int_{\frac{1}{\sigma}}^{+\infty} \alpha(a) a^{\frac{2}{p}} \mathrm{d}a \left(\int_0^{\tilde{\chi}} \mathrm{d}\chi' (\sinh\chi')^{\frac{1}{2}} (\sinh p\chi')^{\frac{1}{p} - \frac{1}{2}} (\cosh p\chi')^{\frac{1}{2}} |\psi(\chi')| \right)^2 .$$

Since the hyperbolic functions involved in the integration on the χ' variable are increasing, we finally end with the estimate :

$$I_3 \leq\sim (\sinh\tilde{\chi})^{\frac{1}{2}} (\sinh p\tilde{\chi})^{(\frac{1}{p} - \frac{1}{2})} (\cosh p\tilde{\chi})^{\frac{1}{2}} \|\psi\|_1^2 \int_{\frac{1}{\sigma}}^{+\infty} \alpha(a) a^{\frac{2}{p}} \mathrm{d}a.$$

and so $\alpha(a)$ should behave at least like $a^{-\beta}$ with $\beta > \frac{2}{p} + 1$ for $a \to \infty$.

The convergence of I_1 and I_3 clearly depends on the choice of measure in the integral over scales. Restricting ourselves to homogeneous measures $\alpha(a) = a^{-\beta}$ and to the range $p > 0$, one can distinguish the following cases :

- $\beta \leqslant \frac{2}{p} + 1$: in this case I_3 does not converge and there are no admissible wavelets.
- $\beta > \frac{2}{p} + 1$: In this case I_1 diverges except when $\int_{H^2_+} \psi \left[\frac{\sinh 2p\chi}{\sinh \chi}\right]^{\frac{1}{2}} = 0$. ■

6.3 An example of Hyperbolic Wavelet

Let us present here a class of admissible vectors which satisfy the admissibility condition. We restrict ourself to the simplest case $p = \frac{1}{2}$. Let us first state a preliminary result.

Proposition 3 *Let $\psi \in L^2(H^2_+, \mathrm{d}\mu)$. Then*

$$\int_{H^2_+} D_a\psi(\chi,\varphi)\mathrm{d}\mu(\chi,\varphi) = a \int_{H^2_+} \psi(\chi,\varphi)\mathrm{d}\mu(\chi,\varphi). \tag{78}$$

Proof: We have to compute the following integral

$$I = \int_{H^2_+} D_a\psi(\chi,\varphi)\mathrm{d}\mu(\chi,\varphi) = \int_{H^2_+} \lambda^{\frac{1}{2}}(a,\chi)\psi(\chi_{\frac{1}{a}},\varphi)\mathrm{d}\mu(\chi,\varphi).$$

By change of variable $\chi_{\frac{1}{a}} = \chi'$, we get

$$\begin{aligned} I &= \int_{H^2_+} \lambda^{\frac{1}{2}}(a,\chi'_a)\,\psi(\chi',\varphi)\,\lambda(a^{-1},\chi')\mathrm{d}\mu(\chi',\varphi) \\ &= \int_{H^2_+} \lambda^{\frac{1}{2}}(a^{-1},\chi')\psi(\chi',\varphi)\mathrm{d}\mu(\chi',\varphi), \end{aligned}$$

and having $\lambda^{\frac{1}{2}}(a^{-1},\chi') = a$, which follows directly from (45), we get

$$I = a\int_{H^2_+} \psi(\chi',\varphi)\mathrm{d}\mu(\chi',\varphi),$$

which proves the proposition. ■

Using this result, we can build the hyperbolic "difference" wavelet (difference-of-Gaussian, or DOG wavelet). Given a square-integrable function ψ, we define

$$f^{\vartheta}_{\psi}(\chi,\varphi) = \psi(\chi,\varphi) - \frac{1}{\vartheta}D_{\vartheta}\psi(\chi,\varphi), \quad \vartheta > 1.$$

More precisely, using the hyperbolic function $\psi = e^{-\sinh^2\frac{\chi}{2}}$, we dilate it using the conic projection and obtain

$$D_\vartheta\psi = \frac{1}{\vartheta}e^{-\frac{1}{\vartheta^2}\sinh^2\frac{\chi}{2}}, \tag{79}$$

we get:

$$f_\psi^\vartheta(\chi,\varphi) = e^{-\sinh^2\frac{\chi}{2}} - \frac{1}{\vartheta^2}e^{-\frac{1}{\vartheta^2}\sinh^2\frac{\chi}{2}}. \tag{80}$$

Now, applying a dilation operator on (80) we get

$$D_a f^\vartheta = \frac{1}{a}e^{-\frac{1}{a^2}\sinh^2\frac{\chi}{2}} - \frac{1}{a\vartheta^2}e^{-\frac{1}{a^2\vartheta^2}\sinh^2\frac{\chi}{2}}. \tag{81}$$

One particular example of *hyperbolic DOG wavelet* at $\vartheta = 2$ is:

$$f_\psi^2(\chi,\varphi) = \frac{1}{a}e^{-\frac{1}{a^2}\sinh^2\frac{\chi}{2}} - \frac{1}{4a}e^{-\frac{1}{4a^2}\sinh^2\frac{\chi}{2}}.$$

The resulting hyperbolic DOG wavelet at different values of the scale a and the position (χ,φ) on the hyperboloid is shown on Figures 9 , while Figures 10 and 11 show the same wavelet but viewed on the unit disk.

Of course, similar admissible DOG wavelets can be constructed for generic $p > 0$.

6.4 An example of Continuous Wavelet Transform on the Hyperboloid

For concluding this section we provide an example of the continuous wavelet transform applied on a synthetic signal-a hyperbolic triangle. The signal is projected on the unit disc and the visualization of its CWT at different scale a is depicted in Figure 12.

7 Euclidean limit

Since the hyperboloid is locally flat, the associated wavelet transform should match the usual 2-D CWT in the plane at small scales, i. e, for large curvature radiuses. In this section we recall some basic facts emphasizing those notions.

Let $\mathcal{H}_\rho \equiv L^2(H^2_{+\rho}, \mathrm{d}\mu_\rho)$ be the Hilbert space of square integrable functions on a hyperboloid of radius ρ,

$$\int_{H^2_\rho} |f(\chi,\varphi)|^2\rho^2\sinh\chi\mathrm{d}\chi\mathrm{d}\varphi < \infty, \tag{82}$$

and $\mathcal{H} = L^2(\mathbb{R}^2, \mathrm{d}^2\vec{x})$ be the Hilbert space of square integrable functions on the plane.

One can easily adapt the Fourier-Helgason transform by updating $\mathcal{E}_{\nu,\xi}(x)$ for any ρ [Alonso et al., 2002]:

$$\mathcal{E}^\rho_{\nu,\xi}(x) = \left(\frac{x_0 - \hat{n}\vec{x}}{\rho}\right)^{-\frac{1}{2}-i\nu\rho}, \tag{83}$$

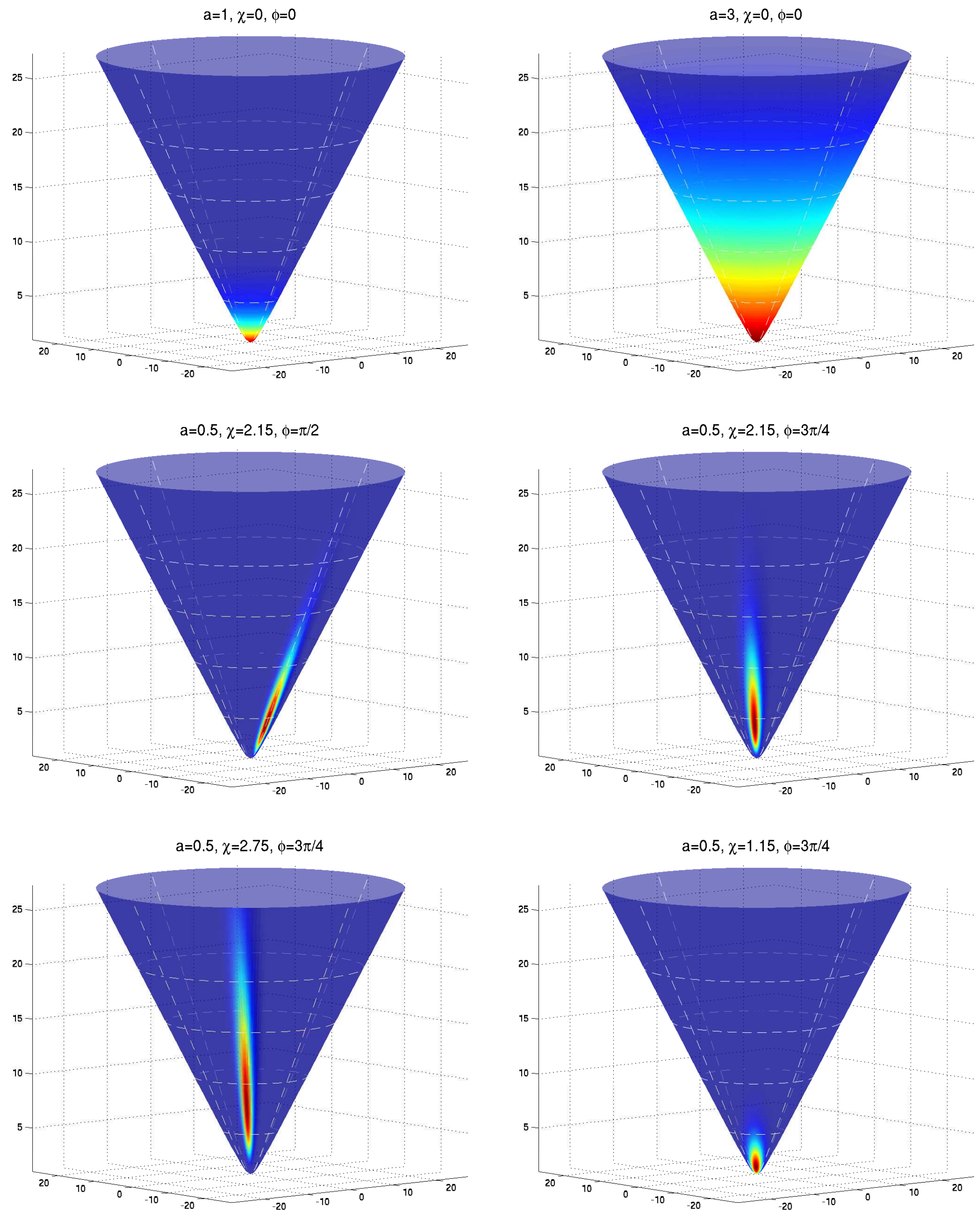

Figure 9: The hyperbolic DOG wavelet f_{ψ}^{ϑ}, for $\vartheta = 2$ at different scales a and positions (χ, φ).

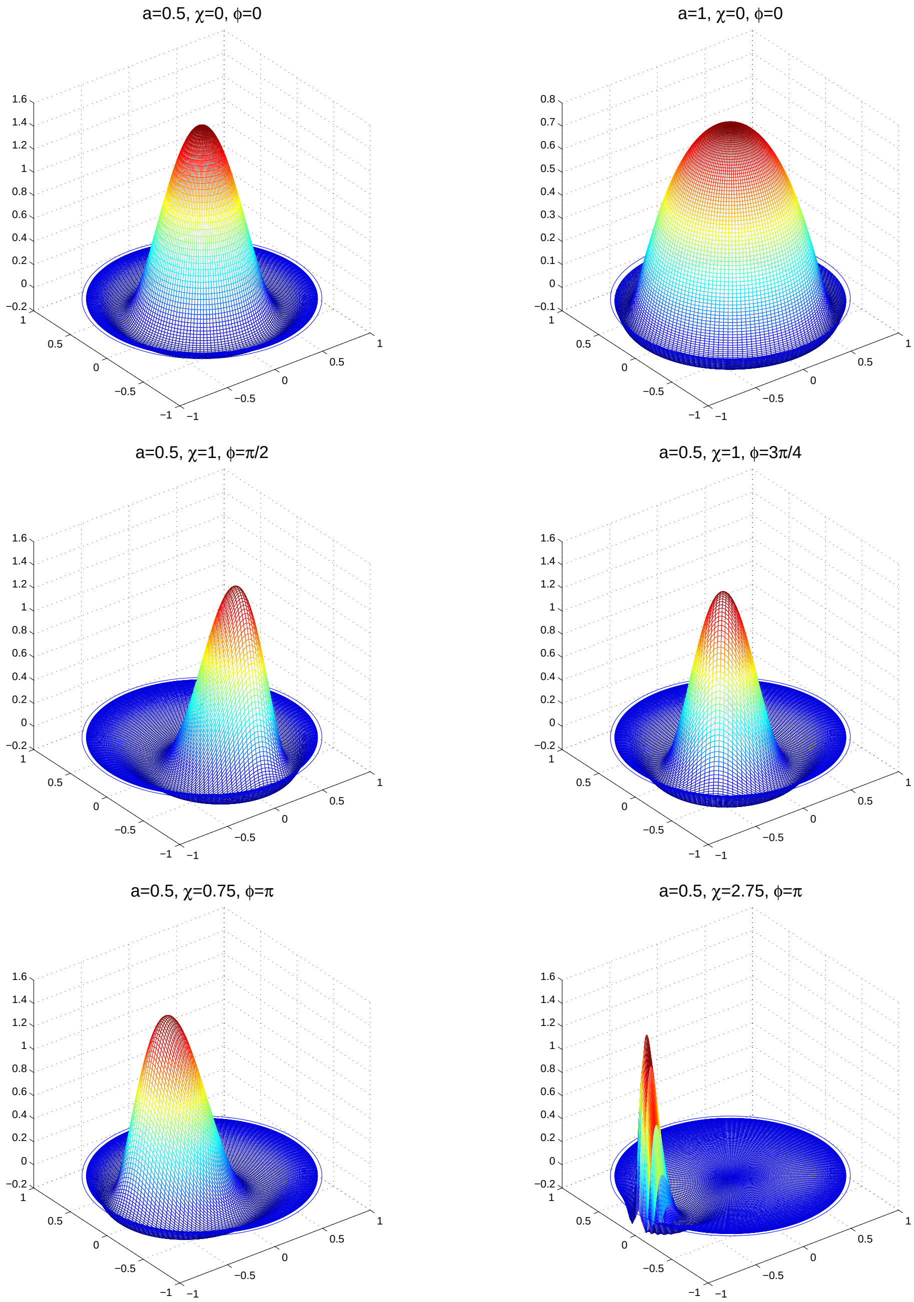

Figure 10: The hyperbolic DOG wavelet f_ψ^ϑ, for $\vartheta = 2$ at different scales a and positions (χ, φ), viewed on the unit disk in 3-D perspective.

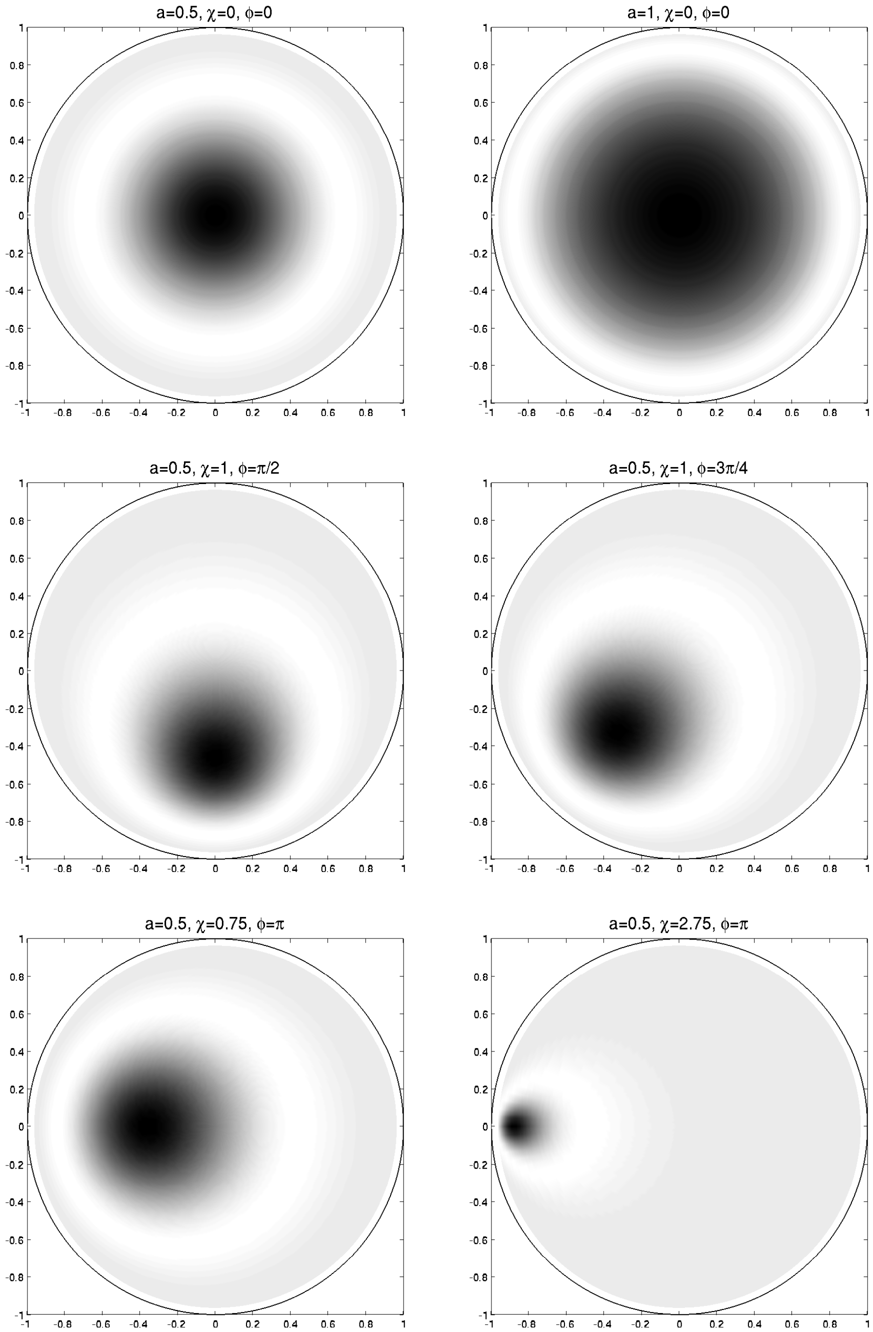

Figure 11: The hyperbolic DOG wavelet f_ψ^ϑ in the disk, for $\vartheta = 2$ at different scales a and positions (χ, φ).

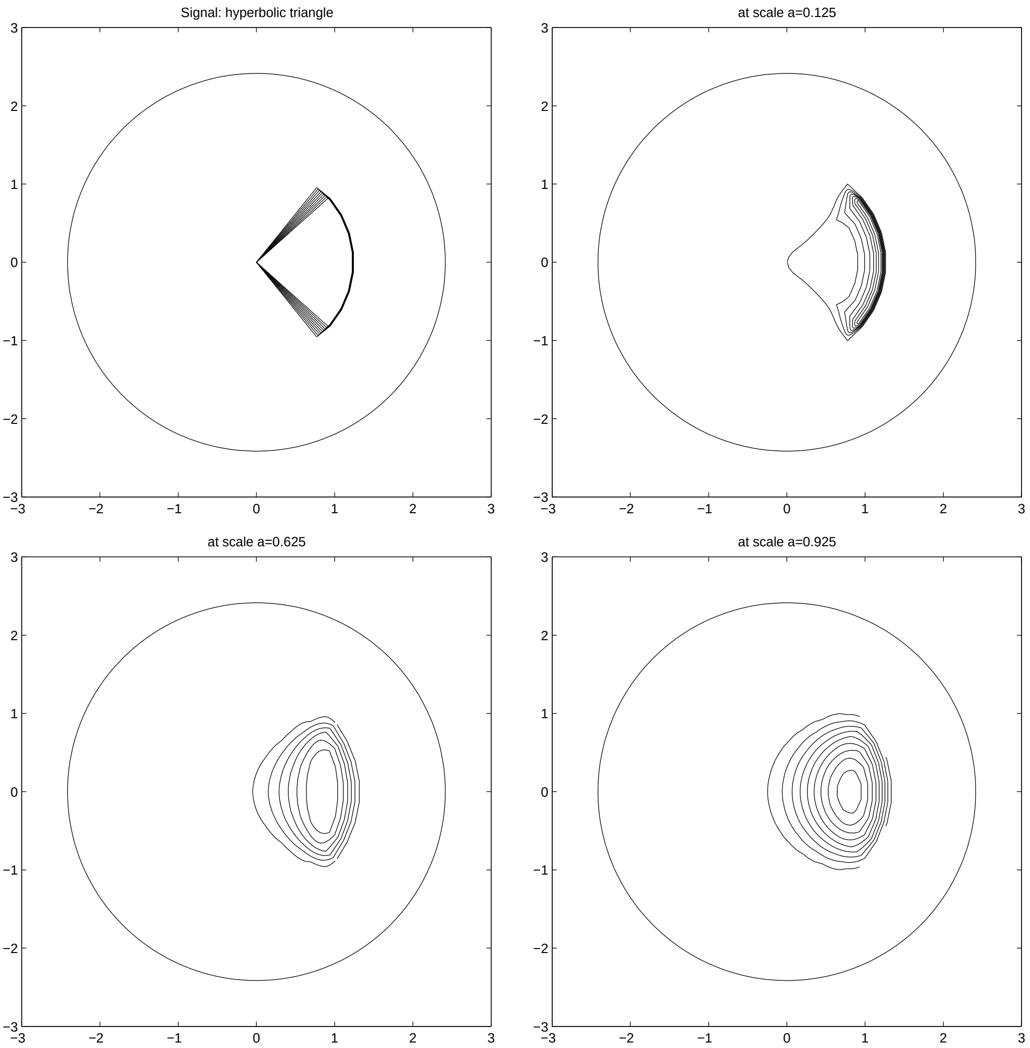

Figure 12: Continuous wavelet transform with $p = \frac{1}{2}$ of an hyperbolic triangle at different scales a

for $x \in H^2_{+\rho}$, $(x^2 = \rho^2)$. The Inönü-Wigner contraction limit of the Lorentz to the Euclidean group $SO(2,1)_+ \to ISO(2)_+$ is the limit at $\rho \to \infty$ for (83) with $x_0 \approx \rho$, $\vec{x}^2 \ll \rho^2$, i.e

$$\lim_{\rho\to\infty} \mathcal{E}^\rho_{\nu,\xi}(x) = \lim_{\rho\to\infty} \left(\frac{x_0 - \hat{n}\vec{x}}{\rho}\right)^{-\frac{1}{2}-i\nu\rho} \tag{84}$$

$$\approx \lim_{\rho\to\infty} \left(1 - \frac{\hat{n}\vec{x}}{\rho}\right)^{-i\nu\rho} = \exp{(i\nu\hat{n}\vec{x})}. \tag{85}$$

The Fourier-Helgason transform on the hyperboloid of radius ρ reads :

$$\hat{\psi}^\rho(\nu,\xi) = \frac{\rho}{2\pi}\int_{\vec{x}} \psi(\vec{x})\mathcal{E}_{\nu,\xi}(\vec{x})\frac{\mathrm{d}^2\vec{x}}{x_0} \tag{86}$$

and since $x_0 \approx \rho$ for $\rho \to \infty$, we obtain

$$\lim_{\rho\to\infty} \hat{\psi}^\rho(\nu,\xi) = \frac{1}{2\pi}\int_{\vec{x}} \psi(\vec{x})\exp{(i\nu\hat{n}\vec{x})}\mathrm{d}^2\vec{x} \tag{87}$$

$$= \hat{\psi}(\vec{k}), \tag{88}$$

which is the Fourier transform in the plane.

This relation shows that the geometric and algebraic breakdown $SO(2,1)_+ \to ISO(2)_+$ is mirrored at the functional level. Indeed, condition (65) with $\alpha(a) = a^{-3}$ asymptotically converges to its euclidean counterpart. Along the same line, the necessary condition of the hyperbolic wavelet contracts to the 2-D euclidean one:

$$\lim_{\rho\to\infty}\int_{H^2} \psi^\rho(\chi,\varphi)\mathrm{d}\mu(\chi,\varphi) \to \int_{\mathbb{R}^2} \psi(\vec{x})\mathrm{d}^2\vec{x}. \tag{89}$$

A much finer analysis would be necessary to understand if this association holds at the level of the necessary and sufficient condition (65), but this is out of the scope of this paper.

8 Conclusions

In this paper we have presented a constructive theory for the continuous wavelet transform on the hyperboloid $H^2_+ \in \mathbb{R}^3_+$. First we have defined the affine transformations on the hyperboloid and proposed different schemes for dilating H^2_+. After selecting the dilation of H^2_+ through conic projection, we have introduced the notion of convolution on this manifold. Using the hyperbolic convolution we have constructed the continuous wavelet transform and derived the corresponding admissibility condition. An example of hyperbolic DOG wavelet has been given. Finally, we have used the Inönü-Wigner contraction limit of the Lorentz to the Euclidean group $SO_0(2,1)_+ \to ISO(2)_+$ to check the consistency of the CWT on the hyperboloid with that one on the plane.

Interesting directions for future work include the design of a fast convolution algorithm for an efficient implementation of the transform and discretization of the theory so as to obtain frames of hyperbolic wavelets.

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