Fuzzy de Sitter space-times via coherent states quantization

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Abstract

A construction of the 2d and 4d fuzzy de Sitter hyperboloids is carried out by using a (vector) coherent state quantization. We get a natural discretization of the dS "time" axis based on the spectrum of Casimir operators of the respective maximal compact subgroups $SO(2)$ and $SO(4)$ of the de Sitter groups $SO_0(1,2)$ and $SO_0(1,4)$. The continuous limit at infinite spins is examined.

1 Introduction

The Madore construction of the fuzzy sphere [1] is based on the replacement of coordinate functions of the sphere by components of the angular momentum operator in a $(2j+1)$-dimensional UIR of $SU(2)$. In this way, the commutative algebra of functions on S^2, viewed as restrictions of smooth functions on $\mathbb{R}^3$, becomes the non-commutative algebra of $(2j+1)\times(2j+1)$-matrices, with corresponding differential calculus. The commutative limit is recovered at $j\to\infty$ while another parameter, say ρ, goes to zero with the constraint $j\,\rho = 1$ (or R for a sphere of radius R). The aim of the present work is to achieve a similar construction for the $2d$ and $4d$ de Sitter hyperboloids. The method is based on a generalization of coherent state quantization *à la* Klauder-Berezin (see [2, 3] and references therein). We recall that the de Sitter space-time is the unique maximally symmetric solution of the vacuum Einstein's equations with positive cosmological constant Λ. This constant is linked to the constant Ricci curvature 4Λ of this space-time. There exists a fundamental length $H^{-1} := \sqrt{3/(c\Lambda)}$.

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The isometry group of the de Sitter manifold is the ten-parameter de Sitter group $SO_0(1,4)$, the latter is a deformation of the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$.

2 Coherent state quantization: the general framework

Let X be a set equipped with the measure $\mu(dx)$ and $L^2(X,\mu)$ its associated Hilbert space of square integrable functions $f(x)$ on X. Among the elements of $L^2(X,\mu)$ let us select an orthonormal set $\{\phi_n(x),\ n = 1,2,\dots,N\}$, N being finite or infinite, which spans, by definition, a separable Hilbert subspace $\mathcal{H}$. This set is constrained to obey: $0 < \mathcal{N}(x) := \sum_n |\phi_n(x)|^2 < \infty$. Let us then consider the family of states $\{|x\rangle\}_{x\in X}$ in $\mathcal{H}$ through the following linear superposition:

$$|x\rangle := \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_n \overline{\phi_n(x)} |\phi_n\rangle. \tag{1}$$

This defines an injective map (which should be continuous w.r.t. some topology affected to X) $X \ni x \mapsto |x\rangle \in \mathcal{H}$. These coherent states are normalized and provide a resolution of the unity in $\mathcal{H}$:

$$\langle x \,|\, x\rangle = 1, \quad \int_X |x\rangle\langle x|\, \mathcal{N}(x)\, \mu(dx) = \mathbb{I}_{\mathcal{H}}. \tag{2}$$

A *classical* observable is a function $f(x)$ on X having specific properties. Its quantization *à la* Berezin-Klauder-“Toeplitz” consists in associating to $f(x)$ the operator

$$A_f := \int_X f(x) |x\rangle\langle x|\, \mathcal{N}(x)\, \mu(dx). \tag{3}$$

For instance, the application to the sphere $X = S^2$ with normalized measure $\mu(dx) = \sin\theta\ d\theta\ d\phi/4\pi$ is carried out through the choice as orthonormal set the set of *spin spherical harmonics* ${}_\sigma Y_{jm}(\hat{\mathbf{r}})$ for fixed σ and j. One obtains [4] in this way a family of inequivalent (with respect to quantization) fuzzy spheres, labeled by the the spin parameter $0 < |\sigma| \leq j$, $j \in \mathbb{N}_*/2$. Note that the spin is necessary in order to get a nontrivial quantization of the cartesian coordinates.

3 Application to the $2d$ de Sitter space-time

De Sitter space is seen as a one-sheeted hyperboloid embedded in a three-dimensional Minkowski space:

$$M_H = \{x \in \mathbb{R}^3 \ : \ x^2 = \eta_{\alpha\beta}\, x^\alpha x^\beta = (x^0)^2 - (x^1)^2 - (x^2)^2 = -H^{-2}\}. \tag{4}$$

The de Sitter group is $SO_0(1,2)$ or its double covering $SU(1,1) \simeq SL(2,\mathbb{R})$. Its Lie algebra is spanned by the three Killing vectors $K_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha$ (K_{12}: compact, for "space translations", K_{02}: non compact, for "time translations", K_{01}: non compact, for Lorentz boosts). These Killing vectors are represented as (essentially) self-adjoint operators in a Hilbert space of functions on M_H, square integrable with respect to some invariant inner (Klein-Gordon type) product.

The quadratic Casimir operator has eigenvalues which determine the UIR's :

$$Q = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} = -j(j+1)\mathbb{I} = \Big(\rho^2 + \frac{1}{4}\Big)\mathbb{I} \tag{5}$$

where $j = -\frac{1}{2} + i\rho$, $\rho \in \mathbb{R}^+$ for the principal series.

Comparing the geometric constraint (4) to the group theoretical one (5) (in the principal series) suggests the fuzzy correspondence [5]:

$$x^\alpha \mapsto \widehat{x^\alpha} = \frac{r}{2} \varepsilon^{\alpha\beta\gamma} M_{\beta\gamma}, \ \textit{i.e.}\ \widehat{x^0} = rM_{21}, \ \widehat{x^1} = rM_{02}, \ \widehat{x^2} = rM_{10}.$$

r being a constant with length dimension. The following commutation rules are expected

$$[\widehat{x^0}, \widehat{x^1}] = ir\widehat{x^2}, \ [\widehat{x^0}, \widehat{x^2}] = -ir\widehat{x^1}, \ [\widehat{x^1}, \widehat{x^2}] = ir\widehat{x^0}, \tag{6}$$

with $\eta_{\alpha\beta}\widehat{x^\alpha}\widehat{x^\beta} = -r^2(\rho^2 + \frac{1}{4})\mathbb{I}$, and its "commutative classical limit", $r \to 0$, $\rho \to \infty$, $r\rho = H^{-1}$.

Let us now proceed to the CS quantization of the 2d dS hyperboloid. The "observation" set X is the hyperboloid M_H. Convenient global coordinates are those of the topologically equivalent cylindrical structure: (τ, θ), $\tau \in \mathbb{R}$, $0 \leq \theta < 2\pi$, through the parametrization, $x^0 = r\tau$, $x^1 = r\tau \cos\theta - H^{-1}\sin\theta$, $x^2 = r\tau \sin\theta + H^{-1}\cos\theta$, with the invariant measure: $\mu(dx) = \frac{1}{2\pi}\, d\tau\, d\theta$. The functions $\phi_m(x)$ forming the orthonormal system needed to construct coherent states are suitably weighted Fourier exponentials:

$$\phi_m(x) = \left(\frac{\epsilon}{\pi}\right)^{1/4} e^{-\frac{\epsilon}{2}(\tau - m)^2}\, e^{im\theta}, \ m \in \mathbb{Z}, \tag{7}$$

where the parameter $\epsilon > 0$ can be arbitrarily small and represents a necessary regularization. Through the superposition (1) the coherent states read

$$|\tau, \theta\rangle = \frac{1}{\sqrt{\mathcal{N}(\tau)}} \left(\frac{\epsilon}{\pi}\right)^{1/4} \sum_{m\in\mathbb{Z}} e^{-\frac{\epsilon}{2}(\tau-m)^2}\, e^{-im\theta}|m\rangle, \tag{8}$$

where $|\phi_m\rangle \simeq |m\rangle$. The normalization factor $\mathcal{N}(\tau) = \sqrt{\frac{\epsilon}{\pi}}\sum_{m\in\mathbb{Z}} e^{-\epsilon(\tau-m)^2} < \infty$ is a periodic train of normalized Gaussians and is proportional to an elliptic Theta function.

The CS quantization scheme (3) yields the quantum operator A_f, acting on $\mathcal{H}$ and associated to the classical observable $f(x)$. For the most basic one, associated to the coordinate τ, one gets

$$A_\tau = \int_X \tau\, |\tau, \theta\rangle\langle\tau, \theta|\mathcal{N}(\tau)\, \mu(dx) = \sum_{m\in\mathbb{Z}} m|m\rangle\langle m|. \tag{9}$$

This operator reads in angular position representation (Fourier series): $A_\tau = -i\frac{\partial}{\partial\theta}$, and is easily identified as the compact representative M_{12} of the Killing vector K_{12} in the principal series UIR. Thus, the "time" component x^0 is naturally quantized, with spectrum $r\mathbb{Z}$ through $x^0 \mapsto \widehat{x^0} = -rM_{12}$. For the two other ambient coordinates one gets:

$$\widehat{x^1} = \frac{re^{-\frac{\epsilon}{4}}}{2}\sum_{m\in\mathbb{Z}}\left\{p_m|m{+}1\rangle\langle m|{+}h.c\right\},\ \widehat{x^2} = \frac{re^{-\frac{\epsilon}{4}}}{2i}\sum_{m\in\mathbb{Z}}\left\{p_m|m{+}1\rangle\langle m|{-}h.c\right\},$$

with $p_m = (m + \frac{1}{2} + i\rho)$. Commutation rules are those of $so(1, 2)$, that is those of (6) with a local modification to $[\widehat{x^1}, \widehat{x^2}] = -ire^{-\frac{\epsilon}{2}}\widehat{x^0}$. The commutative limit at $r \to 0$ is apparent. It is proved that the same holds for higher degree polynomials in the ambient space coordinates.

4 Application to the 4d de Sitter space-time

The extension of the method to the 4d-de Sitter geometry and kinematics involves the universal covering of $SO_0(1, 4)$, namely, the symplectic $Sp(2, 2)$ group, needed for half-integer spins. In a given UIR of the latter, the ten Killing vectors are represented as (essentially) self-adjoint operators in Hilbert space of (spinor-)tensor valued functions on the de Sitter space-time M_H, square integrable with respect to some invariant inner (Klein-Gordon type) product : $K_{\alpha\beta} \to L_{\alpha\beta}$. There are now two Casimir operators whose eigenvalues determine the UIR's:

$$Q^{(1)} = -\frac{1}{2}L_{\alpha\beta}L^{\alpha\beta},\ Q^{(2)} = -W_\alpha W^\alpha,\ W^\alpha := -\frac{1}{8}\epsilon^{\alpha\beta\gamma\delta\eta}L_{\beta\gamma}L_{\delta\eta}.$$

Similarly to the 2-dimensional case, the principal series is involved in the construction of the fuzzy de Sitter space-time. Indeed, by comparing both constraints, the geometric one: $\eta_{\alpha\beta}x^\alpha x^\beta = -H^{-2}$ and the group theoretical one, involving the *quartic* Casimir (in the principal series with spin $s > 0$): $Q^{(2)} = -W^\alpha W_\alpha = \left(\nu^2 + \frac{1}{4}\right) s(s+1)\,\mathbb{I}$ suggests the correspondence [5]: $x^\alpha \mapsto \widehat{x^\alpha} = rW^\alpha$, and the "commutative classical limit" : $r \to 0, \nu \to \infty$, $rs\sqrt{\nu^2 + \frac{1}{4}} = H^{-1}$.

For the CS quantization of the 4d-dS hyperboloid, suitable global coordinates are those of the topologically equivalent $\mathbb{R} \times S^3$ structure: (τ, ξ), $\tau \in \mathbb{R}$, $\xi \in S^3$, through the following parametrization, $x^0 = r\tau$, $\mathbf{x} = (x^1, x^2, x^3, x^4)^\dagger = r\tau\,\xi + H^{-1}\,\xi^\perp$, where $\xi^\perp \in S^3$ and $\xi \cdot \xi^\perp = 0$, with the invariant measure: $\mu(dx) = d\tau\,\mu(d\xi)$. We now consider the spectrum $\{\tau_i \mid i \in \mathbb{Z}\}$ of the compact "dS fuzzy time" operator rW^0 in the Hilbert space $L^2_{\mathbb{C}^{2s+1}}(S^3)$ which carries the principal series UIR $U_{s,\nu}$, $s > 0$. This spectrum is discrete. Let us denote by $\{\mathcal{Z}_{\mathcal{J}}(\xi)\}$, where $\mathcal{J}$ represents a set of indices including in some way the index i, an orthonormal basis of $L^2_{\mathbb{C}^{2s+1}}(S^3)$ made up with the eigenvectors of W^0. The functions $\phi_{\mathcal{J}}(x)$, forming the orthonormal system needed to construct coherent states, are suitably weighted Fourier exponentials:

$$\phi_{\mathcal{J}}(x) = \left(\frac{\epsilon}{\pi}\right)^{1/4} e^{-\frac{\epsilon}{2}(\tau-\tau_i)^2}\,\mathcal{Z}_{\mathcal{J}}(\xi), \tag{10}$$

where $\epsilon > 0$ can be arbitrarily small. The resulting vector coherent states read as

$$|\tau, \xi\rangle = \frac{1}{\sqrt{\mathcal{N}(\tau,\xi)}}\left(\frac{\epsilon}{\pi}\right)^{1/4} \sum_{\mathcal{J}} e^{-\frac{\epsilon}{2}(\tau-\tau_i)^2}\,\overline{\mathcal{Z}_{\mathcal{J}}(\xi)}|\mathcal{J}\rangle, \tag{11}$$

with normalization factor

$$\mathcal{N}(x) \equiv \mathcal{N}(\tau,\xi) = \sqrt{\frac{\epsilon}{\pi}} \sum_{\mathcal{J}} e^{-\epsilon(\tau-\tau_i)^2}\,\mathcal{Z}^\dagger_{\mathcal{J}}(\xi)\mathcal{Z}_{\mathcal{J}}(\xi) < \infty.$$

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