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Abstract

We study a hierarchy of discrete Boltzmann models (DBMs) with speeds $0, 1, \sqrt{2}$ when, in addition to binary collisions, ternary and quaternary multiple collisions are included: i) the square $9v_i$ model, ii) an associated three dimensional $15v_i$ model. Firstly we find, for shock waves, that the two equilibrium states are the same for binary alone collisions or not. We deduce, from the H-Theorem, a criterion for any multiple collision term. Secondly, from the knowledge of only the two equilibrium states associated to "shock profiles" solutions we can predict whether or not overshoots for the ratios P/M (P for pressure, M for mass and P/M for internal energy) are possible. In the arbitrary parameter space of the two equilibrium states we are able to predict the subdomains where both overshoots can occur or not and the strength of the effect. These subdomains are characterized by the singularities of the propagation speed ζ. Comparing with the square $8v_i$ model, without rest particles, a great difference occurs for $\zeta \simeq 0$. These predictions are independent of the fact that multiple collisions are present or not and can be generalized to any other DBM. Finally we construct exact similarity shock waves when ternary collisions are present, observe thinner shock profiles and verify the previous predictions on the P/M behaviours.

I Introduction

For the DBMs[1], the inclusion of multiple collisions is not new[2−1], however an increasing number of papers appear in this field[1−3−4]. Concerning the exact solutions, only a few models with ternary collisions[5] have been studied. Recently[6], for two models with speeds $1, \sqrt{2}$, the square $8v_i$ model and an associated $14v_i$ model, exact solutions were constructed with ternary and quaternary collisions included. However let us consider, for such models, collisions of the p-th order with in the loss and gain terms, respectively m, m' particles of speed 1 and $p - m$, $p - m'$ particles of speed $\sqrt{a} \neq 1$. The microscopic energy conservation law requires $m + a(p - m) = m' + a(p - m')$, or $m = m'$. As was recently emphasized by Ernst[7], any collision for such models conserves the number of slow and fast particles. This spurious conservation disappears if we add a rest particle with speed zero. Firstly, adding one rest particle to our previous two models we consider (fig.1a) both the square[8] $9v_i$ and the associated $15v_i$ models for which binary or multiple collisions can change the number of slow or fast particles. In contrast to some other models with multiple collisions, for shock waves the equilibrium states are the same as the binary ones. Applying the H-Theorem, we give a criterion for any multiple collision term. Secondly, for shock waves, we seek to understand whether physical effects like overshoots can be predicted from the knowledge of only the two equilibrium states. We want also to establish the difference

between models with or without rest particles, in particular between the $8v_i, 9v_i$ models. Thirdly we extend the previous construction of exact solutions[6], and check whether the predicted physical effects are observed.

The first model[8−9−10−11] is the square $d = 2$ (d dimension of the space) $9v_i$ model with three speeds $0, 1, \sqrt{2}$ and we choose, for the shock solutions, the spatial x-coordinate along one median of the square, the x-projections of the velocities being $-1, 0, 1$. There remain six independent densities $N_1, M_1, M_3, R, M_2, N_2$, associated to the x-projections $1, 1, 0, 0, -1, -1$ and to the velocities ($x_1 = x, x_2$) in the plane:

$$N_1 : (1, \pm 1), \ N_2 : (-1, \pm 1), \ M_1 : (1, 0), \ M_2 : (-1, 0), \ M_3 : (0, \pm 1), \ R : (0, 0).$$

Recently[12] many multispeed DBMs with binary collisions alone, have been classified following the $(1+1)$- dimensional restriction of their multidimensional PDE satisfied by the microscopic densities. Hierarchies of models, differing only by coefficients which depend on the spatial d dimension, have been found. As with the previous $8v_i, 14v_i$ models[6], we study such a hierarchy of models when multiple collisions are included.

The second model[12] is a $d = 3, 15v_i$ model which in the ($x_1 = x, x_2, x_3$) three dimensional space is the superposition of two square models in both the x_1, x_2 and x_1, x_3 planes. For the $x = x_1$ restriction of this model we still have the same six independent densities associated to the following velocities in the space:

$$N_1 : (1, \pm 1, 0), (1, 0, \pm 1), \ N_2 : (-1, \pm 1, 0), (-1, 0, \pm 1),$$

$$M_1 : (1, 0, 0), \ M_2 : (-1, 0, 0), \ M_3 : (0, \pm 1, 0), (0, 0, \pm 1), \quad R : (0, 0, 0).$$

For both models these six densities satisfy three linear relations equivalent to the mass, momentum and energy conservation laws and three independent nonlinear equations with (cf. fig1b) binary, ternary and quaternary collisions: ($p_\pm := \partial_t \pm \partial_x$, $d_* := d - 1$)

$$2d_*(p_+ N_1 + p_- N_2) = R_t, \quad p_+ M_1 + p_- M_2 + 2(R_t + d_* M_{3t}) = 0$$

$$p_+ M_1 - p_- M_2 + 2d_*(p_+ N_1 - p_- N_2) = 0, \quad p_+ N_1 = Q_{N1} = Q_{N1b} + Q_{N1t} + Q_{N1q}$$

$$R_t = Q_R = Q_{Rb} + Q_{Rt} + Q_{Rq}, \quad M_{3t} = Q_{M3} = Q_{M3b} + Q_{M3t} + Q_{M3q} \qquad (1.1)$$

For binary collisions (cf. fig.1b) with cross sections σ_{bi} we get:

$$Q_{N1b} = \sigma_{b1} B_1 + \sigma_{b3} B_{31}, \quad Q_{Rb} = 2d_* \sigma_{b3}(B_{31} + B_{32}), Q_{M3b} = \sigma_{b2} B_2 - \sigma_{b3}(B_{31} + B_{32})$$

$$B_1 = N_1 M_2 - N_2 M_1, \quad B_2 = M_1 M_2 - M_3^2, \quad B_{3i} = M_i M_3 - R N_i, \quad i = 1, 2 \qquad (1.2)$$

The macroscopic conservative quantities: mass M, momentum J and energy E are linear combinations of the microscopic densities, contrary to the velocity U and the pressure P:

$$M = M_1 + M_2 + 2d_*(M_3 + N_1 + N_2) + R, \quad J = M_1 - M_2 + 2d_*(N_1 - N_2) \quad (1.3)$$

$$2E = M_1 + M_2 + 2d_* M_3 + 4d_*(N_1 + N_2), \quad U = J/M, \quad P = 2E - MU^2 \qquad (1.4)$$

In section2 we seek the new collision terms (figs.1b), when higher order ternary and quaternary collisions are included, and find two different classes. In the first class, which

we call spectatorless collisions, the particles present in the loss term are missing in the gain term. In the second class, for instance the pseudotriple, pseudoquadruple,.. we add to these spectatorless binary, ternary,.. collision terms one particle, two,.. present in both loss and gain terms... We consider only the pseudotriple collisions where in the rhs of the binary (1.2) terms we replace σ_{bi} by $\sigma_{bi} + M\sigma_{pi}$. Historically this type of multiple collisions was the first introduced[2−13]. For the spectatorless collisions (cf. Tables1-2) we introduce *elementary collision terms $Q = G - L$ (where both gain and loss terms are only products of the densities)* and from the microscopic conservation laws find for the ternary collisions : $T_j, T_{2j}, T_{3j}, T_{4j}, j = 1, 2$ and $T_{5k}, k = 1, ..4$. and for the quaternary collisions : $Q_j, j = 1, 2, 4, 5$, $Q_{kj}, k = 3, 6, 8, 11, j = 1, 2$, $Q_{9k}, k = 1, 2, 3$, $Q_{10k}, Q_{12k}, k = 1, 2, 3$. These Q terms contain linearly one or two binary terms, with coefficients density dependent, so that they vanish for the same values of the equilibrium states defined by the B_i, B_{3i} terms.

We must check that the coefficients of any Q term into the nonlinear (1.2) equations are correct. So for both the $9v_i, 8v_i$, $15v_i, 14v_i$ models we give a criterion, such that any Q term leads to a negative contribution to the H-Theorem. Let $L/G = N_1^{\overline{\lambda}} M_3^{\overline{\mu}} R^{\overline{\nu}}...$ and in the nonlinear equations: $p_+ N_1 = ..\lambda Q$, $M_{3t} = ..\mu Q$, $R_t = ..\nu R$. *Our criterion $\lambda/\overline{\lambda} = \mu/\overline{\mu} = \nu/(2d_*\overline{\nu})$, has been checked for all Q terms of figs.1 and Tables1-2.*

In section 3 we firstly study the Rankine-Hugoniot (R-H) relations which contain both the three conservation laws for densities functions of a similarity variable η

$$N_1(\eta), \quad N_2(\eta), \quad M_1(\eta), \quad M_2(\eta), \quad M_3(\eta), \quad R(\eta), \quad \eta = x - \zeta t \qquad (1.6)$$

with propagation speed ζ and the relations coming from the vanishing of the collision terms for the two equilibrium states. Assuming that these states are independent of the cross-sections, *the vanishing of three independent binary collision terms imply the vanishing of the multiple ones, and therefore is sufficient for the determination of the asymptotic states.* We firstly study simple R-H relations where at the upstream state only one density (infinite Mach shock) or two (semi-infinite Mach shock) are different of zero. These solutions which depend on one scaling parameter and one (infinite shock) or two (semi-infinite) arbitrary parameters can be written down analytically. Secondly we establish conditions for the existence or not of overshoots for P/M (internal energy) which previously was considered as a quantity for the temperature[13], in particular for the $9v_i$ DBM[9−10]. However, although the DBMs have not yet been derived rigorously from the continuous Boltzmann theory, Cercignani[14] has given arguments showing that the temperature deduced from either P/M or the standard derivation from the entropy, could be valid only for models with an infinite number of velocities. Nevertheless P/M has an intrinsic physical significance for shock waves which are mainly defined by events with increase (or decrease) of both P, M across the shock. So the increase or decrease of P/M across the shock gives further physical information. Furthermore we have verified in many exact or numerical solutions, for many DBMs, the existence of possible overshoots with particular conditions for the propation speed ζ. Consider for instance models (like the $8v_i$ model) with projections of the velocities ± 1 along the x-axis. Then we have found, for the positive solutions with $|\zeta| < 1$, the possible overshoots for values not too far from ± 1, while they are monotonic for values close to 0 (far from the singularities ± 1). Similarly for models[11] (like the $9v_i$ model) with three singularities along the x-axis $\pm 1, 0$, it was found that the P/M

overshoots occur for $|\zeta|$ not to far from $\pm 1, 0$ while they do not exist for instance for the value $\zeta = \pm 1/2$. The following question arises: *Can we predict, from the knowledge of the R-H relations, the existence or not of such phenomena?* The answer is yes and we predict, in the parameter space of the R-H equilibrium states, the subdomains where this can happen or not. We emphasize that this study can be done for any DBMs.

In section 4 we construct, for binary p=2 and ternary p=3 collisions, a class of exact similarity shock waves solutions which are functions of the variable $\eta = x - \zeta t$

$$N_i = s_i - n_i/D^q, M_i = p_i - m_i/D^q, R = r_{00} - r/D^q, D = 1 + exp(\gamma^{(p)}\eta), q(p-1) = 1 \ (1.6)$$

where the R-H relations give the densities $s_i, s_i - n_i, p_i, p_i - m_i, r_{00}, r_{00} - r$ of the Maxwellian states, while the nonlinear equations will give new relations, for new parameters, $\gamma^{(p)}$ and the cross-sections. We observe thinner shock profiles when multiple collisions are present and verify the R-H predictions for P/M.

2 Multiple Collisions, Criterion for the elementary collision terms

2.1 Ternary collisions

For the $9v_i$ model, we write down the elementary ternary collision terms associated to the pictures $T_i, i = 1, ..5$ of fig.1b. We must take into account all symmetries of the model. For instance T_2, T_4 give respectively two different elementary terms T_{21}, T_{22} and T_{41}, T_{42} due to the exchange $x_1 \longleftrightarrow -x_1$ or $(N_1, M_1, M_3, R) \longleftrightarrow (N_2, M_2, M_3, R)$ and $(B_1, B_2, B_{31}) \longleftrightarrow (-B_1, B_2, B_{32})$. Similarly T_3, T_5 give four terms $T_{3i}, T_{5i}, i = 1, 2$ with $x_1 \to -x_1$ and two others $i = 3, 4$ due to the rotations of $\pi/2$ of the axes.

Table 1: Ternary spectatorless collisions $i = 1, 2, j = 1, 2, i \neq j$

$T_1 = -M_3 B_1, T_2 : T_{2i} = N_i M_3^2 - N_j M_i^2 = (-1)^j M_i B_1 - N_j B_2, T_3 : T_{3i} = -N_i B_2, T_{3,2+i} = -T_{2i}, T_4 : T_{4i} = N_j B_{3i}, T_5 : T_{5i} = M_3^3 - M_i N_j R = M_i B_{3j} - M_3 B_2, T_{5,2+i} = M_j^2 M_i - M_3 R N_j = M_3 B_{3j} + M_j B_2$

In order to check the coefficients of any Q term, into the N_i, M_i, R equations, we will verify the linear relations(1.1) and in section2.3 give a criterion for the nonlinear N_1, M_3, R equations. Our conventions are the following: We choose for N_1, M_3, N_2 the densities associated to the velocities $(1, 1), (0, 1), (-1, 1)$ in the (x_1, x_2) plane and $(1, 0, 1)$, $(0,1,0)$,$(-1,0,1)$ for $d = 3$ with for Q the coefficients $1, -1$ if they correspond to loss or gain terms. We notice that M_1, R, M_2 belonging, for $d = 3$,to the two (x_1, x_2) and (x_1, x_3) squares, it follows that for these densities the corresponding coefficients of Q are multiplied by d_*. Furthermore we take the multiplicity $1, 2, 3$ if the associated velocity is alone, double and triple. As illustration we choose two examples $Q = T_1, T_{21}$ and verify (1.1). We consider both T_1 as in fig.1b and $-T_1$ for the associated collision $x_1 \to -x_1$. The Q coefficients for $N_1, N_2, M_1, M_2, M_3, R$ are: $0 + 1, -1 + 0, -1 - 1, 1 + 1, 1 - 1, 0, 0$. For T_{21} in fig.1, Table1 and the associated collision $x_2 \to -x_2$ with M_1, M_3 being double, we get $-1 + 0, 1 + 0, 2 + 2, 0 + 0, 0 - 2, 0$ for $N_1, N_2, M_1, M_2, M_3, R$.

We notice that all ternary collision terms contain linearly the binary ones and so, for shock waves vanish for the same equilibrium states. When the rest particle is not present, these collisions are those of the $8v_i$ model and we verify that they conserve the number of slow and fast particles. We sketch briefly the method in order to determine all

spectatorless collisions and begin when the rest particles are present. Let us call [0],[1],[2] the particles associated to $|v_i^2| = 0, 1, 2$. Forgetting momentum conservation and keeping energy conservation plus spectatorless collisions we get for the possible collisions: $[0]+[1]+[2] \to [1]+[1]+[2]$ and $[0]+[2]+[2] \to [1]+[1]+[2]$. With the momentum conservation and a lot of tedious analytical calculations we get the T_4, T_5 collisions of fig.1,Table1. Similarly for the $8v_i$ collisions we first get $[1]+[2]+[2]$ and $[1]+[1]+[2]$ (with conservation of slow and fast particles) but momentum conservation excludes the first type of collisions. We rewrite the nonlinear (1.2) equations when ternary spectatorless collisions are present:

$$p_+N_1 = Q_{N1b} - B_1[\sigma_{b1} + M_3\sigma_{t1} + (M_1 + M_2)\sigma_{t23}] + B_{31}[\sigma_{b3} + N_2\sigma_{t4} + \sigma_{t5}(M_2 + M_3)]$$

$$+B_2[(\sigma_{t23}(N_1 - N_2) + \sigma_{t5}(M_1 - M_3)] = Q_{N1}, \qquad R_t/2d_* = Q_R = Q_{Rb} + B_{31}$$

$$[\sigma_{b3} + s_5(M_2 + M_3) + \sigma_{t4}N2] + B_{32}[\sigma_{b3} + \sigma_5(M_1 + M_3) + \sigma_{t4}] + B_2\sigma_{t5}(M_1 + M_2 - 2M_3)$$

$$M_{3t} = B_2[\sigma_{b2} + 2(\sigma_{t23} + \sigma_{t3}(N_1 + N_2)) + \sigma_{t5}(M_1 + M_2 + 6M_3)] + B_1 2\sigma_{t23}(M_2 - M_1)$$

$$-B_{31}[\sigma_{b3} + N_2\sigma_{t4} + \sigma_{t5}(3M_2 - M_3)] - B_{32}[\sigma_{b3} + N_1\sigma_{t4} + \sigma_{t5}(3M_1 - M_3)] = Q_{M3} \quad (2.1)$$

where σ_{ti} are the cross sections for T_i and $\sigma_{t23} = \sigma_{t2} + \sigma_{t3}$. We recall that if pseudotriple collisions are included then $\sigma_{bi} \to \sigma_{bi} + M\sigma_{pi}$.

2.2 Quaternary collisions

For the determination of the possible collisions (cf. fig.1b) we apply the same method as for the ternary ones. We first use energy conservation plus spectatorless collisions and later on use momentum conservation. When the rest particles are absent we obtain the $8v_i$ collisions (with conservation of [1] and [2] particles) while when they are introduced we find collisions with one and two rest particles.

Table2: Quaternary spectatorless collisions $i = 1, 2, j = 1, 2, i \neq j$

$Q_1 = M_1^2 N_2^2 - M_2^2 N_1^2 = -(M_2 N_1 + M_1 N_2)B_1, Q_2 = -N_1 N_2 B_2, Q_3 : Q_{3i} = -N_i M_3 B_1,$

$Q_4 = M_3^4 - M_1^2 M_2^2 = -(M_3^2 + M_1^2 M_2^2)B_2, Q_5 = Q_1$

$Q_6 : Q_{6i} = N_j T_{2i}, Q_7 = M_1 M_2 M_3^2 - N_1 N_2 R^2 = M_2 M_3 B_{31} + R N_1 B_{32}, Q_8 : Q_{8i} =$

$M_i^2 M_3^2 - R^2 N_i^2 = (M_1 M_3 + R N_i)B_{3i} Q_9 : Q_{9i} = Q_{8i}, Q_{93} = Q_7, Q_{10} : Q_{10i} = N_i T_{5i}$

$Q_{10i+2} = N_i T_{5i+2}, Q_{11} : Q_{11i} = N_i M_3 B_{3i}, Q_{12} : Q_{12i} = Q_{10i}, Q_{12,i+2} = M_j^2 N_i - M_3 N_j^2 R =$

$N_j T_{5,2+i} + (-1)^j M_j^2 B_1, Q_{13} : Q_{13i} = N_1 N_2 B_{3i}$

Here also, due to the symmetries of the model, for each drawing of fig.1, different elementary collisions are associated. The collision terms contain linearly the binary ones and, for shock waves, vanish for the two equilibrium binary states.

2.3 Criterion for the elementary collision Q terms, cf. AppendixA

Firstly (Lemmas1-2-3) for any elementary collision Q term, L/G can be written (cf.(A.1)) as the products (with arbitrary powers) of the L/G associated to B_i and one of the B_{3i}. The equilibrium states for binary or multiple collisions are the same.

Secondly, for the H-Theorem, the expression, which must be negative, can be written as a superposition of terms associated to each Q. We define: $G = R^\gamma N_1^{\alpha_1} M_3^{\beta_3}...$, $L = R^{\overline{\gamma}}$ $N_1^{\overline{\alpha}_1} M_3^{\overline{\beta}_3}..$, $R_t = ... + \nu Q$, $p_+ N_1 = ... + \lambda Q$, $M_{3t} = ... + \mu Q$ and prove (Theorem1) that Q gives a negative contribution if: $\lambda/(\overline{\alpha}_1 - \alpha) = \mu/(\overline{\beta}_3 - \beta_3) = \nu/2d_*(\overline{\gamma} - \gamma)$.

With this criteron verified, cf.(A.5), we can check the nonlinear N_1, M_3, R equations.

3: R-H relations, possible P/M overshoots

3.1 Determination of the two equilibrium states (AppendixB)

The main result is that the equilibrium states are the same whether the binary collisions are alone or multiple collisions are included, leading to the same solutions for the R-H relations. We assume that the densities are functions of a similarity variable $\eta = x - \zeta t$ and define for $(N_i, i = 1, 2, M_i, i = 1, 2, 3, R)$ two Maxwellian states when $|\eta| = \infty$:

$$(i) : (n_{0i}.., m_{0i}.., r_0), \quad (ii) : (s_i.., p_i.., r_{00}), \quad s_i = n_{0i} + n_i, p_i = m_{0i} + m_i, \ r_{00} = r_0 + r \quad (3.1)$$

The three linear conservation laws (1.1) give three relations for the n_i, m_i, r parameters, while the vanishing of the four binary collision terms B_i, B_{3i} for the (i) and (ii) states give only six independent relations:

$$\zeta/2d_* = (1 + \zeta)n_2 - (1 - \zeta)n_1, \quad 2\zeta(r + d_* m_3) = (1 - \zeta)m_1 - (1 + \zeta)m_2$$

$$(1 - \zeta)(m_1 + 2d_* n_1) + (1 + \zeta)(m_2 + 2d_* n_2) = 0, \quad n_{01} m_{02} = n_{02} m_{01}, \quad m_{01} m_{02} = m_{03}^2$$

$$n_{01} r_0 = m_{01} m_{03}, \quad s_1 p_2 = s_2 p_1, \quad p_1 p_2 = p_3^2, \quad s_1 r_{00} = p_1 p_3 \quad (3.2)$$

For the determination of the two Maxwellian states (3.1) we have thirteen parameters: $\zeta, n_{0i}, m_{0i}, \ n_i, m_i, \ r_0, r$ and nine relations (3.2). We choose $n_{01} = 1$ as the scaling parameter so that the general solutions depend on three arbitrary parameters which need the resolution of two coupled cubic equations. We want to obtain analytical solutions and choose simpler solutions. At the downstream states the densities are zero except $p_2 \neq 0$ and s_2 is either 0 (infinite Mach shock) or $\neq 0$ (semi-infinite shock). We report briefly the results of the study done in AppendicesB1-2-3. For the semi-infinite shock, the solutions depend on two arbitrary parameters: $m_{01} > 0, 0 < \zeta < \zeta_{sup} = 2/\sqrt{4 + m_{01}^2(m_{01}/d_* + 1)}$. For the infinite shock, due to the relation $\zeta = \zeta_{sup}$, it remains one parameter.

In AppendixB3 we recall the results for the $8v_i, 14v_i$ models without rest particle R. For the semi-infinite shock we have the same two parameters with a similar condition $0 < \zeta < \zeta_{sup}, \ \zeta_{sup} = 1/\sqrt{1 + (2/m_{01} + 1/d_*)^2}$ while for the infinite case it remains only one parameter with $\zeta = \zeta_{sup}$.

3.2 Predictions for the possible P/M overshoots

To the two equilibrium states (i), (ii) we associated macroscopic quantities: $(m_0, m_{00} = m_0 + m)$ for the mass, $(j_0, j_{00} = j_0 + j)$ for the momentum and $(e_0, e_{00} = e_0 + e)$ for the energy which are written down in AppendixB4 as linear combinations of the asymptotic densities. We assume that to these macroscopic densities there exist associated "shock profiles" η-dependent functions:

$$M = m_{00} - m D^{-1}(\eta), \quad J = j_{00} - j D^{-1}, \quad E = e_{00} - e D^{-1}, \quad m_{00} = m_0 + m, e_{00} = e_0 + e,$$

$j_{00} = j_0 + j$, $D(\eta)$ monotonic η-dependent function, $\eta \in [-\infty, \infty]$, $D(\eta) \in [1, \infty]$ (3.3)

where the $D(\eta)$, different for binary collisions or binary plus ternary collisions..., are always satisfying (3.3). We easily find for the derivative of P/M:

$$[M^3 \partial_\eta (P/M)]/2\partial_\eta D^{-1} = \Lambda + \Omega/D(\eta)$$

$$\Lambda = m_{00}(me_{00} - m_{00}e) - j_{00}(mj_{00} - m_{00}j), \Omega = m(m_{00}e - me_{00}) + j(mj_{00} - jm_{00}) \quad (3.4)$$

with Λ, Ω determined from the two equilibrium (i),(ii) states.

Theorem 2: *P/M is monotonic or nonmonotonic depending whether $\Lambda(\Lambda + \Omega) \geq 0$ or < 0.*

For the "shock profiles" solutions, even without an explicit knowledge of $D(\eta)$, we can predict, from the two equilibrium states, the existence or not of P/M overshoots. The ratios P/M are $2e_0/m_0 - (j_0/m_0)^2$ at the (i) state and $2e_{00}/m_{00} - (j_{00}/m_{00})^2$ at the (ii) state. Let us normalize the macroscopic quantities at the downstream state, here for the infinite and semi-infinite cases: $(m_0, j_0, e_0, 2e_0/m_0 - (j_0/m_0)^2)$ associated to $(M, J, E, P/M)$. The extremum of P/M is given by the η value for which $\Lambda + \Omega/D(\eta) = 0$ and for this value we can predict the strength $STr[P/M] = [sup(P/M)]/[P/M]_{down}$ of the P/M bump:

$$sup(P/M) = 2(e_{00} + e\Lambda/\Omega)(m_{00} + m\Lambda/\Omega)^{-1} - [j_{00} + j\Lambda/\Omega]/(m_{00} + m\Lambda/\Omega)]^2 \quad (3.5)$$

Theorem 3: *For any DBM for which "shock profiles" solutions exist, the existence or not of an P/M overshoot as well as its strength are only functions of the two equilibrium states. Furthermore these results are independent of the fact that only binary collisions occur or multiple collisions are included.*

Finally let us require, for the infinite shock solutions, that the mass ratio across the shock satisfies $m_0/m_{00} = d + 1$ for the $d = 2, 3$ models (AppendixB5). For the $9v_i$ model we find $\zeta = 1/2, m_{01} = 2$ and we predict a monotonic P/M curve. For the $15v_i$ model the result is numerical: $\zeta = 0.637, m_{01} = 1.76, STr = 1.013$ and predict a negligible P/M overshoot. For the $8v_i, 14v_i$ models we find respectively $m_0/m_{00} > 5.08, 5.83 > d + 1$.

3.3 Applications to the $9v_i, 15v_i, 8v_i$ models

In figs.2a-b-c-d for the $9v_i, 15v_i, 8v_i, 14v_i$ models, we present the ζ, m_{01} (or $m_{01}/(1 + m_{01})$) domains for which the R-H solutions (semi-infinite and infinite shocks) are positive and the contour-maps for P/M. The domains are limited (partly in figs.2c-d) by the border-line S_∞ which corresponds to the infinite shock. For P/M the strength is 1 in the domain I which means a monotonic behaviour while $1 < STr < 1.04$ in the domain II, $1.04 < STr < 1.25$ in III and as illustration we present also the line $STr = 1.6$. *The main result is that nonmonotonic P/M behaviours are possible for ζ values around 1 while around 0 this is possible only for the models $9v_i, 15v_i$ with rest particles.* The ζ values 0 and 1 correspond to mathematical singularities for which the formalism can blow up (cf. (1.1),(2.1)). So, for physical considerations, we must not look to values very closed to these singularities. We must also consider domains for which the possible overshoots are significant, let us say: $STr > 1.04$. We observe, with some universality in these models, that for $0.8 < \zeta < 0.9$ we can expect such physically reasonable P/M effects. The singularitiy $\zeta = 0$ comes from two terms: both the rest particle $\zeta dR/d\eta$ and $\zeta dM_3/d\eta$.

Contrary to the singularity associated to R, the M_3 one comes from a velocity projection on the x-axis and does not exist in the $d = 2, 3$ dimensional spaces. For the $8v_i, 14v_i$ models, only the M_3 term is present and monotonic P/M are predicted. On the contrary for the models with rest particles, cf. figs2a-b, for $\zeta \simeq 0$, domains with $STr > 1$ exist, but they are very close to 0 and so appear more as mathematical effects than physical ones. Notice that for the $9v_i$ model with $\zeta = 1/2$, at the middle of the two singularities $0, 1$ we find P/M monotonic in agreement with the calculation of AppendixB where we show that at the boundary $m_{01} = 2$, then $\Lambda + \Omega = 0$. In the next section we construct exact solutions which will confirm these predictions, deduced only from the equilibrium states.

4 Similarity Shock Waves solutions

4.1 Application of the section3 R-H relations

Substituting the similarity densities (1.6) into (1.1-2), (2.1) we obtain both the R-H relations for the (i),(ii) states which were constructed in section3 and new relations, from the nonlinear equations, which contain the cross sections and $\gamma^{(p)}$ as new parameters. For the semi-infinite solutions with $p_2 \neq 0, s_2 \neq 0$, we rewrite both the densities, the binary terms and the linear terms of the nonlinear (2.1) equations:

$$D^{-q} = N_1 = R/r_0 = M_3/m_{03}, N_2 = s_2 - n_2 D^{-q}, M_2 = p_2 - m_2 D^{-q}$$

$$D = 1 + exp(\gamma^{(p)}\eta), B_1/a_{01} = B_2/a_{02} = B_{32}/a_{03} = \hat{D}_q := D^{-q}(1 - D^{-q}), B_{31} = 0$$

binary $q = 1, p = 2$, binary plus ternary $q = 1/2, p = 3$, $a_{01} = p_2 - s_2 m_{01}$, $a_{03} = m_{03}a_{01}$, $a_{02} = m_{01}p_2$

$$p_+ N_1/q(1 - \zeta)\gamma = R_t/q\zeta r\gamma = M_{3t}/q\zeta m_3\gamma = D^{-(q+1)} - D^{-q} \tag{4.1}$$

4.2 Relations coming from the nonlinear (2.1) equations: AppendixC

The linear terms of (2.1) contain terms proportional to D^{-q} and D^{-q-1} with opposite coefficients (cf.(4.1)). We study the structure of the nonlinear terms Q_{N1}, Q_R, Q_{M3}. They contain linearly the binary terms (proportional to $\hat{D}_q$) multiplied by factors which reduce to constants if only binary collisions occur. In contrast, if ternary collisions are present, these factors contain linealy D^{-q}. Consequently if only binary collisions are present the nonlinear terms contain D^{-q}, D^{-2q}, $q = 1$ with opposite constants and finally the three (2.1) equations will give only three new relations. On the contrary for the $q = 1/2$ ternary case the rhs of the (2.1) equations contain terms, like the lhs, proportional to $D^{-3/2}, D^{-1/2}$ with opposite coefficients, but in addition terms D^{-1} not present in the lhs. In this case we obtain six relations (C.1-2-3-4-5-6).

In AppendixC2 we study the case of binary collisions alone. From (C.1-3-5) we find (cf. (C.7)) for the semi-infinite solutions: $\gamma^{(2)} > 0$, $\sigma_{bi} > 0$. *In AppendixC3 we study the case of binary and pseudotriple collisions.* From the six first (C.i) relations we deduce the parameters σ_{bi}, which are the same as in the binary case, and $\gamma^{(3)}$, σ_{pi} which are positive if $m + m_{00} = 2m_{00} - m_0 < 0$. For the infinite shock solutions this condition is satisfied but not always for the larger semi-infinite class. *In Appendix C4 we study the case of binary and ternary spectatorless solutions.* Choosing $\sigma_{b1} = 1$ and σ_{b3}, σ_{t5} arbitrary, from the six first (C.i) relations, we get $\gamma^{(3)}$, σ_{tj}, $j = 1, 2, 3, 4$ and σ_{b2}. The positivity for the cross sections is verified numerically. In AppendixC5 we give the results for the $8v_i, 14v_i$ models

with pseudotriple collisions included. In AppendixC6 we write down the shock thickness $w^{(p)}$ associated to quadratic $p = 2, q = 1$ and cubic $p = 3, q = 1/2$ collision terms.

4.3 Numerical Calculations for Compressive shocks

In figs.3a-b-c, fig3d for the $9v_i, 15v_i$ models we present $M, P/M$ curves for infinite Mach shock solutions and in fig.3e for the $8v_i$ model, a semi-infinite shock solution. The shocks are compressive and, using the weak shock polynomials associated to the equilibrium states[11], we have verified (results not reported) the subsonic and supersonic inequalities. We do not discuss the Whitham stability conditions of the equilibrium states. We plot the curves for binary alone collisions (called b), plus pseudotriple collisions ($b + p$) and spectatorless ternary collisions ($b + t$). Our goal is to verify whether or not the predictions given in section3 are satisfied. We give the arbitrary ζ, m_{01} values, choose $\sigma_{b1} = 1$, report the other cross sections values and the ratios of the two thickness $w^{(3)}/w^{(2)}$. We recall that for b and $b + p$ the σ_{bi} are the same.

figs.	ζ	m_{01}	coll.	σ_{b2}	σ_{b3}	coll	σ_{p1}	σ_{p2}	σ_{p3}	$w^{(3)}/w^{(2)}$
3a	.9	0.73	b	3.5	2.97	$b+p$	0.35	1.21	1.03	0.55
3b	.5	2.	b	2	1	$b+p$	3/16	3/8	3/16	$3\sqrt{3}/16 = .3$
3c	.046	12.	b	1.17	2.9	$b+p$	2.10^{-3}	2.10^{-3}	7.10^{-3}	0.45
3d	.946	0.6	b	3.83	2.64					
3e	.046	12.	b	4.10^{-3}	0					

figs.	coll.	σ_{b2}	σ_{b3}	σ_{t1}	σ_{t2}	σ_{t3}	σ_{t4}	σ_{t5}	$w^{(3)}/w^{(2)}$
3a	$b+t$	44.	137.	310	0.005	104.	137.	10^{-7}	.015
3d	$b+t$	86.	28.	165.	0.035	32.8	28.	10^{-5}	.061

Firstly, in agreement with section3 theoretical results, the existence or nonexistence of P/M overshoots is independent of the inclusion of multiple collisions. We only observe thinner shock profiles for multiple collisions. Secondly the theoretical predictions of overshoots for ζ not too far from the singularities 1 in figs.3a-d and 0 in fig.3c is verified. The nonexistence of overshoots for ζ far from these singularities is also verified in fig.3b with $\zeta = 1/2$. The difference between the $9v_i$ and the $8v_i$ models is the overshoot in fig.3c which does not exist in fig.3e for the $8v_i$ model (without rest particles), for the same ζ, m_{01} values.

5: Discussions

The first motivation of this paper was to include all possible ternary and quaternary collisions to the $9v_i$ model which, in contrast to some other models, is physical (without spurious conservation laws) at the binary level. It is not surprising that all multiple collision terms contain linearly the binary ones. Consequently, if we except thinner shock profiles, the behaviours of the macroscopic quantities seem very similar.

The second motivation was to try to understand more deeply the existence of physical P/M (internal energy) overshoots for the "thermal"[7] $9v_i$ model[8] and compare with the "athermal"[7] $8v_i$ model[15]. It is interesting that the signature of the effects can be detected at the R-H relations level with the sole knowledge of the equilibrium states. For the "thermal" model, the existence of the rest particle modifies completely the P/M contour-maps for $\zeta \simeq 0$. We observe that the influence of the nonzero speed particles

(here ± 1), on these contour maps, is deeper than the influence of the rest particle. The reader can easily check for $\zeta = 1/2$, $9v_i$ that the effect does not always exist. For the ζ values there exist two types of singularities, either "real" for velocities present on the x-axis ± 1 and 0 if rest particles are present, or "artificial", $\zeta = 0$, coming from projections of velocities. *For "real" singularities, we have found domains with $STr > 1$ while for the "artificial" $\zeta = 0$ singularity we have observed, up to now, monotonic P/M. Are these results particular to these models or more general?.* We report preliminary results for other models: (i) Infinite shock for the square $4v_i$ model and the $6v_i$ Broadwell model with for $\zeta \geq 0$ a "real" singularity at $\zeta = 1$ and an "artificial" one at 0. For these $d = 2,3$ models we observe $STr > 1$ for respectively $\zeta > 1/2$, $\zeta > 1/3$ with $m_0/m_{00} \geq 3, \geq 4$ and monotonic behaviour for $\zeta < 1/2, 1/3$. (ii) For the square $8v_i, 9v_i$ models with the x-axis along the diagonal and $\zeta \simeq 0$, we still observe P/M overshoots only if the rest particle is present. (iii) Generalizations of the $8v_i, 9v_i$ DBMs exist[12] in higher R^p spaces. In R^3 there exist the $18v_i, 19v_i$ DBMs[8] for which P/M effects were observed[12] but it remains to study $\zeta \simeq 0$. (iv) Adding an internal energy to the rest particle[16], there exist other "thermal" and "athermal" DBMs.

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Acknowledgment

AppendixA: Criterion for the elementary collision terms: $9v_i, 8v_i, 15v_i, 14v_i$

For any elementary collision term $Q = G - L$, with G, L products of the densities, we associate L/G. For B_1, B_2, B_{3i} we get $L/G = N_2 M_1 / N_1 M_2$, $M_3^2 / M_1 M_2$, $R N_i / M_3 M_i$.

Lemma1: L/G for B_{3i} can be obtained from L/G of $B_1, B_{3j}, i \neq j$. For B_{3i} we get $R N_i / M_3 M_i = (R N_j / M_j M_3)(M_j N_i / N_j M_i)$. Consequently the equilibrium states for binary collisions can be determined from B_1, B_2 and one B_{3i}.

Lemma2: For any Q term of Tables1-2, L/G is the product of L/G for B_i and one B_{3i}. The result is obvious for the Q which are binary terms multiplied by a density dependent factor. It remains T_{2i}: $(N_j M_i / N_i M_j)$ $(M_i M_j / M_3^2)$, T_{5i}: $(N_j R / M_j M_3)$ $(M_i M_j / M_3)$, $T_{5,2+i}$: $(R N_j / M_j M_3)$ $(M_3^2 / M_i M_j)$, Q_7: $(R N_1 / M_3 M_1)$ $(R N_2 / M_3 M_2)$, $Q_{12,2+i}$: $(R N_j / M_3 M_j)$ $(M_i N_j / M_j N_i)$ $(M_3^2 / M_i M_j)$.
Consequently, due to $R^2 N_2^2 / M_1 M_2^3 = (R N_2 / M_2 M_3)^2 (M_3^2 / M_1 M_2)$ we get:

Lemma3: Let a, b, c be real numbers, then for any Q of Tables1-2, L/G is of the type:

$$L/G = (N_1 M_2 / N_2 M_1)^a (M_3^2 / M_1 M_2)^{b/2} (R^2 N_2^2 / M_1 M_2^3)^{c/4} = N_1^a M_3^b R^{c/2} ... \qquad (A.1)$$

and is characterized by the powers $a, b, c/2$ of N_1, M_3, R.
From the H-functional $H = \sum_{i=1}^{2} (2 d_* N_i \log N_i + M_i \log M_i) + 2 d_* M_3 \log M_3 + R \log R$ we get:

$$\partial_t H + \partial_x(...) = \sum_{i=1}^{2} (2 d_* Q_{N_i} \log N_i + Q_{M_i} \log M_i) + 2 d_* Q_{M_3} \log M_3 + Q_R \log R \qquad (A.2)$$

Taking into account the linear (1.1) relations we rewrite the rsh:

$$2 d_* Q_{N1} \log(N_1 M_2 / N_2 M_1) + d_* Q_{M3} \log(M_3^2 / M_1 M_2) + Q_R / 2 \log(R^2 N_2^2 / M_1 M_2^3) \qquad (A.3)$$

which is a sum of terms corresponding to all possible elementary Q terms. For the binary collisions Q_{N1b}, Q_{M3b}, Q_{Rb} give a negative contribution and a condition for the H-Theorem is that all Q multiple collision terms give also negative values.

Theorem 1: We define for $Q = G - L$: $G = R^\gamma \prod_i N_i^{\alpha_i} M_i^{\beta_i}$, $L = R^{\overline{\gamma}} \prod_i N_i^{\overline{\alpha}_i} M_i^{\overline{\beta}_i}$, $R_t = ... + \nu Q$, $p_+ N_1 = ... + \lambda Q$, $M_{3t} = ... + \mu Q$. The contribution of Q to (A.3) being $Q \log[(N_1 M_2 / N_2 M_1)^\lambda \quad (M_3 / \sqrt{M_1 M_2})^\mu \quad (R N_2 / M_2^3 \sqrt{M_1})^{\nu/2 d_*}]$ is, from (A.1), negative if

$$\lambda / (\overline{\alpha}_1 - \alpha) = \mu / (\overline{\beta}_3 - \beta_3) = \nu / 2 d_* (\overline{\gamma} - \gamma) \qquad (A.4)$$

Due to $d = 2 \to 3$, $\nu \to d_* \nu$, λ, μ unchanged, we check (A.4) for $d = 2$.

Lemma4: In Tables1-2 all Q collision terms, give a negative contribution to (A.3).
(i) The Q terms, proportional to a binary one must have positive factors. T_1, Q_1, Q_{3i} in Q_{N1} are proportional to $-B_1$ with factors $M_3, M_2 N_1 + M_1 N_2, N_i M_3$. $-T_{3i}, -Q_2, -Q_4$, in Q_{M3} are proportional to B_2 with factors $N_i, N_1 N_2, M_3^2 + M_1^2 M_2^2$. For B_{31} present in Q_{N1}, B_{3i} in Q_R and $-B_{3i}$ in Q_{M3}, the Q terms: T_{41} for N_1 and T_{41}, Q_{8i}, Q_{13i} for M_3, R have positive factors. (ii) The Q terms with two binary collision terms must satisfy (A.4).

$$
\begin{vmatrix}
Q & \lambda & \overline{\alpha}_1 - \alpha & \mu & \overline{\beta}_3 - \beta_3 & \nu/2 & \overline{\gamma} - \gamma \\
T_{2i}, i = 1,2 & (-1)^i & (-1)^i & -2 & -2 & 0 & 0 \\
T_{5i}, i = 1,2 & i-1 & i-1 & -3 & -3 & 1 & 1 \\
T_{5,2+i}, i = 1,2 & i-1 & i-1 & 1 & 1 & 1 & 1 \\
Q_7 & 1 & 1 & -2 & -2 & 2 & 2 \\
Q_{12,2+i} & i(-1)^i & i(-1)^i & 1 & 1 & 1 & 1
\end{vmatrix} \qquad (A.5)
$$

For the $8v_i, 14v_i$ models we study $T_1, T_2, T_3, Q_1, ..Q_6$ and the two first (A.1-4)terms.

AppendixB: R-H solutions and Predictions for the P/M overshoots

B.1: Semi-infinite Mach shock R-H relations for the $9v_i, 15v_i$ models

We solve the (3.2) relations with the conditions:$s_1 = p_1 = p_3 = r_{00} = 0$, $p_2 \neq 0, s_2 \neq 0$. Firstly the (i) densities states are deduced from the n_{02}, m_{01} parameters:

$$m_{02} = m_{01} n_{02}, \quad m_{03} = m_{01}\sqrt{n_{02}}, \quad r_0 = m_{01}^2\sqrt{n_{02}} \qquad (B.1)$$

Secondly from two linear (1,1) relations for $r = -r_0 = 2d_*((1+\zeta)n_2 + (1-\zeta))/\zeta$ and $-m_3 = m_{03} = [(1-\zeta)(m_{01}/d_* + 3) + (1+\zeta)n_2]/\zeta$ and (B.1) we get both n_{02}, n_2, s_2:

$$\sqrt{n_{02}} = 2(1-\zeta)/\zeta m_{01}, n_2(1+\zeta) = (\zeta-1)(1+m_{01}/d_*), (1+\zeta)s_2 = (1-\zeta)[4(\zeta^{-2}-1)/m_{01}^2 -1 - m_{01}/d_*] \qquad (B.2)$$

Thirdly, from the last linear (1.1) relation we get m_2 and p_2:

$$m_2(1+\zeta) = (1-\zeta)(3m_{01} + 4d_*), p_2(1+\zeta) = (1-\zeta)\ [4(\zeta^{-2}-1)/m_{01} + 3m_{01} + 4d_*] \ (B.3)$$

Lemma5: Positive (i) and (ii) states can be determined from the two parameters m_{01}, ζ satisfying the constraints:

$$m_{01} > 0, \qquad 0 < \zeta < \zeta_{sup} = 2/\sqrt{4 + m_{01}^2(m_{01}/d_* + 1)} \qquad (B.4)$$

From (B.2-1) both $n_{02} > 0$ and all densities of the (i) state are positive. From (B.2-3-4) $n_2 < 0$ but $s_2 > 0, m_2 > 0, p_2 > 0$. All densities of the (ii) states are nonnegative.

B2: Infinite shock for the $9v_i, 15v_i$ models

Lemma6: Positive (i) and (ii) states are determined from the arbitrary $0 < \zeta < 1$ and $m_{01} > 0$ solution of the cubic equation ($s_2 = 0$ or $\zeta = \zeta_{sup}$ in (B.4)):

$$m_{01}^3/d_* + m_{01}^2 + 4(1 - \zeta^{-2}) = 0, \rightarrow p_2(1+\zeta)d_* = (1-\zeta)(m_{01} + 2d_*)^2 > 0 \qquad (B.5)$$

In particular for the $9v_i$ model with $\zeta = 1/2$: $m_{0j} = 2$, $n_{0i} = 1$, $r_0 = 4$, $p_2 = 16/3$.

B3: Semi-infinite and Infinite shock for the $8v_i, 14v_i$ models

For the semi-infinite shock R-H relations we put $r_0 = r = r_{00} = 0$ and get successively with the same m_{01}, ζ arbitrary parameters and in particular the same $-m_3 = m_{03}$ relation:

$$n_2(1+\zeta) = \zeta - 1, \zeta\sqrt{n_{02}} = (1-\zeta)(2/m_{01} + d_*^{-1}), \quad m_2 = (1-\zeta)\ (m_{01} + 4d_*)\ /(1+\zeta) > 0, \ p_2 > 0, \quad s_2 = (1-\zeta)[(\zeta^{-2} - 1)\ (1/d_* + 2/m_{01})^2 - 1]/(1+\zeta) > 0 \text{ if } \quad 0 < \zeta < \zeta_{sup} = 1/\sqrt{1 + (2/m_{01} + 1/d_*)^{-2}} \qquad (B.6)$$

For the infinite shock, with only one parameter $p_2 \neq 0$, we have: $\zeta = \zeta_{sup}$ with $1/\sqrt{1 + d_*^2} < \zeta < 1, n_{02} = (1-\zeta)/(1+\zeta), m_{01} = 2\sqrt{1 - \zeta^2}/[\zeta - \sqrt{1 - \zeta^2}/d_*] \qquad (B.7)$

B4: Macroscopic values for the equilibrium states

We write the M, J, E (i) and (ii) values for the semi-infinite solutions:

$(i): m_0 = \sum_{i=1}^2 (m_{0i} + 2d_* n_{0i}) + 2d_* m_{03} + r_0, 2e_0 = \sum_{i=1}^2 (m_{0i} + 4d_* n_{0i}) + 2d_* m_{03}$

$j_0 = m_{01} - m_{02} + 2d_*(n_{01} - n_{02}), \quad (ii): m_{00} = p_2 + 2d_* s_2 = -j_{00}, 2e_{00} = p_2 + 4d_* s_2 \ (B.8)$

In the infinite case we put $s_2 = 0$. For the $9v_i$ model with $\zeta = 1/2, m_{01} = 2$ we get:

$m_0 = 16, m_{00} = 16/3, j_0 = 0, j_{00} = -16/3, e_0 = 8, e_{00} = 8/3, \Lambda = -\Omega = 16^3/9, \Lambda + \Omega = 0$ and deduce for the shock profiles (3.4) that $P/M = 1 - [(D-1)/(D+2)]^2$ is monotonic. Finally we notice that the "shock profiles" solutions $M = m_{00} - mD^{-1}(\eta)$ are equivalent to $M = m_0 + m\Delta^{-1}(\eta)$ with the change $D\Delta = D + \Delta$.

B.5: Mass Ratio across the shock for the Infinite shock

For the $9v_i, 15v_i$ models, requiring $m_0/m_{00} = d + 1$ we find in addition to (B.5), $m_{01} = 2d_*[-1 + 1/d\zeta(1-\zeta)]$ giving an exact value $\zeta = 1/2, m_{01} = 2$ for the d=2 model. For the $8v_i, 14v_i$ models we find from (B.7): $m_0/m_{00} = 1 + 1/\zeta(1-\zeta) > 3 + 2\sqrt{2} > 3$ for the d=2 model and $> (9 + 5\sqrt{5})/4 > 4$ for the d=3 model.

AppendixC: Similarity Solutions for the $9v_i, 15v_i$ models

C1: General Relations coming from (2.1)

To (2.1) we add the pseudotriple collisions $\sigma_{bi} \to \sigma_{bi} + \sigma_{pi}(m_{00} - mD^{-q})$ and get relations from the coefficients of D^{-q-1}, D^{-2q} which are different only for $q = 1/2$.

$p_+N_1 = Q_{N1} \to q = 1/2, 1: \quad q\gamma(1 - \zeta) - a_{01}(\sigma_{b1} + m_{00}\sigma_{p1}) = \sigma_{t23}(a_{01}p_2 + a_{02}s_2) \quad (C.1)$

$q = 1/2: a_{01}[s_{b1} + (m_{00} + m)\sigma_{p1} + m_3\sigma_{t1}] + \sigma_{t23}[a_{01}(m_1 + m_2 + p_2) + a_{02}(s_2 + n_2 - n_1)] + a_{02}\sigma_{t5}(m_3 - m_1) = 0 \quad (C.2)$

$R_t = Q_R \to q = 1/2, 1: \quad -qr\zeta\gamma/2d_* = a_{02}\sigma_{t5}p_2 + a_{03}(\sigma_{b3} + m_{00}\sigma_{p3}) \quad (C.3)$

$q = 1/2: \sigma_{t4} = \sigma_{b3} + (m_{00} + m)\sigma_{p3} + \sigma_{t5}[m_1 + m_3 + a_{02}(p_2 + m_1 + m_2 - 2m_3)/a_{03}] \ (C.4)$

$M_{3t} = Q_{M3} \to q = 1/2, 1: \quad m_3 q\zeta\gamma + a_{02}[\sigma_{b2} + \sigma_{p2}m_{00} + 2(\sigma_{t23} + \sigma_{t3})s_2 + \sigma_{t5}p_2] + 2a_{01}p_2\sigma_{t23} = a_{03}(\sigma_{b3} + m_{00}\sigma_{p3}) \quad (C.5)$

$q = 1/2, a_{02}[\sigma_{b2} + \sigma_{p2}(m_{00} + m) + 2(\sigma_{t23} + \sigma_{t3})(s_2 + n_1 + n_2) + \sigma_{t5}(p_2 + m_1 + m_2 + 6m_3)] + 2a_{01}\sigma_{t23}(m_2 - m_1) = a_{03}[\sigma_{b3} + (m_{00} + m)\sigma_{p3} - \sigma_{t4} + \sigma_{t5}(3m_1 - m_3)] \quad (C.6)$

We notice that $a_{01} = p_2 - s_2m_{01} = m_2 - m_{01}n_2 > 0$ (Theorem 2) and also $a_{0j} > 0, j = 2, 3$.

C.2: Binary collisions alone $q = 1, p = 2$

Lemma7: For binary collisions we choose $\sigma_{b1} = 1$ and get $\sigma_{bi} > 0, \gamma^{(2)} > 0$.

We have three relations (C.1-3-5), four parameters, get $\gamma^{(2)} = a_{01}/(1 - \zeta) > 0$ and

$\sigma_{b2} = a_{01}\zeta[r_0/2d_* + m_{03}]/a_{02}(1 - \zeta) > 0, \quad \sigma_{b3} = r_0\zeta/2d_*(1 - \zeta)m_{03} > 0 \quad (C.7)$

C.3: Binary and Pseudotriple collisions $q = 1/2, p = 3$

In the six first (C.i) relations we put $\sigma_{ti} = 0$ and it remains seven parameters.

Lemma8: All the $\sigma_{bi}, \sigma_{pi}, \gamma$ parameters are determined from $\sigma_{b1} = 1$ and they are positive if $m_{00} + m < 0$. We successively get:

σ_{bi} as in (C.7), $\sigma_{pi} = -\sigma_{bi}/(m_{00} + m), \quad \gamma^{(3)} = \gamma^{(2)}2m/(m_{00} + m) \quad (C.8)$

For the infinite shock solutions we get both $m(1+\zeta) = -2(m_{01}+2d_*)\zeta < 0$ and $m_{00}+m = (m_{01}+2d_*)[m_{01}/d_*+2-2(1+\zeta^2)/\zeta]/(1+\zeta) < 0$ and in particular for $\zeta = 1/2$: $\gamma^{(2)} = 32/3$, $\gamma^{(3)} = 4\gamma^{(2)}, \sigma_{pi} = \sigma_{bi}3/16$.

C.4: Binary and Spectatorless Ternary collisions $q = 1/2, p = 3$

In the six first (C.i) relations we put $\sigma_{pi} = 0$ and it remains nine parameters.

Lemma9: With $\sigma_{b1} = 1$ and σ_{b3}, σ_{t5} arbitrary we determine all parameters. From (C.4-3-1-2) we successively get σ_{t4}, $\gamma^{(3)}$, σ_{t23}, σ_{t1} and finally σ_{b2}, σ_{t3} from (C.5-6).

C.5: Binary and Pseudotriple collisions for the $8v_i, 14v_i$ models

We put $R = \sigma_{b3} = \sigma_{p3} = a_{03} = 0$, $\sigma_{b1} = 1$ and obtain from the (C.i) relations the same $\gamma^{(2)}, \gamma^{(3)}$, σ_{pi} as in C.2-3 with the only change:$\sigma_{b2} = m_{03}a_{01}\zeta/ a_{02}(1 - \zeta)$. We still have $\sigma_{bi} > 0, \gamma^{(2)} > 0$ and still $\sigma_{pi} > 0, i = 1, 2$ if $m_{00} + m < 0$.

C.6: Mass M and shock thickness $w^{(p)}$

For DBMs with quadratic $q = 1, p = 2$ or cubic $q = 1/2, p = 3$ nonlinearities, we write down both the mass $M = m_{00} - mD^{-q}, D = 1 + exp(\gamma^{(p)}\eta)$ and

$$w^{(p)} = |m|/max|dM/d\eta| = (1 + q^{-1})^{q+1}/|\gamma^{(p)}|, \ w^{(2)} = 4/|\gamma^{(2)}|, \ w^{(3)} = 3\sqrt{3}/|\gamma^{(3)}| \quad (C.9)$$

For instance for the pseudotriple collisions we get: $w^{(3)}/w^{(2)} = 3\sqrt{3}[(m_{00} + m)/2m]/4$.