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Abstract

We show that longwavelength excitations of the quark-gluon plasma are described by simple kinetic equations which represent the exact equations of motion at leading order in g. Properties of the so-called "hard thermal loops", i.e. the dominant contributions to amplitudes with soft external lines, find in this approach a natural explanation. In particular, their generating functional appears here as the effective action describing long wavelength excitations of the plasma.

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Significant progress has been achieved recently in understanding long wavelength excitations of a quark-gluon plasma[1, 2, 3]. In equilibrum, at high temperature, such a plasma may be viewed as a gas of weakly interacting, massless quarks and gluons. When coupled to weak and slowly varying perturbations, this system may acquire a collective behaviour on a length scale $\sim 1/gT$, where T is the temperature and g the coupling constant, assumed to be small. In this letter, we present a consistent and physically intuitive description of such long wavelength phenomena, based on a set of coupled mean field and kinetic equations, to which the exact equations of motion reduce in leading order in g[4, 5].

The kinetic equations encompass all the so-called "hard thermal loops" and clarify the nature of their remarkable properties, left largely unexplained by their original derivation in terms of Feynman diagrams[6, 7, 2]. These hard thermal loops (HTL) are the dominant corrections, at high temperature, to amplitudes involving soft external lines. (Following the usual terminology, we call an energy or a momentum "soft" when it is of order gT, and "hard" when it is of order T; at equilibrum, most particles are hard.) As they are of the same order of magnitude as the corresponding non-vanishing tree level amplitudes, HTL need to be resummed consistently in higher order calculations[1, 2].

Our equations isolate consistently the dominant terms in g in the hierarchy of equations which describe the response of the plasma to weak and slowly varying disturbances, i.e. varying on a scale of order $1/gT$. In doing so, we treat bosons and fermions on the same footing and introduce an average fermionic field $\psi(X)$ in parallel with the average gauge field $A_\mu(X)$. A noteworthy feature of the present problem is that g, besides measuring the interaction strength, controls the wavelength of the soft space-time variations and, for consistency, also the strength of the mean fields. One finds for example that, in order for the deviations away from equilibrium to stay small, the gauge field strength tensor should be at most of order gT^2; then $gA \sim gT$ is of the same order as the derivative of a "slowly varying" quantity[4, 5].

The dominant interactions which determine the response of the plasma are those which take place between the hard particles and the soft mean fields. In leading order, we can neglect the direct interaction between the hard particles, which allows us to truncate the equations at the level of the 2-point functions. The motion of a given hard particle is only slightly perturbed by its interaction with a soft mean field. However, because the mean fields vary over distances much larger than the interparticle distance ($\sim 1/T$), they affect coherently many hard particles, giving rise to collective "polarization" phenomena. These show up as "induced sources" which add to the external ones in determining the properties of the mean fields.

The equations for the quark and gauge average fields are

$$i\not{D}_X\psi(X) = \eta(X) + \eta^{ind}(X), \tag{1}$$

and

$$[\, D^\nu_X,\, F_{\nu\mu}(X)\,]^a - g\bar\psi(X)\gamma_\mu t^a\psi(X) = \, j^a_\mu(X) + j^{ind\,a}_\mu(X). \tag{2}$$

Here η and j^a_μ are external sources, $a = 1, \ldots, N^2-1$ are color indices for the adjoint representation of the $SU(N)$ gauge group, while $\mu,\, \nu = 0, \ldots, d-1$ are space-time indices ($d = 4$ throughout this work). The covariant derivative is $D_\mu \equiv \partial_\mu + igt^aA^a_\mu$ and $F_{\mu\nu} \equiv [D_\mu, D_\nu]/(ig)$. When using a covariant gauge, one should also consider equations for ghost mean fields. These are not written here as it turns out that they are trivial, i.e there are no induced sources for the ghost mean fields[5]. After functional differentiation, the induced sources yield the one particle irreducible amplitudes with soft external lines. Thus for example the fermion self-energy is given by $\Sigma(x,y) = \delta\eta^{ind}(x)/\delta\psi(y)$, the gluon polarization tensor by $\Pi^{ab}_{\mu\nu}(x,y) = \delta j^{ind\,a}_\mu(x)/\delta A^\nu_b(y)$, etc...

In leading order, the induced sources can be expressed entirely in terms of 2-point functions. For example, $\eta^{ind}(x) = g\gamma^\nu t_a\langle A^a_\nu(x)\psi(x)\rangle_c$, where the subscript c indicates a connected expectation value which, in leading order, involves only the hard particles. The abnormal quark-gluon propagator which enters η^{ind} is nonvanishing only in the presence of the fermionic mean field ψ. It vanishes in equilibrum, as does the induced color current j^{ind}. This current receives contributions from both fermionic and bosonic particles, and accordingly may be written as $j^{ind} = j_{\rm f} + j_{\rm b}$. The quark contribution is $j^\mu_{{\rm f}\,a} = g\langle\bar\psi(x)\gamma^\mu t_a\psi(x)\rangle_c$. In an arbitrary covariant gauge, the bosonic piece involves contributions from both hard gluons and hard ghosts [5]. The polarization phenomena may be induced either by a gauge field A, or by a fermionic one, ψ. Correspondingly, we set $j_{\rm f} = j^A_{\rm f} + j^\psi_{\rm f}$ and $j_{\rm b} = j^A_{\rm b} + j^\psi_{\rm b}$. Here, for instance, $j^A_{\rm f}$ is the color current associated to the collective motion of hard fermions induced by the soft mean field A. It is independent of the fermionic field ψ. In contrast, we shall see that j^ψ has generally a dependence on A imposed by gauge covariance. Note also that the ghosts do not contribute to $j^\psi_{\rm b}$, as they have no direct interaction with the fermionic mean field.

In order to implement the condition that the average fields are slowly varying, it is convenient to use the Wigner transform of the 2-point functions, such as

$$\mathcal{S}(k,X) \equiv \int d^4s e^{ik\cdot s}\langle {\rm T}\psi(X+\frac{s}{2})\bar\psi(X-\frac{s}{2})\rangle_c, \tag{3}$$

for the quark propagator. Note that, in contrast to other authors, we do not insist on defining manifestly gauge covariant Wigner functions [8]. Covariance will be recovered

later, when calculating physical quantities. The induced sources are then integrals over k of Wigner functions, which we refer to as k-space densities. For example:

$$\eta^{ind}(X) = g \int \frac{d^4k}{(2\pi)^4} t^a \gamma^\nu \mathcal{K}^a_\nu(k,X), \tag{4}$$

where $\mathcal{K}^a_\nu(k,X)$ is the Wigner transform of the quark-gluon propagator $K^a_\nu(x,y) \equiv \langle \mathrm{T}\psi(x)A^a_\nu(y)\rangle_c$. Thus $\not{\mathcal{K}}(k,X) \equiv t^a \not{\mathcal{K}}^a(k,X) \equiv t^a\gamma^\nu \mathcal{K}^a_\nu(k,X)$ is the k-space density for $\eta^{ind}(X)$. The densities for the induced currents are denoted by $\mathcal{J}^A_{\mathrm{f,b}}(k,X)$ and $\mathcal{J}^\psi_{\mathrm{f,b}}(k,X)$. For example, the fermionic density is $\mathcal{J}^{\ a}_{\mathrm{f}\,\mu}(k,X) = \mathrm{Tr}\gamma_\mu t^a \mathcal{S}(k,X)$, where the trace refers to both spin and color indices.

At leading order in the coupling g, the equations for the densities of the induced sources read[5]

$$(k\cdot D_X)\,\not{\mathcal{K}}(k,X) = -i\frac{g}{2}(d-2)C_f(\Delta(k)+\tilde{\Delta}(k))\not{k}\psi(X), \tag{5}$$

$$\Big[k\cdot D_X,\, \mathcal{J}^A_{\mathrm{f}\,\mu}(k,X)\Big]^a = 2g\, N_f\, k_\mu\, k^\rho F^a_{\rho\nu}\partial^\nu\tilde{\Delta}(k), \tag{6}$$

$$\begin{aligned}\Big[k\cdot D_X,\, \mathcal{J}^\psi_{\mathrm{f}\,\mu}(k,X)\Big]^a &= ig\,k_\mu\left\{\bar\psi(X)\,t^a\,\not{\mathcal{K}}(k,X) - \not{\mathcal{H}}(k,X)\,t^a\,\psi(X)\right\}\\ &- g\,k_\mu\, f^{abc}\left\{\bar\psi(X)\,t^b\,\not{\mathcal{K}}^c(k,X) - \not{\mathcal{H}}^b(k,X)\,t^c\,\psi(X)\right\},\end{aligned} \tag{7}$$

$$\Big[k\cdot D_X,\, \mathcal{J}^\psi_{\mathrm{b}\,\mu}(k,X)\Big]^a = g\,k_\mu\, f^{abc}\left\{\bar\psi(X)\,t^b\,\not{\mathcal{K}}^c(k,X) - \not{\mathcal{H}}^b(k,X)\,t^c\,\psi(X)\right\}, \tag{8}$$

$$\Big[k\cdot D_X,\, \mathcal{J}^A_{\mathrm{b}\,\mu}(k,X)\Big]^a = g\,N(d-2)\,k_\mu\,k^\rho F^a_{\rho\nu}\partial^\nu\Delta(k). \tag{9}$$

In these equations, $\not{\mathcal{H}}(k,X) = \not{\mathcal{K}}^\dagger(k,X)\gamma^0$, $C_f \equiv (N^2-1)/2N$ is the quark Casimir, N_f is the number of quark flavors and f^{abc} denote the structure constants of $SU(N)$. Furthermore, $\Delta(k) \equiv \rho_0(k)N(k_0)$ and $\tilde{\Delta}(k) \equiv \rho_0(k)n(k_0)$, where $\rho_0(k) = 2\pi\epsilon(k_0)\delta(k^2)$ is the spectral function for free massless particles, and $N(k_0)$ and $n(k_0)$ denote respectively boson and fermion occupation factors. The factor $(d-2)$ in Eqs. (5) and (9) reflects the fact that only the transverse gluons effectively contribute to the densities.

Eqs. (5-9) have a number of interesting properties: **i)** They are independent of the gauge fixing parameter λ which enters calculations in general covariant gauges [4, 5]. **ii)** In their right hand sides, all possible vacuum contributions cancel. **iii)** They transform covariantly under a local gauge transformation of the mean fields A_μ, ψ and $\bar\psi$. The densities $\mathcal{J}^\psi_{\mathrm{f}}$, $\mathcal{J}^\psi_{\mathrm{b}}$ and $\not{\mathcal{K}}$ involve Wigner transforms which are gauge covariant in leading

order. The current induced by a gauge field involves k-space densities, $\mathcal{J}_{\rm f}^A$ and $\mathcal{J}_{\rm b}^A$, which are derived from non covariant Wigner functions. However, these densities are defined up to a total derivative with respect to k which does not contribute to the integrated current. We have used this freedom in order to make the densities explicitly covariant [4, 5]. **iv)** The symmetry between Eqs. (6) and (9) reflects the fact that hard quarks and gluons respond similarly to a soft gauge field. The same symmetry is apparent in Eq. (5) expressing the effect of the fermionic mean field on the hard particles. **v)** Note finally the presence of the factor $\delta(k^2)$ in the r.h.s of these equations. This reflects the elementary dynamics of the hard particles: they remain on their unperturbed mass shell and only undergo essentially forward scattering on the mean fields.

As these remarks strongly suggest, the motion of the hard particles described by Eqs. (5-9) exhibits many features of classical dynamics. This becomes more transparent if one makes explicit the structure of the various densities implied by these equations. For instance, Eq. (6) implies

$$\mathcal{J}^A_{{\rm f}\,\mu}(k,X) = 2\,k_\mu\,N_f\,t^a(2\pi\delta(k^2))\left[\theta(k^0)\delta n^a_+(\vec{k},X) + \theta(-k^0)\delta n^a_-(-\vec{k},X)\right], \tag{10}$$

where $\delta n_\pm = \delta n^a_\pm t^a$ are fluctuations in the quark color densities induced by the gauge field. These fluctuations satisfy (with $\epsilon_k \equiv |\vec{k}|$)

$$\left[v\cdot D_X, \delta n_\pm(\vec{k},X)\right]^a = \mp g\vec{v}\cdot\vec{E}^a(X)\frac{dn(\epsilon_k)}{d\epsilon_k}. \tag{11}$$

In the abelian case, this equation coincides with the linearized Vlasov equation. Here, the color electric field not only modifies the motion of the particle, but also induces a "precession" of the densities in color space. The other equations may be given similar interpretation. Thus, Eq. (5) for the quark-gluon Wigner function $\not{\mathcal{K}}$ describes fluctuations where, under the action of a soft fermionic mean field, quarks are converted into gluons and vice-versa.

The total current induced by a gauge field A is $j^A = j^A_{\rm f} + j^A_{\rm b}$. Its k-space density, $\mathcal{J}^A \equiv \mathcal{J}^A_{\rm f} + \mathcal{J}^A_{\rm b}$, satisfies

$$\left[k\cdot D_X,\, \mathcal{J}^A_\mu(k,X)\right] = g\;k_\mu\,k\cdot F(X)\cdot\partial\left(2N_f\tilde{\Delta}(k) + N(d-2)\Delta(k)\right). \tag{12}$$

This equation generalizes the Vlasov equation to nonabelian plasmas. Previous attempts to derive such an equation led to more intricate results. However, they were based on different approximation schemes which mix leading and non leading contributions in g and, as such, are not entirely consistent[8]. One can also combine Eqs. (7) and (8) into a single equation for the total current density induced by the fermionic fields, $\mathcal{J}^\psi \equiv \mathcal{J}^\psi_{\rm f} + \mathcal{J}^\psi_{\rm b}$,

$$\left[k\cdot D_X,\, \mathcal{J}^\psi_\mu(k,X)\right] = ig\,k_\mu\,t^a\,\left\{\bar{\psi}(X)\,t^a\,\not{\mathcal{K}}(k,X) - \not{\mathcal{H}}(k,X)\,t^a\,\psi(X)\right\}. \tag{13}$$

This equation is similar to the corresponding one in the abelian case [4]. In doing the sum of Eqs. (7) and (8), the typical non-abelian effects cancel; these are contained for example in the second braces in Eq. (7), and involve the 3-gluon vertex leading to gauge field insertions on the hard gluon lines. This kind of cancellation was first noted by Taylor and Wong [9] in relation with the HTL's for amplitudes involving one pair of quarks and any number of soft gluons (albeit their proof is only explicit up to three external gluons).

We have thus reduced the set of equations (5-9) to three fundamental equations, namely Eqs. (5), (12) and (13) for the densities of the induced sources. From the gauge-covariant character of these equations, it follows that, under local gauge transformations, η^{ind} transforms like ψ, while j^{ind} transforms like $F_{\mu\nu}$. Provided the external sources are chosen so as to satisfy the same property, the mean fields equations (1) and (2) are then gauge covariant, as are the classical equations of motion derived from the QCD action.

Eqs. (5,12,13) contain all the information on the generalized polarizability of the plasma. After solving Eq. (5), we compute the fermionic induced source according to Eq. (4) and obtain

$$\begin{aligned} \eta^{ind}(X) &= -i\omega_0^2 \int \frac{d\Omega}{4\pi} \not{v} \int_0^\infty du U(X, X-vu)\psi(X-vu) \\ &\equiv \int d^4Y \delta\Sigma_A(X,Y)\psi(Y), \end{aligned} \tag{14}$$

with $\omega_0^2 \equiv C_f(g^2T^2/8)$ and $v^\mu \equiv (1, \vec{v})$, $\vec{v} \equiv \vec{k}/\epsilon_k$. The angular integral runs over all directions of $\vec{v}$, and $U(x,y)$ is the parallel transporter along a straight line joining x and y[4]. The kernel $\delta\Sigma_A$ is the self energy of a soft fermion propagating in a background gauge field. The current induced by fermionic mean fields results from Eq. (13):

$$\begin{aligned} j_\mu^\psi(X) &= g\omega_0^2 t^a \int \frac{d\Omega}{4\pi} v_\mu v_\nu \int_0^\infty dt \int_0^\infty ds \\ &\qquad \bar{\psi}(X-vt)\gamma^\nu U(X-vt, X)\, t^a\, U(X, X-vs)\psi(X-vs) \\ &\equiv g\, t^a \int d^4Y_1 d^4Y_2 \bar{\psi}(Y_1)\delta\Gamma^A_{\mu,\,a}(X; Y_1, Y_2)\psi(Y_2). \end{aligned} \tag{15}$$

The correction $\delta\Gamma^A$ to the quark-gluon vertex may be easily read out from this equation. Finally, the current induced by soft gauge fields is determined from Eq. (12) to be

$$j_\mu^A(X) = 3\,\omega_p^2 \int \frac{d\Omega}{4\pi} v_\mu \int_0^\infty du\, U(X, X-vu)\, F_{0\,j}(X-vu) v^j\, U(X-vu, X). \tag{16}$$

Here $\omega_p^2 \equiv (g^2T^2/9)(N + N_f/2)$ is the plasma frequency. By successive functional differentiation with respect to A of Eq. (16) we derive corrections to the equilibrum amplitudes for soft gluon fields. All the amplitudes obtained in this way, as well as those contained in $\delta\Sigma_A$ and in $\delta\Gamma^A_\mu$, coincide with the HTL's of the diagrammatic approach.

Eqs. (14-16) imply the following covariant conservation laws for the induced currents[5]:

$$\left[D^\mu,\, j^A_\mu\right] = 0, \tag{17}$$

and

$$\left[D^\mu,\, j^\psi_\mu\right] = igt^a\left(\bar\psi t^a \eta^{ind} - \bar\eta^{ind} t^a \psi\right). \tag{18}$$

By differentiating these equations with respect to the fields, one obtains relations between HTL refered as "QED-like Ward identities" in Ref.[7, 2]. Finally, by using these relations, together with the Jacobi identity $[D^\mu,[D^\nu, F_{\nu\mu}]] = 0$, we see that the external sources must satisfy

$$[D^\mu,\, j_\mu] = igt^a\left(\bar\psi t^a \eta - \bar\eta t^a \psi\right). \tag{19}$$

in order for the mean fields equations (1,2) to be consistent.

By eliminating the induced sources from Eqs. (1,2), using their explicit expressions (14-16), one obtains nonlinear equations of motion which generalize the Maxwell equations in a polarizable medium. In particular, for vanishing external sources, these equations describe the normal modes of the plasma. Note that in general, as a consequence of gauge covariance, quark and gluon modes mix.

These non linear equations for the mean fields can be generated by the minimal action principle applied to an effective action $S_{eff} = S_0 + S_{ind}$. Here S_0 is the classical QCD action, while S_{ind} contains the effects of the interactions between the soft fields and the hard particles of the plasma. It follows that S_{ind} must satify $\delta S_{ind}/\delta\bar\psi(X) = \eta^{ind}(X)$ and $\delta S_{ind}/\delta A^\mu_a(X) = j^{ind\,a}_\mu(X)$. These conditions are satisfied by $S_{ind} = S_f + S_g$, with [5]

$$S_f \;=\; -i\omega_0^2 \int \frac{d\Omega}{4\pi}\int d^4X \int_0^\infty du \bar\psi(X) \not{v} U(X, X-vu)\psi(X-vu), \tag{20}$$

and

$$\begin{aligned} S_g \;=\; & \frac{3}{2}\,\omega_p^2 \int \frac{d\Omega}{4\pi}\int d^4X \int_0^\infty du \int_u^\infty du' \\ & \mathrm{tr}\left\{v^\mu F_{\mu\lambda}(X)\, U(X, X-vu')\, v_\nu F^{\nu\lambda}(X-vu')\, U(X-vu', X)\right\}, \end{aligned} \tag{21}$$

where the trace acts on color indices only. This gauge invariant action coincides with the generating functional for HTL's derived in [9] on the basis of gauge invariance. Here S_{eff} has a different, more physical, interpretation: it is the classical action describing long wavelength excitations in the hot quark-gluon plasma, at leading order in the coupling g.

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