# Collective fermionic excitations in systems with a large chemical potential

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**Abstract**

We study fermionic excitations in a cold ultrarelativistic plasma. We construct explicitly the quantum states associated with the two branches which develop in the spectrum as the chemical potential is raised. The collective nature of the long wavelength excitations is cleary exhibited.

# 1 Introduction

The spectrum of fermion excitations in a hot relativistic plasma composed of electrons, positrons and photons (or quarks, antiquarks and gluons) presents an interesting structure [1-8]. In particular, for small fermion masses $m$ and momenta $p$, i.e. $m, p \lesssim gT$ where $g$ is the coupling constant and $T$ the temperature, it is split. One of the two branches becomes the ordinary fermionic branch at large momentum or large mass. The other has features of a collective excitation, and for this reason has been sometimes dubbed the "plasmino" [9], by analogy with the plasmon oscillation. However, one should keep in mind that, when $p \lesssim gT$, both branches have a collective character and, as we shall see, the whole structure of the spectrum at small momentum reflects a collective behavior of the system. A recent calculation indicates, at least within the one-loop approximation, that this structure in the spectrum does not occur as an abrupt transition, but develops gradually as the temperature is raised [8].

Similar effects are expected to occur at zero temperature, but large chemical potential $\mu$, i.e. when $g\mu \gtrsim m, p$[6]. This is the situation we explore in this paper. There are many physics motivations for studying the spectrum of fermions in very dense matter, ranging from the physics of dense stars, to that of heavy ion collisions. These have stimulated recent works[10]. There are also current investigations on more formal questions related to QED with a chemical potential[11]. However, the main purpose of the present paper is simply to get a better physical understanding of the fermion spectrum, especially at low momentum where collective phenomena develop. We do this, in particular, by providing an explicit calculation of the quantum states corresponding to the fermionic excitations, which is possible at zero temperature.

Some time ago, Weldon presented a physical interpretation for the splitting of the spectrum at high temperature, based on an analysis of the quantum numbers of the modes[12], and an analogy with BCS theory of superconductivity[4] . However, that did not totally clarify the simple physics underlying the emergence of collective fermionic excitations in the high temperature plasma. In fact the analogy with BCS superconductivity is misleading. The formation of Cooper pairs involve particles localized in momentum space at the vicinity of the Fermi surface, and it produces an instability of the Fermi sea leading to a reorganization of the system as a whole. Here, as we shall see, the particles in the vicinity of the Fermi surface are only

weakly perturbed, and the Fermi sea is stable.

We begin in the next section by summarizing some properties of the fermion self-energy at finite temperature and chemical potential, to leading order in the coupling strength $g$. The spectrum of fermion excitations is presented, and some known results are recalled. We show in particular that, in the ultrarelativistic limit, most properties of the spectrum depend on $\mu$ and $T$ only through the combination $M^2 \propto g^2(\mu^2 + \pi^2 T^2)$. In section 3, we discuss some general properties of the one-loop fermion self-energy at zero temperature and finite chemical potential. Analytic results are obtained in two simple cases, $m = 0$ and $p = 0$. We discuss the approach to the ultrarelativistic limit and the emergence of a particle–antiparticle symmetry at large $\mu$. We show that, as it is the case at finite temperature, the structure in the spectrum of quasiparticles develops gradually as the chemical potential is raised, due to the coupling of the quasiparticles to low lying states involving hard electrons and photons. We argue that the imaginary part of the self energy may not be interpreted as a damping of the quasiparticles. The main results of this paper are contained in section 4 where we present a detailed calculation of the quantum states corresponding to the two branches in the excitation spectrum at small momentum. The collective nature of the two branches is clearly exhibited, and various features of the spectrum are analyzed. Our conclusions are summarized in the last section which contains also a discussion of various aspects of the present work, such as its relevance to the case of finite temperature, and also to QCD, the relation with kinetic theory developed in [13], and finally the relation with a mode analogous to the plasmino, the "plasmaron" found in some calculations of the non relativistic electron gas[14, 15, 16].

# 2 Fermionic modes in ultrarelativistic plasmas

In the presence of a chemical potential associated with the conserved charge $Q = \int \bar{\psi}\gamma_0\psi \mathrm{d}^3x$, the Dirac hamiltonian $\boldsymbol{\alpha}\cdot\mathbf{p} + m\gamma_0$ becomes $\boldsymbol{\alpha}\cdot\mathbf{p} + m\gamma_0 - \mu$ ( $\boldsymbol{\alpha} = \gamma_0\boldsymbol{\gamma}$, $\gamma_0$ and $\boldsymbol{\gamma}$ being the usual Dirac matrices). Thus, the net effect of the chemical potential is to shift all the single particle energies by $-\mu$. Correspondingly, the inverse of the free Dirac propagator may be written

$$\begin{aligned} G_0^{-1}(\mathbf{p},\omega) &= -(\omega+\mu)\gamma_0 + \boldsymbol{\gamma}\cdot\mathbf{p} + m \\ &= (\epsilon_p - \mu - \omega)\gamma_0\Lambda^+_{\mathbf{p}} - (\epsilon_p + \mu + \omega)\gamma_0\Lambda^-_{\mathbf{p}} \end{aligned} \tag{1}$$

where

$$\Lambda^{\pm}_{\mathbf{p}} = \frac{1}{2\epsilon_p}\left(\epsilon_p \pm (\boldsymbol{\alpha}\cdot\mathbf{p} + m\gamma_0)\right) \tag{2}$$

are the projectors on positive and negative energy solutions respectively ($(\Lambda^{\pm})^2 = \Lambda^{\pm}$, $\Lambda^+ + \Lambda^- = 1$), $\epsilon_p = \sqrt{p^2+m^2}$, and $\omega$ is a complex variable whose real part measures a single particle energy with respect to the chemical potential. In the following, we shall in fact absorb $\mu$ into $\omega$, i.e. we substitute $\omega$ to $\omega+\mu$ in Eq.(1) so that $\mathrm{Re}\,\omega$ will measure the energy with respect to the zero of energy in the absence of chemical potential. If needed, the Feynman propagator is obtained as usual by letting $\omega$ approach the real axis from above when $\mathrm{Re}\,\omega > \mu$, and from below when $\mathrm{Re}\,\omega < \mu$ [17, 18].

For simplicity, we consider in the main text the interaction of a Dirac fermion $\psi$ and a massless scalar field $\phi$ with a Yukawa coupling $g\phi\bar{\psi}\psi$. However, most of the discussion extends to QED (see Appendix B) and for this reason we shall often refer to the fermion as an "electron" and to the scalar particle as a "photon". In presence of the interaction, the inverse propagator is $G^{-1} = G_0^{-1} + \Sigma$, and, owing to rotational invariance, the mass operator $\Sigma$ may be written in the form

$$\Sigma(\mathbf{p},\omega) = a(p,\omega)\gamma^0 + b(p,\omega)\boldsymbol{\gamma}\cdot\mathbf{p} + c(p,\omega) \tag{3}$$

where $a, b$ and $c$ are three independent functions of the length $p$ of the momentum, and the energy $\omega$.

To order $g^2$ the mass operator at finite temperature and chemical potential is the sum of two terms $\Sigma(\mathbf{p},\omega) = \Sigma_1(\mathbf{p},\omega) + \Sigma_0(\mathbf{p},\omega)$, where $\Sigma_0(\mathbf{p},\omega)$ is the vacuum contribution, that is the mass operator for $T = \mu = 0$, and $\Sigma_1(\mathbf{p},\omega)$ is the correction due to finite $\mu$, finite $T$, effects. The latter is given by

$$\begin{aligned}\Sigma_1(\mathbf{p},\omega) = g^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\Big\{&\Lambda^+_{\mathbf{p+k}}\frac{n_k - f^-_{\mathbf{p+k}}}{\omega - \epsilon_{\mathbf{p+k}} - \omega_k} + \Lambda^-_{\mathbf{p+k}}\frac{f^+_{\mathbf{p+k}} + n_k}{\omega + \epsilon_{\mathbf{p+k}} - \omega_k} \\ &+\Lambda^+_{\mathbf{p+k}}\frac{f^-_{\mathbf{p+k}} + n_k}{\omega - \epsilon_{\mathbf{p+k}} + \omega_k} + \Lambda^-_{\mathbf{p+k}}\frac{n_k - f^+_{\mathbf{p+k}}}{\omega + \epsilon_{\mathbf{p+k}} + \omega_k}\Big\}\gamma_0,\end{aligned} \tag{4}$$

where

$$f^{\pm}_p = \frac{1}{e^{\beta(\epsilon_p \pm \mu)} + 1} \qquad\qquad n_k = \frac{1}{e^{\beta\omega_k} - 1} \tag{5}$$

are the occupation numbers respectively for the electrons ($f_p^-$), the positrons ($f_p^+$), and the photons ($n_k$). The photon energy is $\omega_k = k$, except in the calculation of the vacuum contribution (see below and Appendix A) where we set $\omega_k = \sqrt{k^2+\lambda^2}$, $\lambda$ playing the role of an infrared cut-off. Note that in writing Eq.(4), we ignore contributions involving possibly non–vanishing (in fact infrared divergent) expectation values of $\bar{\psi}\psi$ (or $\bar{\psi}\gamma^0\psi$ in the case of QED). However, such contributions to $\Sigma$ are independent of energy and momentum and produce simply a constant shift in the single particle energies. Therefore, they can be absorbed in a redefinition of the chemical potential. Now, since this plays no role in the present discussion, we simply ignore these constant contributions. (In the usual treatment of the non-relativistic electron gas, one assumes that such contributions are cancelled by those arising from a positive background of positive charges. In the case of QCD, such contributions vanish because of color neutrality). Thus, in zeroth order, the chemical potential will be taken equal to the Fermi energy, i.e. $\mu = \sqrt{m^2+k_F^2}$, where $k_F$ denotes the Fermi momentum, related as usual to the density of the conserved charge.

The vacuum ($T = \mu = 0$) contribution to the mass operator has a structure similar to Eq.(4),

$$\Sigma_0(\mathbf{p},\omega) \;=\; g^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\Big\{\Lambda^+_{\mathbf{p+k}}\frac{1}{\omega-\epsilon_{\mathbf{p+k}}-\omega_k} + \Lambda^-_{\mathbf{p+k}}\frac{1}{\omega+\epsilon_{\mathbf{p+k}}+\omega_k}\Big\}\gamma_0. \tag{6}$$

The integrations over momentum are diverging and require renormalization (see Appendix A). However, as we shall see shortly, in the ultrarelativistic limit where $m \sim g\mu, g\,T$, and for soft modes with $\omega, p \sim g\mu, gT$, the contribution of $\Sigma_0$ is of higher order in $g$ than the dominant terms of $\Sigma_1$, and it will not be discussed further in this section.

In the limit where $m$, $\omega$, $p \ll \mu$, $T$, the integrand in Eq. (4) contributes mostly in the region $k \sim \mu$ or $T$ where we can replace $\epsilon_{\mathbf{p+k}}$ by $k + p\cos\theta$, with $\theta$ the angle between $\mathbf{p}$ and $\mathbf{k}$. Furthermore, the first and the last term in Eq. (4), for which the energy denominators are of order $k \sim \mu, T$, are small compared to the second and third terms where the energy denominators are approximately $\omega \pm p\cos\theta \ll T$, $\mu$. The same reasoning shows that the vacuum contribution, which has the same denominators as the first and last terms of $\Sigma_1$, can be discarded in a first approximation.

Keeping only the dominant terms, one obtains then

$$\begin{aligned}\Sigma(\mathbf{p},\omega) &\approx g^2\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\Big\{\Lambda^-_{\mathbf{p+k}}\frac{f^+_{\mathbf{p+k}}+n_k}{\omega+\epsilon_{\mathbf{p+k}}-\omega_k}+\Lambda^+_{\mathbf{p+k}}\frac{f^-_{\mathbf{p+k}}+n_k}{\omega-\epsilon_{\mathbf{p+k}}+\omega_k}\Big\}\gamma_0\\ &\approx \frac{g^2}{16\pi^2}\int_0^\infty k\mathrm{d}k\left(f^-_k+f^+_k+2n_k\right)\int_{-1}^1\frac{d\cos\theta}{\omega-p\cos\theta}\left(\gamma^0-\frac{\boldsymbol{\gamma}\cdot\mathbf{p}}{p}\cos\theta\right).\end{aligned} \tag{7}$$

The integrals over $k$ and over $\theta$ are decoupled and can be done exactly. Note that the $k$-integration extends to the region of soft momenta for which the approximation done is non longer valid. Therefore, the expression (7) should be used only in situations where the weight of such soft momenta is negligible. A straightforward calculation gives the following results for the functions $a$, $b$ and $c$ defined in Eq.(3):

$$\begin{aligned}\mathrm{Re}\ a(\mathbf{p},\omega) &= \frac{M^2}{2p}\ln\left|\frac{\omega+p}{\omega-p}\right|\\ \mathrm{Re}\ b(\mathbf{p},\omega) &= \frac{M^2}{p^2}\left(1-\frac{\omega}{2p}\ln\left|\frac{\omega+p}{\omega-p}\right|\right)\\ \mathrm{Re}\ c(\mathbf{p},\omega) &= 0\end{aligned} \tag{8}$$

with $\omega$ real and

$$M=\frac{g}{4\pi}\sqrt{\mu^2+\pi^2T^2}. \tag{9}$$

In the ultrarelativistic limit, the self-energy $\Sigma$ depends on the chemical potential and the temperature only through this parameter $M$.

The functions $a$ and $b$ have imaginary parts when $|\omega|<p$, given by

$$\begin{aligned}\mathrm{Im}\ a(p,\omega) &= -\pi\frac{M^2}{2p}\\ \mathrm{Im}\ b(p,\omega) &= \pi\frac{M^2\omega}{2p^3}\end{aligned} \tag{10}$$

if $-p<\omega<p$ and 0 otherwise. The states to which the soft fermion couples, and which are responsible for this imaginary part, may be associated to virtual transitions where hard particles scatter on the soft one, with little deflection, and a possible change in their quantum numbers. For example, a hard electron can

annihilate on a soft positron and "turn into" a hard photon. The kinematics of such processes is somewhat analogous to that involved in the Bremsstrahlung of soft photons by fast electrons. Here, the soft particles are electrons or positrons and the hard particles are electrons, positrons or photons. These virtual transitions have $\omega \sim p\cos\theta$ and their number is constant, independent of $p$; thus their density increases as $1/p$ when $p$ decreases. In fact, this density is measured by the imaginary part of $a \propto M^2/2p$. Essentially all particles can contribute to these transitions, as reflected in the momentum integral in Eq. (7). Note that when $p \to 0$, $b \to 0$, and $a(p,\omega) \to M^2/\omega$, so that, in this limit, $\mathrm{Im}\, a(\omega) \propto \delta(\omega)$. The existence of an imaginary part in the region $|\omega| < p$ is reminiscent of the familiar Landau damping of electromagnetic waves in ordinary plasmas. Here, however, the imaginary part is of the same order of magnitude as the real part, i.e. $\mathcal{O}(M)$, and excitations with such energies ($|\omega| < p$) would be quickly damped.

Well defined quasiparticles exist for $|\omega| > p$. Their energies are the poles of the propagator, and are therefore given by the solutions of the equation $\det(G^{-1}) = 0$, where

$$G^{-1}(\mathbf{p},\omega) = (a-\omega)\,\gamma_0 + (1+b)\,\boldsymbol{\gamma}\cdot\mathbf{p} + (c+m), \tag{11}$$

or

$$\omega - a = \varepsilon\sqrt{(1+b)^2\,\mathbf{p}^2 + (c+m)^2}, \tag{12}$$

with $\varepsilon = \pm 1$. Using the explicit form of the functions $a$, $b$ and $c$ just derived, one can write Eq.(12) in the form

$$\frac{\omega^2 - p^2 - m^2}{p^2} = \mathrm{h}\left(\frac{\omega}{p}\right)\mathrm{h}\left(-\frac{\omega}{p}\right)\left(\frac{M}{p}\right)^4 + 2\left(\frac{M}{p}\right)^2 \tag{13}$$

with

$$\mathrm{h}(x) \equiv 1 - \frac{x+1}{2}\ln\left|\frac{x+1}{x-1}\right|. \tag{14}$$

The solution of Eq. (13) gives $M/p$ as a function of $\omega/p$, from which one obtains the dispersion relation $\omega(p)$ in parametric form for a given $M$. Note that Eq.(13) is even in $\omega$, and its solutions come in pairs with opposite signs. Thus, in the ultrarelativistic limit, there is an apparent particle-antiparticle symmetry for the soft quasiparticles excitation energies. This is at first sight surprising since such a

symmetry is explicitly broken by the presence of the chemical potential. We discuss this point further in the next section. We note here that if $\omega$ is a solution of Eq. (13) with $\varepsilon = 1$, then $-\omega$ is a solution with $\varepsilon = -1$. Furthermore, for a given $\varepsilon$, there are two solutions of opposite signs. We call $\omega_+$ and $\omega_-$ the two solutions corresponding to $\varepsilon = 1$, with $\omega_+ > 0$, $\omega_- < 0$. We call "normal" the branches $\varepsilon\omega_+$ and "abnormal" the branches $\varepsilon\omega_-$. Thus, for $\omega > 0$, the branch $\omega_+$ is the normal one, the branch $-\omega_-$ the abnormal one. This seemingly complicated notation will be fully justified in section 4 after we have clarified the origin of the observed pattern of states. As we shall see, $\varepsilon$ can be identified with the eigenvalue of $\gamma_0$ in the case where $p = 0$, and with that of $\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}$ when $m = 0$ ($\hat{\mathbf{p}} = \mathbf{p}/p$).

The spectrum is displayed in Figs.1. If $m$ is small enough compared to $M$, the group velocity on the abnormal positive branch, $-\mathrm{d}\omega_-/\mathrm{d}p$, is negative for small $p$, and $-\omega_-(p)$ presents a minimum for a finite value of the momentum. The condition under which this phenomenon occurs, $m/M < \sqrt{14/(3\sqrt{5}) - 2} \approx 0.295$, can be derived easily by expanding Eq.(13) in powers of $p/\omega$. For $m = 0$, we have

$$\omega_\pm(p) = \pm M \left[ 1 \pm \frac{1}{3}\frac{p}{M} + \ldots \right]. \tag{15}$$

Thus, in particular, the slopes of the two branches $\omega_\pm$ are the same at small $p$, $d\omega_\pm/dp = 1/3$ when $m = 0$. As $M$ becomes smaller than $m$, the normal branch $\varepsilon\omega_+(p)$ goes over to the normal fermion dispersion relation $\varepsilon\sqrt{p^2 + m^2}$, while the abnormal branch $\varepsilon\omega_-(p)$ approaches $-\varepsilon p$. The asymptotic behavior for $p \gg M$ and $m = 0$, is

$$\begin{aligned} \omega_+(p) &\sim p + \frac{M^2}{p} \\ \omega_-(p) &\sim -p\left[1 + \exp(-2p^2/M^2)\right] \end{aligned} \tag{16}$$

The residues at the quasiparticle poles can also be calculated. They have simple expressions in the two limiting cases where $p = 0$ or $m = 0$. Then, the projectors (2) reduce to $(1 \pm \gamma_0)/2$ when $p = 0$, and to $(1 \pm \boldsymbol{\alpha} \cdot \hat{\mathbf{p}})/2$ when $m = 0$, and we can write , in the appropriate subspace,

$$G(\mathbf{p}, \omega)\gamma_0 = \frac{1}{(\varepsilon\epsilon_{\mathbf{p}} - \omega) + \Sigma_\varepsilon(\omega)} \approx \frac{z_\pm}{\varepsilon\omega_\pm - \omega} \tag{17}$$

where $\varepsilon$ denotes the eigenvalue, $\pm 1$, of $\gamma_0$ in the case $\mathbf{p} = 0$, or that of $\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}$ when $m = 0$, $\Sigma_\varepsilon \equiv a + \varepsilon c$ when $p = 0$, and $\Sigma_\varepsilon \equiv a + \varepsilon b p$ when $m = 0$. The residues are

given by $z_\pm^{-1} = 1 - \partial\Sigma_\varepsilon/\partial\omega|_{\omega_\pm}$ and they depend only on the character of the mode, normal or abnormal, irrespective of the sign of $\omega$, i.e. of $\varepsilon$. When $m = 0$,

$$z_\pm = \frac{\omega_\pm^2 - p^2}{2M^2}. \tag{18}$$

When $p = 0$,

$$z_\pm = \frac{1}{1 + M^2/\omega_\pm^2}. \tag{19}$$

The variations with $p$ of the residues are displayed in Figs.1. When $m \neq 0$, the residue of the abnormal branch is always smaller than that of the normal branch. It becomes negligible when either $m \gg M$ or $p \gg M$. For $p = 0$ we have $z_+ + z_- = 1$. For $p \neq 0$, $z_+ + z_- < 1$; this is because the spectral weight has a smooth component between $\omega = -p$ and $\omega = p$. We will comment on this in section 4.

# 3 One loop results at $T = 0$

We now study the full one loop correction to the fermion propagator at $T = 0$ and $\mu \neq 0$. In this case, the mass operator has a simple analytic expression when either $p = 0$ or $m = 0$. The corresponding formulae are collected in Appendix B. Some of the discussion in this section parallels that in Ref.[8], where a similar study was presented at finite temperature and zero chemical potential. It is instructive to see how the structure in the spectrum develops as the chemical potential is raised and the ultrarelativistic limit is approached. However, one should keep in mind that the full one–loop calculation is not entirely consistent, as revealed for example by the order of magnitude of some contributions to the imaginary part. We shall discuss the physical meaning of these imaginary parts and see why it is not always meaningful to associate those in the vicinity of the quasiparticle poles with the damping of quasiparticles.

## 3.1 The mass operator at $T = 0$

At zero temperature, there are no bosons, i.e. $n_k = 0$, nor antifermions, i.e. $f_p^+ = 0$, and all single particle states with energies less than $\mu$ are occupied, i.e. $f_p^- =$

$\theta(\mu - \epsilon_p)$. The mass operator takes the form

$$\Sigma(\mathbf{p},\omega) = g^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\Big\{ \quad \Lambda^+_{\mathbf{p+k}} \quad \frac{1-f^-_{\mathbf{p+k}}}{\omega - \epsilon_{\mathbf{p+k}} - \omega_k} \\ + \quad \Lambda^+_{\mathbf{p+k}} \quad \frac{f^-_{\mathbf{p+k}}}{\omega - \epsilon_{\mathbf{p+k}} + \omega_k} + \Lambda^-_{\mathbf{p+k}}\frac{1}{\omega + \epsilon_{\mathbf{p+k}} + \omega_k}\Big\}\gamma_0, \tag{20}$$

where we have kept together the vacuum contribution and the finite chemical potential correction, without writing explicitly the renormalization counterterms (see Appendix A for details). Note that the imaginary part, which we discuss below, is finite and not affected by the renormalization. In order to facilitate the foregoing discussion, we label by (i), (ii) and (iii) the three contributions to $\Sigma(\mathbf{p},\omega)$ in Eq. (20). These are illustrated by the diagrams of Figure 2. In these diagrams, the labels on the lines correspond to the momentum and the energy of the single particle state to which the line refers. A fermionic line going upwards represents an electron above the Fermi surface. A fermionic line going downwards represents a hole in the Fermi sea if the energy is positive (case (ii)), and a positron (a hole in the Dirac sea) if the energy is negative (case (iii)). Thus, reading the diagram (ii) in Fig.2 from bottom to top, one finds a hole excitation with energy $-\omega$ and momentum $-\mathbf{p}$ which makes a virtual transition to a state containing a photon with momentum $\mathbf{k}$ and energy $k$, and a hole in the single particle state with momentum $\mathbf{p}+\mathbf{k}$ and energy $\epsilon_{\mathbf{p+k}}$.

It is perhaps useful at this point to recall that the poles of the Green's function occur for values of $\omega$ equal to $\pm(E^{N\pm1}_{\mathbf{p}} - E^N_0)$, where $E^{N\pm1}_{\mathbf{p}}$ is the energy of a state with $N \pm 1$ particles, and total momentum $\mathbf{p}$, while $E^N_0$ is the energy of the ground state with $N$ particles. (Note that in this discussion we count particles with respect to the fully occupied Dirac sea, and similarly for the energies.) Perturbative calculations are meaningful as long as $E^N_0 - E^{N-1}_0 < \mu < E^{N+1}_0 - E^N_0$ [18]. Values of $\omega$ such that $\mathrm{Re}\,\omega > \mu$ correspond to possible energies of states with $N+1$ particles, while values of $\omega$ such that $\mathrm{Re}\,\omega < \mu$ correspond to the negative of energies of states with $N-1$ particles. For example, the state obtained by removing a particle of momentum $\mathbf{p}$ in the unperturbed Fermi sea has a momentum $-\mathbf{p}$, and an energy $-\omega = E^{N-1}_{\mathbf{p}} - E^N_0 = -\epsilon_{\mathbf{p}}$, with $\omega < \mu$. The same reasoning applies to positrons considered as holes in the Dirac sea. Removing an electron in the Dirac sea in a single particle state of momentum $\mathbf{p}$ and energy $-\epsilon_{\mathbf{p}}$, produces a state with $N-1$ particles, a momentum $-\mathbf{p}$, and an energy $-(-\epsilon_{\mathbf{p}}) = \epsilon_{\mathbf{p}}$.

The mass operator acquires imaginary parts in several energy domains where the energy denominators in Eq. (20) vanish for $\omega$ real. To discuss these, we note the following inequalities

$$\epsilon_{\mathbf{p+k}} + k \geq \epsilon_{\mathbf{p}}, \qquad 0 \leq \epsilon_{\mathbf{p+k}} - k \leq \epsilon_{\mathbf{p}}. \tag{21}$$

It is then easily seen that, quite generally, the imaginary part of (i) vanishes when $\omega < \mu$. This follows from the fact that the occupation factor in the numerator forces $\epsilon_{\mathbf{p+k}} > \mu$, so that the denominator $\omega - \epsilon_{\mathbf{p+k}} - k$ cannot vanish if $\omega < \mu$. Similarly, the occupation factor $f^-_{\mathbf{p+k}}$ in the numerator of (ii) forces $\epsilon_{\mathbf{p+k}} < \mu$ so that the imaginary part of (ii) vanishes when $\omega > \mu$. Finally, using the first of the inequalities (21) one gets that the imaginary part of (iii) vanishes when $\omega > -m$.

Now, the precise domains in which the imaginary part does not vanish depend on $p$. These are illustrated in Figure 3. The last term of Eq. (20), (iii), contributes an imaginary part when $\omega < -\epsilon_{\mathbf{p}}$, as easily deduced by using the first of inequalities (21). This corresponds physically to the process by which a positron (off its energy shell) with energy $|\omega|$ and momentum $\mathbf{p}$ decays into a positron of momentum $\mathbf{p}+\mathbf{k}$ and a photon of momentum $-\mathbf{k}$, on their respective mass shells; this process may occur in the vacuum, for example under the action of a weak perturbation which displaces the positron slightly off its mass shell, and it is not affected by the Fermi sea of electrons. We shall refer to the continuum of states into which a positron can decay as the positron-photon continuum. It corresponds to region (III) in Fig.3.

The first term in Eq. (20), labelled (i), has, for $p > k_F$, an imaginary part when $\omega > \epsilon_{\mathbf{p}}$. This imaginary part lies in the vicinity of the electron mass shell, and reflects the possible decay of an electron above the Fermi surface into another electron with less energy and a photon. As in the positron case just discussed, this process may also occur in the vacuum; however, here, the presence of the Fermi sea eliminates some of the states available for the decay in free space, because of the Pauli principle reflected by the occupation factor $1 - f_{\mathbf{p+k}}$ in Eq. (20). In the case $p < k_F$, this factor actually reduces the energy domain allowed by the kinematics, i.e. $\omega > \epsilon_{\mathbf{p}}$, to $\omega > \mu + k_F - p$, and indeed leads to a cancellation of the imaginary part in the region $\epsilon_{\mathbf{p}} < \omega < \mu + k_F - p$. Then, for $p < k_F$, the imaginary part of (i) is non zero in a domain ($\omega > \mu + k_F - p$) which can be far from $\epsilon_{\mathbf{p}}$ and it is not immediately related to a possible decay of the single particle excitation. The continuum of states discussed in this paragraph will be referred to as the electron-photon continuum. It corresponds to region (I) in Fig.3.

The existence of states of the $N+1$ particle system with momentum less than the Fermi momentum $k_F$ reflects modifications of the Fermi sea due to the interactions. Because of these, the single particle states are neither fully occupied below the Fermi surface, nor completely empty above it. These modifications of the Fermi sea are implicitly taken into account, to order $g^2$, in the expression (20) of the mass operator. The interplay between the single particle properties and these ground state corrections is perhaps best understood by considering the energy shift of a single particle state obtained in second order perturbation theory. This is obtained by solving Dyson's equation for the pole of the propagator, setting $\omega = \epsilon_{\mathbf{p}}$ in the mass operator $\Sigma(\mathbf{p}, \omega)$. The first term in $\Sigma(\mathbf{p}, \epsilon_{\mathbf{p}})$, i.e. (i), gives an attractive contribution, while the second term, i.e. (ii), gives a repulsive contribution. This is easily seen by using inequalities (21): the denominator of (i) is always negative, while that of (ii) is always positive. To understand this better, consider a particle state above the Fermi surface, i.e. with $p > k_F$, and recall that the poles of the Green's function correspond, for $\omega > \mu$, to the energy difference $E^{N+1}_{\mathbf{p}} - E^N_0$, where $E^{N+1}_{\mathbf{p}}$ is the energy of the system with $N+1$ particles and momentum $\mathbf{p}$. In carrying out the second order calculation of this energy difference, one is lead to write the total correction as $\Delta\epsilon_{\mathbf{p}} + \Delta E^{N+1}_0 - \Delta E^N_0$. The first term, $\Delta\epsilon_{\mathbf{p}}$, is the correction obtained from (i); it is negative and can indeed be interpreted as the second order correction to the energy of the single particle state $\mathbf{p}$, due to its coupling to states with one electron (above the Fermi surface) and one photon. On the other hand, the quantity $\Delta E^{N+1}_0$ represents a correction to the the energy of the Fermi sea, taking into account the presence of the extra particle. The full second order correction is given by $\Delta E^N_0$, and is negative. Because of the Pauli principle, the presence of the extra particle suppresses some of the intermediate states (those having a particle with momentum $\mathbf{p}$ above the Fermi surface) involved in the full second order correction to $E^N_0$; this is illustrated in Fig. 4. As a result, $\Delta E^{N+1}_0 > \Delta E^N_0$, and the contribution of (ii), which can be identified to $\Delta E^{N+1}_0 - \Delta E^N_0$, is positive. A similar interpretation may be given for states below the Fermi surface, exchanging the role of contributions (i) and (ii).

We come now to the second term in Eq. (20), labelled (ii). In contrast to (i) and (iii) where vacuum contributions explicitly enter, this term has no counterpart in the vacuum and is entirely due to the Fermi sea of electrons. It contains the dominant contribution in the ultrarelativistic limit. It has for $p < k_F$ an imaginary part in the region $\mu - k_F - p < \omega < \epsilon_{\mathbf{p}}$. This imaginary part measures the density of states with momentum $\mathbf{p}$ containing one hole and one photon with respect to the unperturbed

Fermi sea. Such states may couple with a hole at the bottom of the Fermi sea, or a hole at the top of the Dirac sea, i.e. with a positron. A pictorial representation of all these states, which will play a central role in our discussion (in particular in section 4), is given in Fig 5. One may view the corresponding virtual transitions as processes by which a hole in the Fermi sea, or a positron, gets "filled up" by electrons falling into it from above in the Fermi sea, emitting at the same time a photon. If $p > k_F$, (ii) has an imaginary part in the domain $\mu - p - k_F < \omega < \mu - p + k_F$. The existence of states of the $N-1$ particle system having a momentum greater than the Fermi momentum reflects, similarly to the case discussed above, a modification of the Fermi sea due to interactions. The continuum of states corresponding to the virtual transitions described in this paragraph will be refered to as the hole-photon continuum. This correspond to region (II) of Fig.3.

In the ultrarelativistic limit, the hole-photon continuum is entirely contained in the region $|\omega| > p$ (see section 2, the discussion after Eq.(10). However, the above discussion shows that the imaginary part does not strictly vanish outside this interval. The additional contribution, in the interval $p < \omega < \epsilon_{\mathbf{p}}$, can be calculated assuming $\mu \gg \omega, p, m$ and $\omega - p \gg g^2\mu$. The result is,

$$\begin{aligned} \mathrm{Im}\ a &= -\pi \frac{g^2}{32\pi^2} \frac{\omega\,[m^4 - (\omega^2 - p^2)^2]}{(\omega^2 - p^2)^2} \\ \mathrm{Im}\ b &= -\pi \frac{g^2}{32\pi^2} \frac{m^4 - (\omega^2 - p^2)^2}{(\omega^2 - p^2)^2} \\ \mathrm{Im}\ c &= -\pi \frac{g^2}{32\pi^2} \frac{2m\,[m^2 - (\omega^2 - p^2)]}{\omega^2 - p^2} \end{aligned} \tag{22}$$

The same expressions hold also, up to a global change of sign of $\mathrm{Im}\, b$, in the region $\omega < -\epsilon_{\mathbf{p}}$ correponding to the positron-photon continuum. When $p$, $m$ and $\omega$ are of order $g\mu$, $\mathrm{Im}\,\Sigma(\mathbf{p}, \omega)$, as given by Eq.(22), is of order $g^3\mu$, i.e. smaller by two powers of $g$ than when $|\omega| < p$. This comes from the fact that almost all the states in the Fermi sea contribute to the hole-photon continuum, while only a restricted phase space is involved in the calculation leading to Eqs.(22). Indeed the phase space vanishes as $\omega$ approaches $\epsilon_{\mathbf{p}}$, as may be seen on these expressions (22). Note that Eqs.(22) do not involve the chemical potential. This is to be contrasted to the case $T \neq 0$, where the phase space depends explicitly on the temperature through the statistical factors. Thus at $T \neq 0$ the imaginary part of the one-loop mass operator is of order $g^2 T$ (instead of $g^3\mu$ here), because the number of soft photons of energy $\sim gT$ is of order $1/g$.

We have seen in section 2 (see also Fig.1) that, in the ultrarelativistic limit, there are four quasiparticles of a given momentum $p < k_F$, whose energies are the solutions of Eq.(12). Some of these quasiparticles appear in regions where Im$\Sigma$ is small but non vanishing. One may then be tempted to associate these small imaginary parts to a weak damping of the quasiparticles. Because of them, the quasiparticle energies are indeed shifted to complex values, $\omega = \omega_\pm - i\gamma_\pm$, where $\gamma_\pm$ is always positive ($\gamma_\pm = -z_\pm \mathrm{Im}\Sigma(\omega_\pm)$). (It is interesting in this respect to notice that without the vacuum contribution, there would be some imaginary part for $\epsilon_{\mathbf{p}} < \omega < \mu$ coming from the term labelled (i), with the wrong sign.) The normal quasiparticle branch with $\varepsilon = 1$ always satisfies $\omega_+ > \epsilon_{\mathbf{p}}$ and thus has no imaginary part (the branch $\omega_+$ is above the hole-photon continuum). On the other hand, the normal branch $-\omega_+ < -\epsilon_{\mathbf{p}}$ is always damped (it is always in the positron-photon continuum). As to the abnormal branches, there are two possibilities. Either $-\omega_- > \epsilon_{\mathbf{p}}$ (case $M \gg m$ in Fig.1); then the branch $-\omega_-$ is above the hole-photon continuum, and the branch $\omega_- < 0$ is inside the positron-photon continuum. Or $-\omega_- < \epsilon_{\mathbf{p}}$ (case $M \approx m$ in Fig.1), and then it is the mode $-\omega_- > 0$ which is damped, being found inside the hole-photon continuum. Now one may question whether the fact that a branch of the spectrum lies inside a continuum of states implies necessarily a damping of the corresponding single particle excitation. When $M$ is sufficiently large compared to $m$, possible damping would occur because of a mixing of the single particle excitation with the positron-photon continuum . However, a consistent calculation of the damping should take into account the fact that the edge of this continuum, taken here to be the unperturbed positron energy $\epsilon_{\mathbf{p}}$, is strongly modified by the interactions, which affect actually both the propagation of the fermions and that of the photons. Taking as a minimal approximation the modification of the fermion energy in the calculation of the damping, one would obtain that the damping is forbiden by the kinematics (the branch $-\omega_+$ becomes in this approximation the edge of the continuum). The difficulty encountered here is identical to that noted in a related context (the damping of gluonic excitations in a quark-gluon plasma) by several authors[19, 20, 21]. Technically, it finds its origin in the fact that the imaginary parts, as obtained from the one loop approximation, being of order $g^3\mu$, are inconsistent since higher loop diagrams can contribute to the same order. Consistent calculations require resummations as explained in Refs.[22].

In order to get further insight into the properties of the quasiparticles, and the characteristic features of the ultrarelativistic limit, we now briefly consider successively the particular cases where $p = 0$ and $m = 0$.

## 3.2 Massive fermion at rest

The equation which determines the poles of the propagator (in the region where the imaginary part is small), namely Eq.(12), becomes, when $p = 0$

$$\omega - \varepsilon m = \mathrm{Re}(a_1 + \varepsilon c_1), \tag{23}$$

where we have ignored the vacuum contribution, and $\varepsilon$ denotes the eigenvalue of $\gamma^0$. Analytical expressions for $\mathrm{Re}\, a_1$ and $\mathrm{Re}\, c_1$, as well as for the corresponding imaginary parts can be found in Appendix B. The quantity $\mathrm{Re}(a_1 + \varepsilon c_1)$ is plotted in Figs.6, as a function of $\omega/m$ for various values of $\mu/m$. In general, when $\mu \neq 0$, there is no symmetry under the transformation $\omega \to -\omega$, $\varepsilon \to -\varepsilon$. However, this symmetry emerges at large $\mu$, as is already apparent in Figs.6 (the function $\mathrm{Re}\, a_1$ becomes odd in $\omega$ and $\mathrm{Re}\, c_1$ becomes negligible at large $\mu$). A graphical solution of Eq. (23) is obtained by taking the intersection of the curve $\mathrm{Re}(a_1 + \varepsilon c_1)$ with the straight line $\omega - \varepsilon m$. One sees that the normal unperturbed pole at $\omega = \varepsilon m$ is shifted to the right ($\varepsilon = 1$) or to the left ($\varepsilon = -1$). Furthermore, when $\varepsilon = 1$, two additional poles appear on both sides of the logarithmic divergence at $\omega = \mu - k_F$, even if $\mu$ is arbitrarily small. When $\varepsilon = -1$, a similar situation occurs with two new poles slightly above $\omega = \mu - k_F$, but only if $\mu$ is large enough. The physical relevance of these extra solutions may be asserted by considering the imaginary part of $a + \varepsilon c$ (see Eq. (B.3)). One sees that the imaginary part remains large in the vicinity of the pole closest and above $\mu - k_F$. This pole correspond to an excitation which is killed by Landau damping. The other two poles correspond to physical excitations with negligible widths. As we have seen earlier, it is consistent to ignore the small imaginary parts in the ultrarelativistic limit. A consistent calculation of the damping rates of the quasiparticles requires going beyond the one loop apparoximation.

The structure of the self–energy can be understood from the dispersion relation

$$\mathrm{Re}\, \Sigma(\omega) = \mathrm{P} \int \frac{d\omega'}{2\pi} \frac{\mathrm{Im}\, \Sigma(\omega')}{\omega' - \omega} \tag{24}$$

where the symbol P in front of the integral indicates that the principal value is to be taken. The region contributing to the dominant imaginary part is the small $\omega$ region where the imaginary part goes as $\omega^{-3}$. This behavior reflects the increase of the phase space for processes in which a hole at the bottom of the Fermi sea couples with holes near the Fermi surface and photons with momentum $\sim k_F$. When $\mu$ grows, the lower bound of the allowed phase space $\mu - k_F \sim m^2/(2\mu) \to 0$, and

the number of such transitions becomes increasingly important. Thus, at large $\mu$, the imaginary part is approximately given by $\mathrm{Im}\Sigma \sim -(g^2m^4/32\pi\omega^3)$. Using the dispersion relation (24), and keeping only the dominant contribution coming, when $\omega \sim g\mu$, from the lower bound of the integral $\sim m^2/(2\mu)$, one gets $\mathrm{Re}\,\Sigma \sim M^2/\omega = \mathrm{Re}\,a(0,\omega)$ ($M = g\mu/4\pi$), in agreement with Eq.(8). Thus the leading behavior of $\mathrm{Re}\,\Sigma$ is determined by the hole-photon continuum, to which all electrons in the Fermi sea can contribute. This explains the emergence of the symmetry $\omega \to -\omega$ in the ultrarelativistic limit. Most of the particles involved in the virtual transitions making the hole-photon continuum are hard particles. In pictorial terms, an electron with momentum $p \sim \mu \gg m$ falling into a hole with momentum $p \sim g\mu$, does not "see" the mass gap, i.e. it does not feel the difference between a hole at the bottom of the Fermi sea and a hole at the top of the Dirac sea.

In the high density limit Eq. (B.2) has only two solutions for either value of $\varepsilon$ (the third solution, slightly above $\mu - k_F$, disappears in this limit), that is to say four solutions. The positive solutions are

$$\pm\omega_\pm = \pm\frac{m}{2} + \sqrt{M^2 + \left(\frac{m}{2}\right)^2}. \tag{25}$$

Eq.(B.4) shows that the imaginary part vanishes for both positive solutions if $M > m\sqrt{2}$, and only for the solution $\varepsilon = 1$ if $M < m\sqrt{2}$. As to the negative solutions, $-\omega$ has no imaginary part when $\omega$ has one and vice-versa. As already mentioned, this imaginary part is small and can be ignored.

## 3.3 Massless fermion with a finite momentum

When $m = 0$, Eq.(12) becomes (as in the previous case we ignore the imaginary parts and the vacuum contribution)

$$\omega - \varepsilon p = \mathrm{Re}\,(a_1 + \varepsilon b_1 p) \tag{26}$$

and $\varepsilon$ denotes the eigenvalue of $\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}$ ($\hat{\mathbf{p}} = \mathbf{p}/p$). The exact expressions for the real and the imaginary parts of $a_1$ and $b_1 p$ are given in appendix B, and $\mathrm{Re}(a_1 + \varepsilon b_1 p)$ is displayed in Figs.7 as a function of $\omega$. One sees that the imaginary part of $a + \varepsilon b p$ is continuous everywhere except at $\omega = -\varepsilon p$, as can be checked in Eq. (B.7). This discontinuity at the edge of the hole-photon continuum is an effect of the zero mass. The imaginary part is significant only if $|\omega| < p$. The number of processes

contributing to it is an increasing function of $\mu$. We also note that, as in the previous section, the real part of $a_1 + \varepsilon b_1 p$ is, in general, not odd under the changes of sign of both $\varepsilon$ and $\omega$. However, it becomes so in the high $\mu$ limit, for reasons which have already been explained. Eq. (26) can be solved graphically in Fig.7, exactly in the same way as Eq. (23) in Fig.6. One sees that for a small value of $g$, Eq. (26) always has two solutions in addition to the solution close to $\omega = \varepsilon p$. These new poles are on both sides of $\omega = -\varepsilon p$, and the pole which is inside the hole-photon continuum is Landau damped if $\mu \gg p$. Thus we have finally one new physical pole for either value of $\varepsilon$ when $m = 0$, as we had for $p = 0$.

Finally, let us mention that the states in the vicinity of the Fermi surface are not much affected by the interactions in this order. This is illustrated in Figs.8. The scale on these figures shows indeed that $\Sigma$ is a small quantity when $p \sim k_F$ (it is completely perturbative, i.e. of order $g^2$). The lines $\omega \mp p$ are almost vertical lines in these plots, indicating that single particle states are shifted from their unperturbed positions by negligible amounts. As for the imaginary part, it vanishes for $\omega = \mu$ as is evident on Eq.(B.7).

We end this section by summarizing the physical picture which emerges from this analysis. As we have seen, most of the interesting physics is due to the hole-photon continuum. When $\mu$ is large, this continuum involves virtual transitions in which hard electrons "fall" into holes at the bottom of the Fermi sea, or at the top of the Dirac sea (soft positrons). The number of such transitions grows as $\mu^2$, reflecting the fact that most particles of the Fermi sea participate and suggesting the collective character of the quasiparticles. We shall now complete this picture by presenting an explicit calculation of the quantum states corresponding to the quasiparticles.

# 4 Explicit calculation of the states

In this section we present an explicit construction of the quantum states associated with each branch of the excitation spectrum. We shall work mainly in the ultrarelativistic limit. As we shall see, many of the features which were left unexplained in the previous analysis will find here a natural interpretation.

We decompose the fermion and scalar fields into plane waves normalized in a

volume $\Omega$

$$\psi(\mathbf{x}) = \frac{1}{\sqrt{\Omega}} \sum_{\lambda,\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}}\, w_{\lambda,\mathbf{p}} b_{\lambda,\mathbf{p}} \tag{27}$$

$$\phi(\mathbf{x}) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_k}} \left( e^{i\mathbf{k}\cdot\mathbf{x}}\, a_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{x}}\, a^{\dagger}_{\mathbf{k}} \right) \tag{28}$$

In the decomposition of the fermion field $\psi(\mathbf{x})$, we do not separate at this stage positive and negative energy solutions of the Dirac equation, and consider a positron as a hole in the sea of negative energy electrons. In Eq.(27), we therefore sum over positive and negative energy solutions and the index $\lambda$ takes four values, two for the spin and two for the sign of energy. This convention allows us to treat the Dirac and Fermi seas on the same footing: all single particle states with energy less than $\mu$ are occupied in the unperturbed ground state $|\Phi_0\rangle$. The fermion spinors are normalized according to $w^{\dagger}_{\lambda,\mathbf{p}} w_{\lambda',\mathbf{p}'} = \delta_{\lambda,\lambda'}\delta_{\mathbf{p},\mathbf{p}'}$ and they obey the free Dirac equation

$$\left[\boldsymbol{\alpha}\cdot\mathbf{p} + \gamma_0 m\right] w_{\lambda,\mathbf{p}} = \epsilon_{\lambda,\mathbf{p}} w_{\lambda,\mathbf{p}} \tag{29}$$

The interaction hamiltonian is

$$\begin{aligned} H_{int} &= g \int \phi(\mathbf{x}) \bar{\psi}(\mathbf{x}) \psi(\mathbf{x}) d^3\mathbf{x} \\ &= \frac{g}{\sqrt{\Omega}} \sum_{\mathbf{p},\mathbf{k},\lambda,\nu} \frac{1}{\sqrt{2\omega_k}} \bar{w}_{\nu,\mathbf{p}+\mathbf{k}} w_{\lambda,\mathbf{p}} b^{\dagger}_{\nu,\mathbf{p}+\mathbf{k}} b_{\lambda,\mathbf{p}} \left( a_{\mathbf{k}} + a^{\dagger}_{-\mathbf{k}} \right) \end{aligned} \tag{30}$$

We shall show that, in the ultrarelativistic limit, the quantum states corresponding to soft fermionic excitations can be obtained by diagonalizing the total hamiltonian $H = H_0 + H_{int}$, where $H_0$ is the free hamiltonian for electrons and photons, in the following basis of states

$$b_{\lambda,\mathbf{p}}|\Phi_0\rangle \qquad\qquad b_{\nu,\mathbf{p}+\mathbf{k}} a^{\dagger}_{\mathbf{k}} |\Phi_0\rangle, \tag{31}$$

where $\epsilon_{\lambda,\mathbf{p}} < \mu$, $\epsilon_{\nu,\mathbf{p}+\mathbf{k}} < \mu$ and, as said earlier, $|\Phi_0\rangle$ is the unperturbed ground state with energy $E_0$. All the states (31) have momentum $-\mathbf{p}$. The first state is a hole with momentum $-\mathbf{p}$. The second represents a state composed of a hole with momentum $-\mathbf{p}-\mathbf{k}$ accompanied by a photon with momentum $\mathbf{k}$. These are eigenstates of $H_0$ with eigenvalues $E_0 - \epsilon_{\lambda,\mathbf{p}}$ and $E_0 - \epsilon_{\nu,\mathbf{p}+\mathbf{k}} + \omega_k$ respectively. A

pictorial representation of these excitations is given in Fig.5. The single particle energy $\epsilon_{\lambda,\mathbf{p}}$ is, as mentioned above, positive or negative depending on $\lambda$, namely $\epsilon_{\lambda,\mathbf{p}} = \epsilon_p$ for $\lambda = 1, 2$ and $\epsilon_{\lambda,\mathbf{p}} = -\epsilon_p$ for $\lambda = 3, 4$.

We call $X_{\lambda,\mathbf{p}}$ and $Y_{\nu,\mathbf{p},\mathbf{k}}$, respectively, the amplitudes associated with the two types of states (31) in the decomposition of the eigenstates $|\Psi_{-\mathbf{p}}\rangle$ of $H$ on the unperturbed basis

$$|\Psi_{-\mathbf{p}}\rangle = \sum_{\lambda} X_{\lambda,\mathbf{p}} b_{\lambda,\mathbf{p}} |\Phi_0\rangle + \sum_{\mathbf{k},\nu} Y_{\nu,\mathbf{p},\mathbf{k}} b_{\nu,\mathbf{p}+\mathbf{k}} a^{\dagger}_{\mathbf{k}} |\Phi_0\rangle. \tag{32}$$

Note that in restricting ourselves to the states (31) we are performing a truncation of the Hilbert space. In particular, the states with one hole and one photon are coupled to states involving two photons, two holes and one electron above the Fermi sea. This truncation is in line with the high density limit and is legitimate since the neglected states would contribute in higher order in $g$ to the properties of the single hole state that we are interested in. The eigenvalue problem $(H_0 + H_{int})|\Psi_{-\mathbf{p}}\rangle = -\omega|\Psi_{-\mathbf{p}}\rangle$ then takes the form

$$\begin{aligned} &(\omega - \epsilon_{\lambda,p}) X_{\lambda,\mathbf{p}} - g \sum_{\mathbf{k},\nu} \frac{1}{\sqrt{2\omega_k \Omega}} \bar{w}_{\nu,\mathbf{p}+\mathbf{k}} w_{\lambda,\mathbf{p}} Y_{\nu,\mathbf{p},\mathbf{k}} = 0 \\ &-\frac{g}{\sqrt{2\omega_k \Omega}} \sum_{\lambda} \bar{w}_{\lambda,\mathbf{p}} w_{\nu,\mathbf{p}+\mathbf{k}} X_{\lambda,\mathbf{p}} + (\omega - \epsilon_{\nu,\mathbf{p}+\mathbf{k}} + \omega_k) Y_{\nu,\mathbf{p},\mathbf{k}} = 0 \end{aligned} \tag{33}$$

where $-\omega$ measures the energy of a state with one particle less than $|\Phi_0\rangle$ with respect to $E_0$ (see the discussion at the beginning of section 3). In the first equation, the sum runs over all single particle states such that $\epsilon_{\nu,\mathbf{p}+\mathbf{k}} < \mu$.

In order to calculate the eigenvalues $\omega$, we use the second equation to eliminate the amplitudes $Y_{\nu,\mathbf{p},\mathbf{k}}$. One gets, after a straightforward calculation,

$$(\omega - \epsilon_{\lambda,\mathbf{p}}) X_{\lambda,\mathbf{p}} - \sum_{\lambda'} X_{\lambda',\mathbf{p}} \bar{w}_{\lambda',\mathbf{p}} M(\mathbf{p},\omega) w_{\lambda,\mathbf{p}} = 0, \tag{34}$$

where $M(\mathbf{p},\omega)$ is given by

$$M(\mathbf{p},\omega) = g^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left\{ \frac{\Lambda^{+}_{\mathbf{p}+\mathbf{k}} f^{-}_{\mathbf{p}+\mathbf{k}}}{\omega - \epsilon_{\mathbf{p}+\mathbf{k}} + \omega_k} + \frac{\Lambda^{-}_{\mathbf{p}+\mathbf{k}}}{\omega + \epsilon_{\mathbf{p}+\mathbf{k}} + \omega_k} \right\} \gamma_0. \tag{35}$$

Here $\epsilon_{\mathbf{p}+\mathbf{k}} = |\epsilon_{\nu,\mathbf{p}+\mathbf{k}}|$ and $\Lambda^{\pm}_{\mathbf{p}}$, introduced in Eq.(2) projects onto states with positive or negative energy, i.e. $\Lambda^{+}_{\mathbf{p}} = \sum_{\nu=1,2} w_{\nu,\mathbf{p}} w^{\dagger}_{\nu,\mathbf{p}}$, and $\Lambda^{-}_{\mathbf{p}} = \sum_{\nu=3,4} w_{\nu,\mathbf{p}} w^{\dagger}_{\nu,\mathbf{p}}$ (see

Eq. (2)). The occupation factor $f^-_{\mathbf{p}+\mathbf{k}}$ is 1 for a particle in the Fermi sea and 0 otherwise. The expression (35) coincides with the sum of the terms (ii) and (iii) in the expression (20) of the mass operator $\Sigma(\mathbf{p},\omega)$. The term (i) in (20) accounts, for small $p$ and $\omega$, for modifications of the Fermi sea, as discussed in the beginning of section 3. Such modifications are negligible in the ultrarelativistic limit, and indeed they are not taken into account by the eigenvalue problem (33). We shall come back to these at the end of this section. The last term of (35), which involves positron intermediate states, is also negligible in the ultrarelativistic limit, and we shall consequently ignore it in the following. By multiplying Eq. (34) by $\bar{w}_{\lambda,\mathbf{p}}$ and summing over $\lambda$, one easily gets

$$\gamma_0 G^{-1}(\mathbf{p},\omega)\sum_\lambda X^*_{\lambda,\mathbf{p}} w_{\lambda,\mathbf{p}} = 0. \tag{36}$$

The values of $\omega$ for which Eqs. (33) are satisfied are therefore the zeros of $G^{-1}$, that is, as announced earlier, they coincide with the poles of the Green function calculated here in the ultrarelativistic limit.

Using results established in section 2, we can write Eq. (36) as follows

$$(\omega - a)\chi = \left[(1+b)\boldsymbol{\alpha}\cdot\mathbf{p} + m\gamma_0\right]\chi, \tag{37}$$

where $\chi = \sum_\lambda X^*_{\lambda,\mathbf{p}} w_{\lambda,\mathbf{p}}$. The eigenvalues are given by Eq. (12). Let us recall that there are two pairs of solutions which depend on the parameter $\varepsilon = \pm 1$, with the normal solutions such that $\omega\varepsilon > 0$. Besides, each solution possesses a two-fold spin degeneracy since the spin operator $\boldsymbol{\Sigma} = \gamma_5\boldsymbol{\alpha}$ commutes with both $\boldsymbol{\alpha}\cdot\mathbf{p}$ and $\gamma_0$.

The spinors $w_{\lambda,\mathbf{p}}$ are eigenstates of the free Dirac hamiltonian (see Eq. (29)). In either case, $p = 0$ or $m = 0$, one can choose the solution of Eq. (37) to be one of these, i.e. choose $\chi = w_{\lambda,\mathbf{p}}$. Namely, when $m = 0$, we can choose $\chi$ to be an eigenstate of $\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}$ with eigenvalue $\varepsilon$. When $p = 0$, we can choose $\chi$ to be an eigenstate of $\gamma_0$ with eigenvalue $\varepsilon$. In the general case where neither $m$ nor $p$ vanish, the presence of the factor $(1+b)$ in Eq. (37) induces a mixing which depends on $\omega$ and which needs to be calculated explicitly for each eigenvalue. This will be done later, and we shall consider first these particular cases where $m = 0$ or $p = 0$. Then, for each eigenvalue, only one amplitude, say $X_{\lambda,\mathbf{p}}$, is non vanishing. Setting $X_{\lambda,\mathbf{p}} = 1$, i.e. deferring the normalisation of the states to the end of the calculation, we obtain the following simple expression for the amplitudes $Y_{\nu,\mathbf{p},\mathbf{k}}$

$$Y_{\nu,\mathbf{p},\mathbf{k}} = \frac{g}{\sqrt{2k\Omega}}\frac{\bar{w}_{\lambda,\mathbf{p}} w_{\nu,\mathbf{k}}}{\omega - p\cos\theta}, \tag{38}$$

where we have used the fact that $k \gg p, m$ to simplify the denominator, and $\theta$ is the angle between $\mathbf{k}$ and $\mathbf{p}$.

In order to calculate explicitly the overlap $\bar{w}_{\lambda,\mathbf{p}} w_{\nu,\mathbf{k}}$, we choose the chiral representation of the $\gamma$ matrices in which the spinors take the form

$$\begin{aligned} u_{\lambda,\mathbf{p}} &= \frac{1}{2\sqrt{\epsilon_p(m+\epsilon_p)}} \begin{pmatrix} (\epsilon_p + m + \boldsymbol{\sigma}\cdot\mathbf{p})\phi_{\lambda\hat{\mathbf{p}}} \\ (-\epsilon_p - m + \boldsymbol{\sigma}\cdot\mathbf{p})\phi_{\lambda\hat{\mathbf{p}}} \end{pmatrix} \\ v_{\lambda,\mathbf{p}} &= \frac{1}{2\sqrt{\epsilon_p(m+\epsilon_p)}} \begin{pmatrix} (\epsilon_p + m - \boldsymbol{\sigma}\cdot\mathbf{p})\phi_{\lambda\hat{\mathbf{p}}} \\ (\epsilon_p + m + \boldsymbol{\sigma}\cdot\mathbf{p})\phi_{\lambda\hat{\mathbf{p}}} \end{pmatrix} \end{aligned} \tag{39}$$

where $\hat{\mathbf{p}} = \mathbf{p}/p$. These spinors are eigenstates of helicity with helicity $\lambda$ if the two components spinors are, i.e. if $(\boldsymbol{\sigma}\cdot\hat{\mathbf{p}})\phi_{\lambda\hat{p}} = \lambda\phi_{\lambda\hat{\mathbf{p}}}$. In the limit $m = 0$ they become eigenstates of $\gamma_5$, with eigenvalues $\lambda$ for $u_\lambda$ and $-\lambda$ for $v_\lambda$. In the limit $p = 0$ they are eigenstates of $\gamma_0$ with eigenvalues $+1$ for $u$ and $-1$ for $v$. Note that in both cases the spinor $u_{\lambda,\mathbf{p}}$ is associated to $\varepsilon = 1$ while the spinor $v_{\lambda,\mathbf{p}}$ is associated to $\varepsilon = -1$. Since we now distinguish explicitly positive and negative energy solutions, the index $\lambda$ takes only two values, i.e. $u_{1,-1} = w_{1,2}$, $v_{1,-1} = w_{3,4}$. Accordingly, we change the decomposition of the fermion field into

$$\psi(\mathbf{x}) = \frac{1}{\sqrt{\Omega}} \sum_{\lambda,\mathbf{p}} \left( e^{-ip\cdot x}\, u_{\lambda,\mathbf{p}} b_{\lambda,\mathbf{p}} + e^{ip\cdot x}\, v_{\lambda,\mathbf{p}} d^\dagger_{\lambda,-\mathbf{p}} \right), \tag{40}$$

where $d^\dagger_{\lambda,-\mathbf{p}}$ denotes the creation operator for a positron with helicity $\lambda$ and momentum $-\mathbf{p}$. Also, from now on, $|\Phi_0\rangle$ denotes the Fermi sea (i.e. excluding the Dirac sea).

Let us first consider the case $m = 0$. Then we have

$$\bar{u}_{\lambda,\mathbf{p}} u_{\nu,\mathbf{k}} = \phi^\dagger_{\lambda\hat{\mathbf{p}}}\phi_{\nu\hat{k}}\delta_{\nu,-\lambda} \qquad \bar{v}_{\lambda,\mathbf{p}} u_{\nu,\mathbf{k}} = -\lambda\phi^\dagger_{\lambda\hat{\mathbf{p}}}\phi_{\nu\hat{k}}\delta_{\nu,\lambda}. \tag{41}$$

Only spinors with opposite chirality contribute in the coupling to a scalar field. From these relations, one immediately deduces the form of the eigenstates

$$\begin{aligned} |\Psi^u_{-\mathbf{p}}\rangle_{\pm,\lambda} &= \sqrt{z_\pm} \left( b_{\lambda,\mathbf{p}} + g \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_k \Omega}} \frac{\phi^\dagger_{\lambda\hat{\mathbf{p}}}\phi_{-\lambda\hat{k}}}{\omega_\pm - p\,\cos\theta} b_{-\lambda,\mathbf{p}+\mathbf{k}} a^\dagger_{\mathbf{k}} \right) |\Phi_0\rangle \\ |\Psi^v_{-\mathbf{p}}\rangle_{\pm,\lambda} &= \sqrt{z_\pm} \left( d^\dagger_{\lambda,-\mathbf{p}} + g \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_k \Omega}} \frac{-\lambda\,\phi^\dagger_{\lambda\hat{\mathbf{p}}}\phi_{\lambda\hat{k}}}{-\omega_\pm - p\,\cos\theta} b_{\lambda,\mathbf{p}+\mathbf{k}} a^\dagger_{\mathbf{k}} \right) |\Phi_0\rangle, \end{aligned} \tag{42}$$

where $|\Psi^u_{-\mathbf{p}}\rangle$ (resp. $|\Psi^v_{-\mathbf{p}}\rangle$) is the eigenstate corresponding to $\varepsilon = +1$ (resp. $\varepsilon = -1$), the subscript $+$ (resp. $-$) refers to the normal (resp. abnormal) branch, and $z_\pm$ is a normalization constant. The norm of the states can be calculated directly from Eq.(42) by replacing the sum over $\mathbf{k}$ by an integral, as in Eq.(35), and noting that $\left|\phi^\dagger_{\lambda\hat{\mathbf{p}}}\phi_{\lambda\hat{k}}\right| = \cos(\theta/2)$ and $\left|\phi^\dagger_{\lambda\hat{\mathbf{p}}}\phi_{-\lambda\hat{k}}\right| = \sin(\theta/2)$. One then obtains $z_\pm = (\omega^2_\pm - p^2)/2M^2$, which coincides with the residues given by Eq.(18). It may also be verified by an explicit calculation, using the results of section 2, that all the states are orthogonal.

One sees on Eqs. (42) that, as a result of the interactions, the "bare" hole $b_{\lambda,\mathbf{p}}|\Phi_0\rangle$ and the "bare" positron $d^\dagger_{\lambda,-\mathbf{p}}|\Phi_0\rangle$ couple *independently* to a a coherent superposition of hole-photon states. There is no direct mixing between the hole and the positron states. Such a mixing does in fact occur when $m \neq 0$ and $p \neq 0$, as we shall see later, but even then, this is not a dominant feature. The basic mechanism at work is best understood by concentrating on what happens to one of the single particle states, the hole state for example, as the interaction is switched on. This is illustrated in Fig.7 where, on the left hand side is drawn the dispersion relation $\omega(p)$ for the hole state in absence of interaction, while in the right hand side is the split dispersion relation $\omega_\pm(p)$ which results when the interaction is taken into account. The shaded area represents the phase space occupied by the continuum of hole-photon states to which the hole couples. We have represented only the dominant part of the density of states. As we have seen in section 3.1, the density of state is non vanishing in the region $p < \omega < \epsilon_{\mathbf{p}}$, but it does vanish as $\omega \to \epsilon_{\mathbf{p}}$. As it is clear from this figure, this coupling to the continuum shifts some of the single particle strength from the positive energy, where it normally belongs, to the negative energies where it appears as an "abnormal" branch. In the limit where $m, p \ll \mu$, the strength becomes equally distributed among the two branches. The branches corresponding to the positrons are symmetrical to those just described for the hole. The interpretation of the two branches with positive $\omega$ is then clear. The normal branch is that part of the electron strength which is pushed up by the interaction with the continuum; the abnormal branch is that part of the positron strength which is pushed up to positive energies by the same interactions. We note that the energy shift, being of order $g\mu$ rather than $g^2\mu$, is non perturbative. Most of the electrons in the Fermi sea contribute to this phenomenon, and indeed the structure of the states just described has many features of collective excitations in many body systems.

We now discuss some properties of the spectrum in the light of these remarks.

When $m = 0$ and $p \to 0$, $\omega \to M$ (see Eq. (15)) and $z_\pm \to 1/2$ (see Eq. (18)) and all four states have a similar structure, i.e. the single particle strength is equally distributed among the four of them. When $p$ increases, the abnormal branch quickly becomes $\omega_- \simeq -p$ with a residue $z_- \to 0$, while the normal branch goes over to the usual fermion dispersion relation with $z_+ \to 1$. At the same time, the relative weight of the hole-photon component, $(1 - z_\pm)/z_\pm$ decreases for the normal branch, but increases for the abnormal one. As revealed by a simple analysis, when $p \gtrsim M$, only the states at the edges of the hole-photon continuum contribute to the abnormal branches. These states are of the form $b_{\lambda,\mathbf{p}+\mathbf{k}} a^\dagger_{\mathbf{k}} |\Phi_0\rangle$, with $\mathbf{p}$ and $\mathbf{k}$ antiparallel when $\omega \sim -p$, and with $\mathbf{p}$ and $\mathbf{k}$ parallel when $\omega \sim p$.

The two states (42), i.e. the dressed hole and the dressed positron, have the same fermionic charge $-1$ with respect to the unperturbed ground state. This may be verified explicitly by taking the expectation values of the operator $Q \equiv \int : \psi^\dagger \psi : d^3\mathbf{x} = \sum_{\lambda,p} \left( b^\dagger_{\lambda,p} b_{\lambda,p} - d^\dagger_{\lambda,p} d_{\lambda,p} \right)$ in either state. One may write this expectation value as

$$Q_\pm = -(z_\pm + (1 - z_\pm)) \tag{43}$$

where the first contribution is the "direct" contribution, i.e. that of the bare hole (or the bare positron), while the second contribution is the "induced" part, that is the charge carried by the hole-photon component. When $p$ grows, $z_+ \to 1$ and $z_- \to 0$, as already mentioned. Thus, as $p$ grows, on the normal branch the direct component grows, while on the abnormal branch it is the induced component which grows. Thus, when $p \gtrsim M$, the single particle state completely decouples from the abnormal branch.

One gets another illustration of this phenomenon by calculating the expectation value of the current operator $\mathbf{I} = \int \psi^\dagger \boldsymbol{\alpha} \psi d^3\mathbf{x} = \sum_{\lambda,p} \hat{\mathbf{p}} \left( b^\dagger_{\lambda,p} b_{\lambda,p} - d^\dagger_{\lambda,p} d_{\lambda,p} \right)$. In a bare hole state $b_{\lambda,\mathbf{p}} |\Phi_0\rangle$ this is $\mathbf{I} = -\hat{\mathbf{p}}$, opposite to that of the bare positron with the same momentum, i.e. $d^\dagger_{\lambda,-\mathbf{p}}$. This difference may be understood by noting that, viewed as a hole in the Dirac sea, the positron is associated to a single particle state whose energy, $-\epsilon_{\mathbf{p}}$ decreases with increasing momentum $\mathbf{p}$, whereas the single particle state corresponding to the hole in the Fermi sea has an energy $+\epsilon_{\mathbf{p}}$ which increases with increasing momentum. This difference translates into a difference in the signs of the velocities $d\epsilon_p/dp$ of the two excitations. Taking the expectation

value of the current operator in the states(42) one obtains

$$\mathbf{I}^{\varepsilon}_{\pm} = -\varepsilon\hat{\mathbf{p}}\, z_{\pm}\left[1 + \frac{M^2}{2}\int_{-1}^{1} \mathrm{d}\cos\theta \frac{\cos\theta(1-\cos\theta)}{(\omega_{\pm} - p\cos\theta)^2}\right], \tag{44}$$

where we have neglected $p$ compared to $k$ in evaluating the second terms. It can be verified that these expressions coincide with the velocity of the excitations as deduced from the dispersion relation $\omega_{\pm}(p)$. For example, for the hole excitation, we have

$$\frac{d\omega_{\pm}(p)}{d\mathbf{p}} = z_{\pm}\left(\frac{d\epsilon_p}{d\mathbf{p}} + \frac{\partial\Sigma_u(\omega_{\pm},\mathbf{p})}{\partial\mathbf{p}}\right) \tag{45}$$

with $z_{\pm} = 1/\left(1 - \partial\Sigma_u(\omega,\mathbf{p})/\partial\omega\right)$, $\Sigma_u = a + bp$ denotes the mass operator in the eigenspace corresponding to $\varepsilon = 1$, and $\mathbf{I}^{\varepsilon}_{\pm} = -\varepsilon(d\omega_{\pm}/d\mathbf{p})$. By carrying out the integrals in (44), and using the results of section 2, one can rewrite the currents as follows

$$\mathbf{I}^{\varepsilon}_{\pm} = -\varepsilon\hat{\mathbf{p}}\left[2z_{\pm} + \frac{\omega_{\pm}}{p}(1 - 2z_{\pm})\right] \tag{46}$$

As we did earlier in the case of the charge, we can interpret Eq.(46) as the sum of a "direct" contribution, that carried by the bare single particle, i.e. $-\varepsilon\hat{\mathbf{p}}z_{\pm}$, and an "induced" contribution coming from the hole-photon component of the states (42). The hole photon part of the quantum states gives a contribution to the current which is opposite to that of the current carried by the bare particle. When $p \to 0$, $\mathbf{I}^{\varepsilon}_{\pm} = -\varepsilon\hat{\mathbf{p}}/3$. Thus, at very low momentum, the sign of the current is the same as for the corresponding bare particle: the negative group velocity on the abnormal branch with $\omega > 0$ simply reflects the fact that the corresponding bare particle is a *positron* state dressed by the medium, in spite of the fact that it has positive $\omega$. When $p$ becomes large, the current goes to 1 on both positive $\omega$ branches. However this is achieved in different ways on the normal and the abnormal branch.On the normal branch, the residue $z_+ \to 1$ and, at large $p$ the current is carried entirely by the bare hole. On the abnormal branch, $z_- \to 0$, and the current is no longer carried by the positron component, but only by the hole-photon component. This explains the minimum (for $\omega > 0$) in the abnormal branch of the dispersion relation. Indeed, as we have seen, at small $p$ the current is mostly that of the bare positron. As $p$ increases, the weight of the hole-photon component increases until it completely dominates. At this point, the current is that of the hole in the hole-photon state,

and we have seen that the states with $\omega \sim p$ correspond to holes with momentum parallel to $\mathbf{p}$, and therefore to current opposite to that of the positron.

We now turn to the case of a massive fermion at rest ($p = 0$). Then

$$\bar{u}_{\lambda 0} u_{\nu,\mathbf{k}} = \frac{1}{\sqrt{2}} \phi^\dagger_\lambda \phi_{\nu \hat{k}} \qquad\qquad \bar{v}_{\lambda 0} u_{\nu,\mathbf{k}} = \frac{-\nu}{\sqrt{2}} \phi^\dagger_\lambda \phi_{\nu \hat{k}} \tag{47}$$

The eigenvectors are therefore

$$\begin{aligned} |\Psi^u_0\rangle_{\pm,\lambda} &= \sqrt{z_\pm} \left( b_{\lambda,0} + \frac{g}{\omega_\pm \sqrt{2}} \sum_{\nu,\mathbf{k}} \frac{\phi^\dagger_\lambda \phi_{\nu \hat{k}}}{\sqrt{2\omega_k \Omega}} b_{\nu,\mathbf{k}} a^\dagger_{\mathbf{k}} \right) |\Phi_0\rangle \\ |\Psi^v_0\rangle_{\pm,\lambda} &= \sqrt{z_\pm} \left( d^\dagger_{\lambda,0} + \frac{g}{\omega_\pm \sqrt{2}} \sum_{\nu,\mathbf{k}} \frac{-\nu\, \phi^\dagger_\lambda \phi_{\nu \hat{k}}}{\sqrt{2\omega_k \Omega}} b_{\nu,\mathbf{k}} a^\dagger_{\mathbf{k}} \right) |\Phi_0\rangle \end{aligned} \tag{48}$$

The normalisation constant $z_\pm$ can be calculated directly, as for $m = 0$. However, both helicity states now contribute to the sum and one must use $\sum_\nu \left| \phi^\dagger_\lambda \phi_{\nu \hat{k}} \right|^2 = 1$. This gives immediately $z_\pm^{-1} = 1 + M^2/\omega_\pm^2$, in agreement with Eq.(19). The orthogonality of states with different $\omega$ can also be checked explicitly. One could also repeat here the discussion given above in the case $m = 0$ concerning the decoupling of the single particle excitation which occurs when $m \gtrsim M$.

A remarkable fact in Eq.(48) is that $\omega$ factorizes out of the hole-photon component, so that, up to a trivial multiplication factor $1/\omega_\pm$, the same coherent superposition of hole-photon states contributes to both the normal and the abnormal branch. Thus for example, we may write the eigenstate corresponding to the normal branch $\omega_+$ in the form

$$|\Psi^u_0\rangle_{+,\lambda} = \sqrt{z_+} b_{\lambda,0} |\Phi_0\rangle + \sqrt{1 - z_+} |\Phi_1\rangle \tag{49}$$

where $|\Phi_1\rangle$ is the collective state. The eigenstate corresponding to the abnormal branch $\omega_-$ is also a linear combination of $b_{\lambda,0}|\Phi_0\rangle$ and $|\Phi_1\rangle$. Since $\omega_+ \neq \omega_-$, the eigenstates are orthogonal to each other and we may write, up to a phase,

$$|\Psi^u_0\rangle_{-,\lambda} = \sqrt{1 - z_+} b_{\lambda,0} |\Phi_0\rangle - \sqrt{z_+} |\Phi_1\rangle \tag{50}$$

and thus $z_- = 1 - z_+$. The explicit construction of the eigenstates thus explains naturally why the residues add up to unity when $p = 0$ (see section 2). On the

other hand, for a massless fermion with $\mathbf{p} \neq 0$, Eq.(42) shows that the hole-photon component is different for the two branches and the previous argument cannot be used. Indeed, the sum of the residues is smaller than unity in this case, some of the specral weight being displaced in the region $-p < \omega < p$.

Let us finally consider the general case when $m$ and $p$ are both different from zero. Then the solutions $\chi$ of Eq.(37) are linear superpositions of the solutions $w_{\lambda,\mathbf{p}}$ of the free Dirac hamiltonian. For instance, a solution with $\varepsilon = +1$ and helicity $\lambda$ can be written as a superposition of $u_{\lambda,\mathbf{p}}$ and $v_{\lambda,\mathbf{p}}$ defined in Eq.(39):

$$\chi = \sqrt{1 - r_\pm}\; u_{\lambda,\mathbf{p}} + \mathrm{e}^{i\varphi}\sqrt{r_\pm}\; v_{\lambda,\mathbf{p}} \tag{51}$$

where $r_+$ (resp. $r_-$) measures the mixing between the hole state $b_{\lambda,\mathbf{p}}|\Phi_0\rangle$ and the positron state $d^\dagger_{\lambda,-\mathbf{p}}|\Phi_0\rangle$ on the normal (resp. abnormal) branch. For a state with $\varepsilon = -1$, $u$ and $v$ must be interchanged in the latter equation. Thus the eigenstates are of the form

$$|\Psi^u_{-\mathbf{p}}\rangle_{\pm,\lambda} = \sqrt{z_\pm}\left(\sqrt{1 - r_\pm}\; b_{\lambda,\mathbf{p}}|\Phi_0\rangle + \sqrt{r_\pm}\; d^\dagger_{\lambda,-\mathbf{p}}|\Phi_0\rangle\right) + \sqrt{1 - z_\pm}|\Phi_1\rangle \tag{52}$$

if $\varepsilon = +1$, and $b$ and $d^\dagger$ must be interchanged if $\varepsilon = -1$. In Eq.(52), $z_\pm$ is the residue at the quasiparticle pole and $|\Phi_1\rangle$ is a linear superposition of states $b_{\nu,\mathbf{p}+\mathbf{k}}a^\dagger_{\mathbf{k}}|\Phi_0\rangle$ as in Eqs.(42) and (48). From Eq.(52) we deduce that the probability that a hole state $b_{\lambda,\mathbf{p}}|\Phi_0\rangle$ be found in the eigenstate $|\Psi^u_{-\mathbf{p}}\rangle_{\pm,\lambda}$ (resp. $|\Psi^v_{-\mathbf{p}}\rangle_{\pm,\lambda}$) is $z_\pm(1 - r_\pm)$ (resp. $z_\pm r_\pm$).

The mixing coefficients $r_\pm$ can be calculated easily by noting that the matrices $\gamma^0$ and $\boldsymbol{\alpha}\cdot\hat{\mathbf{p}}$ appearing in Eqs. (29) and (37) obey the same commutation relations as two arbitrary Pauli matrices, say $\sigma_1$ and $\sigma_2$. On may then associate a solution to Eq. (29) to a spinor pointing in a direction making an angle $\gamma$ with the 1-axis, with $\tan\gamma = p/m$, and the corresponding solution to Eq. (37) to a spinor pointing in a direction making an angle $\gamma'$ with the 1-axis, with $\tan\gamma' = (1+b)p/m$. The overlap between the two spinors is then $\left|w^\dagger w_{\lambda,\mathbf{p}}\right| = \sqrt{1 - r_\pm} = \cos((\gamma - \gamma')/2)$ or, explicitly,

$$r_\pm = \frac{1}{2}\left[1 - \frac{1 + (1+b)p^2/m^2}{\sqrt{1 + p^2/m^2}\sqrt{1 + (1+b)^2p^2/m^2}}\right]. \tag{53}$$

It depends on $\omega$ through $b$. For the normal branch (resp. the abnormal branch), $b$ varies from 0 to $-1/3$ (resp. from $-\infty$ to $-1/3$) when the chemical potential

increases from 0 to $+\infty$. Thus the bounds on the mixing coefficient $r_\pm$ are

$$0 < r_+ < \frac{1}{2}\left[1 - \frac{1+(2/3)p^2/m^2}{\sqrt{1+p^2/m^2}\sqrt{1+(4/9)p^2/m^2}}\right] < r_- < \frac{1}{2}\left[1 + \frac{p/m}{\sqrt{1+p^2/m^2}}\right]. \quad (54)$$

The maximum value of $r_+$ for the normal branch is reached when $p/m = \sqrt{3/2}$ and infinite density: $r_+^{\max} = 1/2 - \sqrt{6}/5 \simeq 10^{-2}$, which is rather low. On the other hand, for the abnormal branch the mixing can exceed $1/2$ if the density is not too high; however, the residue $z_-$ of the abnormal branch is then very small, so that the product $z_- r_-$, which is the probability that a hole state $b_{\lambda,\mathbf{p}}|\Phi_0\rangle$ be found in the state $|\Psi^v\rangle_{-,\lambda}$, remains small, at most of the order of $10^{-2}$.

This mixing between holes and positrons can be easily understood in the ultra–high density limit $M \gg m, p$. In this limit Eq.(12) gives $\omega \to \pm M$ for both values of $\varepsilon$ (see Fig.1a). Thus there is a degeneracy at $\omega = M$ (resp. $-M$) between the branch $\omega_+$ (normal hole branch) and the branch $-\omega_-$ (abnormal positron branch). This degeneracy is lifted by terms proportional to $p$ and $m$, which act like a perturbation in the eigenvalue equation (37) and therefore induce mixing between the two branches. Writing Eq.(37) in the form $(H_0(\omega)+V)\chi = \omega\chi$ with $H_0(\omega) = a \simeq M^2/\omega$ and $V = (2/3)\boldsymbol{\alpha}\cdot\mathbf{p} + \gamma^0 m$ (we have used the fact that $b \to -1/3$ for $|\omega| \gg p$), and expanding around $\omega = M$ one gets

$$\left[\left(\frac{dH_0}{d\omega}\right)_{\omega=M}(\omega - M) + V\right]\chi = (\omega - M)\chi. \quad (55)$$

Since $(dH_0/d\omega)_{\omega=M} = -1$, the problem reduces to diagonalizing $V$. In particular, one gets immediately the eigenvalues $\omega = M \pm \sqrt{p^2/9 + m^2/4}$. Note that $r_+ = r_-$ in this limit, and that the mixing vanishes if either $m = 0$ or $p = 0$, as expected.

The calculation of the eigenstates that we have performed in this section, which is equivalent to solving Dyson's equation for the poles of the Green function in the ultrarelativistic limit, assumes that the modifications of the ground state can be ignored. More generally, the solution of Dyson's equation for the full one loop self energy is equivalent to the following eigenvalue problem

$$(\omega - \epsilon_{\lambda,p})X_{\lambda,\mathbf{p}} - \sum_{k\nu}\frac{g}{\sqrt{2\omega_k\Omega}}\bar{w}_{\nu,\mathbf{p}+\mathbf{k}}w_{\lambda,\mathbf{p}}Y_{\nu,\mathbf{p},\mathbf{k}} - \sum_{k\nu}\frac{g}{\sqrt{2\omega_k\Omega}}\bar{w}_{\nu,\mathbf{p}+\mathbf{k}}w_{\lambda,\mathbf{p}}Z_{\nu,\mathbf{p},\mathbf{k}} = 0$$

$$-\frac{g}{\sqrt{2\omega_k\Omega}}\sum_\lambda \bar{w}_{\nu,\mathbf{p}+\mathbf{k}}w_{\lambda,\mathbf{p}}X_{\lambda,\mathbf{p}} + (\omega - \epsilon_{\nu,\mathbf{p}+\mathbf{k}} + \omega_k)Y_{\nu,\mathbf{p},\mathbf{k}} = 0$$

$$-\frac{g}{\sqrt{2\omega_k \Omega}} \sum_\lambda \bar{w}_{\nu,\mathbf{p+k}} w_{\lambda,\mathbf{p}} X_{\lambda,\mathbf{p}} + (\omega - \epsilon_{\nu,\mathbf{p+k}} - \omega_k) Z_{\nu,\mathbf{p},\mathbf{k}} = 0 \tag{56}$$

where, in the first equation the sum runs only over the states such that $\epsilon_{\mathbf{p+k}} < \mu$ for the $Y$ amplitudes, and such that $\epsilon_{\mathbf{p+k}} > \mu$ for the $Z$ amplitudes. That this system of equations is equivalent to Dyson's equation can be easily verified by eliminating the amplitudes $Y_{\nu,\mathbf{p},\mathbf{k}}$ and $Z_{\nu,\mathbf{p},\mathbf{k}}$ with the help of the last two equations, and comparing the resulting equation for $X_{\lambda,\mathbf{p}}$ with Dyson's equation. This comparison allows us to identify the mass operator, and one recovers indeed Eq. (20). The equations (56) take into account the modifications of the Fermi sea due to interactions, at this order of perturbation theory. In particular, because of the interactions, single particle levels are not fully occupied below the Fermi level, and there are non vanishing amplitudes of the form

$$Z_{\nu,\mathbf{p},\mathbf{k}} = \langle \Psi_{\nu,\mathbf{p},\mathbf{k}} | b^\dagger_{\nu,\mathbf{p+k}} a_{\mathbf{k}} | \Psi_0 \rangle, \tag{57}$$

where $|\Psi_0\rangle$ and $\langle \Psi_{\nu,\mathbf{p},\mathbf{k}}|$ denote true eigenstates, that is including perturbative corrections. As for the amplitudes $X$ and $Y$ they are given as before by

$$X_{\nu,\mathbf{k}} = \langle \Psi_{\nu,\mathbf{p}} | b_{\nu,\mathbf{p}} | \Psi_0 \rangle \qquad\qquad Y_{\nu,\mathbf{p},\mathbf{k}} = \langle \Psi_{\nu,\mathbf{p},\mathbf{k}} | b_{\nu,\mathbf{p+k}} a^\dagger_{\mathbf{k}} | \Psi_0 \rangle, \tag{58}$$

but now, the state $|\Psi_0\rangle$ contains perturbative corrections, i.e. it differs from the unperturbed Fermi sea $|\Phi_0\rangle$. It is easily seen on Eq. (56) that in the ultrarelativistic limit, and for soft modes, the amplitudes $Z_{\nu,\mathbf{p},\mathbf{k}}$ become negligible as compared to the $Y_{\nu,\mathbf{p},\mathbf{k}}$ ones.

# 5 Discussion

We have, in the main text, described a model plasma composed of Dirac fermions interacting with a scalar field. Most of our results extend however to a dense system of electrons (see Appendix C), or to dense quark matter. In such high density systems, particle masses are negligible in a first approximation, and the only energy scale is provided by the chemical potential, $\mu$. Most particles in such a system have momenta of order $\mu$. Long wavelength excitations develop on the scale $g\mu$, and have a collective character. Bosonic modes, not discussed in this paper, correspond to the usual plasmons involving particle-hole excitations localized in the vicinity of the Fermi surface. We have concentrated here on fermionic modes, which can be excited

by processes adding small momentum positrons or removing electrons at the bottom of the Fermi sea.

The physical picture which emerges from our calculation is quite in line with the work of Ref.[13] where the equations of motions describing the long wavelength excitations of an ultrarelativistic plasma at high temperature are reduced to a set of coupled mean field and kinetic equations incorporating the dominant medium effects as "polarization" phenomena. In either case of large temperature or large chemical potential, the polarization is entirely due to the hard particles with momenta of order $T$ or $\mu$. In the zero temperature case, the dominant process is that in which a hard electron annihilates with the soft hole, or the soft positron, and turns into a hard photon. Thus, the quantum numbers of the hard particles fluctuate, with a typical period of order $1/g\mu$. This is accompanied by a corresponding oscillation in the number of soft particles, which is interpreted as an oscillation of the average soft fermionic field in Ref.[13]. The present study allows us to give a simple illustration of this in terms of the quantum states that we have constructed. Consider indeed a perturbation which consists in adding soft positrons to, or creating soft holes in, the unperturbed Fermi sea at time $t = 0$, say. The system will then evolve into a state $|\Phi(t)\rangle$ which will contain components on the states $|\Psi^e\rangle_\pm$ constructed in section 4:

$$|\Phi(t)\rangle = |\Phi_0\rangle + \alpha(t)|\Psi^e\rangle_\pm \tag{59}$$

where $\alpha(t)$ is a small time dependent amplitude. The expectation value of the fermionic field in such a state is proportional to the amplitudes $X_{\lambda,\mathbf{p}}$ introduced in section 4, while the amplitudes $Y_{\lambda,\mathbf{p},\mathbf{k}}$ contribute to the "induced source" for the fermion field.

The explicit construction of quasiparticle states in section 4 clearly shows how new modes emerge: the free electron hole is coupled to a large number of states which do not interact together and have an energy comparable to that of the unperturbed hole excitation. To outline this structure, let us consider a simplified situation where a quantum state $|0\rangle$ with energy $E_0$ (corresponding to $\epsilon_p$ in the previous section) is coupled with equal strength $V$ to a large number $N$ of states $|i\rangle$, uniformly distributed throughout the energy interval $[-\Delta E/2, \Delta E/2]$. In the calculation performed in section 4, $\Delta E = 2p$, $N$ is of order $\Omega\mu^3$ in a volume $\Omega$ and $V$ is of order $g/\sqrt{\Omega\mu}$, i.e. $V\sqrt{N} \sim g\mu$. The equation determining the modes $\omega$ is

$$\omega - E_0 = -\frac{NV^2}{\Delta E} \log \left| \frac{\omega + \Delta E/2}{\omega - \Delta E/2} \right| \tag{60}$$

A substantial energy shift and/or a new quasiparticle will occur only if $V\sqrt{N}$ is at least of the same order of magnitude as both $E_0$ (which is the difference between the energy of $|0\rangle$ and that of the centroid of the states $|i\rangle$ to which $|0\rangle$ couples) and $\Delta E$ (the energy interval over which the states $|i\rangle$ are spread). For the calculation of section 4, this is equivalent to saying that $g\mu$ must be at least as large as $m$ and $p$.

One can then wonder whether a similar situation could occur in other systems. In fact, a structure similar to that studied here has been found in the study of the electron gas at intermediate densitites[14, 16]. There, the coupling of an electron hole to collective plasmons gives rise to a peak in the imaginary part of the hole self-energy which is sufficiently strong to produce a new quasiparticle, dubbed the "plasmaron". However, in the case of the non relativistic electron gas, the validity of the perturbative approach is less clear, and it was indeed found that vertex corrections could very well suppress the structure[15].

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## APPENDIX A : The vacuum part of the self–energy

At zero temperature and zero chemical potential, the mass operator $\Sigma_0(\mathbf{p},\omega)$ may be written as

$$\Sigma_0(\mathbf{p},\omega) = a_0(p^2,p_0)\gamma_0 + b_0(p^2)\boldsymbol{\gamma}\cdot\mathbf{p} + c_0(p^2) \tag{A.1}$$

where, because of Lorentz invariance, the functions $a_0/p_0, b_0$ and $c_0$ depend only on the 4-momentum squared, $p^2 = p_0^2 - \mathbf{p}^2$, and $b_0(p^2) = -a_0(p^2,p_0)/p_0$. We shall limit the discussion to the case $\mathbf{p}=0$ (the result for $\mathbf{p}\neq 0$ may be obtained from Lorentz invariance). The expressions for $a_0$ and $c_0$ may be read off Eqs. (6), and they involve ultraviolet diverging integrals over momenta. Adding the usual counterterms of mass and wave function renormalization, we write the renormalized inverse propagator as

$$\begin{aligned}\frac{G_R^{-1}(\omega)}{Z_2} &= G_0^{-1}(\omega) + \Sigma_0(\omega) - \delta m \\ &= \left(-\omega + a_0(\omega)\right)\gamma^0 + \left(m + c_0(\omega) - \delta m\right)\end{aligned} \tag{A.2}$$

The counterterms are fixed by the on–shell renormalization conditions

$$\begin{aligned}\delta m &= a_0(m) + c_0(m) \\ Z_2^{-1} - 1 &= 1 - \left.\frac{\partial a_0}{\partial\omega}\right|_{\omega=m} - \left.\frac{\partial c_0}{\partial\omega}\right|_{\omega=m}.\end{aligned} \tag{A.3}$$

In doing the calculations of $Z_2$, it is necessary to give the photon a small mass $\lambda$ to avoid the infrared divergence. The Green function can finally be written in the form

$$G_R^{-1}(\omega) = \left(-\omega + a_R(\omega)\right) + m + c_R(\omega), \tag{A.4}$$

where $a_R(\omega)$ and $c_R(\omega)$ are the renormalized contributions of the vacuum to the mass operator, which must be added to $a_1(\omega)$ and $c_1(\omega)$ in the finite density case:

$$\begin{aligned}a_R(\omega) &= \frac{g^2}{16\pi^2}\left[\frac{\omega^2-m^2}{2\omega}\left(1 + \frac{\omega^2+m^2}{\omega^2}\ln\left|1-\frac{\omega^2}{m^2}\right|\right) - \omega\left(5 + 4\ln\frac{\lambda}{m}\right)\right] \\ c_R(\omega) &= \frac{g^2 m}{16\pi^2}\left[\left(1-\frac{m^2}{\omega^2}\right)\ln\left|1-\frac{\omega^2}{m^2}\right| + \left(5 + 4\ln\frac{\lambda}{m}\right)\right],\end{aligned} \tag{A.5}$$

and we have kept only the dominant terms in $\lambda/m$ as $\lambda \to 0$.

Note that the renormalization does not affect the imaginary part which is finite. Note also that in calculating the frequencies of the modes at high density, one can ignore $a_R$ and $c_R$. Indeed, when $\omega \sim g\mu$, their contribution is of order $g^3\mu$ while that of $\Sigma_1$ is of order $g\mu$ . Thus, in particular, the term in $\lambda$ which is infrared divergent and, in gauge theories, gauge dependent, does not contribute in the high density limit. Presumably it would disappear in a consistent calculation of quasiparticle energies at order $g^3\mu$, i.e. beyond one-loop .

## APPENDIX B : Exact expressions for the one–loop mass operator at $T = 0$

In this appendix we give the exact expression of the mass operator (20), valid when either $p$ or $m$ vanishes.

If $p = 0$, the finite density contribution to the mass operator can be written

$$\Sigma_1(\mathbf{p} = 0, \omega) = a_1(\omega)\gamma^0 + c_1(\omega) \tag{B.1}$$

The real parts of $a_1$ and $c_1$ are

$$\begin{aligned}
\mathrm{Re}\,[a_1(\omega)] &= \frac{g^2 m}{16\pi^2}\left[\frac{\sqrt{\mu^2 - m^2}}{m}\left(1 + \frac{m^2}{\omega^2} + \frac{\mu}{\omega}\right) + \frac{m^3}{\omega^3}\cosh^{-1}\left(\frac{\mu}{m}\right)\right. \\
&\qquad \left. + \frac{\omega}{2m}\left(1 - \frac{m^4}{\omega^4}\right)\ln\left|\frac{\omega - \mu - k_F}{\omega - \mu + \sqrt{\mu^2 - m^2}}\right|\right] \\
\mathrm{Re}\,[c_1(\omega)] &= \frac{g^2 m}{8\pi^2}\left[\frac{k_F}{\omega} + \frac{m^2}{\omega^2}\cosh^{-1}\left(\frac{\mu}{m}\right)\right. \\
&\qquad \left. + \frac{1}{2}\left(1 - \frac{m^2}{\omega^2}\right)\ln\left|\frac{\omega - \mu - k_F}{\omega - \mu + \sqrt{\mu^2 - m^2}}\right|\right]
\end{aligned} \tag{B.2}$$

with $k_F = k_F$ the Fermi momentum. The imaginary parts are (including the vacuum contribution)

$$\begin{aligned}
\mathrm{Im}\,[a_1(\omega)] &= -\pi\frac{g^2}{16\pi^2}\,\frac{m^4 - \omega^4}{2\omega^3} \\
\mathrm{Im}\,[c_1(\omega)] &= -\pi\frac{g^2 m}{16\pi^2}\,\frac{m^2 - \omega^2}{\omega^2}
\end{aligned} \tag{B.3}$$

if

$$\omega < -m \quad \text{or} \quad \mu - k_F < \omega < m \tag{B.4}$$

and 0 otherwise. Note that this equation agrees with Eq.(22) when we set $p = 0$ in the latter. We note the relation $\mathrm{Im}(a + c)/\mathrm{Im}(a - c) = (\omega + m)^2/(\omega - m)^2$, identical to that found at high $T$ and $\mu = 0$ [8]. The imaginary part is independent of $\mu$, except for the domain where it is non vanishing. In the high density limit, this domain is the interval $m^2/(2\mu) < \omega < m$; the limit $m \to 0$ is thus singular.

If $m = 0$, $k_F = \mu$, the finite density contribution to the mass operator can be written

$$\Sigma_1(\mathbf{p},\omega) = a_1(\mathbf{p},\omega)\gamma^0 + b_1(\mathbf{p},\omega)\boldsymbol{\gamma}\cdot\mathbf{p} \tag{B.5}$$

The functions $a_1$ and $b_1$ can be calculated analytically by choosing $(\mathbf{p}-\mathbf{k})$ as integration variable in Eq. (4). The real parts are:

$$\begin{aligned}
\mathrm{Re}\,[a_1(p,\omega)] &= \frac{g^2}{32\pi^2 p}\Bigg\{p\mu + \frac{(2\mu-\omega-p)(2\mu+\omega+p)}{4}\ln\left|\frac{\omega+p}{2\mu-\omega-p}\right| \\
&\qquad - \frac{(2\mu-\omega+p)(2\mu+\omega-p)}{4}\ln\left|\frac{\omega-p}{2\mu-\omega+p}\right|\Bigg\} \\
\mathrm{Re}\,[b_1(p,\omega)] &= \frac{g^2}{16\pi^2 p^2}\Bigg\{\mu^2 + \frac{2\mu-\omega-p}{4p}\left[\omega^2-p^2-\frac{\omega}{2}(2\mu+\omega+p)\right]\ln\left|\frac{\omega+p}{2\mu-\omega-p}\right| \\
&\qquad - \frac{2\mu-\omega+p}{4p}\left[\omega^2-p^2-\frac{\omega}{2}(2\mu+\omega-p)\right]\ln\left|\frac{\omega-p}{2\mu-\omega+p}\right|\Bigg\}
\end{aligned} \tag{B.6}$$

As to the imaginary parts, several cases must be distinguished:

$$\begin{aligned}
III: \qquad \mathrm{Im}\,[a(p,\omega)] &= \pi\frac{g^2}{32\pi^2}\omega \\
\mathrm{Im}\,[b(p,\omega)] &= -\pi\frac{g^2}{32\pi^2} \\
II: \qquad \mathrm{Im}\,[a(p,\omega)] &= -\pi\frac{g^2}{32\pi^2}\frac{(2\mu-\omega-p)(2\mu+\omega+p)}{4p} \\
\mathrm{Im}\,[b(p,\omega)] &= -\pi\frac{g^2}{32\pi^2}\frac{2\mu-\omega-p}{2p^3}\left[\omega^2-p^2-\frac{\omega}{2}(2\mu+\omega+p)\right] \\
IV: \qquad \mathrm{Im}\,[a(p,\omega)] &= \mathrm{Im}\,[b(p,\omega)] = 0 \\
Ia: \qquad \mathrm{Im}\,[a(p,\omega)] &= \pi\frac{g^2}{32\pi^2}\frac{(2\mu-\omega-p)(2\mu+\omega+p)}{4p} \\
\mathrm{Im}\,[b(p,\omega)] &= \pi\frac{g^2}{32\pi^2}\frac{2\mu-\omega-p}{2p^3}\left[\omega^2-p^2-\frac{\omega}{2}(2\mu+\omega+p)\right] \\
Ib: \qquad \mathrm{Im}\,[a(p,\omega)] &= -\pi\frac{g^2}{32\pi^2}\omega \\
\mathrm{Im}\,[b(p,\omega)] &= \pi\frac{g^2}{32\pi^2}
\end{aligned} \tag{B.7}$$

where the various regions are indicated on Fig.10 ($III : \omega < -p; II : -p < \omega < \min(p, 2\mu - p); IV : \min(p, 2\mu - p) < \omega < \max(p, 2\mu - p); Ia : \max(p, 2\mu - p) < \omega < 2\mu + p; Ib : \omega > 2\mu + p$). These expressions include the contribution of the vacuum to the imaginary parts. One checks easily that these equations give the same result in the limit $p \to 0$ as Eqs.(B.2) and (B.3) when $m \to 0$. Furthermore, if $\mu \gg \omega, p$, one recovers Eqs.(8) and Eq.(10). Note that the imaginary part is everywhere negative.

## APPENDIX C : QED

The results obtained in sections 2 and 4 in the case where fermions are coupled to a massless scalar field can be easily generalized to an electromagnetic coupling. The corresponding modifications are summarized in this appendix.

In the Coulomb gauge, the electromagnetic field is decomposed in

$$A^\mu(x) = \frac{1}{\sqrt{\Omega}} \sum_{\mathbf{k}} \sum_{\rho=1,2} \frac{1}{\sqrt{2\omega_k}} \left( e^{-ik\cdot x}\, \epsilon^\mu_{\rho,\mathbf{k}} a_{\rho,\mathbf{k}} + e^{ik\cdot x}\, \epsilon^{\mu,*}_{\rho,\mathbf{k}} a^\dagger_{\rho,\mathbf{k}} \right), \tag{C.1}$$

where $\rho = 1, 2$ is a polarization index and $\epsilon^\mu_{\rho,\mathbf{k}}$ are normalized polarization vectors : $\epsilon^0_{\rho,\mathbf{k}} = 0$, $\boldsymbol{\epsilon}_{\rho,\mathbf{k}} \cdot \mathbf{k} = 0$ and $\boldsymbol{\epsilon}_{\rho,\mathbf{k}} \cdot \boldsymbol{\epsilon}_{\rho',\mathbf{k}} = \delta_{\rho,\rho'}$. The interaction hamiltonian is now

$$\begin{aligned} H_{int} &= g \int \bar{\psi}(x) \not{A}(x) \psi(x) d^3x \\ &= \frac{g}{\sqrt{\Omega}} \sum_{\mathbf{p},\mathbf{k},\lambda,\nu,\rho} \frac{1}{\sqrt{2\omega_k}} \left( \bar{w}_{\nu,\mathbf{p}+\mathbf{k}} \not{\epsilon}_{\rho,\mathbf{k}} w_{\lambda,\mathbf{p}} a_{\rho,\mathbf{k}} + \bar{w}_{\nu,\mathbf{p}+\mathbf{k}} \not{\epsilon}^*_{\rho,\mathbf{k}} w_{\lambda,\mathbf{p}} a^\dagger_{\rho,-\mathbf{k}} \right) b^\dagger_{\nu,\mathbf{p}+\mathbf{k}} b_{\lambda,\mathbf{p}} \end{aligned} \tag{C.2}$$

and the result of the one loop calculation may be obtained, in the ultrarelativistic limit, by diagonalizing $H$ in the basis

$$b_{\lambda,\mathbf{p}}|\Phi_0\rangle \qquad\qquad b_{\nu,\mathbf{p}+\mathbf{k}} a^\dagger_{\rho,\mathbf{k}}|\Phi_0\rangle. \tag{C.3}$$

The corresponding amplitudes will be denoted by $X_{\lambda,\mathbf{p}}$ and $Y_{\lambda,\rho,\mathbf{p},\mathbf{k}}$. The eigenvalue equation takes the form (34) with

$$M(\mathbf{p},\bar{\omega}) = g^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left\{ \frac{\not{\epsilon}^*_{\rho,\mathbf{k}} \Lambda^+_{\mathbf{p}+\mathbf{k}} \gamma^0 \not{\epsilon}_{\rho,\mathbf{k}} f_{\mathbf{p}+\mathbf{k}}}{-\bar{\omega} - \epsilon_{\mathbf{p}+\mathbf{k}} + \omega_k} + \frac{\not{\epsilon}^*_{\rho,\mathbf{k}} \Lambda^-_{\mathbf{p}+\mathbf{k}} \gamma^0 \not{\epsilon}_{\rho,\mathbf{k}}}{-\bar{\omega} + \epsilon_{\mathbf{p}+\mathbf{k}} + \omega_k} \right\}. \tag{C.4}$$

In the ultrarelativistic limit ($k \gg m, p$), the term proportional to $m$ in $\Lambda^\pm(\mathbf{p}+\mathbf{k})$ can be neglected, and $\mathbf{p}+\mathbf{k}$ replaced by $\mathbf{k}$. Then, since $\boldsymbol{\epsilon}_{\rho,\mathbf{k}} \cdot \mathbf{k} = 0$, $\not{\epsilon}_{\rho,\mathbf{k}} = -\boldsymbol{\gamma} \cdot \boldsymbol{\epsilon}_{\rho,\mathbf{k}}$ anticommutes with $\boldsymbol{\gamma} \cdot \mathbf{k}$ and thus with $\Lambda^\pm_{\mathbf{k}} \gamma^0$. Then, using $\sum_\rho \not{\epsilon}^*_{\rho,\mathbf{k}} \not{\epsilon}_{\rho,\mathbf{k}} = -2$, one gets

$$M(\mathbf{p},\bar{\omega})_{\text{Coulomb}} = 2 M(\mathbf{p},\bar{\omega})_{\text{scalar}}. \tag{C.5}$$

Thus, the only difference with the Yukawa coupling is a global factor of two in the mass operator, which corresponds to the fact that there are two spin states for the

bosons instead of one. The same result holds more generally for the mass operator $\Sigma$ in the ultrarelativistic limit, even at finite $T$, so that the results of section 2 are also valid in the case of QED, if the definition of $M$ in Eq.(9) is replaced by

$$M = \frac{g\sqrt{2}}{4\pi}\sqrt{\mu^2 + \pi^2 T^2}. \tag{C.6}$$

Let us now construct the eigenstates explicitly. The difference with the scalar field lies in that the spinor overlap $\bar{w}_{\nu,\mathbf{p}+\mathbf{k}} w_{\lambda,\mathbf{p}}$ in Eq.(33) must be replaced by $\bar{w}_{\nu,\mathbf{p}+\mathbf{k}} \not{\epsilon}^*_{\rho,\mathbf{k}} w_{\lambda,\mathbf{p}}$. If $m = 0$,

$$\bar{u}_{\lambda,\mathbf{p}} \not{\epsilon}^*_{\rho,\mathbf{k}} u_{\nu,\mathbf{k}} = -\lambda \phi^\dagger_{\lambda\hat{p}} \boldsymbol{\epsilon}_{\rho,\mathbf{k}} \cdot \boldsymbol{\sigma} \phi_{\nu\hat{k}} \delta_{\nu,\lambda} \quad \bar{v}_{\lambda,\mathbf{p}} \not{\epsilon}^*_{\rho,\mathbf{k}} u_{\nu,\mathbf{k}} = -\phi^\dagger_{\lambda\hat{p}} \boldsymbol{\epsilon}_{\rho,\mathbf{k}} \cdot \boldsymbol{\sigma} \phi_{\nu\hat{k}} \delta_{\nu,-\lambda}. \tag{C.7}$$

The electromagnetic interaction, unlike the scalar coupling, conserves chirality. The eigenstates are therefore

$$\begin{aligned}
|\Psi^u_{-\mathbf{p}}\rangle_{\pm,\lambda} &= \sqrt{z_\pm}\left(b_{\lambda,\mathbf{p}} + g\sum_{\rho,\mathbf{k}} \frac{1}{\sqrt{2\omega_k \Omega}} \frac{-\lambda\, \phi^\dagger_{\lambda\hat{p}} \boldsymbol{\epsilon}^*_{\rho,\mathbf{k}} \cdot \boldsymbol{\sigma} \phi_{\lambda\hat{k}}}{\omega_\pm - p\,\cos\theta} b_{\lambda,\mathbf{p}+\mathbf{k}} a^\dagger_{\rho,\mathbf{k}}\right)|\Phi_0\rangle \\
|\Psi^v_{-\mathbf{p}}\rangle_{\pm,\lambda} &= \sqrt{z_\pm}\left(d^\dagger_{\lambda,-\mathbf{p}} + g\sum_{\rho,\mathbf{k}} \frac{1}{\sqrt{2\omega_k \Omega}} \frac{-\phi^\dagger_{\lambda\hat{p}} \boldsymbol{\epsilon}^*_{\rho,\mathbf{k}} \cdot \boldsymbol{\sigma} \phi_{-\lambda\hat{k}}}{\omega_\pm - p\,\cos\theta} b_{-\lambda,\mathbf{p}+\mathbf{k}} a^\dagger_{\rho,\mathbf{k}}\right)|\Phi_0\rangle.
\end{aligned} \tag{C.8}$$

In order to calculate the normalization, the electric current, etc, one must evaluate the sum over polarization states $\sum_\rho \left|\phi^\dagger_{\lambda\hat{p}} \boldsymbol{\epsilon}^*_{\rho,\mathbf{k}} \cdot \boldsymbol{\sigma} \phi_{\pm\lambda\hat{k}}\right| = 1 \mp \cos\theta$. These expressions are similar to those obtained in the case of a Yukawa coupling, except for a factor of 2 which is absorbed in the definition of $M$. So the results are finally the same.

In the case $\mathbf{p} = 0$, a similar calculation gives

$$\begin{aligned}
|\Psi^u_{-\mathbf{p}}\rangle_{\pm,\lambda} &= \sqrt{z_\pm}\left(b_{\lambda,0} + \frac{g}{\omega_\pm\sqrt{2}} \sum_{\nu,\rho,\mathbf{k}} \frac{-\nu\, \phi^\dagger_{\lambda} \boldsymbol{\epsilon}^*_{\rho,\mathbf{k}} \cdot \boldsymbol{\sigma} \phi_{\nu\hat{k}}}{\sqrt{2\omega_k \Omega}} b_{\nu,\mathbf{k}} a^\dagger_{\mathbf{k}}\right)|\Phi_0\rangle \\
|\Psi^v_{-\mathbf{p}}\rangle_{\pm,\lambda} &= \sqrt{z_\pm}\left(d^\dagger_{\lambda,0} + \frac{g}{\omega_\pm\sqrt{2}} \sum_{\nu,\rho,\mathbf{k}} \frac{-\phi^\dagger_{\lambda} \boldsymbol{\epsilon}^*_{\rho,\mathbf{k}} \cdot \boldsymbol{\sigma} \phi_{\nu\hat{k}}}{\sqrt{2\omega_k \Omega}} b_{\nu,\mathbf{k}} a^\dagger_{\rho,\mathbf{k}}\right)|\Phi_0\rangle.
\end{aligned} \tag{C.9}$$

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## FIGURE CAPTIONS

**Fig.1:**

The spectrum of fermionic excitation in an ultrarelativistic plasma for two cases, $M = 10m$ (Fig.1a) and $M = m$ (Fig.1b). The full lines correspond to $\varepsilon = 1$, the dashed lines to $\varepsilon = -1$. The lower part of the figures displays the residues at the quasiparticle poles: $z_+$ is the residue of the normal branches and $z_-$ is the residue of the abnormal branches.

**Fig.2:**

Graphs corresponding to the three contributions labelled (i),(ii) and (iii) in the expression (20) of the mass operator.

**Fig.3:**

Energy domains where the imaginary part of $\Sigma$, given by Eq.(20), is non vanishing. The three regions (I), (II) and (III) correspond respectively to the electron-photon continuum, the hole-photon continuum and the positron-photon continuum, and are the regions in which, respectively, the contributions (i),(ii) and (iii) to Im$\Sigma$ are non vanishing.

**Fig.4:**

Diagrams contributing to the second order correction to the energy of a hole (i) with momentum $p < k_F$ , and a particle (ii) with momentum $p > k_F$, and reflecting modifications of the Fermi sea. The diagram (0) represents the second order correction to the energy of the Fermi sea, i.e. $\Delta E_0^N$ (see text). It is the sum over the fermion momenta in this diagram which is restricted by the presence of the extra particle (in (ii)) or the extra hole (in (i)).

**Fig.5:**

Schematic representation of some important excitations of the ultrarelativistic plasma at zero temperature. (a) A hole at the bottom of the Fermi sea. (b) A positron

with $p = 0$, i.e. a hole at the top of the Dirac sea. (c) A hole left by a "hard" electron with momentum $p \lesssim k_F$. This hole is accompanied by a hard photon (not drawn).

**Fig.6:**

a) *solid lines*: real part and imaginary part of the mass operator for an eigenstate of $\gamma^0$ with the eigenvalue $\varepsilon = 1$. The calculations are done using Eq. (B.2), Eq. (B.3) and Eq. (B.4) (the vacuum contribution to the real part is thus omitted). The imaginary part vanishes stricly on this graph for $-m < \omega < 0$ and for $\omega > m$. It is small but not zero for $\omega < -m$. The labels on the curves (5,10,20) are the corresponding values of $\mu/m$. In this figure as well as in Figs. 4 and 5, $g^2/4\pi = 1/137$. *Vertical dash-dotted lines*: indicate the value $\omega = \mu - k_F$ where the mass operator is divergent. The other divergence, at $\omega = \mu + k_F$ is outside the range of the abscissa. *Dashed line*: line $\omega - m$, whose intersection with the solid line gives the elementary modes according to Eq. (23).

b) Same as a) for an eigenstate of $\gamma^0$ with the value $\varepsilon = -1$. The dashed line is now $\omega + m$ according to Eq. (23).

**Fig.7:**

a) *solid lines*: real part and imaginary part of the mass operator for an eigenstate of $\boldsymbol{\alpha} \cdot \hat{p}$ with the eigenvalue $\varepsilon = 1$. The calculations are done using Eq. (B.6) and Eq. (B.7). The imaginary part vanishes strictly on this graph for $\omega > p$. It is small but non vanishing for $\omega < -p$. The labels (10,20,40) on the curves are the corresponding values of $\mu/p$. *Dashed line*: line $\omega - p$, whose intersection with the solid line gives the elementary modes according to Eq. (26).

b) Same as a) for an eigenstate of $\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}$ with the value $\varepsilon = -1$.

**Fig.8:**

Same as Fig.7 with $p$ very close to the Fermi surface ($k_F/p = 0.8, 1, 1.2$).

**Fig.9:**

The dispersion relation for the hole excitation. Left-hand side: no interaction. Right-hand side: with interaction. The shaded area represents the most important part of the continuum of hole–photon states to which the hole couples.

**Fig.10:**

The energy domains where the imaginary part of the mass operator is non vanishing, in the limit $m = 0$ (see Eq.(B.7)).