Conditioning SLEs and loop erased random walks

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Abstract

We discuss properties of dipolar SLE_κ under conditioning. We show that $\kappa = 2$, which describes continuum limits of loop erased random walks, is characterized as being the only value of κ such that dipolar SLE conditioned to stop on an interval coincides with dipolar SLE on that interval. We illustrate this property by computing a new bulk passage probability for SLE_2.

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1 Introduction

Schramm-Loewner evolution (SLE) comes in several flavors. Chordal SLE describes random curves between two points on the boundary of a simply connected domain. Radial SLE describes random curves between an interior point and a boundary point. Dipolar SLE describes random curves between a boundary point and a subarc of the boundary. These various types of SLE are closely related to each other and to the more general $SLE_{\kappa,\rho}$ processes [1, 2, 3]

We consider dipolar SLE from a boundary point z to a subarc A of the boundary and ask what happens if we condition it to end in a subarc $A' \subset A$, or to end at a point $w \in A$. We will show that the result is an $SLE_{\kappa,\rho}$ process. In the particular case of $\kappa = 2$, when the dipolar SLE is conditioned to end at a fixed point the distribution of the resulting curve is just chordal SLE.

Let us observe however that even if a discrete model is known to (or conjectured to) converge to chordal SLE_κ in the continuum limit, there is at the moment very little understanding how to modify it to a discrete process candidate to converge to an $SLE_{\kappa,\rho}$ in the continuum limit. Even the definition of "dipolar self avoiding walk" escapes us at the moment.

Certain lattice models of random curves have special properties that allow us to identify the associated κ in the continuum limit assuming the continuum limit is an SLE. The restriction property of the self avoiding walk implies that $\kappa = 8/3$ as the only possible candidate for the scaling limit. The locality property of percolation identifies $\kappa = 6$ as the only possible scaling limit.

The loop-erased random walk (LERW) has another property that together with our result for $\kappa = 2$ implies that if the continuum limit is SLE, then κ must be 2. For concreteness, we consider the model in the infinite strip. Consider a random walk starting at the origin and conditioned to exit the strip on the upper boundary. Loop-erasing this model gives what we will refer to as the dipolar LERW. If we take the original random walk which was conditioned to exit through the upper boundary and further condition it to exit through a point w in the upper boundary, the result is the same as conditioning a random walk starting at the origin to exit the strip through w. So if we condition the dipolar LERW to end at w, then the result is the same as the loop erasure of a random walk starting at 0 and conditioned to exit at w. The latter is just the chordal LERW. In short, conditioning the dipolar LERW to end at a particular point gives the chordal LERW. As we will see, SLE has this property only for $\kappa = 2$.

The LERW has been proved to converge to SLE_2 in the scaling limit [4, 5]. Since the LERW trivially has the property that conditioning the dipolar version to end at a fixed point give the chordal version, the proofs of convergence of the LERW to SLE_2 have as a corollary the result that conditioning dipolar SLE_2 to end at a fixed point gives chordal SLE_2. Nonetheless, it is desirable to have a direct proof that SLE_2 has this property.

Properties that SLE satisfies for only one value of the parameter κ are useful for identifying the scaling limit of discrete or lattice models that satisfy the property. Recently, claims that SLE samples describe lines in a variety of 2d systems have appeared : these claims concern zero vorticity lines in turbulence [6], nodal lines of chaotic or random wave function [7] and interfaces in ground states of spin glasses [8, 9]. The first two cases seem to involve SLE_6 (i.e. percolation). This is far from fully justified, even if some heuristic theoretical arguments make this plausible. The case of spin glasses is even more intricate. The value of κ computed numerically is not too far (but apparently reliably different) from 2. Moreover there are several definitions of the notion of interface. For some of the definitions conformal invariance has been ruled out so that only for one definition is the relationship with SLE still a plausible guess.

Of course numerical studies are made in finite geometries (even with finite systems) and the problem of finite size effects has to be taken into account. In particular dipolar SLE, which is well adapted to a strip geometry, seems to give much more accurate numerical evaluations of κ and of the quadratic variation of the driving process than chordal SLE.

SLE has been studied by researchers coming from a variety of disciplines ranging from probability to conformal field theory. To make this article accessible to these various groups, we derive our results in several ways. In addition, for several results that have already appeared in the literature we give another derivation from a different point of view.

2   Avatars of dipolar and chordal SLE

2.1   Preliminaries

In the half plane geometry $\mathbb{H} \equiv \{z, \text{ Im } z \in]0, +\infty[\}$, a growth process of a piece of curve starting from a point on the lower boundary Im $z = 0$ can be uniformized by a map $g(z)$ having the point at infinity as a fixed point with

the normalization: $g(z) \simeq z + 2c/z + o(1/z)$ for $z \to \infty$. It is customary to call this the hydrodynamic normalization. The number c is real, nonnegative, and increases during the growth. Then, with a time parameterization s of the growing trace such that $c_s \equiv s$, the uniformizing map satisfies

$$\frac{dg_s(z)}{ds} = \frac{2}{g_s(z) - V_s}, \qquad g_0(z) = z$$

Chordal SLE from 0 to ∞ is the stochastic growth process obtained when $V_s = \sqrt{\kappa} B_s$ where B_s is a standard Brownian motion.

In the strip geometry $\mathbb{S} \equiv \{z, \text{ Im } z \in]0, \pi[\}$ a growth process of a piece of curve starting from a point on the lower boundary $\text{Im } z = 0$ can be uniformized by a map $g(z)$ fixing the two points at infinity. This leaves the possibility to translate g by a real constant. But $g(z) \simeq z + c_\pm + o(1)$ for $z \to \pm\infty$, and g can be fully normalized by requiring that $c_+ + c_- = 0$. We propose to call this the strip symmetric normalization. Then, with a time parameterization s such that $c_+ = s$ of the growing trace, the uniformizing map satisfies

$$\frac{dg_s(z)}{ds} = \frac{1}{\tanh(g_s(z) - V_s)/2}, \qquad g_0(z) = z$$

Throughout the paper $\tanh x/2$ stands for $\tanh(x/2)$. Dipolar SLE_κ from 0 to the upper boundary $\text{Im } z = \pi$ (as defined in [10]) is the stochastic growth process obtained when $V_s = \sqrt{\kappa} B_s$ where B_s is a standard Brownian motion.

2.2 Change of domain

Chordal SLE is particularly simple in the half plane geometry with the hydrodynamic normalization and dipolar SLE is particularly simple in the strip geometry with the symmetric normalization. But both are conformally invariant processes, and as such could be described in any domain (with two marked points for the chordal case, or with one marked boundary interval and one marked point in its boundary complement in the dipolar case) and moreover with any choice of normalization of the Loewner map.

We illustrate this issue with four examples. The first one is well known. We treat the second one in detail, using a commutative diagram technique introduced in the study of locality and restriction for SLE, see e.g. [11], and sketch the treatment of the fourth. The same method can be used in all

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four cases, leading to straightforward but slightly painful computations. We shall see later how ideas from statistical mechanics and conformal field theory followed by a simple application of Girsanov's theorem allow to simplify the computations.

Chordal SLE from 0 **to** a **in the upper half plane** with the hydrodynamic normalization is described by the system of stochastic differential equations:

$$\frac{dg_s(z)}{ds} = \frac{2}{g_s(z) - V_s}, \quad \frac{dA_s}{ds} = \frac{2}{A_s - V_s},$$

$$dV_s = \sqrt{\kappa} dW_s + (\kappa - 6) \frac{ds}{V_s - A_s},$$

with initial conditions $g_0(z) = z$, $V_0 = O$ and $A_0 = a$.

Chordal SLE from 0 **to** $i\pi + a$ **in the strip** with the symmetric normalization is described by the system of stochastic differential equations:

$$\frac{dg_s(z)}{ds} = \frac{1}{\tanh(g_s(z) - V_s)/2}, \quad \frac{dA_s}{ds} = \tanh(A_s - V_s)/2,$$

$$dV_s = \sqrt{\kappa} dW_s + (\kappa/2 - 3) ds \tanh(V_s - A_s)/2$$

with initial conditions $g_0(z) = z$, $V_0 = O$ and $A_0 = a$.

Dipolar SLE from 0 **to** $[a, b]$ **in the upper half plane** with the hydrodynamic normalization is described by the system of stochastic differential equations:

$$\frac{dg_s(z)}{ds} = \frac{2}{g_s(z) - V_s}, \quad \frac{dA_s}{ds} = \frac{2}{A_s - V_s}, \quad \frac{dB_s}{ds} = \frac{2}{B_s - V_s},$$

$$dV_s = \sqrt{\kappa} dW_s + (\kappa/2 - 3) \left(\frac{ds}{V_s - A_s} + \frac{ds}{V_s - B_s} \right),$$

with initial conditions $g_0(z) = z$, $V_0 = O$, $A_0 = a$ and $B_0 = b$.

Dipolar SLE from 0 **to** $[i\pi + a, i\pi + b]$ **in the strip** with the symmetric normalization is described by the system of stochastic differential equations:

$$\begin{aligned}
\frac{dg_s(z)}{ds} &= \frac{1}{\tanh(g_s(z) - V_s)/2}, \\
\frac{dA_s}{ds} &= \tanh(A_s - V_s)/2, \quad \frac{dB_s}{ds} = \tanh(B_s - V_s)/2, \\
dV_s &= \sqrt{\kappa} dW_s + (\kappa/2 - 3) ds \frac{\tanh(V_s - A_s)/2 + \tanh(V_s - B_s)/2}{2}
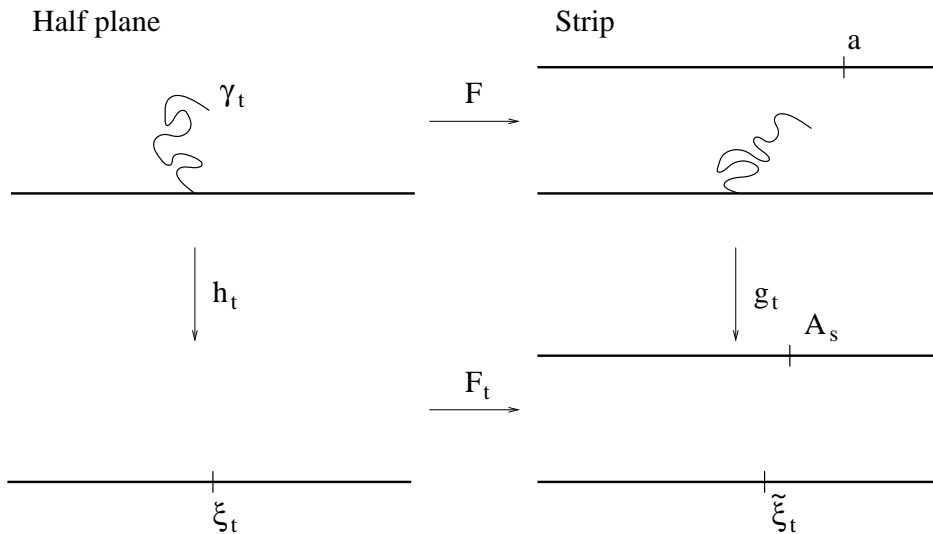\end{aligned}$$

Figure 1: The basic commutative diagram

with initial conditions $g_0(z) = z$, $V_0 = O$, $A_0 = a$ and $B_0 = b$.

In all the above systems, W_s is a normalized Brownian motion. Before we explain the second case, let us make a few remarks.

– The drifts all vanish at $\kappa = 6$, which is a manifestation of the locality of percolation.

– For $\kappa \leq 4$ the above equations give a complete description. For $\kappa > 4$, the first two equations give only a partial description : the first is a description of chordal SLE from 0 to a in the upper half plane only up to the first time the process separates a from ∞ by hitting the real line in the interval $[a, \infty]$ that does not contain 0; the second is a description of chordal SLE from 0 to $i\pi + a$ in the strip only up to the first time the process hits the line Im $z = \pi$. In both cases, the trouble comes solely from the choice of normalization.

With this proviso in mind, we go on to write down chordal SLE in the strip with the symmetric normalization. The starting point will be 0 and the end point will be $i\pi + a$ where a is a real number. An invaluable tool to make computations straightforward is a commutative diagram, just as in the study of locality and restriction for SLE [11].

Uniformize chordal SLE in the upper-half plane from 0 to ∞ with trace

$\gamma_{[0,t]}$ up to time t by $h_t : \mathbb{H}\backslash\gamma_{[0,t]} \to \mathbb{H}$ in the hydrodynamical normalization:

$$\frac{dh_t(w)}{dt} = \frac{2}{h_t(w) - \xi_t}, \quad \xi_t = \sqrt{\kappa}B_t.$$

Map the upper half plane to the strip by any conformal map $F : \mathbb{H} \to \mathbb{S}$ such that $F(0) = 0$ and $F(\infty) = i\pi + a$ to get SLE in $\mathbb{S}$ from 0 to $i\pi + a$. The formula for F is

$$\tanh(F(w) - a)/2 + \tanh a/2 = lw.$$

where $l \in \mathbb{R}^{+*}$ is arbitrary (we have given only two conditions to normalize F). Let g_t be the uniformizing map $g_t : \mathbb{S}\backslash F(\gamma_{[0,t]}) \to \mathbb{S}$ with the symmetric normalization as above except that the time parameterization is not at our disposal

$$\frac{dg_t(z)}{dt} = \frac{a_t}{\tanh(g_t(z) - \tilde{\xi}_t)/2}.$$

Finally let $F_t : \mathbb{H} \to \mathbb{S}$ be the map "closing the square",

$$g_t \circ F = F_t \circ h_t.$$

Take the time derivative of $g_t \circ F = F_t \circ h_t$ and afterwards substitute w for h_t to get

$$\frac{a_t}{\tanh(F_t(w) - \tilde{\xi}_t)/2} = \frac{dF_t(w)}{dt} + F'_t(w)\frac{2}{w - \xi_t}.$$

Now $F_t(w)$ is non singular at $w = \xi_t$ so that the pole on the right hand-side is cancelled by a pole on the left hand-side:

$$\frac{a_t}{\tanh(F_t(w) - \tilde{\xi}_t)/2} = F'_t(\xi_t)\frac{2}{w - \xi_t} + O(1) \text{ when } w \to \xi_t,$$

leading to $\tilde{\xi}_t = F_t(\xi_t)$ and $a_t = F'_t(\xi_t)^2$. Continuing the expansion one step further yields

$$-a_t F''_t(\xi_t)/F'_t(\xi_t)^2 = \frac{dF_t}{dt}(\xi_t) + 2F''_t(\xi_t) \text{ i.e. } \frac{dF_t}{dt}(\xi_t) = -3F''_t(\xi_t).$$

Now use Ito's formula (in a slightly extended context as usual in this computation) to get

$$d\tilde{\xi}_t = d(F_t(\xi_t)) = dF_t(\xi_t) + F'_t(\xi_t)d\xi_t + \frac{\kappa}{2}F''_t(\xi_t)dt,$$

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i.e. $d\tilde{\xi}_t = F'_t(\xi_t)d\xi_t + (\kappa/2 - 3)F''_t(\xi_t)dt$. Define $s(t) \equiv \int_0^t F'_u(\xi_u)^2 du$ and $\sqrt{\kappa}W_{s(t)} \equiv \int_0^t F'_u(\xi_u)d\xi_u$ so that W_s is a standard Brownian motion with time parameter s. Set $V_{s(t)} \equiv F_t(\xi_t)$ so

$$dV_s = \sqrt{\kappa}dW_s + (\kappa/2 - 3)\frac{F''_{t(s)}(\xi_{t(s)})}{F'_{t(s)}(\xi_{t(s)})^2}ds.$$

This equation is in fact valid for all four cases, but of course F_t closes a different commutative diagram in each case. Computations are simplified a little bit when one realizes that if $F(w) = z$ has inverse $\Phi(z) = w$ then

$$\frac{F''(w)}{F'(w)^2} = -(\log \Phi'(z))'.$$

In this identity, the $'$ indicates derivative with respect to w on the left hand side but with respect to z on the right hand side. So if we write $\Phi_s(z) = w$ for the inverse of $F_t(w) = z$ (where s and t are related by the time change) we get the important intermediate formula :

$$dV_s = \sqrt{\kappa}dW_s + (3 - \kappa/2)(\log \Phi'_s(z))'_{|z=V_s}ds. \tag{1}$$

Let $A_s \equiv g_{t(s)}(i\pi + a) - i\pi$ be the trajectory of a under the Loewner evolution. From $g_t \circ F = F_t \circ h_t$, $h_t(\infty) = \infty$ and $F(\infty) = i\pi + a$ we learn that $g_t(i\pi + a) = F_t(\infty) = i\pi + A_s$. Combined with $F_t(\xi_t) = V_{s(t)}$ we find

$$\tanh(F_{t(s)}(w) - A_s)/2 + \tanh(A_s - V_s)/2 = l_s(w - \xi_{t(s)}),$$

or

$$\tanh(z - A_s)/2 + \tanh(A_s - V_s)/2 = l_s(\Phi_s(z) - \xi_{t(s)}),$$

where l_s is an unspecified but irrelevant parameter.

The computation of the drift is now easy (the reader should compare with a direct use of the formula involving F_t) :

$$((\log \Phi'_s(z))'_{|z=V_s} = -\tanh(V_s - A_s)/2.$$

This finishes to establish the formula announced above for chordal SLE in the strip from 0 to $i\pi + a$ in the symmetric normalization.

We end this section with some remarks on how to write down dipolar SLE from 0 to $[i\pi + a, i\pi + b]$ in the strip with the symmetric normalization.

The steps above could be reproduced, leading to equation (1), but with a function Φ_s mapping the strip to the strip, V_s to $\xi_{t(s)}$, $i\pi + A_s$ to $-\infty$ and $i\pi + B_s$ to $+\infty$. This fixes Φ_s uniquely. A convenient representation is

$$\coth(\Phi_s(z) - \xi_{t(s)})/2 = \nu + \mu \coth(z - V_s)/2$$

with $-1 = \nu + \mu \tanh(A_s - V_s)/2$, $1 = \nu + \mu \tanh(B_s - V_s)/2$.

To compute the drift we have to be able to take two derivatives with respect to z and then put $z = V_s$ so it is enough to use the second order expansion:

$$\frac{\Phi_s(z) - \xi_{t(s)}}{2} = \frac{z - V_s}{2\mu} - \nu \left(\frac{z - V_s}{2\mu} \right)^2 + \cdots$$

for $z \to V_s$. We infer that

$$(\log \Phi'_s(z))'_{|z=V_s} = -\nu/\mu = \frac{\tanh(V_s - B_s)/2 + \tanh(V_s - B_s)/2}{2},$$

leading to the announced formula.

3 Conditioning

We aim at a description of dipolar SLE in a domain (call it D) from a boundary point (call it 0) to a boundary interval (call it I) conditioned to hit either a subinterval J of I or, via a limiting procedure, a point in I.

3.1 Naïve considerations

Girsanov's theorem is the general tool to tackle these problems, but in this section we want to illustrate what goes on using only elementary manipulations, with the hope that the explicit argument shows in the clearest way what is involved, compensating at least partly the lack of elegance of the approach.

We could in principle make computations in any domain, with any normalization of the uniformizing maps. For illustration, we choose D to be the strip $\mathbb{S}$, I to be upper boundary $\text{Im } z = \pi$, and J to be the interval $]i\pi + a, i\pi + b[$ where $a < b$ are real numbers. Before conditioning, the process is described by the equations

$$\frac{dg_s(z)}{ds} = \frac{1}{\tanh(g_s(z) - V_s)/2}, \qquad g_0(z) = z$$

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where $V_s = \sqrt{\kappa}W_s$ and W_s is a standard Brownian motion. The inverse image $g_s^{-1}(\mathbb{S})$ is $\mathbb{S}\backslash\gamma(]0,s])$. In particular, the inverse image of $g_s^{-1}(V_s)$ –defined as $\lim_{\varepsilon\mapsto 0^+} g_s^{-1}(V_s + i\varepsilon)-$ is $\gamma(s)$.

The martingale We condition on the event that the curve ends in an interval J. We will denote this event by J as well, no confusion should arise. The first thing we have to know is the probability of J. This is a routine computation if one uses martingales, but this is also the key to make contact with Girsanov's theorem later. Write $p(J)$ for the hitting probability of dipolar SLE and observe that $p(J) = f(a) - f(b)$ where $f(c)$ is the probability to hit on the right of c. So it is enough to deal with the case when $J =]i\pi + a, i\pi + \infty[$. Observe that $A_s \equiv g_s(i\pi + a) - i\pi$ satisfies

$$\frac{dA_s}{ds} = \tanh(A_s - V_s)/2, \quad A_0 = a.$$

Our claim is the following: for $t \geq 0$, $\mathbf{E}\left[1_J|\mathcal{F}_t\right] = f(A_t - V_t)$. Recall that an $\mathcal{F}_t$-measurable event is an event whose realization can be decided from the knowledge of the process $W_\bullet$ for times up to t. So the left hand side is the probability to hit in J knowing the process $W_\bullet$ up to time t. Intuitively, this knowledge is equivalent to the knowledge of g_t. Dipolar SLE is conformally invariant, and in particular, for $s \geq t$, the statistics of $g_t(\gamma(]t,s]) - V_t$, which is supposed to hit on the right of $A_t - V_t$, is the same as the statistics of a $\gamma(]0,s-t])$. The hitting probability of this process is $f(A_t - V_t)$ as announced.

The equation $\mathbf{E}\left[1_J|\mathcal{F}_t\right] = f(A_t - V_t)$ means that $f(A_t - V_t)$ is a –so called "closed"– martingale. In particular the Ito derivative of the stochastic process $f(A_t - V_t)$ has no drift. A routine use of Ito's formula yields

$$\frac{\kappa}{2}f''(a) + f'(a)\tanh a/2 = 0,$$

whose only solution with an acceptable probabilistic interpretation is proportional to $\int_a^{+\infty} dx \cosh^{-4/\kappa} x/2$, leading to

$$p(J) = \frac{\int_a^b dx \cosh^{-4/\kappa}(x/2)}{\int_{-\infty}^{+\infty} dx \cosh^{-4/\kappa}(x/2)}$$

for the hitting probability of the interval $J =]i\pi + a, i\pi + b[$, a formula established in [10].

Markov property Let p denote the probability distribution for dipolar SLE (more precisely for the driving process V_s, which is proportional to a

Brownian motion), and let $\tilde{p}$ denote the probability distribution for dipolar SLE conditioned to hit the upper boundary in J (again more precisely for the driving process V_s, which after conditioning is not simply a Brownian motion anymore). Define $J_s \equiv g_s(J) =]i\pi + A_s, i\pi + B_s[$.

Observe that the knowledge of V_t, A_t, B_t for all t's is a (redundant) description of the growth process. It is redundant because the knowledge of V_t for all t's would suffice. For $A < B$ and $t \geq 0$, let $E_{t,V,A,B}$ denote the set of driving functions $V_\bullet$ such that $V_t \in [V, V + dV]$ $A_t \in [A, A + dA]$ and $B_t \in [B, B + dB]$. This is an $\mathcal{F}_t$-measurable set. Let E be any $\mathcal{F}_t$-measurable subset of $E_{t,V,A,B}$, and for $t \leq s$ compute $\tilde{p}(E$ and $E_{s,V',A',B'})$. By definition, this is

$$\frac{p(E \text{ and } E_{s,V',A',B'} \text{ and } \gamma \text{ hits in }]a,b[)}{p(\gamma \text{ hits in }]a,b[)}.$$

But under p, the process $(V_\bullet, A_\bullet, B_\bullet)$ is Markovian, so that

$$p(E \text{ and } E_{s,V',A',B'} \text{ and } \gamma \text{ hits in }]a,b[) = p(E)p(E')p(J')$$

where E' is the event that starting from V, A, B at time t, $V_s \in [V', V' + dV']$, $A_s \in [A', A' + dA']$, $B_s \in [B', B' + dB']$ and J' is the event that dipolar SLE started at V_s on the lower boundary hits the upper boundary in the interval $J' =]A', B'[$, or by translation invariance, that dipolar SLE started at 0 on the lower boundary hits the upper boundary in the interval $J' =]A' - V', B' - V'[$. We have already computed $p(J')$. Hence

$$\tilde{p}(E \text{ and } E_{s,V',A',B'}) = p(E)p(E')\frac{p(\gamma \text{ hits in }]A' - V', B' - V'[)}{p(\gamma \text{ hits in }]a,b[)}.$$

A simpler argument yields

$$\tilde{p}(E) = p(E)\frac{p(\gamma \text{ hits in }]A - V, B - V[)}{p(\gamma \text{ hits in }]a,b[)},$$

and by comparison,

$$\tilde{p}(E \text{ and } E_{s,V',A',B'}) = \tilde{p}(E)p(E')\frac{p(\gamma \text{ hits in }]A' - V', B' - V'[)}{p(\gamma \text{ hits in }]A - V, B - V[)}$$

for any $\mathcal{F}_t$-measurable event $E \subset E_{t,V,A,B}$.

This proves that under $\tilde{p}$, the process V_t, A_t, B_t is Markovian and, from the definition of E', that the transition probability density under $\tilde{p}$ to go

from (V, A, B) at time t to (V', A', B') at time s is simply the transition probability density under p to go from (V, A, B) at time t to (V', A', B') at time s weighted by the known ratio $\frac{p(\gamma \text{ hits in }]A'-V', B'-V'[)}{p(\gamma \text{ hits in }]A-V, B-V[)}$. Under p, V_t alone is a Markov process, so there is some price to pay to save the Markov property.

Computation of the drift Note that the transition probability density under p to go from (V, A, B) at time t to (V', A', B') at time s is translation invariant and time homogeneous, so that we can assume without loss of generality that $V = 0$ and $t = 0$. Now we assume that $s = \varepsilon$ and $V' = \Delta$ are small and we compute the probability distribution of Δ. Under p, the density of Δ is $\frac{1}{\sqrt{2\pi\kappa\varepsilon}} e^{-\Delta^2/(2\kappa\varepsilon)}$, and the argument in the exponential is of order 1 only when $\Delta \sim \sqrt{\varepsilon}$. The conditioning ratio $\frac{p(\gamma \text{ hits in }]A'-V', B'-V'[)}{p(\gamma \text{ hits in }]A-V, B-V[)}$ can be expanded using this scaling, and only terms up to order Δ need to be kept. As $A' - A$ and $B' - B$ are of order ε we find

$$\frac{p(\gamma \text{ hits in }]A' - V', B' - V'[)}{p(\gamma \text{ hits in }]A - V, B - V[)} = 1 - \Delta F(A, B) + 0(\varepsilon, \Delta^2),$$

where

$$F(A, B) = \frac{\cosh^{-4/\kappa} B/2 - \cosh^{-4/\kappa} A/2}{\int_A^B dx \cosh^{-4/\kappa} x/2}$$

So the transition probability density under $\tilde{p}$ of V_ε starting from $V_0 = 0$ is

$$\frac{1}{\sqrt{2\pi\kappa\varepsilon}} e^{-(\Delta + \kappa\varepsilon F(A,B))^2/(2\kappa\varepsilon)}(1 + 0(\varepsilon, \Delta^2)).$$

The meaning of this is that for infinitesimal ε, V_ε is a Gaussian random variable of mean $-\kappa\varepsilon F(A, B)$ and standard deviation $\sqrt{\kappa\varepsilon}$. Using translation invariance and time homogeneity, this is exactly the meaning of the stochastic differential equation

$$dV_s = \sqrt{\kappa}dW_s - \kappa F(A_s - V_s, B_s - V_s)ds.$$

When $B \to A$, $F(A, B)$ has a finite limit, $-\frac{2}{\kappa} \tanh A/2$.

We summarize the central results of this section:

Dipolar SLE from 0 to the upper boundary in the strip, conditioned to hit in $[i\pi + a, i\pi + b]$ with the symmetric normalization is

described by the system of stochastic differential equations:

$$\frac{dg_s(z)}{ds} = \frac{1}{\tanh(g_s(z) - V_s)/2},$$

$$\frac{dA_s}{ds} = \tanh(A_s - V_s)/2, \quad \frac{dB_s}{ds} = \tanh(B_s - V_s)/2,$$

$$dV_s = \sqrt{\kappa}dW_s - \kappa \left(\frac{\cosh^{-4/\kappa}(B_s - V_s)/2 - \cosh^{-4/\kappa}(A_s - V_s)/2}{\int_{A_s-V_s}^{B_s-V_s} dx \cosh^{-4/\kappa}(x/2)} \right)$$

with initial conditions $g_0(z) = z$, $V_0 = O$, $A_0 = a$ and $B_0 = b$.

Dipolar SLE from 0 to the upper boundary in the strip conditioned to hit at $i\pi + a$ with the symmetric normalization is described by the system of stochastic differential equations:

$$\frac{dg_s(z)}{ds} = \frac{1}{\tanh(g_s(z) - V_s)/2}, \quad \frac{dA_s}{ds} = \tanh(A_s - V_s)/2,$$

$$dV_s = \sqrt{\kappa}dW_s - 2\tanh(V_s - A_s)/2$$

with initial conditions $g_0(z) = z$, $V_0 = O$ and $A_0 = a$.

As a corollary, dipolar SLE conditioned to hit in a subinterval is dipolar SLE if and only if $\kappa = 2$, and dipolar SLE conditioned to hit at a given point is chordal SLE if and only if $\kappa = 2$.

This is obtained by direct comparison with the formulæ in subsection 2.2.

3.2   An elementary application at $\kappa = 2$

The above result is a nice characterization of the value $\kappa = 2$. We show in this section that it allows us to compute in an elementary way the probability that a dipolar SLE_2 trace passes to the right of a given bulk point. The same strategy can be used to compute other physical observables of interest, see [12].

The probability that a dipolar SLE_κ trace (or hull) ends to the right of a given boundary point of the hitting interval is obtained by routine martingale techniques, see e.g. [10].

For $\kappa < 8$, the probability that the chordal SLE_κ trace passes to the right of a given bulk point z (for $4 < \kappa < 8$ we means that when z is swallowed

it goes to the negative real axis) is again obtained via a routine martingale technique see [13].

For $\kappa \geq 4$, the probability that a dipolar SLE_κ hull contains the bulk point z and the probability that it passes to the right of z can be computed (somewhat miraculously) again by routine martingale techniques by making the ansatz that they are harmonic functions of z and then checking that appropriate boundary conditions can be imposed, see again [10]. But the harmonic ansatz fails for $\kappa < 4$.

For $\kappa = 2$, we obtain the probability that a dipolar SLE_2 trace passes to the right of a given bulk point by the following steps.

We start from Schramm's result at $\kappa = 2$ for the chordal case in the upper half plane: if $\vartheta \in [0, \pi]$ is the argument of $z \in \mathbb{H}$, the probability that chordal SLE_2 passes to the right of z is

$$\frac{\vartheta}{\pi} - \frac{\sin 2\vartheta}{2\pi}.$$

We consider a conformal map from $\mathbb{H}$ to the strip $\mathbb{S}$ sending 0 to 0 and ∞ to the point $a + i\pi$ on the hitting interval, to get the probability that chordal SLE_2 in the strip from 0 to $a + i\pi$ passes to the right of $w \in \mathbb{S}$. Cumbersome but elementary algebra shows that this probability is obtained by substitution of $\vartheta(a, w)$ for ϑ in the above formula, where

$$\vartheta(a, w) \equiv \arctan \frac{\sin v}{\sinh u - (\cosh u - \cos v) \tanh a/2},$$

the function arctan takes values in $[0, \pi]$ and $w = u + iv$ is the decomposition into real and imaginary parts.

The trick now is that because dipolar SLE_2 conditioned to hit at a is nothing but chordal SLE_2 aiming at a, the probability $p_2(w)$ that dipolar SLE_2 passes to the right of w can be represented as the integral over a of the dipolar hitting point density at a times the probability that chordal SLE_2 aiming at a passes to the right of w. Explicitly:

$$p_2(w) = \int_{-\infty}^{+\infty} \frac{da}{4 \cosh^2 a/2} \left(\frac{\vartheta(a, w)}{\pi} - \frac{\sin 2\vartheta(a, w)}{2\pi} \right).$$

Choosing $\tanh a/2$ as the new integration variable leads to an elementary integral because the differential of $x \arctan 1/x$ is exactly $dx(\arctan 1/x -$

$\frac{1}{2} \sin 2 \arctan 1/x)$. Finally:

$$p_2(w) = \frac{(e^u - \cos v) \arctan \frac{\sin v}{e^u - \cos v} - (\cos v - e^{-u}) \arctan \frac{\sin v}{\cos v - e^{-u}}}{2\pi(\cosh u - \cos v)}.$$

with $w = u + iv$ the decomposition into real and imaginary parts.

For other values of κ, we would have to work with dipolar SLE conditioned to hit a point, and compute probability that the trace passes to the right of w for this process. This is likely to be even more complicated than $p_\kappa(w)$, the probability we are aiming at: it involves four boundary points (the starting point, the hitting interval and the conditioned endpoint) whereas $p_\kappa(w)$ involves only three. The miracle of $\kappa = 2$ is that after conditioning the hitting interval plays no role anymore. Another natural explanation for this comes from conformal field theory.

3.3   Another computation using martingales

Reference [1] contains several computations analogous to the above ones (some are even assigned as exercises). In this section give another derivation of the result from subsection 3.1.

For definiteness, we consider dipolar SLE in the half plane from 0 to $[-1, 1]^c$. (By $[-1, 1]^c$ we mean the complement of $[-1, 1]$ with respect to the real axis.) The hitting density is proportional to

$$p(x) = x^{-4/\kappa}(x^2 - 1)^{-1+2/\kappa}$$

We use the following martingale. For real forcing points $x_1, \cdots, x_n$ with weights $\rho_1, \cdots, \rho_n$, it is

$$M_t(x_1, \cdots, x_n; \rho_1, \cdots, \rho_n) = \prod_{j=1}^{n} L(x_j, \rho_j) \prod_{1 \le i < j \le n} Q(x_i, x_j; \rho_i, \rho_j)$$

where

$$L(x_j, \rho_j) = |g_t'(x_j)|^{\alpha_j} |X_t^{x_j}|^{\rho_j/\kappa}$$

$$Q(x_i, x_j; \rho_i, \rho_j) = |X_t^{x_i} - X_t^{x_j}|^{\rho_i \rho_j/(2\kappa)}$$

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$$\alpha_j = (8 - 2\kappa + 2\rho_j)\rho_j/(8\kappa)$$

Here X_t^x denotes $g_t(x) - W_t$. Let $\hat{M}_t$ be this martingale normalized so its expectation is 1. Note that M_0 is a constant, so $\hat{M}_t = M_t/M_0$. If we take chordal SLE in the half plane from 0 to ∞ and weight it by $\hat{M}_t$, then we get the SLE$_{\kappa,\rho}$ process with forcing points at the x_i and weights ρ_i. This martingale appears in [1, 18][1].

Dipolar SLE in the upper half plane from 0 to $[-1, 1]^c$ is the same as SLE$_{\kappa,\rho}$ with two force points at -1 and $+1$ with both weights equal to $(\kappa - 6)/2$. This is the same as taking chordal SLE in the upper half plane from 0 to ∞ and weighting it by the martingale $M_t(-1, 1; (\kappa - 6)/2, (\kappa - 6)/2)$. We want to show that if we condition this to end at some $a \in [-1, 1]$, then we get an SLE$_{\kappa,\rho}$ process with three force points. One is at a with weight -4 and the other two are at ± 1, both with weight $\frac{\kappa-2}{2}$. We will show this by showing that

$$\int_{[-1,1]^c} \hat{M}_t(a, -1, 1; -4, \frac{\kappa - 2}{2}, \frac{\kappa - 2}{2})\, p(a)\, da = \hat{M}_t(-1, 1; \frac{\kappa - 6}{2}, \frac{\kappa - 6}{2})$$

This equation is trivially true at $t = 0$. We note that $M_0(a, -1, 1; -4, \frac{\kappa-2}{2}, \frac{\kappa-2}{2}) = cp(a)$. for some constant c which only depends on κ, and $M_0(-1, 1; \frac{\kappa-6}{2}, \frac{\kappa-6}{2})$ only depends on κ. So with some rearranging, the above is the same as

$$\int_{[-1,1]^c} L(a, -4)\, Q(a, -1; -4, \frac{\kappa - 2}{2})\, Q(a, 1; -4, \frac{\kappa - 2}{2})\, da$$
$$= C\, \frac{L(-1, \frac{\kappa-6}{2})\, L(1, \frac{\kappa-6}{2})\, Q(-1, 1; \frac{\kappa-6}{2}, \frac{\kappa-6}{2})}{L(-1, \frac{\kappa-2}{2})\, L(1, \frac{\kappa-2}{2})\, Q(-1, 1; \frac{\kappa-2}{2}, \frac{\kappa-2}{2})} \tag{2}$$

For $\rho = (\kappa - 2)/2$ and $\rho = (\kappa - 6)/2$ we get the same value of α. Thus

$$\frac{L(p, \frac{\kappa-6}{2})}{L(p, \frac{\kappa-2}{2})} = |X_t^p|^{-2/\kappa}$$

for $p = \pm 1$. We find that

$$\frac{Q(-1, 1; \frac{\kappa-6}{2}, \frac{\kappa-6}{2})}{Q(-1, 1; \frac{\kappa-2}{2}, \frac{\kappa-2}{2})} = |X_t^{-1} - X_t^1|^{-1+4/\kappa}$$

---

[1]Schramm and Wilson comment that the martingale was also discoverd independently by M. Biskup.

We also have

$$L(a, -4) = |g_t'(a)| \, |X_t^a|^{-4/\kappa}$$

and for $p = \pm 1$,

$$Q(a, p; -4, \frac{\kappa - 2}{2}) = |X_t^a - X_t^p|^{(2-\kappa)/\kappa}$$

Thus eq.(2) becomes

$$\int_{[-1,1]^c} |g_t'(a)| \, |X_t^a|^{-4/\kappa} |X_t^a - X_t^{-1}|^{(2-\kappa)/\kappa} |X_t^a - X_t^1|^{(2-\kappa)/\kappa} \, da$$
$$= \quad c \, |X_t^{-1}|^{-2/\kappa} |X_t^1|^{-2/\kappa} |X_t^{-1} - X_t^1|^{-1+4/\kappa}$$

Note that $|g_t'(a)| = \frac{dX_t^a}{da}$. X_t^a is a differentiable function of a and we can do a change of variables to change the above to

$$\int_{[P,Q]^c} X^{-4/\kappa} \, [(X - P)(Q - X)]^{(2-\kappa)/\kappa} \, dX = c \, P^{-2/\kappa} Q^{-2/\kappa} (Q - P)^{-1+4/\kappa}$$

where X is shorthand for X_t^a, P is for X_t^{-1} and Q for X_t^1. This identity follows from the substitution

$$X = PQ \frac{z + 1}{Pz + Q}$$

followed by some nice cancellations.

4   Girsanov's theorem

Girsanov's theorem gives a unified view of the computations of the two preceding sections. Let us first briefly recall the theorem and then go on with illustrations.

Take a probability space with a filtration $(\Omega, \mathcal{F}, \mathcal{F}_t, p)$ and let $M_t, \mathcal{F}_t$ be a nonnegative martingale on it such that $M_0 = 1$.

If X is $\mathcal{F}_s$-measurable and $t \geq s$ then basic rules of conditional expectations yield $\mathbf{E}[XM_t] = \mathbf{E}[XM_s]$ so that one can make a consistent definition $\tilde{\mathbf{E}}[X] \equiv \mathbf{E}[XM_t]$ whenever X is $\mathcal{F}_t$ measurable. Then $\tilde{\mathbf{E}}[\cdots]$ is easily

seen to be a positive linear functional with $\tilde{\mathbf{E}}[1] = 1$. Hence the definition $\tilde{p}_t(A) \equiv \tilde{\mathbf{E}}[\mathbf{1}_A]$ for $A \in \mathcal{F}_t$ makes $(\Omega, \mathcal{F}_t, \tilde{p}_t)$ a probability space.

Now take for $(\Omega, \mathcal{F}, p)$ a probability space carrying a (continuous) Brownian motion B_t with its filtration $\mathcal{F}_t$. If M_t is a continuous martingale, then Itô's representation formula says M_t satisfies the stochastic integral equation $M_t = 1 + \int_0^t M_s X_s dB_s$ for some process X_s (which satisfies a number of technical conditions, in particular X_s is $\mathcal{F}_s$-measurable for each s). In fact one can write $M_t = \exp\left[\int_0^t X_s\, dB_s - \frac{1}{2}\int_0^t X_s^2\, ds\right]$ as shown by an application of Itô's formula.

Girsanov's theorem states that the process $V_t \equiv B_t - \int_0^t X_s ds$ is a Brownian motion on $[0, T]$ for $(\Omega, \mathcal{F}_T, \tilde{p}_T)$ for each $T > 0$ (see for instance [14] for a readable mathematical introduction, [15] for a more exhaustive mathematical reference or [16] for a heuristic proof).

A simple special case is $M_t \equiv e^{xB_t - tx^2/2}$, which is a martingale on the Brownian motion space satisfying the conditions above for a constant $X_s = x$. Girsanov's theorem yields that under $\tilde{p}_T$, $V_t = B_t - xt$ is a Brownian motion on $[0, T]$, so that under $\tilde{p}_T$ the original process B_t is now a Brownian motion with a constant drift. This makes clear that the restriction to finite T in Girsanov's theorem is crucial: if we could take $T \to \infty$, we would have $B_t/t \to x$ for large t with probability 1 for $\tilde{p}$ while this event has probability 0 for p.

4.1   Comparison between different kinds of SLEs

Our computations in subsection 2.2 all dealt with the following situation : for a certain variant of SLE on a certain domain and with a certain normalization of the Loewner uniformizing map the driving function satisfied a known stochastic differential equation (in fact it was a Brownian motion), and we aimed at the stochastic differential equation for another variant.

Physical arguments allow to get this from Girsanov's theorem. SLE curves can be seen in general as interfaces in conformally invariant field theories which are continuum limits of statistical mechanics systems where an interface can already be identified in the discrete setting due to boundary conditions. Statistical mechanics relies on the computation of partition functions, and in the case at hand, the partition function when an initial segment of the interface is fixed is (up to normalization) the probability that the interface

starts with this initial segment. Changing the boundary conditions changes the partition function i.e. the distribution of the interface. The ratio of two such partition functions for the same initial segment of interface then appears as a discrete version of a Radon-Nykodim derivative, and indeed a slight extension of the arguments in [10] yields the fact the ratio of partition functions is a martingale when the initial segment gets larger [2].

A discrete partition function in an arbitrary geometry is hard to compute, to say the least. However the conformally invariant continuum limit (when it exists) is much more accessible. Its relation to the discrete system involves removing certain divergences, and the limit in arbitrary geometry (pieces of the boundary are SLEs with some change of boundary conditions at the tip) can be quite singular. But ratios of partition functions behave smoothly exactly when one variant of interface involved is absolutely continuous with respect to the other, and the limiting ratio is expected to be the Radon-Nykodim derivative for the corresponding probability measures in the continuum.

Let us illustrate this on an example and then say a few words on the general framework.

The partition function $Z_{\mathbb{D}}^{\mathrm{dip}}(x,a,b)$ for dipolar SLE in domain $\mathbb{D}$ from a boundary point x to a boundary interval $[a,b]$ (not containing x) is given by a three point correlation function $\langle \phi_{h_{0,1/2}}(a)\phi_{h_{1,2}}(x)\phi_{h_{0,1/2}}(b)\rangle_{\mathbb{D}}$ where ϕ_h is a customary notation for a primary conformal (boundary) field of weight h, and $h_{r,s} \equiv \frac{(\kappa r - 4s)^2 - (\kappa-4)^2}{16\kappa}$. So $h_{0,1/2} = \frac{(\kappa-2)(6-\kappa)}{16\kappa}$ and $h_{1,2} = \frac{(6-\kappa)}{2\kappa}$.

Such three point functions are completely fixed up to normalization by conformal invariance because all quadruples $(\mathbb{D}, x, a, b)$ are conformally equivalent. If $\mathbb{D}$ is the upper half plane $\mathbb{H}$ (then a, b, x are real numbers),

$$\langle \phi_h(a)\phi_{h'}(x)\phi_{h''}(b)\rangle_{\mathbb{H}} \propto |b-a|^{h'-h-h''}|x-a|^{h''-h-h'}|x-b|^{h-h''-h'}.$$

More generally, if the boundary of $\mathbb{D}$ is smooth at a, b, x and x and g maps $\mathbb{D}$ to $\mathbb{H}$,

$$\langle \phi_h(a)\phi_{h'}(x)\phi_{h''}(b)\rangle_{\mathbb{D}} = \langle \phi_h(g(a))\phi_{h'}(g(x))\phi_{h''}(g(b))\rangle_{\mathbb{H}}|g'(a)|^h|g'(x)|^{h'}|g'(b)|^{h''},$$

emphasizing the fact that partition functions are sections of tensor density bundles.

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[2]This extension however points to the importance that the variant of interface involved in the numerator is absolutely continuous with respect to the interface involved in the denominator.

In the case at hand,

$$Z_{\mathbb{H}}^{\text{dip}}(x,a,b) \propto |b-a|^{(\kappa-6)^2/(8\kappa)}|x-a|^{(\kappa-6)/(2\kappa)}|x-b|^{(\kappa-6)/(2\kappa)}$$

and

$$Z_{\mathbb{D}}^{\text{dip}}(x,a,b) = Z_{\mathbb{H}}^{\text{dip}}(g(x),g(a),g(b))|g'(a)|^{\frac{(\kappa-2)(6-\kappa)}{16\kappa}}|g'(x)|^{\frac{(6-\kappa)}{2\kappa}}|g'(b)|^{\frac{(\kappa-2)(6-\kappa)}{16\kappa}}.$$

Chordal SLE in $\mathbb{D}$ from x to y has partition function $Z_{\mathbb{D}}^{\text{chord}}(x,y) = \langle\phi_{h_{1,2}}(x)\phi_{h_{1,2}}(y)\rangle_{\mathbb{D}}$ which is even simpler. Taking $\mathbb{D}$ to be the upper half plane $\mathbb{H}$,

$$Z_{\mathbb{H}}^{\text{chord}}(x,y) = |x-y|^{-2h_{1,2}} = |x-y|^{(\kappa-6)/(\kappa)},$$

and in general

$$Z_{\mathbb{D}}^{\text{chord}}(x,y) = Z_{\mathbb{H}}^{\text{chord}}(g(x),g(y))|g'(x)|^{\frac{(6-\kappa)}{2\kappa}}|g'(y)|^{\frac{(6-\kappa)}{2\kappa}}.$$

Note that if $y \to \infty$, one has to change local coordinate to express the section, and $Z_{\mathbb{H}}^{\text{chord}}(x,\infty) = 1$.

In the identity

$$\frac{Z_{\mathbb{D}}^{\text{chord}}(x,y)|g'(y)|^{\frac{(\kappa-6)}{2\kappa}}}{Z_{\mathbb{H}}^{\text{chord}}(g(x),g(y))} = \frac{Z_{\mathbb{D}}^{\text{dip}}(x,a,b)|g'(a)|^{\frac{(\kappa-2)(\kappa-6)}{16\kappa}}|g'(b)|^{\frac{(\kappa-2)(\kappa-6)}{16\kappa}}}{Z_{\mathbb{H}}^{\text{dip}}(g(x),g(a),g(b))},$$

the derivative of g at x disappears, and the formula makes sense even if $\mathbb{D}$ is not smooth at x.

Suppose now that $\mathbb{D}$ is the domain obtained after the interface has grown for some time (and x is at the tip), capacity is used to measure time, and the uniformizing map is taken in the hydrodynamical normalization i.e. $\mathbb{D} = \mathbb{H}_t \equiv \mathbb{H}\backslash\gamma_{]0,t]}$ is uniformized by g_t satisfying $dg_t(z) = 2dt/(g_t(z) - V_t)$ where $V_t \equiv g_t(\gamma_t)$. Let $A_t \equiv g_t(a)$, $B_t \equiv g_t(b)$. Take $y = \infty$ and observe that $g_t(\infty) = \infty$ and $g_t'(\infty) = 1$. Then the above ratio becomes

$$Z_{\mathbb{H}_t}^{\text{chord}}(\gamma_t,\infty) = Z_{\mathbb{H}_t}^{\text{dip}}(\gamma_t,a,b)\frac{|g_t'(a)|^{\frac{(\kappa-2)(\kappa-6)}{16\kappa}}|g_t'(b)|^{\frac{(\kappa-2)(\kappa-6)}{16\kappa}}}{Z_{\mathbb{H}}^{\text{dip}}(V_t,A_t,B_t)}$$

Two strategies are now available. The first one looks at the ratio

$$\frac{Z_{\mathbb{H}_t}^{\text{dip}}(\gamma_t,a,b)}{Z_{\mathbb{H}_t}^{\text{chord}}(\gamma_t,\infty)} \equiv \frac{Z_{\mathbb{H}}^{\text{dip}}(V_t,A_t,B_t)}{|g_t'(a)|^{\frac{(\kappa-2)(\kappa-6)}{16\kappa}}|g_t'(b)|^{\frac{(\kappa-2)(\kappa-6)}{16\kappa}}}.$$

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A general argument from conformal field theory guaranties that this ration is a martingale of chordal SLE, call it M_t (of course this can also be checked by a straightforward computation). The naive continuum limit argument given above suggests that M_t is exactly the Radon-Nykodim derivative of dipolar SLE from $x = V_0$ to $[a, b]$ with respect to chordal SLE. For chordal SLE, $\kappa^{-1/2} V_t$ is a Brownian motion, and the other building blocks in the formula for M_t are of bounded variation (i.e. do not contribute to dV_t terms in dM_t), so that

$$dM_t = M_t \partial_x \log Z_{\mathbb{H}}^{\mathrm{dip}}(x = V_t, A_t, B_t) dV_t.$$

Then Girsanov's theorem yields the stochastic differential equation of dipolar SLE:

$$dV_t = \sqrt{\kappa} dW_t + \kappa \partial_x \log Z_{\mathbb{H}}^{\mathrm{dip}}(x = V_t, A_t, B_t) dt,$$

which coincides with the already obtained formula. From this viewpoint, the appearance of a logarithmic derivative in the drift is totally natural, and comparison with the initial formula implies an amusing relationship between the partition function $Z_{\mathbb{H}}^{\mathrm{dip}}(x, a, b)$ and any conformal map $\Phi(z)$ mapping the upper-half plane to the strip a to $-\infty$ and b to $+\infty$ (two such maps differ by an additive constant): $Z_{\mathbb{H}}^{\mathrm{dip}}(x, a, b) \Phi'(x)^{h_{1,2}}$ is x-independent.

The second one is to look at the inverse ratio

$$\frac{Z_{\mathbb{H}_t}^{\mathrm{chord}}(\gamma_t, \infty)}{Z_{\mathbb{H}_t}^{\mathrm{dip}}(\gamma_t, a, b)}.$$

This time CFT ensures that this is a martingale of dipolar SLE. Assuming that dipolar is absolutely continuous with respect to chordal i.e. that $\kappa^{-1/2} V_t$ is a Brownian motion plus a drift, this drift is fixed by the martingale property. Of course the result is in agreement with the other approaches. The computation is more painful because one needs to look at the second order term in Ito's formula. However, there is one benefit: only the martingale property is used, and not the fact that the ratio is a Radon-Nykodim derivative. Indeed one could replace the ratio $Z_{\mathbb{H}_t}^{\mathrm{chord}}(\gamma_t, \infty)/Z_{\mathbb{H}_t}^{\mathrm{dip}}(\gamma_t, a, b)$ by any correlation function of dipolar SLE in $\mathbb{H}_t$ with hitting interval $[a, b]$, which is again a tautological martingale, to compute the drift.

This second strategy was used in [17] to get the form of the stochastic differential system describing multiple SLE's, and in [18] to give the conformal field theory approach to SLE(κ, ρ's) and generalizations thereof. Indeed, the

martingale used in section 3.3 is the simplest partition function[3] describing the effect of marked points (in physics language, these could be impurities) on SLE interfaces and was identified as such in [18]. From the CFT viewpoint, this partition function leads to an $\mathrm{SLE}_{\kappa,\rho}$ tautological martingale by a general argument, and no explicit computation is required.

4.2 Conditioning

Take a probability space with a filtration $(\Omega, \mathcal{F}, \mathcal{F}_t, p)$ and an event $J \subset \Omega$ such that $p(J) = \mathbf{E}\left[\mathbf{1}_J\right] \neq 0$ and let $\tilde{p}, \tilde{\mathbf{E}}\left[\cdots\right]$ be the probability and expectation on $(\Omega, \mathcal{F})$ obtained by conditioning p on J.

If Y is a random variable, $\tilde{\mathbb{E}}(Y) = \frac{\mathbf{E}[Y\mathbf{1}_J]}{\mathbf{E}[\mathbf{1}_J]}$. In particular, if Y is $\mathcal{F}_s$ measurable $\mathbf{E}\left[Y\mathbf{1}_J\right] = \mathbf{E}\left[\mathbf{E}\left[Y\mathbf{1}_J|\mathcal{F}_s\right]\right] = \mathbf{E}\left[Y\mathbf{E}\left[\mathbf{1}_J|\mathcal{F}_s\right]\right]$. Then $M_s \equiv \frac{\mathbf{E}[\mathbf{1}_J|\mathcal{F}_s]}{\mathbf{E}[\mathbf{1}_J]}$ is a closed bounded positive martingale with $M_0 = 1$ and $\tilde{\mathbf{E}}\left[Y\right] = \mathbf{E}\left[YM_s\right]$. From the previous subsection, it is natural so view the numerator as a partition function Z_t.

If $(\Omega, \mathcal{F}, p)$ is a probability space carrying a (continuous) Brownian motion B_t with its filtration $\mathcal{F}_t$ and $M_s \equiv \frac{\mathbf{E}[\mathbf{1}_J|\mathcal{F}_s]}{\mathbf{E}[\mathbf{1}_J]}$ is a continuous martingale, Girsanov's theorem applies.

In our illustrative example, J is "the dipolar SLE process from 0 to the upper boundary in the strip hits in the interval $[i\pi + a, i\pi + b]$". We have computed the corresponding martingale: $M_t = Z(V_t, A_t, B_t)/Z(0, a, b)$ where

$$Z(V, A, B) \equiv \int_{A-V}^{B-V} dx \cosh^{-4/\kappa} x/2.$$

The martingale M_t is obviously continuous and Girsanov's theorem avoids the previous elementary but clumsy discussion to yield the drift in one stroke, and then the Markov property follows immediately.

Again, the processes A_t and B_t are differentiable with respect to t so that in the equation $M_t = 1 + \int_0^t M_s X_s dV_s$ only the variation of $Z(V_s, A_s, B_s)$ with respect to V_s contributes to X_s and we find $X_s = \partial_V \log Z(V_s, A_s, B_s)$: the drift is again the variation of the appropriate free energy with respect to the starting point of the curve.

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[3]In the coulomb gas formalism of conformal field theory, it involves no screening charges, hence appears as a simple product.

5 Conclusions

In this paper we have shown that if $\kappa = 2$, conditioning dipolar SLE to hit a subinterval yields dipolar SLE_2 towards that subinterval. As a limiting case, conditioning dipolar SLE_2 to hit at a point yields chordal SLE_2. We have also shown that these properties characterize the value $\kappa = 2$ by computing explicitly the drift term that arises when dipolar SLE_κ for general κ is conditioned to hit a subinterval. This drift term corresponds to some $\mathrm{SLE}_{\kappa,\rho}$. We have shown that this drift can be computed by standard martingale techniques or by conformal field theory inspired methods.

This can be seen as a characterization of $\kappa = 2$ analogous to the characterization of $\kappa = 8/3$ by the restriction property or $\kappa = 6$ by locality.

The relationship with lattice models leads to puzzling facts however. First of all, there is no obvious interpretation of the dipolar geometry for all lattice models. Loop erased random walks, the Ising model for spin clusters and percolation are among the few favorable cases. For loop erased random walks, the fact that conditioning of dipolar to hit a subinterval is dipolar to the subinterval is true almost by definition. For the Ising model and percolation, the boundary condition on the hexagonal lattice are $+$ and $-$ on each side of the starting point and *free* on the boundary interval. It is easy to understand then that conditioning to hit a subinterval is a non trivial operation. But the case of self avoiding walks is irritating. The obvious candidate for the dipolar geometry would be to consider all the simple walks joining a boundary point to a boundary interval and weight each step by the critical fugacity μ_c. This prescription has the property that conditioning to end at a given point yields the chordal case. So we conclude that this naive definition cannot converge to dipolar $\mathrm{SLE}_{8/3}$, most likely because it corresponds to non-conformally invariant boundary conditions.

References

[1] O. Schramm, D. Wilson, *SLE coordinate change*, New York Journal of Mathematics **11** (2005) 659-669. `ArXiv:math.PR/0505368`

[2] W. Werner, *Girsanov transformation for* $\mathrm{SLE}(\kappa, \rho)$ *processes, intersection exponents and hiding exponents*, Ann. Fac. Sci. Toulouse **13** (2004) 121-147. `ArXiv:math.PR/0302115`

[3] J. Dubédat, $SLE(\kappa, \rho)$ *martingales and duality*, Ann. Probab. **33** (2005) 223-243. `ArXiv:math.PR/0303128`

[4] G. Lawler, O. Schramm, W. Werner, *Conformal invariance of planar loop-erased walk and uniform spanning trees*, Ann. Prob. **32** (2004) 939-996. `arXiv:math.PR/0112234`

[5] D. Zhan, *The scaling limits of planar LERW in finitely connected domains*, `ArXiv:math/0610304`

[6] D. Bernard, G. Boffetta, A. Celani and G. Falkovich, *Conformal invariance in two-dimensional turbulence*, Nature Physics **2** (2006) 124. `ArXiv:nlin/0602017`

[7] E. Bogomolny, R. Dubertrand and C. Schmit, *SLE description of the nodal lines of random wave functions*, J. Phys. A: Math. Theor. **40** (2007) 381-395. `ArXiv:nlin/0609017`

[8] C. Amuroso, A. K. Hartmann, M. B. Hastings and M. A. Moore, *Conformal Invariance and SLE in Two-Dimensional Ising Spin Glasses*, Phys. Rev. Lett. **97** (2006) 267202. `ArXiv:cond-mat/0601711`

[9] D. Bernard, P. Le Doussal A. A. and Middleton, *Possible description of domain walls in two-dimensional spin glasses by stochastic Loewner evolutions*, Phys. Rev. **B 76** (2007) 020403. `ArXiv:cond-mat/0611433`

[10] M. Bauer, D. Bernard and J. Houdayer, *Dipolar SLEs*, J. Stat. Mech. (2005) P03001. `ArXiv:math-ph/0411038`

[11] G. Lawler, O. Schramm and W. Werner, *Conformal restriction: the chordal case*, J. Amer. Math. Soc. **16** (2003) 917-955. `ArXiv:math.PR/0209343`

[12] C. Hagendorf, private communication.

[13] O. Schramm, *A percolation formula*, Electron. Comm. Probab. Vol. **6** (2001) 115-120 . `ArXiv:math.PR/0107096`

[14] B. Öksendal, *Stochastic differential equations : an introduction with applications* 5th Ed., Springer Universitext , Springer (2003).

[15] I. Karatzas and S.E. Shreve, *Brownian motion and stochastic calculus* 2nd Ed., GTM 113, Springer (1991).

[16] M. Bauer and D. Bernard, *2D growth processes: SLE and Loewner chains* Physics Reports **432**, (2006) 115-221. `ArXiv:math-ph/0602049`

[17] M. Bauer, D. Bernard, K. Kytölä, *Multiple Schramm-Loewner evolutions and statistical mechanics martingales*, J. Stat. Phys. **120** (2005) 1125-1163. `ArXiv:math-ph/0503024`

[18] K. Kytölä, *On conformal field theories of* $\text{SLE}(\kappa, \rho)$, J. of Stat. Phys, **123** (2006) 1169-1181. `ArXiv:math-ph/0504057`