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Bertrand Eynard[a] and Nicolas Orantin[b]

[a] Institut de Physique Théorique, CEA, IPhT
F-91191 Gif-sur-Yvette, France
CNRS, URA 2306, F-91191 Gif-sur-Yvette, France
bertrand.eynard@cea.fr

[b] Theory department, CERN
Geneva 23, CH-1211 Switzerland
nicolas.orantin@cern.ch

Abstract

We review the method of symplectic invariants recently introduced to solve matrix models loop equations, and further extended beyond the context of matrix models. For any given spectral curve, one defines a sequence of differential forms, and a sequence of complex numbers F_g. We recall the definition of the invariants F_g, and we explain their main properties, in particular symplectic invariance, integrability, modularity,... Then, we give several examples of applications, in particular matrix models, enumeration of discrete surfaces (maps), algebraic geometry and topological strings, non-intersecting brownian motions,...

Contents

1   Introduction

Recently, it was understood how to solve, order by order in the so-called "topological expansion", the loop equations (Schwinger-Dyson equations) for matrix integrals [47]. The solution brought an unexpectedly rich structure [60], which, did not only solve the 1-matrix model, but which also solved multi-matrix models, as well as their limits. Later, it was understood that this structure also appears in other matrix models, and in problems of enumerative geometry, not directly related to matrix models.

Thus, there is an underlying structure which can be defined beyond the context of matrix models, and relies only on the intrinsic algebro-geometric properties of a plane curve, called the spectral curve.

In other words, for any regular (to be defined below) complex plane curve $\mathcal{E} = \{y(x)\}$ (whether it is related to a matrix model or not), we can define a sequence of numbers $F_g(\mathcal{E})$, $g = 0, 1, 2, \ldots, \infty$. Those numbers F_g are called the **symplectic invariants** of the spectral curve $\mathcal{E}$ (first introduced in [60]). The reason is because two spectral curves $\mathcal{E}$ and $\tilde{\mathcal{E}}$ which can be deduced from one another by a symplectic transformation (i.e. they have the same wedge product $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$), have the same F_g's. F_g is called the symplectic invariant of degree $2 - 2g$, because under a rescaling $y \to \lambda y$, F_g scales as $F_g \to \lambda^{2-2g} F_g$ (except F_1 which is logarithmic).

Moreover, for a spectral curve $\mathcal{E} = \{y(x)\}$, we define not only its symplectic invariants F_g's, we also define a doubly infinite sequence of symmetric meromorphic forms $\omega_n^{(g)}(x_1, \ldots, x_n)[\mathcal{E}]$, $n \in \mathbb{N}, g \in \mathbb{N}$, and such that $F_g = \omega_0^{(g)}$. For $n \geq 1$, those forms are not symplectic invariants, but they have many nice properties. They allow to compute the derivatives of the F_g's with respect to any parameter on which $\mathcal{E}$ could depend.

Those geometric objects are interesting, in particular for their applications to various problems of enumerative geometry (each problem corresponding to a given spectral curve $\mathcal{E}$), but also on their own. Indeed they have remarkable properties for arbitrary spectral curves, i.e. even for spectral curves not known to correspond to any enumerative geometry problem.

In particular, they are related to the Kodaira-Spencer field theory, to Frobenius manifolds, to the WDVV special geometry, topological strings and Dijkgraaf-Vafa conjecture. They are expected to be the B-model partition function, and through mirror symmetry, the F_g's are thus expected to be the generating functions of Gromov-Witten invariants of genus g for some toric geometries.

As another example of interesting properties, the F_g's have nice modular behaviors, and, for instance, they provide a solution to holomorphic anomaly equations.

They also contain an integrable structure, related to "multicomponent KP" hierarchy, e.g. they satisfy determinantal formulae, Hirota equations,...

Another nice property, is that they can be computed by a simple diagrammatic method, which makes them really easy to use. For instance the holomorphic anomaly equations can be proved only by drawing diagrams.

Regarding the applications, we will consider the following examples:
- Enumeration of discrete surfaces, possibly carrying colors on their faces (Ising model), as well as the asymptotics of large discrete surfaces.
- For the curve $y = \frac{1}{2\pi} \sin(2\pi\sqrt{x})$, the F_g's compute the Weyl-Petersson volumes.
- We will consider also the Kontsevich spectral curve, related to the Kontsevich integral, for which the F_g's are generating functions for intersection numbers of Chern classes of cotangent bundles at marked points, and the $W_n^{(g)}$'s are generating functions of Mumford κ classes (in some sense by "forgetting" some marked points).
- For the curve $y = \text{Argcosh}(x)$, and deformations of that curve, the F_g's are generating functions for counting partitions with the Plancherel measure, related to the computation of Hurwitz numbers.
- q-deformed versions of Plancherel measure sums of partitions can also be computed with symplectic invariants of some appropriate spectral curve, which, not so surprisingly, is the (singular locus of the) mirror of a toric Calabi-Yau manifold. This is consistent with the conjecture that the F_g's are related to Gromov-Witten invariants. Indeed, Gromov-Witten invariants of toric Calabi-Yau 3-folds can be computed, using the topological vertex, as sums of partitions, typically q-deformed Plancherel sums, for the simplest examples of toric Calabi-Yau 3-folds.

2 Symplectic invariants of spectral curves

Symplectic invariants were introduced in [60], as a common framework for the solution of loop equations of several matrix models: 1-matrix, 2-matrix, matrix with external fields,..., as well as their double scaling limits. Then it was discovered that they have many nice properties, in particular symplectic invariance, and that they appear in other problems of enumerative geometry, not necessarily related to random matrices.

Here we only briefly summarize the construction of [60], without proofs, and we refer the reader to the original article for more details.

2.1 Spectral curves

In this article, we define a spectral curve as follows[1]:

Definition 2.1 *A spectral curve $\mathcal{E} = (\mathcal{L}, x, y)$, is the data of a compact Riemann surface $\mathcal{L}$, and two analytical functions x and y on some open domain in $\mathcal{L}$.*

In some sense, we consider a parametric representation of the spectral curve $y(x)$, where the space of the parameter z is a Riemann surface $\mathcal{L}$.

---

[1]This definition is not exactly the one usually encountered in integrable systems [14], in fact it turns out that the plane curve we are considering here, is the "classical limit" of the full spectral curve. We call it spectral curve by abuse of language, and because it has become customary to do so.

Definition 2.2 *If $\mathcal{L}$ is a compact Riemann surface of genus $\bar{g}$, and x and y are meromorphic functions on $\mathcal{L}$, we say that the spectral curve is algebraic. If in addition, $\mathcal{L}$ is the Riemann sphere ($\mathcal{L} = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, i.e. of genus $\bar{g} = 0$), we say that the spectral curve is rational.*

Indeed, for an algebraic spectral curve, it is always possible to find a polynomial relationship between x and y:

$$\text{Pol}(x, y) = 0. \tag{2-1}$$

For a rational spectral curve, the polynomial equation $\text{Pol}(x, y) = 0$, can be parameterized with two rational functions $x(z)$ and $y(z)$ of a complex variable z.

Definition 2.3 *A spectral curve $(\mathcal{L}, x, y)$ is called regular if:*
- *The differential form dx has a finite number of zeroes $dx(a_i) = 0$, and all zeroes of dx are simple zeroes.*
- *The differential dy does not vanish at the zeroes of dx, i.e. $dy(a_i) \neq 0$.*

This means that near $x(a_i)$, y behaves locally like a square-root $y(z) \sim y(a_i) + C \sqrt{x(z) - x(a_i)}$, or in other words, that the curve $y(x)$ has a vertical tangent at a_i.

From now on, we assume that we are considering only regular spectral curves. Symplectic invariants are defined only for regular spectral curves, and they diverge when the spectral curve becomes singular. Examples of singular spectral curves are considered in section 4.8, they play a central role in the double scaling limit in chapter 8.

Definition 2.4 *We say that two spectral curves $\mathcal{E} = (\mathcal{L}, x, y)$ and $\tilde{\mathcal{E}} = (\tilde{\mathcal{L}}, \tilde{x}, \tilde{y})$ are symplectically equivalent if there is a conformal mapping $\mathcal{L} \to \tilde{\mathcal{L}}$, and if under this mapping $dx \wedge dy \to d\tilde{x} \wedge d\tilde{y}$. The group of symplectomorphisms is generated by:*

- $\tilde{x} = x$, $\tilde{y} = y + R(x)$, $R(x) = $ *rational function of* x.

- $\tilde{x} = \frac{ax+b}{cx+d}$, $\tilde{y} = \frac{(cx+d)^2}{ad-bc} y$.

- $\tilde{x} = f(x)$, $\tilde{y} = \frac{1}{f'(x)} y$, *where f is analytical and injective in the image of x.*

- $\tilde{x} = y$, $\tilde{y} = -x$.

All those transformations conserve the symplectic form on $\mathcal{L}$, whence the name:

$$d\tilde{x} \wedge d\tilde{y} = dx \wedge dy. \tag{2-2}$$

The main property of the F_g's we are going to define, is that they are symplectic invariants, i.e. two curves which are symplecticaly equivalent, have the same F_g's.

2.1.1 Examples of spectral curves

Interesting examples of spectral curves may come from several areas of physics or mathematics, and are related to some problems of enumerative geometry. We will study in details some examples between section 5 and 11. Here, in order to illustrate our notion of spectral curve, we give some examples of spectral curves of interest extracted from those applications.

For the readers familiar with matrix models, the spectral curve under consideration here, can be thought of, as the "equilibrium density of eigenvalues of the random matrix". It is not to be confused with the large N density of eigenvalues, although, for many simple cases the two may coincide[2]. In the most simple matrix models, the spectral curve is algebraic. For formal random matrix models, designed as combinatorics generating funcions for counting discrete surfaces, the spectral curve is shown to be rational (see section 7).

In the context of string theory, the spectral curve is often given by a transcendental equation of the form $H(e^x, e^y) = 0$, where H is a polynomial. It is not an algebraic spectral curve, but is closely related to an algebraic curve. In that case, dx and dy are abelian meromorphic differentials on the compact Riemann surface $\mathcal{L}$ corresponding to H (see section 11).

The origin of all the examples below are described with more details in sections 5 to 11.

- The following curve is a rational spectral curve:

$$\mathcal{L} = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \quad , \quad x(z) = z^2 - 2 \quad , \quad y(z) = z^3 - 3z. \quad (2\text{-}3)$$

It satisfies the algebraic equation $y^2 - 2 = x^3 - 3x$. It is an hyperelliptical curve of genus $\bar{g} = 0$. This spectral curve is related to the so-called "pure gravity Liouville field theory". It will often be called the "pure gravity" spectral curve, or also the $(3, 2)$ spectral curve, because pure gravity is the $(3, 2)$ minimal conformal field theory, it has central charge $c = 0$. See sections 4.8 and 8.

- The curve $y = \sqrt{x}$, is also a rational spectral curve which satisfies $y^2 = x$, and which can be parameterized by:

$$\mathcal{L} = \mathbb{P}^1 \quad , \quad x(z) = z^2 \quad , \quad y(z) = z. \quad (2\text{-}4)$$

This spectral curve arises in the study of the extreme eigenvalues statistics of a random matrix, i.e. in the study of the Tracy-Widom law and of the Airy kernel [111]. It will often be called the "Airy" spectral curve. It will also be called the $(1, 2)$ spectral curve in order to match the classification of minimal conformal field theories. The minimal model $(1, 2)$ has central charge $c = -2$. See section 8.

- The following spectral curve is also a rational spectral curve:

$$\mathcal{L} = \mathbb{P}^1 \quad , \quad x(z) = \gamma \left(z + \frac{1}{z} \right) \quad , \quad y(z) = -\frac{t}{\gamma z} + \frac{t_4 \gamma^3}{z^3} \quad (2\text{-}5)$$

---

[2]The two notions coincide for example for matrix integrals with a polynomial potential. They do not coincide for example when the potential has an explicit dependence on N.

where $\gamma^2 = \frac{1-\sqrt{1-12tt_4}}{6t_4}$. This spectral curve arises in the enumeration of quadrangulated surfaces, i.e. in the formal quartic matrix model. See section 7.4.

- The following spectral curve

$$\mathcal{L} = \mathbb{P}^1 \quad , \quad x(z) = z^2 \quad , \quad y(z) = \frac{1}{2\pi} \sin\left(2\pi z\right) \tag{2-6}$$

is related to the computation of Weyl-Petersson volumes. Notice that it is not algebraic, but it can be parameterized by a complex variable, i.e. by a $\bar{g} = 0$ Riemann surface. See section 10.2.

- The following rational spectral curve

$$\mathcal{L} = \mathbb{P}^1 \quad , \quad x(z) = z^2 \quad , \quad y(z) = z^3 - 3tz \tag{2-7}$$

is singular at $t = 0$. Indeed at $t = 0$, the differential $dy = 3(z^2 - t)dz$ vanishes at $z = 0$ which is the zero of dx. We will see below that the F_g's diverge for singular curves, and thus the function $F_g(t)$ has a singularity at $t = 0$. Just from homogeneity, and by considering the change of variable $z \to \sqrt{t}\,z$, we see that:

$$F_g(t) = t^{5(1-g)}\, F_g(1) \tag{2-8}$$

which indeed diverges at $t = 0$. See section 4.8.

- The following spectral curve depends on two parameters $p \in \mathbb{Z}$ and $z_0 \in \mathbb{C}^*$:

$$\mathcal{L} = \mathbb{P}^1 \quad , \quad \begin{cases} x(z) = \dfrac{(1 - \frac{z}{z_0})(1 - \frac{1}{zz_0})}{(1 + \frac{1}{z_0})^2} \\[4mm] y(z) = \dfrac{1}{x(z)} \left(-\ln z + \dfrac{p}{2} \ln\left(\dfrac{1 - z/z_0}{1 - 1/zz_0}\right) \right) \end{cases} \tag{2-9}$$

It appears in the enumeration of $q-$deformed Plancherel sums of partitions, i.e. in the computation of the Gromov-Witten invariants of the toric Calabi-Yau manifold $X_p = O(-p) \oplus O(p - 2) \to \mathbb{P}^1$. See section 9.3.

This spectral curve is symplectically equivalent to (compute $dx \wedge dy$ in both cases):

$$\mathcal{L} = \mathbb{P}^1 \quad , \quad \begin{cases} x(z) = \ln\left((1 - \dfrac{z}{z_0})(1 - \dfrac{1}{zz_0}) \right) \\[4mm] y(z) = \ln\left(\dfrac{1}{z} \left(\dfrac{1 - z/z_0}{1 - 1/zz_0} \right)^{\frac{p}{2}} \right) \end{cases} \tag{2-10}$$

This last spectral curve is such that e^x and e^y are rational functions of z, and thus by eliminating z, there exists a polynomial $H(e^x, e^y)$ such that:

$$H(e^x, e^y) = 0. \tag{2-11}$$

This equation is precisely the singular locus of the mirror manifold of X_p. The full mirror manifold (not only its singular locus) is the 3 dimensional submanifold of $\mathbb{C}^4$ locally given by $\{(x, y, \omega_+, \omega_-) \in \mathbb{C}^4 \ / \ H(e^x, e^y) = \omega_+\omega_-\}$. See section 11.

- The following spectral curve is of genus $\bar{g} = 1$, it is algebraic but not rational:

$$\mathcal{L} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \quad , \quad x(z) = \wp(z, \tau) \quad , \quad y(z) = \wp'(z, \tau) \tag{2-12}$$

where $\wp$ is the Weierstrass function, and $\mathcal{L}$ is the torus of modulus τ. It is algebraic because the Weierstrass function obeys the differential equation:

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3. \tag{2-13}$$

This spectral curve is called the Seiberg-Witten curve since it first appeared in a solution to $\mathcal{N} = 2$ Supersymmetric Yang-Mills theory proposed by Seiberg and Witten in [107].

2.2 Geometry of the spectral curve

2.2.1 Genus and cycles

The only compact Riemann surface of genus $\bar{g} = 0$ is the Riemann sphere $\mathbb{P}^1 = \mathbb{C}\cup\{\infty\}$. It is simply connected.

A compact Riemann surface $\mathcal{L}$ of genus $\bar{g} \geq 1$, can be equipped with a symplectic basis (not unique) of $2\bar{g}$ non-contractible cycles such that:

$$\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j} \quad , \quad \mathcal{A}_i \cap \mathcal{A}_j = 0 \quad , \quad \mathcal{B}_i \cap \mathcal{B}_j = 0. \tag{2-14}$$

They are such that $\mathcal{L}\setminus (\cup_i\mathcal{A}_i \cup_i \mathcal{B}_i)$ is a simply connected domain of $\mathcal{L}$, which we shall call the fundamental domain.
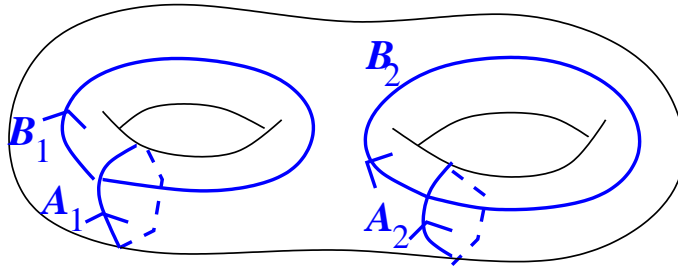
The choice of cycles and of a fundamental domain is rather arbitrary, and is not unique. Many of the quantities we are going to consider depend on that choice.

The quantities which do not depend on that choice are called modular invariant.

On a compact Riemann surface of genus $\bar{g} \geq 1$, there exist holomorphic differential forms (analytical everywhere on $\mathcal{L}$, in particular with no pole). Those holomorphic forms clearly form a vector space (linear combinations are also holomorphic) over $\mathbb{C}$, and this vector space has dimension $\bar{g}$.

When we have a choice of cycles $\mathcal{A}_i, \mathcal{B}_j$, it is possible to choose a basis (which is unique), which we call $du_1, \ldots, du_{\bar{g}}$, and normalized such that:

$$\oint_{\mathcal{A}_i} du_j = \delta_{i,j}. \tag{2-15}$$

Once we have defined those du_j's, we can compute the following Riemann matrix of periods:

$$\tau_{i,j} = \oint_{\mathcal{B}_i} du_j. \tag{2-16}$$

This matrix $\tau_{i,j}$ is symmetric, and its imaginary part is positive definite:

$$\tau_{i,j} = \tau_{j,i} \quad , \quad \mathrm{Im}\,\tau > 0. \tag{2-17}$$

2.2.2   Abel map

Consider an arbitrary origin o in the fundamental domain, and fixed throughout all this article.

For any point z in the fundamental domain, the vector $(u_1(z), \ldots, u_{\bar{g}}(z))$:

$$u_i(z) = \int_o^z du_i \tag{2-18}$$

where the integration path is in the fundamental domain, is called the Abel map of z. It is a vector in $\mathbb{C}^{\bar{g}}$. It depends on the choice of o by an additive constant, and it depends on the choice of the fundamental domain, by a vector in the lattice $\mathbb{Z}^{\bar{g}} + \tau\mathbb{Z}^{\bar{g}}$. The quotient $\mathbb{C}^{\bar{g}}/(\mathbb{Z}^{\bar{g}} + \tau\mathbb{Z}^{\bar{g}})$ is called the Jacobian.

The Abel map, sends points of $\mathcal{L}$ to points in the Jacobian.

2.2.3   Bergmann kernel

Given a choice of cycles, we define the Bergmann kernel:

$$B(z_1, z_2) \tag{2-19}$$

as the unique bilinear differential having one double pole at $z_1 = z_2$ (it is called "2nd kind") and no other pole, and such that, in any local parameter z:

$$B(z_1, z_2) \underset{z_1 \to z_2}{\sim} \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \mathrm{reg} \quad , \quad \forall i = 1, \ldots, \bar{g}, \quad \oint_{\mathcal{A}_i} B(z_1, z_2) = 0. \tag{2-20}$$

One should keep in mind that the Bergmann kernel depends only on $\mathcal{L}$, and not on the functions x and y.

The Bermann kernel can be seen as the derivative of the Green function, i.e. the solution of the heat kernel equation on $\mathcal{L}$.

The Bergmann kernel is clearly unique because the difference of two Bergmann kernels would have no pole, and vanishing $\mathcal{A}$-cycle integrals, therefore it would vanish. It is also interesting to note that it is symmetric in its variables z_1 and z_2.

Examples:

- if $\mathcal{L} = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ =the Riemann Sphere, the Bergmann kernel is a rational expression:

$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}. \tag{2-21}$$

Most of the applications between section 5 and section 11, will be on $\mathcal{L} = \mathbb{P}^1$, and will use this rational Bergmann kernel.

- if $\mathcal{L} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ =Torus of modulus τ, the Bergmann kernel is

$$B(z_1, z_2) = \left(\wp(z_1 - z_2, \tau) + \frac{\pi}{\text{Im}\,\tau}\right) dz_1 dz_2 \tag{2-22}$$

where $\wp$ is the Weierstrass elliptical function.

- if $\mathcal{L}$ is a compact Riemann surface of genus $\bar{g} \geq 1$, of Riemann matrix of periods $\tau_{i,j}$, the Bergmann kernel is

$$B(z_1, z_2) = d_{z_1} d_{z_2} \ln\left(\theta(u(z_1) - u(z_2) - c, \tau)\right) \tag{2-23}$$

where $u(z)$ is the Abel map, c is an odd characteristic, and θ is the Riemann theta function of genus $\bar{g}$ (cf [66, 65] for theta-functions).

2.2.4 Generalized Bergmann kernel

Given an arbitrary symmetric matrix κ of size $\bar{g} \times \bar{g}$, we consider a "deformed" Bergmann kernel:

$$B_\kappa(z_1, z_2) = B(z_1, z_2) + 2i\pi \sum_{i,j=1}^{\bar{g}} \kappa_{i,j} \, du_i(z_1) du_j(z_2). \tag{2-24}$$

If $\kappa = 0$ we recover the usual Bergmann kernel $B_0 = B$.

The reason for introducing this κ, is that a change of basis of cycles and fundamental domain, can be rewritten as a change of κ.

Indeed, perform a $Sp_{2\bar{g}}(\mathbb{Z})$ change of symplectic basis of cycles ($C, D, \tilde{C}, \tilde{D}$ have coefficients in $\mathbb{Z}$ and $CD^t = DC^t$, $\tilde{C}\tilde{D}^t = \tilde{D}\tilde{C}^t$, $C\tilde{D}^t - D\tilde{C}^t = \mathbf{1}$):

$$\mathcal{A}_i = \sum_j C_{i,j}\mathcal{A}'_j + \sum_j D_{i,j}\mathcal{B}'_j \quad , \quad \mathcal{B}_i = \sum_j \tilde{C}_{i,j}\mathcal{A}'_j + \sum_j \tilde{D}_{i,j}\mathcal{B}'_j. \tag{2-25}$$

The Riemann matrix of periods τ' in the new basis $\mathcal{A}', \mathcal{B}'$, is related to the old one by the modular transformation:

$$\tau' = (\tilde{D} - \tau D)^{-1}(\tau C - \tilde{C}) \tag{2-26}$$

and the Bergmann kernel changes as:

$$B_0 \to B'_0 = B_0 + 2i\pi \sum_{i,j=1}^{\bar{g}} \kappa_{i,j} \, du_i(z_1) du_j(z_2) \quad , \quad \kappa = (\tilde{D}D^{-1} - \tau)^{-1} \tag{2-27}$$

in other words, the change of cycles can be reabsorbed as a change of κ.

More generally, the kernel B_κ in a basis $\mathcal{A}, \mathcal{B}$ is equal to $B'_{\kappa'}$ in the basis $\mathcal{A}', \mathcal{B}'$, where:

$$\kappa' = (\tilde{D}^t - D^t \tau) \kappa (\tilde{D} - \tau D) - (\tilde{D}^t - D^t \tau) D. \tag{2-28}$$

From now on, we will always consider B_κ, and we will write B instead of B_κ, i.e. we will omit the κ subscript, unless ambiguity. However, for most of the practical applications, one often chooses $\kappa = 0$.

2.2.5 Schiffer kernel

In particular if we choose κ to be the Zamolodchikov Kähler metric:

$$\kappa = (\overline{\tau} - \tau)^{-1} = \frac{i}{2} \, (\mathrm{Im}\, \tau)^{-1}, \tag{2-29}$$

we see that in the new basis $\mathcal{A}', \mathcal{B}'$, the matrix κ becomes

$$\kappa' = \frac{i}{2} \, (\mathrm{Im}\tau')^{-1}, \tag{2-30}$$

i.e. it takes the same form as in the initial basis. Therefore, with this special value of κ, the Bergmann kernel B_κ is called the Schiffer kernel [16] and it is modular invariant: it does not depend on a choice of cycles. However, the price to pay to have modular invariance, is to have a non analytical dependence in τ, and thus a non analytical dependence in the spectral curve. This incompatibility between analyticity and modular invariance is the origin of the so-called "holomorphic anomaly equation", see section 4.4.2.

Example: if $\mathcal{L} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) =$Torus of modulus τ, the Schiffer kernel is

$$B(z_1, z_2) = \wp(z_1 - z_2, \tau) \, dz_1 dz_2 \tag{2-31}$$

where $\wp$ is the Weierstrass elliptical function. Compare with eq.(2-22).

2.2.6 Branchpoints

Branchpoints are the points with a vertical tangent, they are the **zeroes of** dx. Let us write them a_i, $i = 1, \ldots, \#\mathrm{bp}$:

$$\forall i, \quad dx(a_i) = 0. \tag{2-32}$$

Since we consider a regular spectral curve, all branchpoints are simple zeroes of dx, the curve $y(x)$ behaves locally like a square root $y(z) \sim y(a_i) + C_i \sqrt{x(z) - x(a_i)}$, near a branchpoint a_i, and thus, for any z close to a_i, there is exactly one point $\bar{z} \neq z$ in the vicinity of a_i such that:

$$x(\bar{z}) = x(z). \tag{2-33}$$

$\bar{z}$ is called the **conjugated point** of z. It is defined locally near each branchpoint a_i, and it is not necessarily defined globally.

Examples:

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- **enumeration of maps ≈ 1-matrix model in the 1-cut case**:

   in this case we have $\mathcal{L}$ =Riemann sphere, and (see section 7.3.2):

$$x(z) = \alpha + \gamma(z + 1/z) \qquad , \qquad dx(z) = x'(z)dz = \gamma(1 - z^{-2})\,dz. \qquad (2\text{-}34)$$

   The zeroes of $dx(z)$ are $z = \pm 1$, and we clearly have $\bar{z} = 1/z$:

$$a_1 = 1, \;\; a_2 = -1 \qquad , \qquad \bar{z} = 1/z. \qquad (2\text{-}35)$$

   In this case $\bar{z}$ is defined globally.

- **pure gravity** $(3, 2)$:

   in that case we have $\mathcal{L}$ =Riemann sphere, and

$$x(z) = z^2 - 2 \qquad , \qquad dx(z) = 2z\,dz. \qquad (2\text{-}36)$$

   The only zeroe of $dx(z)$ is $z = 0$, and we have $\bar{z} = -z$:

$$a = 0 \qquad , \qquad \bar{z} = -z. \qquad (2\text{-}37)$$

   In this case $\bar{z}$ is defined globally.

- **Ising model** $(4, 3)$:

   in that case we have $\mathcal{L}$ =Riemann sphere, and

$$x(z) = z^3 - 3z \qquad , \qquad dx(z) = 3(z^2 - 1)\,dz. \qquad (2\text{-}38)$$

   The zeroes of $dx(z)$ are $a_i = \pm 1$, and near $a_i = \pm 1$ we have:

$$a_i = \pm 1 \qquad , \qquad \bar{z} = -\frac{1}{2}(z - a_i\sqrt{12 - 3z^2}). \qquad (2\text{-}39)$$

   In this case $\bar{z}$ is not defined globally, and it depends on a_i.

2.2.7   Recursion Kernel

For any $z_0 \in \mathcal{L}$, and any z close to a branchpoint, we define the recursion kernel:

$$\boxed{K(z_0, z) = \frac{-1}{2} \; \frac{\int_{z'=\bar{z}}^{z} B(z_0, z')}{(y(z) - y(\bar{z}))\,dx(z)}}$$

$$(2\text{-}40)$$

where the integral is taken in a small domain in the vicinity of the concerned branchpoint.

$K(z_0, z)$ is a meromorphic 1-form in the variable z_0, it is defined globally for all $z_0 \in \mathcal{L}$, it has simple poles at $z_0 = z$ and $z_0 = \bar{z}$.

On the contrary, in the variable z, the kernel $K(z_0, z)$ is defined only locally near branchpoints $z \sim a_i$, and it is the inverse of a differential. As we shall see below,

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$K(z_0, z)$ will always be used only in the vicinity of branchpoints, and it will always be multiplied by a quadratic differential in z, so that the product will be a differential form.

Let us notice that $K(z_0, z) = K(z_0, \bar{z})$, and that $K(z_0, z)$ has a simple pole when z approaches the branchpoint. Using De L'Hopital's rule, the leading behavior near the branchpoint is:

$$K(z_0, z) \underset{z \to a_i}{\sim} -\frac{B(z, z_0)}{2\, dy(z) dx(z)} + \text{regular} \ldots \tag{2-41}$$

2.3 Correlation functions

We start by defining a sequence of meromorphic n-forms $\omega_n^{(g)}$ with $n = 1, 2, \ldots$ and $g = 0, 1, 2, \ldots$, called correlators or correlation functions, by the following recursion:

Definition 2.5 *Given a spectral curve* $\mathcal{E} = (\mathcal{L}, x, y)$, *and a matrix* κ *(see section 2.2.4), we define recursively the following meromorphic forms:*

$$\omega_1^{(0)}(z) = -y(z) dx(z) \tag{2-42}$$

$$\omega_2^{(0)}(z_1, z_2) = B(z_1, z_2) \tag{2-43}$$

and if $2g - 2 + n \geq 0$, *and* J *is a collective notation for* n *variables* $J = \{z_1, \ldots, z_n\}$:

$$\omega_{n+1}^{(g)}(z_0, J) = \sum_i \operatorname*{Res}_{z \to a_i} K(z_0, z) \left[\omega_{n+2}^{(g-1)}(z, \bar{z}, J) + \sum_{h=0}^{g} {\sum_{I \subset J}}' \omega_{1+|I|}^{(h)}(z, I) \omega_{1+n-|I|}^{(g-h)}(\bar{z}, J \backslash I) \right] \tag{2-44}$$

where ${\sum}'$ *in the RHS means that we exclude the terms with* $(h, I) = (0, \emptyset)$, *and* (g, J).

This definition is indeed a recursive one, because all the terms in the RHS have a strictly smaller $2g - 2 + n$ *than the LHS.*

The functions $\omega_n^{(g)}$ *with* $2 - 2g - n < 0$ *are called stable, the others are unstable (the only unstable ones are thus* $\omega_1^{(0)}$ *and* $\omega_2^{(0)}$).

$\omega_n^{(g)}(z_1, \ldots, z_n)$ is a meromorphic 1-form on $\mathcal{L}$ in each variables z_i. It can be proved by recursion, that it is in fact a symmetric form. Moreover, if $2 - 2g - n < 0$, its only poles are at branchpoints $z_i \to a_j$, and have no residues:

$$\operatorname*{Res}_{z_1 \to a_i} \omega_n^{(g)}(z_1, z_2, \ldots, z_n) = 0. \tag{2-45}$$

Those properties can be proved by recursion, and we refer the reader to [60].

2.4 Free energies

The previous definition, defines $\omega_n^{(g)}$ only if $n \geq 1$. Now, we define $F_g = \omega_0^{(g)}$, called "Free energies" or "symplectic invariant of degree $2 - 2g$", by the following:

Definition 2.6 Symplectic invariants
We define for $g \geq 2$:

$$\boxed{F_g(\mathcal{E}, \kappa) = \omega_0^{(g)} = \frac{1}{2 - 2g} \sum_i \operatorname*{Res}_{z \to a_i} \Phi(z)\, \omega_1^{(g)}(z)}$$

(2-46)

where Φ is any function defined locally near branchpoints, such that $d\Phi = ydx$ (Φ is defined up to an additive constant, but thanks to eq.(2-45), F_g does not depend on the choice of that constant).

The unstable cases $g = 0$ and $g = 1$ are special. We have:

Definition 2.7 *For $g = 1$ we define*

$$\boxed{F_1 = \frac{1}{24} \ln \left(\tau_B(\{x(a_i)\}) \prod_i y'(a_i) \right)}$$

(2-47)

where we define:

$$y'(a_i) = \lim_{z \to a_i} \frac{y(z) - y(a_i)}{\sqrt{x(z) - x(a_i)}}$$

(2-48)

and τ_B is the Bergmann τ-function of Kokotov-Korotkin [87]. If $x(z)$ is a meromorphic function on $\mathcal{L}$, τ_B depends only on the values of x at its branch points, i.e. $X_i = x(a_i)$. It is defined by:

$$\frac{\partial \ln \tau_B(\{X_i\})}{\partial X_i} = \operatorname*{Res}_{z \to a_i} \frac{B(z, \bar{z})}{dx(z)}.$$

(2-49)

Notice that B here stands for B_κ with arbitrary κ.

The definition of F_0 is more involved, and we refer the reader to [60]. A convenient way to define F_0, is through its 3rd derivatives, using theorem 4.3 below. In fact, all the F_g's with $g \geq 1$ are obtained in terms of local behaviors around branchpoints, but F_0 depends on the whole spectral curve, not only on the vicinity of branchpoints. In the context of topological strings, F_0 is called the prepotential.

3 Diagrammatic representation

The recursive definitions of $\omega_k^{(g)}$ and $F^{(g)}$ can be represented **graphically**.

We represent the $k-$form $\omega_k^{(g)}(p_1, \ldots, p_k)$ as a "blob-like surface" with g holes and k legs (or punctures) labeled with the variables $p_1, \ldots, p_k$, and $F^{(g)} = \omega_0^{(g)}$ with 0 legs and g holes.

$$(3\text{-}1)$$

We represent the Bergmann kernel $B(p,q)$ (which is also $\omega_2^{(0)}$, i.e. a blob with 2 legs and no hole) as a straight non-oriented line between p and q

$$B(p,q) := \quad \mathsf{p} \,\rule[0.5ex]{3em}{0.4pt}\, \mathsf{q} \;. \qquad (3\text{-}2)$$

We represent $K(p,q)$ as a straight arrowed line with the arrow from p towards q, and with a tri-valent vertex whose left leg is q and right leg is $\overline{q}$

$$(3\text{-}3)$$

Graphs

Definition 3.1 *For any $k \geq 0$ and $g \geq 0$ such that $k + 2g \geq 3$, we define:*
Let $\mathcal{G}_{k+1}^{(g)}(p, p_1, \ldots, p_k)$ be the set of connected trivalent graphs defined as follows:

1. *there are $2g + k - 1$ trivalent vertices called vertices.*

2. *there is one 1-valent vertex labelled by p, called the root.*

3. *there are k 1-valent vertices labelled with $p_1, \ldots, p_k$ called the leaves.*

4. *There are $3g + 2k - 1$ edges.*

5. *Edges can be arrowed or non-arrowed. There are $k + g$ non-arrowed edges and $2g + k - 1$ arrowed edges.*

6. *The edge starting at p has an arrow leaving from the root p.*

7. *The k edges ending at the leaves $p_1, \ldots, p_k$ are non-arrowed.*

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8. The arrowed edges form a "spanning[3] planar[4] binary skeleton[5] tree" with root p. The arrows are oriented from root towards leaves. In particular, this induces a partial ordering of all vertices.

9. There are k non-arrowed edges going from a vertex to a leaf, and g non arrowed edges joining two inner vertices. Two inner vertices can be connected by a non arrowed edge only if one is the parent of the other following the arrows along the tree.

10. If an arrowed edge and a non-arrowed inner edge come out of a vertex, then the arrowed edge is the left child. This rule only applies when the non-arrowed edge links this vertex to one of its descendants (not one of its parents).

Example of $\mathcal{G}_1^{(2)}(p)$

As an example, let us build step by step all the graphs of $\mathcal{G}_1^{(2)}(p)$, i.e. $g = 2$ and $k = 0$.

We first draw all planar binary skeleton trees with one root p and $2g + k - 1 = 3$ arrowed edges:

$$\text{(3-4)}$$

Then, we draw $g + k = 2$ non-arrowed edges in all possible ways such that every vertex is trivalent, also satisfying rule 9) of definition 3.1. There is only one possibility for the first tree, and two for the second one:

$$\text{(3-5)}$$
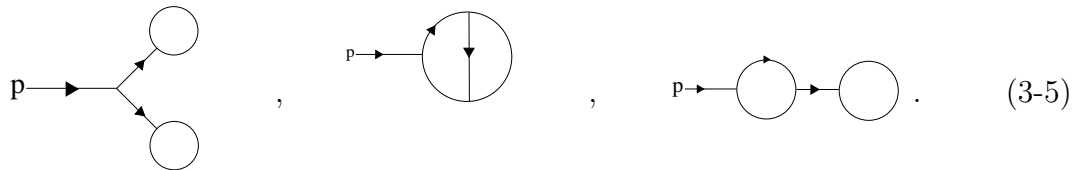
It just remains to specify the left and right children for each vertex. The only possibilities in accordance with rule 10) of def.3.1 are[6]:

---
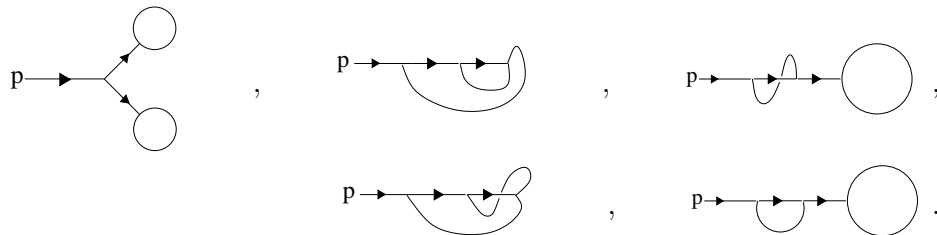
[3]It goes through all vertices.

[4]Planar tree means that the left child and right child are not equivalent. The right child is marked by a black disk on the outgoing edge.

[5]A binary skeleton tree is a binary tree from which we have removed the leaves, i.e. a tree with vertices of valence 1, 2 or 3.

[6]Note that the graphs are not necessarily planar.

In order to simplify the drawing, we can draw a black dot to specify the right child. This way one gets only planar graphs:
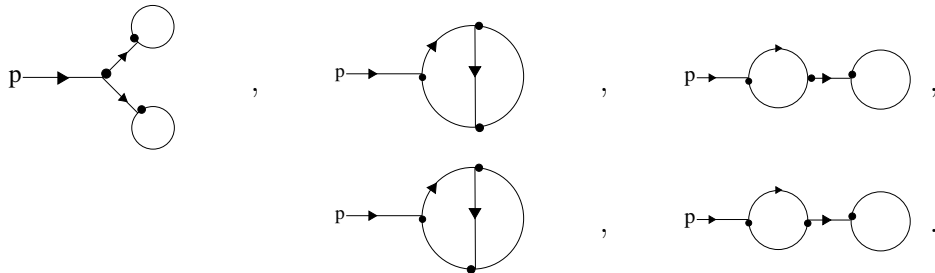
Remark that without the prescriptions 9) and 10), one would get 13 different graphs whereas we only have 5.

Weight of a graph

Consider a graph $G \in \mathcal{G}^{(g)}_{k+1}(p, p_1, \ldots, p_k)$. Then, to each vertex $i = 1, \ldots, 2g + k - 1$ of G, we associate a label $q_i \in \mathcal{L}$, and we associate q_i to the beginning of the left child edge, and $\bar{q}_i$ to the right child edge. Thus, each edge (arrowed or not), links two labels which are points on the spectral curve $\mathcal{L}$.

- To an arrowed edge going from q' towards q, we associate a factor $K(q', q)$.

- To a non arrowed edge going between q' and q we associate a factor $B(q', q)$.

- Following the arrows backwards (i.e. from leaves to root), for each vertex q, we take the sum over all branchpoints a_i of residues at $q \to a_i$.

After taking all the residues, we get the weight of the graph:

$$w(G) \tag{3-8}$$

which is a multilinear form in $p, p_1, \ldots, p_k$.

Similarly, we define weights of linear combinations of graphs by:

$$w(\alpha G_1 + \beta G_2) = \alpha w(G_1) + \beta w(G_2) \tag{3-9}$$

and for a disconnected graph, i.e. a product of two graphs:

$$w(G_1 G_2) = w(G_1) w(G_2). \tag{3-10}$$

Theorem 3.1 *We have:*

$$\omega^{(g)}_{k+1}(p, p_1, \ldots, p_k) = \sum_{G \in \mathcal{G}^{(g)}_{k+1}(p,p_1,\ldots,p_k)} w(G) = w\left(\sum_{G \in \mathcal{G}^{(g)}_{k+1}(p,p_1,\ldots,p_k)} G \right). \tag{3-11}$$

proof:

This is precisely what the recursion equations 2-44 of def.2.5 are doing. Indeed, one can represent them diagrammatically by

$$(3\text{-}12)$$

$\square$

Such graphical notations are very convenient, and are a good support for intuition and even help proving some relationships. It was immediately noticed after [47] that those diagrams look very much like Feynman graphs, and there was a hope that they could be the Feynman's graphs for the Kodaira–Spencer quantum field theory. But they ARE NOT Feynman graphs, because Feynman graphs can't have non-local restrictions like the fact that non oriented lines can join only a vertex and one of its descendent.

Those graphs are merely a notation for the recursive definition 2-44.

Lemma 3.1 Symmetry factor:

The weight of two graphs differing by the exchange of the right and left children of a vertex are the same. Indeed, the distinction between right and left child is just a way of encoding symmetry factors.

proof:

This property follows directly from the fact that $K(z_0, z) = K(z_0, \bar{z})$. $\square$

3.1   Examples.

Let us present some examples of correlation functions and free energy for low orders.

3.1.1   3-point function.

$$
\begin{aligned}
\omega_3^{(0)}(p, p_1, p_2) \quad &= \quad \\
&= \quad \operatorname*{Res}_{q \to \mathbf{a}} K(p, q) \, [B(q, p_1) B(\bar{q}, p_2) + B(\bar{q}, p_1) B(q, p_2)] \\
&= \quad -2 \operatorname*{Res}_{q \to \mathbf{a}} K(p, q) \, [B(q, p_1) B(q, p_2)]
\end{aligned}
$$

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$$= \operatorname*{Res}_{q \to \mathbf{a}} \frac{B(q,p)\, B(q,p_1)\, B(q,p_2)}{dx(q)\, dy(q)}$$

$$= \sum_i \frac{B(a_i,p)\, B(a_i,p_1)\, B(a_i,p_2)}{2\, dy(a_i)\, dz_i(a_i)^2}$$

$$(3-13)$$

where $z_i(z) = \sqrt{x(z) - z(a_i)}$ is a local coordinate near a_i.

3.1.2 4-point function.

$$\omega_4^{(0)}(p,p_1,p_2,p_3) \qquad = \qquad \text{(diagram)} \qquad + \qquad 5 \text{ permutations of } (1,2,3)$$

$$+ \qquad \text{(diagram)} \qquad + \qquad 5 \text{ permutations of } (1,2,3)$$

$$= \operatorname*{Res}_{q \to \mathbf{a}} \operatorname*{Res}_{r \to \mathbf{a}} K(p,q)\, K(q,r)\, [B(\overline{q},p_1)B(r,p_2)B(\overline{r},p_3)$$
$$+ B(\overline{q},p_1)B(\overline{r},p_2)B(r,p_3) + B(\overline{q},p_2)B(r,p_1)B(\overline{r},p_3)$$
$$+ B(\overline{q},p_2)B(\overline{r},p_1)B(r,p_3) + B(\overline{q},p_3)B(r,p_2)B(\overline{r},p_1)$$
$$+ B(\overline{q},p_3)B(\overline{r},p_2)B(r,p_1)]$$
$$+ \operatorname*{Res}_{q \to \mathbf{a}} \operatorname*{Res}_{r \to \mathbf{a}} K(p,q)\, K(\overline{q},r)\, [B(q,p_1)B(r,p_2)B(\overline{r},p_3)$$
$$+ B(q,p_1)B(\overline{r},p_2)B(r,p_3) + B(q,p_2)B(r,p_1)B(\overline{r},p_3)$$
$$+ B(q,p_2)B(\overline{r},p_1)B(r,p_3) + B(q,p_3)B(r,p_2)B(\overline{r},p_1)$$
$$+ B(q,p_3)B(\overline{r},p_2)B(r,p_1)]\,.$$

$$(3-14)$$

3.1.3 1-point function to order 1.

$$\omega_1^{(1)}(p) \qquad = \qquad \text{(diagram)}$$

$$= \operatorname*{Res}_{q \to \mathbf{a}} K(p,q)\, B(q,\overline{q})\,.$$

$$(3-15)$$

3.1.4 1-point function to order 2.

$$\omega_1^{(2)}(p) = \text{(diagram)} + \text{(diagram)}$$

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$$
\begin{aligned}
= \quad & \mathrm{Res}_{q\to\mathbf{a}}\,\mathrm{Res}_{r\to\mathbf{a}}\,\mathrm{Res}_{s\to\mathbf{a}}\,K(p,q)K(q,r)K(r,s)\,B(\overline{q},\overline{r})B(s,\overline{s}) \\
+ & \mathrm{Res}_{q\to\mathbf{a}}\,\mathrm{Res}_{r\to\mathbf{a}}\,\mathrm{Res}_{s\to\mathbf{a}}\,K(p,q)K(q,r)K(\overline{r},s)\,B(\overline{q},r)B(s,\overline{s}) \\
+ & \mathrm{Res}_{q\to\mathbf{a}}\,\mathrm{Res}_{r\to\mathbf{a}}\,\mathrm{Res}_{s\to\mathbf{a}}\,K(p,q)K(q,r)K(r,s)\,B(\overline{q},\overline{s})B(s,\overline{r}) \\
+ & \mathrm{Res}_{q\to\mathbf{a}}\,\mathrm{Res}_{r\to\mathbf{a}}\,\mathrm{Res}_{s\to\mathbf{a}}\,K(p,q)K(q,r)K(r,s)\,B(\overline{q},s)B(\overline{s},\overline{r}) \\
+ & \mathrm{Res}_{q\to\mathbf{a}}\,\mathrm{Res}_{r\to\mathbf{a}}\,\mathrm{Res}_{s\to\mathbf{a}}\,K(p,q)K(\overline{q},r)K(q,s)\,B(\overline{r},r)B(\overline{s},s)
\end{aligned}
$$

$$
= \quad 2 \quad \text{[diagram]} \quad + 2 \quad \text{[diagram]} \quad + \quad \text{[diagram]} \quad (3\text{-}16)
$$

where the last expression is obtained using lemma 3.1.

3.1.5 Free energy F_2.

The second free energy reads

$$
\begin{aligned}
-2F_2 \quad = \quad & 2\,\mathrm{Res}_{p\to\mathbf{a}}\,\mathrm{Res}_{q\to\mathbf{a}}\,\mathrm{Res}_{r\to\mathbf{a}}\,\mathrm{Res}_{s\to\mathbf{a}}\,\Phi(p)\,K(p,q)K(q,r)K(r,s)\,B(\overline{q},\overline{r})B(s,\overline{s}) \\
+ & 2\,\mathrm{Res}_{p\to\mathbf{a}}\,\mathrm{Res}_{q\to\mathbf{a}}\,\mathrm{Res}_{r\to\mathbf{a}}\,\mathrm{Res}_{s\to\mathbf{a}}\,\Phi(p)\,K(p,q)K(q,r)K(r,s)\,B(\overline{q},\overline{s})B(s,\overline{r}) \\
+ & \mathrm{Res}_{p\to\mathbf{a}}\,\mathrm{Res}_{q\to\mathbf{a}}\,\mathrm{Res}_{r\to\mathbf{a}}\,\mathrm{Res}_{s\to\mathbf{a}}\,\Phi(p)\,K(p,q)K(\overline{q},r)K(q,s)\,B(\overline{r},r)B(\overline{s},s).
\end{aligned}
$$

$$(3-17)$$

3.2 Teichmuller pants gluings

Every Riemann surface of genus g with k boundaries can be decomposed into $2g-2+k$ pants whose boundaries are $3g-3+k$ closed geodesics (in the Poincaré metric with constant negative curvature) [73]. The number of ways (in the combinatorial sense) of gluing $2g-2+k$ pants by their boundaries is clearly the same as the number of diagrams of $\mathcal{G}_k^{(g)}$, and each diagram corresponds to one pants decomposition.

Indeed, consider the root boundary labeled by p, and attach a pair of pants to this boundary. Draw an arrowed propagator from the boundary to the first pants. Then, choose one of the other boundaries of the pair of pants (there are thus 2 choices, left or right), it must be glued to another pair of pants (possibly not distinct from the first

one). If this pair of pants was never visited, draw an arrowed propagator, and if it was already visited, draw a non-arrowed propagator. In the end, you get a diagram of $\mathcal{G}_k^{(g)}$. This procedure is bijective (up to symmetry factors), and to a diagram of $\mathcal{G}_k^{(g)}$, one may associate a gluing of pants.

Example with $k = 1$ and $g = 2$:

$$\omega_1^{(2)} = \qquad\qquad + 2 \qquad\qquad + 2 \qquad\qquad .$$
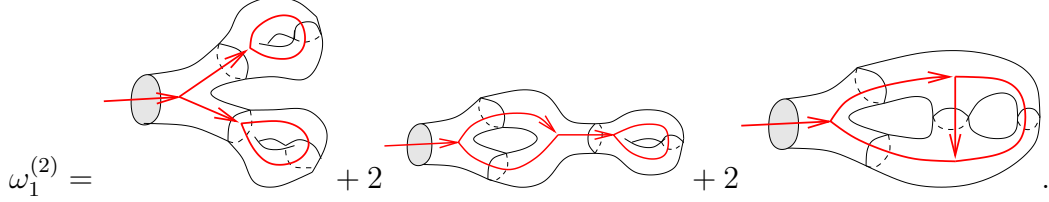
4   Main properties

So, for every regular spectral curve $\mathcal{E} = (\mathcal{L}, x, y)$ (and matrix κ if $\mathcal{L}$ has genus $\bar{g} > 0$) we have defined some meromorphic n-forms $\omega_n^{(g)}$ and some complex numbers $F_g = \omega_0^{(g)}$. They have some remarkable properties (see [60]):

- $\omega_n^{(g)}$ is symmetric in its n variables (this is proved by recursion).

- If $2g - 2 + n > 0$, then $\omega_n^{(g)}(z_1, \ldots, z_n)$ is a meromorphic form (in z_1 for instance) with poles only at the branch-points, of degree at most $6g - 6 + 2n + 2$, and with vanishing residue.

- If two spectral curves $\mathcal{E} = (\mathcal{L}, x, y)$ and $\tilde{\mathcal{E}} = (\widetilde{\mathcal{L}}, \tilde{x}, \tilde{y})$ are symplectically equivalent, they have the same F_g's or $g > 1$ (although they do not have the same $\omega_n^{(g)}$'s in general)

$$dx \wedge dy = \pm\, d\tilde{x} \wedge d\tilde{y} \qquad \rightarrow \qquad F_g(\mathcal{E}, \kappa) = F_g(\tilde{\mathcal{E}}, \kappa). \tag{4-1}$$

- if $\mathcal{L}$ is of genus $\bar{g} = 0$, then $\tau = \exp\left(\sum_{g=0}^{\infty} N^{2-2g} F_g\right)$ is a formal tau function, it obeys Hirota's equation. This theorem can be extended to $\bar{g} > 0$, with additional θ-functions, see section 4.6.

- Dilaton equation, for $2g - 2 + n > 0$:

$$\sum_i \operatorname*{Res}_{z_{n+1} \to a_i} \Phi(z_{n+1})\, \omega_{n+1}^{(g)}(z_1, \ldots, z_n, z_{n+1}) = (2 - 2g - n)\, \omega_n^{(g)}(z_1, \ldots, z_n). \tag{4-2}$$

  This equation just reflects the homogeneity property, i.e. under a rescaling $y \to \lambda y$, we have $\omega_n^{(g)} \to \lambda^{2-2g-n} \omega_n^{(g)}$.

- The derivatives of $\omega_n^{(g)}$ with respect to many parameters on which the spectral curve may depend is computed below in section 4.3.

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- The $\omega_n^{(g)}$'s have many other properties, for instance their modular behaviour satisfies the Holomorphic anomaly equations.

Let us study those properties in deeper details.

4.1 Homogeneity

If one changes the function $y(z) \to \lambda \, y(z)$, i.e. just a rescaling of the curve, then it is clear from eq.(2-40) that the kernel K is changed to K/λ and nothing else is changed. Thus, $\omega_n^{(g)}$ changes as:

$$\omega_n^{(g)} \to \lambda^{2-2g-n} \; \omega_n^{(g)}, \tag{4-3}$$

and in particular

$$F_g(\mathcal{L}, x, \lambda y) = \lambda^{2-2g} \, F_g(\mathcal{L}, x, y). \tag{4-4}$$

This implies that F_g is a homogeneous function of the spectral curve, of degree $2 - 2g$.

In particular, if one choses $\lambda = -1$ one gets (for $g \geq 2$):

$$F_g(\mathcal{L}, x, -y, \kappa) = F_g(\mathcal{L}, x, y, \kappa). \tag{4-5}$$

4.2 Symplectic invariance

It is clear from the definitions, that F_g and $\omega_n^{(g)}$ depend on the spectral curve only through the kernels B and K, and the number and position of branchpoints.

The Bergmann kernel B depends only on the underlying complex structure of the Riemann surface $\mathcal{L}$, thus it remains unchanged if we change the functions x and y, as long as we don't change $\mathcal{L}$.

The kernel K, depends on the functions x and y, only through the combination:

$$(y(z) - y(\bar{z})) \, dx(z). \tag{4-6}$$

Therefore K remains unchanged if we don't change this combination.

In particular, the kernel K, and therefore F_g and $\omega_n^{(g)}$ remain unchanged, if we change:
- $y \to y + R(x)$ where $R(x)$ is some rational function of x.
- $y \to \lambda y$, $x \to x/\lambda$ where $\lambda \in \mathbb{C}^*$.
- $x \to \frac{ax+b}{cx+d}$, $y \to \frac{(cx+d)^2}{ad-bc} \, y$.

Those transformations, form a subgroup of the symplectomorphisms. Indeed, in all those cases, the symplectic form $dx \wedge dy$ is unchanged.

In order to have invariance under the full group of symplectomorphisms, we need to prove the invariance under the $\frac{\pi}{2}$ rotation in the x, y plane, i.e. $x \to y$, $y \to -x$, which also conserves $dx \wedge dy$. Using homogeneity eq.(4-5), we see that this is equivalent to consider the invariance under $x \to y$, $y \to x$.

This transformation however, does not conserve K, it does not conserve the number of branchpoints, and it does not conserve the $\omega_n^{(g)}$'s with $n \geq 1$. However, it was proved

in [61] that it does conserve the F_g's. The proof of [61] is very technical. It is inspired from the loop equations for the 2-matrix model. It amounts to defining some mixed $n + m$-forms $\omega_{n,m}^{(g)}$, where x and y play similar roles, and for which $\omega_{n,0}^{(g)}$ coincides with the $\omega_n^{(g)}$ for the spectral curve $(\mathcal{L}, x, y)$, and $\omega_{0,m}^{(g)}$ coincides with the $\omega_m^{(g)}$ for the spectral curve $(\mathcal{L}, y, x)$. In particular $F_g = \omega_{0,0}^{(g)}$ is both the $F_g = \omega_0^{(g)}$ for the spectral curve $(\mathcal{L}, x, y)$, and the $F_g = \omega_0^{(g)}$ for the spectral curve $(\mathcal{L}, y, x)$. The proof of [61] relies on the fact that $\omega_{n+1,m}^{(g)} + \omega_{n,m+1}^{(g)}$ is an exact form.

That leads to:

Theorem 4.1 *Symplectic invariance*

The F_g's are invariant under the group of symplectomorphisms generated by:
- $y \to y + R(x)$ *where $R(x)$ is some rational function of x.*
- $y \to \lambda y$, $x \to x/\lambda$ *where $\lambda \in \mathbb{C}^*$.*
- $x \to \frac{ax+b}{cx+d}$, $y \to \frac{(cx+d)^2}{ad-bc} y$.
- $x \to y$, $y \to -x$.

In addition, the F_g's are also invariant under:
- $x \to x$, $y \to -y$.

This theorem is a powerful tool which allows to compare the F_g's of models which look a priory very different. We will see examples of applications in section 10.1.

The $\omega_n^{(g)}$'s with $n \geq 1$ are not conserved under symplectic transformation, instead they get shifted by exact forms.

4.3   Derivatives

In this section, we study how the F_g's and $\omega_n^{(g)}$'s change under a change of spectral curve, and in particular under infinitesimal holomorphic changes.

Consider an infinitesimal change $y \to y + \epsilon \delta y$ at fixed x, or in fact it is more appropriate to consider the variation of the differential form ydx:

$$ydx \to ydx + \epsilon\delta(ydx) + O(\epsilon^2) = ydx + \epsilon d\Omega + O(\epsilon^2) \tag{4-7}$$

where $d\Omega$ is an analytical differential form on an open subset of $\mathcal{L}$[7]. If instead of working at fixed x, we prefer to work with some local parameter z, we write:

$$\delta y(z)dx(z) - \delta x(z)dy(z) = d\Omega(z). \tag{4-8}$$

This shows that the set of holomorphic deformations of the spectral curve is equipped with a Poisson structure, but we shall not study it in details in this article, see [43, 44] for the Frobenius manifold structure.

Classification of possible 1-forms $d\Omega$:

The deformation $d\Omega$ is a 1-form. Here we shall consider only meromorphic deformations, and meromorphic 1-forms are classified as 1st kind (no pole), 2nd kind (multiple poles, without residues), and 3rd kind (only simple poles).

---

[7]Note that ydx does not need to be a meromorphic form itself to be able to consider such deformations.

- First kind deformations are holomorphic forms on $\mathcal{L}$, i.e. they are linear combinations of the du_i's (see section 2.2.1):

$$du_i(z) = \frac{1}{2i\pi} \oint_{\overline{\mathcal{B}}_i} B(z, z') \qquad (4\text{-}9)$$

where $\overline{\mathcal{B}}_i = \mathcal{B}_i - \sum_j \tau_{i,j} \mathcal{A}_j$.

- 2nd kind deformations have double or multiple poles. They can be taken as linear combinations of Bergmann kernels or of their derivatives. Choose a point $p \in \mathcal{L}$. If x is regular at p, choose the local parameter $\xi(z) = x(z) - x(p)$, and if x has a pole of degree d at p, choose $\xi(z) = x(z)^{-1/d}$, and define:

$$B_k(z; p) = \operatorname*{Res}_{z' \to p} B(z, z')\, \xi(z')^{-k}. \qquad (4\text{-}10)$$

All 2nd kind differentials are linear combinations of such $B_k(z; p)$.

- 3rd kind differentials have only simple poles, and since the sum of residues must vanish, they must have at least 2 simple poles. Choose two points p_1 and p_2 in the fundamental domain, and define:

$$dS_{p_1, p_2}(z) = \int_{p_2}^{p_1} B(z, z'). \qquad (4\text{-}11)$$

All 3rd kind differentials are linear combinations of such dS.

Theorem 4.2 *A general meromorphic differential form $d\Omega$ with poles p_k's, can be written:*

$$d\Omega(z) = 2i\pi \sum_{i=1}^{\bar{g}} \delta\epsilon_i\, du_i(z) + \sum_k \delta t_k\, dS_{p_k, o}(z) + \sum_k \sum_j \delta t_{k,j}\, B_j(z; p_k). \qquad (4\text{-}12)$$

It can be noticed that the coefficients $\delta\epsilon_i, \delta t_k, \delta t_{k,j}$ are the flat coordinates in the metrics of kernel B, of the corresponding Frobenius manifold structure.

proof:
Indeed, let p_k be the poles of $d\Omega$, and write the negative part of the Laurent series of $d\Omega$ near its poles as:

$$d\Omega(z) \underset{z \to p_k}{\sim} \delta t_k \frac{d\xi(z)}{\xi(z)} - \sum_{j \geq 1} j\, \delta t_{k,j} \frac{d\xi(z)}{\xi(z)^{j+1}}. \qquad (4\text{-}13)$$

We see that

$$d\Omega(z) - \sum_k \delta t_k\, dS_{p_k, o}(z) - \sum_k \sum_j \delta t_{k,j}\, B_j(z; p_k) \qquad (4\text{-}14)$$

is a 1-form which has no poles, thus it is a holomorphic form, and it is a linear combination of the du_i's. $\square$

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If $\kappa = 0$, i.e. if B is the Bergmann kernel, it is normalized on the $\mathcal{A}$-cycles, and we have: $\oint_{\mathcal{A}} dS = 0$, $\oint_{\mathcal{A}} B_\kappa = 0$. Thus, the $\delta\epsilon_i$ are easily computed as $\delta\epsilon_i = \frac{1}{2i\pi} \oint_{\mathcal{A}_i} d\Omega$. However, if $\kappa \neq 0$, this is no longer true, we have $\delta\epsilon = \frac{1+\kappa\tau}{2i\pi} \oint_{\mathcal{A}} d\Omega - \frac{\kappa}{2i\pi} \oint_{\mathcal{B}} d\Omega$, and the variations $\delta t_{k,j}$ or δt_k get mixed with the $\delta\epsilon_i$ through the variations of τ. The good way to undo this mixing, is by defining a covariant variation:

Definition 4.1 *Covariant variation:*

$$D_{d\Omega} \overset{\text{def}}{=} \delta_{d\Omega} + \text{tr} \left(\kappa \, \delta_{d\Omega}\tau \, \kappa \, \frac{\partial}{\partial\kappa} \right) \tag{4-15}$$

where $\delta_{d\Omega}\tau$ is the variation of the Riemann matrix of periods τ under $ydx \to ydx + \epsilon d\Omega$.

Derivatives with respect to κ are studied in details in section 4.4.1 below.

The important point, is that $d\Omega$ can always be written as:

$$d\Omega(z) = \int_{\partial\Omega} B(z, z') \, \Lambda(z') \tag{4-16}$$

where $\partial\Omega$ is some continuous path (a chain or a cycle, which is related to the Poincaré dual of $d\Omega$) on $\mathcal{L}$, and $\Lambda(z')$ is an analytical function defined locally in a vicinity of $\partial\Omega$.

The theorem is then:

Theorem 4.3 Variation of the spectral curve:
Under an infinitesimal deformation $\delta y \, dx - \delta x \, dy = d\Omega(z) = \int_{\partial\Omega} B(z, z') \, \Lambda(z')$, the $\omega_n^{(g)}$'s change by:

$$D_{d\Omega} \, \omega_n^{(g)}(z_1, \ldots, z_n) = \int_{\partial\Omega} \omega_{n+1}^{(g)}(z_1, \ldots, z_n, z') \, \Lambda(z'). \tag{4-17}$$

For example, in particular with $n = 0$ we have:

$$D_{d\Omega} F_g = \int_{\partial\Omega} \omega_1^{(g)}(z') \, \Lambda(z'). \tag{4-18}$$

4.3.1 The loop operator

This theorem can also be restated in terms of the "loop operator", which corresponds to $d\Omega(z) = B(z, z')$. We define:

Definition 4.2 *The loop operator is:*

$$D_{z'} \overset{\text{def}}{=} D_{B(z,z')}. \tag{4-19}$$

It satisfies:

$$D_{z'}\omega_n^{(g)}(z_1, \ldots, z_n) = \omega_{n+1}^{(g)}(z_1, \ldots, z_n, z'). \tag{4-20}$$

The loop operator is a derivation, i.e. $D_{z'}(uv) = uD_{z'}v + vD_{z'}u$, and it is such that: $D_{z_1}D_{z_2} = D_{z_2}D_{z_1}$, and $D_{z_1}\frac{\partial}{\partial z_2} = \frac{\partial}{\partial z_2}D_{z_1}$.

In random matrix theory, the loop operator [6], is most often written as a functional derivative with respect to the potential $V(x)$:

$$\frac{\partial}{\partial V(x(z'))} \overset{\text{def}}{=} D_{z'}. \tag{4-21}$$

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4.3.2 Inverse of the loop operator

The loop operator allows to find $\omega_{n+1}^{(g)}$ in terms of $\omega_n^{(g)}$, i.e. increases n by 1. The inverse operator, which decreases n by 1 can also be written explicitly:

Theorem 4.4 *Let Φ be a primitive of ydx, i.e. a function defined on the fundamental domain such that $d\Phi = ydx$, then we have, if $2 - 2g - n < 0$:*

$$(2 - 2g - n)\, \omega_n^{(g)}(z_1, \ldots, z_n) = \sum_i \operatorname*{Res}_{z \to a_i} \omega_{n+1}^{(g)}(z_1, \ldots, z_n, z)\ \Phi(z). \qquad (4\text{-}22)$$

This theorem is easily proved by recursion on $2g + n - 2$.

This theorem is at the origin of definition 2.6, for $n = 0$.

4.4 Modular properties

In section 2.2.3 we have introduced a deformation of the Bergmann kernel with a symmetric matrix κ of size $\bar{g} \times \bar{g}$. The reason to introduce this deformation, was that it encodes the modular dependence of the Bergmann kernel, i.e. how the Bergmann kernel changes under a change of choice of cycles $\mathcal{A}, \mathcal{B}$. Thus, studying the modular dependence of the F_g's and $\omega_n^{(g)}$'s amounts to studying their dependence on κ.

Also, in section 4.3, we have seen that the covariant derivative involves the computation of derivatives with respect to κ.

4.4.1 Dependence on κ

Since the kernels B and K depend linearly on κ, all the stable $\omega_n^{(g)}$'s and F_g's are polynomials in κ, of degree $3g - 3 + n$.

Notice that $\partial B(z_1, z_2)/\partial \kappa_{i,j} = 2i\pi\ du_i(z_1)\, du_j(z_2)$ is factorized, i.e. a function of z_1 times a function of z_2. This simple observation, together with

$$du_i(z) = \frac{1}{2i\pi} \oint_{\bar{\mathcal{B}}_i} B(z, z') \qquad , \qquad \bar{\mathcal{B}}_i = \mathcal{B}_i - \sum_j \tau_{i,j}\mathcal{A}_j \qquad (4\text{-}23)$$

leads, by an easy recursion, to the following theorem:

Theorem 4.5 *For $2 - 2g - n \leq 0$:*

$$2i\pi\, \partial\omega_n^{(g)}(z_1, \ldots, z_n)/\partial \kappa_{i,j} \quad = \quad \frac{1}{2} \oint_{\bar{\mathcal{B}}_i} dz' \oint_{\bar{\mathcal{B}}_j} dz\ \Big[\omega_{n+2}^{(g-1)}(z_1, \ldots, z_n, z, z')$$

$$+ \sum_{h=0}^{g} \sum_{I \subset J}{}' \omega_{1+|I|}^{(h)}(z, I)\, \omega_{1+n-|I|}^{(g-h)}(z', J/I) \Big]$$

$$(4 - 24)$$

where $J = \{z_1, \ldots, z_n\}$, and $\sum_h \sum_I'$ means as usual that we exclude $(h, I) = (0, \emptyset), (g, J)$.

This theorem can be applied recursively, to compute higher derivatives, and eventually recover a polynomial of κ by its Taylor expansion at $\kappa = 0$, of the form:

$$F_g(\kappa) = \sum_{k=0}^{3g-3} \frac{1}{k!} (\kappa)^k \, \partial^k F_g|_{\kappa=0}. \tag{4-25}$$

According to theorem 4.3 of section 4.3, the $\overline{\mathcal{B}}$-cycle integrals, computed at $\kappa = 0$, are the derivatives with respect to coordinates ϵ_i of eq.(4-12):

$$\kappa = 0 \quad \leftrightarrow \quad \frac{\partial \omega_n^{(g)}(z_1, \ldots, z_n)}{\partial \epsilon_i} = \oint_{\overline{\mathcal{B}}_i} \omega_{n+1}^{(g)}(z_1, \ldots, z_n, z) \tag{4-26}$$

and therefore we have:

$$\partial \omega_n^{(g)}(z_1, \ldots, z_n) / \partial \kappa_{i,j}|_{\kappa=0} \quad = \quad \frac{1}{2} \frac{\partial}{\partial \epsilon_i} \frac{\partial}{\partial \epsilon_j} \, \omega_n^{(g-1)}(z_1, \ldots, z_n)$$
$$+ \frac{1}{2} \sum_{h=0}^{g} \sum_{I \subset J} \frac{\partial}{\partial \epsilon_i} \omega_{1+|I|}^{(h)}(I) \frac{\partial}{\partial \epsilon_j} \omega_{n-|I|}^{(g-h)}(J/I).$$
$$(4-27)$$
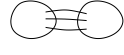
We can thus trade the κ dependance into derivatives with respect to the coordinates ϵ_i. For instance we have at $g = 2$, and with $\bar{g} = 1$:

$$F_2(\kappa) = \overline{F_2} + \frac{\kappa}{2} \left(\partial^2 \overline{F_1} + \partial \overline{F_1} \, \partial \overline{F_1} \right) + \frac{\kappa^2}{8} \left(\partial^4 \overline{F_0} + 4 \, \partial^3 \overline{F_0} \, \partial \overline{F_1} \right) + \frac{\kappa^3}{48} \left(10 \, \partial^3 \overline{F_0} \, \partial^3 \overline{F_0} \right) \tag{4-28}$$

where $\overline{F}_g = F_g(\kappa = 0)$, and $\partial = \partial/\partial \epsilon$.

This result is best interpreted graphically. For example with $g = 2$ we have:

$$F_2(\kappa) \quad = \quad \langle\!\langle \circ \;\; \circ \rangle\!\rangle + \frac{\kappa}{2} \langle\!\langle \circ \rangle\!\rangle + \frac{\kappa}{2} \langle\!\langle \circ \rangle\!\rangle \langle\!\langle \circ \rangle\!\rangle + \frac{\kappa^2}{8} \langle\!\langle \rangle\!\rangle + \frac{\kappa^2}{2} \langle\!\langle \circ \rangle\!\rangle \langle\!\langle \rangle\!\rangle$$
$$+ \frac{\kappa^3}{8} \langle\!\langle \rangle\!\rangle \langle\!\langle \rangle\!\rangle + \frac{\kappa^3}{12} \langle\!\langle \equiv \rangle\!\rangle \tag{4-29}$$

where each line with endpoints (i, j) is a factor $\kappa_{i,j}$, and each connected piece of Riemann surface of genus h, with k punctures $i_1, \ldots, i_k$ is a $\partial^k \overline{F}_h / \partial \epsilon_{i_1} \ldots \partial \epsilon_{i_k}$. Each graph is a possible "stable" degeneracy of a genus g Riemann surface (imagine each link contracted to a point), stability means that each connected component of genus h with k marked points must have $2 - 2h - k < 0$. The prefactor is $1/\#Aut$, i.e. the inverse of the number of automorphisms, for instance in the last graph $\langle\!\langle \equiv \rangle\!\rangle$ we have a $\mathbb{Z}_2$ symmetry by exchanging the 2 spheres, and a σ_3 symmetry from permuting the 3 endpoints of the edges, i.e. $12 = \#(\mathbb{Z}_2 \times \sigma_3)$ automorphisms.

More generally, by a careful analysis of the combinatorics of the ∂_{ϵ_i}'s, one can see (this was done in [63], and coincides with the diagrammatics of [2]) that the Taylor

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expansion eq.(4-25), reconstructs the expansion of a formal Gaussian integral (i.e. order by order in powers of N):

$$
\mathrm{e}^{\sum_g N^{2-2g} F_g(\epsilon,\kappa)} = \int d\eta_1 \ldots \int \partial\eta_{\bar{g}} \; \mathrm{e}^{\sum_g N^{2-2g}\overline{F}_g(\eta)} \; \mathrm{e}^{-N^2 \sum_i (\eta_i-\epsilon_i)\partial_{\epsilon_i}\overline{F}_g}
$$
$$
\mathrm{e}^{-\frac{N^2}{2}\sum_{i,j}(\eta_i-\epsilon_i)(\eta_j-\epsilon_j)\partial_{\epsilon_i}\partial_{\epsilon_j}\overline{F}_g} \; \mathrm{e}^{-\frac{N^2}{4i\pi}\sum_{i,j}(\eta_i-\epsilon_i)(\eta_j-\epsilon_j)(\kappa^{-1})_{i,j}} \tag{4-30}
$$

and the graphical representation above is just the Wick's expansion of the gaussian integral.

This diagrammatic expansion of modular transformations was first derived in [2] in the context of topological strings.

4.4.2 Holomorphic anomaly

In particular, if we choose κ to be the Zamolodchikov Kähler metric $\kappa = (\overline{\tau} - \tau)^{-1}$, we have seen in section 2.2.5, that the Bergmann kernel becomes the Schiffer kernel and is modular invariant, which means that it is independent of the choice of cycles $\mathcal{A}, \mathcal{B}$. Since the only modular dependence of the F_g's and $\omega_n^{(g)}$'s is in the Bergmann kernel, we have:

Theorem 4.6 *If κ is the Zamolodchikov Kähler metric $\kappa = (\overline{\tau} - \tau)^{-1}$, then F_g and $\omega_n^{(g)}$'s are modular invariant.*

The price to pay to have modular invariant F_g's, is that they are no longer analytical functions of τ, i.e. analytical functions of the spectral curve, and in particular they are no longer analytical functions of the ϵ_i's. However, since the only non-analytical dependence is polynomial in κ, and κ^{-1} is linear in $\overline{\tau}$ which is the only non-analytical term, and since we have relationships between derivatives with respect to κ and derivatives with respect to ϵ, by a simple chain rule, we obtain the following theorem [63]:

Theorem 4.7 *The $\omega_n^{(g)}$'s satisfy the Holomorphic anomaly equations*

$$
\frac{\partial \omega_n^{(g)}(J)}{\partial \overline{\epsilon}_i} = \frac{-1}{(2i\pi)^3} \kappa \frac{\partial^3 \overline{F}_0}{\partial \overline{\epsilon}^3} \kappa \frac{1}{2} \left[\frac{\partial^2 \omega_n^{(g-1)}(J)}{\partial \epsilon^2} + \frac{\partial \tau}{\partial \epsilon} \kappa \frac{\partial \omega_n^{(g-1)}(J)}{\partial \epsilon} \right.
$$
$$
\left. + \sum_{h=0}^{g} \sum_{I \subset J} \frac{\partial \omega_{|I|}^{(h)}(I)}{\partial \epsilon} \frac{\partial \omega_{n-|I|}^{(g-h)}(J\backslash I)}{\partial \epsilon} \right]. \tag{4-31}
$$

In particular for $n = 0$:

$$
\frac{\partial F_g}{\partial \overline{\epsilon}_i} = \frac{-1}{(2i\pi)^3} \kappa \frac{\partial^3 \overline{F}_0}{\partial \overline{\epsilon}^3} \kappa \frac{1}{2} \left[\frac{\partial^2 F_{g-1}}{\partial \epsilon^2} + \frac{\partial \tau}{\partial \epsilon} \kappa \frac{\partial F_{g-1}}{\partial \epsilon} + \sum_{h=1}^{g-1} \frac{\partial F_h}{\partial \epsilon} \frac{\partial F_{g-h}}{\partial \epsilon} \right]. \tag{4-32}
$$

This equation was first found by Bershadsky, Cecotti, Ooguri and Vafa (which we refer to as BCOV [17]) in the context of topological string theory. Here we see that the symplectic invariants F_g always satisfy this equation, and it is tempting to believe that the symplectic invariants F_g, should coincide with the string theory amplitudes, i.e. the Gromov-Witten invariants. This question is debated below in section 11. Unfortunately, the holomorphic anomaly equations do not have a unique solution, and although this conjecture is almost surely correct, no proof exists at the present time, apart from a very limited number of cases[8].

Let us briefly sketch the idea of BCOV. String theory partition functions represent "path integrals" over the set of all Riemann surfaces with some conformal invariant weight. In other words, they are integrals over moduli spaces of Riemann surfaces of given topology, and topological strings are integrals with a topological weight, they compute intersection numbers of bundles over moduli spaces (see [91, 115] for introduction to topological strings).

Moduli spaces can be compactified by adding their "boundaries", which correspond to degenerate Riemann surfaces (for instance when a non contractible cycle gets pinched, or when marked points come together). The integrals have thus boundary terms, which can be represented by δ-functions, and δ-functions are not holomorphic. In other words, string theory partition functions contain non-holomorphic terms which count degenerate Riemann surfaces.

On the other hand, if one decides to integrate only on non-degenerate surfaces, one gets holomorphic patition functions, but not modular invariant, because the boundaries of the moduli spaces are associated to a choice of pinched cycles. Modular invariant means independent of a choice of cycles.

To summarize, the holomorphic partition function is obtained after a choice of boundaries, i.e. a choice of a symplectic basis of non contractible cycles $\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}$, and cannot be modular invariant. The modular invariance is restored by adding the boundaries, but this breaks holomorphicity.

There is thus a relationship between holomorphicity and modular invariance.

4.5 Background indepedence and non-perturbative modular invariance

We have seen in the previous section, that the F_g's are not modular invariant, unless we choose $\kappa = (\overline{\tau} - \tau)^{-1}$, i.e. modular invariance can be restored by breaking holomorphicity.

In fact, there is another way of restoring modular invariance, without breaking holomorphicity. It exploits the fact that the modular transformations of F_g's, i.e. eq.(4-24), is very similar to the modular transformation of theta-functions. It was shown in [64], that certain combinations of θ-functions and F_g's, are modular, and reconstruct a non-

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[8]In [22], this conjecture is proposed as a new definition of the type IIB topological string theory. The interested reader may find all the details of this conjecture as well as numerous checks in this paper.

perturbative, modular partition function, which is also a Tau-function (see section 4.6), and which has a background independence property.

Consider a characteristics (μ,ν), and a spectral curve $\mathcal{E} = (\mathcal{L}, x, y)$, choose $\kappa = 0$. Following [55, 64], we introduce a *nonperturbative partition function* by summing over filling fractions, defined by

$$
\begin{aligned}
&Z_{\mathcal{E}}(\mu,\nu;\epsilon)\\
=\quad & e^{\sum_{g\geq 0} N^{2-2g}F_g(\epsilon)} \sum_{k}\sum_{l_i>0}\sum_{h_i>1-\frac{l_i}{2}} \frac{N^{\sum_i(2-2h_i-l_i)}}{k!\,l_1!\,\ldots\,l_k!}\, F_{h_1}^{(l_1)}\ldots F_{h_k}^{(l_k)}\,\Theta_{\mu,\nu}^{(\sum_i l_i)}(NF_0',\tau)\\
=\quad & e^{N^2 F_0+F_1}\,\Theta_{\mu,\nu}\left\{1+\sum_{j=1}^{\infty} N^{-j}\,Z_j(\mu,\nu;\epsilon)\right\}\\
=\quad & e^{N^2 F_0}\,e^{F_1}\,e^{(N^{-2}F_2+N^{-4}F_3+\ldots)}\left\{\Theta_{\mu,\nu}+\frac{1}{N}\left(\Theta_{\mu,\nu}'F_1'+\frac{1}{6}\Theta_{\mu,\nu}'''F_0'''\right)\right.\\
&\left.+\frac{1}{N^2}\left(\frac{1}{2}\Theta_{\mu,\nu}''F_1''+\frac{1}{2}\Theta_{\mu,\nu}''F_1'^2+\frac{1}{24}\Theta_{\mu,\nu}^{(4)}F_0''''+\frac{1}{6}\Theta_{\mu,\nu}^{(4)}F_0'''F_1'+\frac{1}{72}\Theta_{\nu,\mu}^{(6)}F_0'''^2\right)+\ldots\right\}.
\end{aligned}
$$
$(4-33)$

where Z_j is the sum of all terms contributing to order N^{-j}. In this partition function, the F_g's are the symplectic invariants of the spectral curve $\mathcal{E}$, their derivatives are with respect to the background filling fraction ϵ and computed through theorem 4.3, at:

$$
\epsilon = \frac{1}{2i\pi}\oint_{\mathcal{A}} ydx. \tag{4-34}
$$

Notice that the $F_g(\mathcal{E})$'s and their derivatives depend on the choice of a symplectic basis of $2\bar{g}$ one-cycles $\mathcal{A}_i, \mathcal{B}_j$ on $\mathcal{L}$. Finally, the theta function $\Theta_{\mu,\nu}$ of characteristics (μ,ν) is defined by

$$
\Theta_{\mu,\nu}(u,\tau) = \sum_{n\in\mathbb{Z}^{\bar{g}}} e^{(n+\mu-N\epsilon)u}\,e^{i\pi(n+\mu-N\epsilon)\tau(n+\mu-N\epsilon)}\,e^{2i\pi n\nu} \tag{4-35}
$$

and is evaluated at

$$
u = NF_0', \qquad F_0' = \oint_{\mathcal{B}} y(x)dx, \qquad \tau = \frac{1}{2i\pi}F_0''. \tag{4-36}
$$

In (4-33), the derivatives of the theta function (4-35) are w.r.t. u. The derivatives of Θ and the derivatives of F_g, are written with tensorial notations. For instance, $\frac{1}{6}\Theta_{\mu,\nu}^{(4)}F_0'''F_1'$ actually means:

$$
\frac{1}{6}\Theta_{\mu,\nu}^{(4)}F_0'''F_1' \equiv \frac{1}{2!\,3!\,1!}\sum_{i_1,i_2,i_3,i_4}\frac{\partial^4\Theta_{\mu,\nu}}{\partial u_{i_1}\partial u_{i_2}\partial u_{i_3}\partial u_{i_4}}\frac{\partial^3 F_0}{\partial\epsilon_{i_1}\partial\epsilon_{i_2}\partial\epsilon_{i_3}}\frac{\partial F_1}{\partial\epsilon_{i_4}} \tag{4-37}
$$

and the symmetry factor (here $\frac{1}{6}=\frac{2}{2!\,3!\,1!}$) is the number of relabellings of the indices, giving the same pairings, and divided by the order of the group of relabellings, i.e. $k!\,l_1!\ldots l_k!$, as usual in Feynmann graphs.

The Θ function above is closely related to the standard theta function, which is defined by

$$\vartheta[{}^{\mu}_{\nu}](\xi|\tau) = \sum_{\mathbf{n}\in\mathbb{Z}^{\bar{g}}} \exp[i\pi(n+\mu)\tau(n+\mu) + 2i\pi(n+\mu)(\xi+\nu)]. \qquad (4\text{-}38)$$

It it easy to see that these two functions are related as follows

$$\Theta_{\mu,\nu}(u,\tau) = \exp\left[-N^2\left(\epsilon F_0' + \frac{1}{2}\epsilon^2 F_0''\right)\right] \vartheta[{}^{\mu}_{\nu}](\xi|\tau) \qquad (4\text{-}39)$$

where

$$\xi = \frac{N}{2i\pi} \oint_{\mathcal{B}-\tau\mathcal{A}} y(x)dx = N\left(\frac{F_0'}{2i\pi} - \tau\epsilon\right). \qquad (4\text{-}40)$$

4.5.1 Modularity

Theorem 4.8 *All the terms Z_j in eq.(4-33) are modular, i.e. they transform as the characteristics (μ,ν). More precisely, if we make a modular change of cycles $\mathcal{A}, \mathcal{B} \to \tilde{\mathcal{A}}, \tilde{\mathcal{B}}$, we have:*

$$e^{N^2\tilde{F}_0+\tilde{F}_1}\widetilde{\Theta}_{\tilde{\mu},\tilde{\nu}}(\tilde{u},\tilde{\tau}) = \zeta[{}^{\mu}_{\nu}]\, e^{N^2 F_0+F_1}\Theta_{\mu,\nu}(u,\tau) \qquad (4\text{-}41)$$

and for all $j \geq 1$:

$$\tilde{Z}_j(\tilde{\mu},\tilde{\nu}) = Z_j(\mu,\nu). \qquad (4\text{-}42)$$

This theorem was proved in [64], mostly using the diagrammatic representation of section 3, and the diagrammatic representation of [2].

For example the following quantities are modular:

$$Z_1 = \frac{\Theta'_{\mu,\nu}}{\Theta_{\mu,\nu}}F_1' + \frac{1}{6}\frac{\Theta'''_{\mu,\nu}}{\Theta_{\mu,\nu}}F_0''', \qquad (4\text{-}43)$$

$$Z_2 = F_2 + \frac{1}{2}\frac{\Theta''_{\mu,\nu}}{\Theta_{\mu,\nu}}F_1'' + \frac{1}{2}\frac{\Theta''_{\mu,\nu}}{\Theta_{\mu,\nu}}F_1'^2 + \frac{1}{24}\frac{\Theta^{(4)}_{\mu,\nu}}{\Theta_{\mu,\nu}}F_0'''' + \frac{1}{6}\frac{\Theta^{(4)}_{\mu,\nu}}{\Theta_{\mu,\nu}}F_0'''F_1' + \frac{1}{72}\frac{\Theta^{(6)}_{\mu,\nu}}{\Theta_{\mu,\nu}}(F_0''')^2. \quad (4\text{-}44)$$

4.5.2 Background independence

Theorem 4.9 *The partition function eq.(4-33) is independent of the background filling fraction ϵ, i.e., for any two filling fractions ϵ_1 and ϵ_2:*

$$Z_{\mathcal{E}}(\mu,\nu,\epsilon_1) = Z_{\mathcal{E}}(\mu,\nu,\epsilon_2). \qquad (4\text{-}45)$$

This theorem follows directly from the definition eq.(4-33). It has important consequences which we shall not study here [64].

4.6 Integrability

Out of the F_g's, one can define a "formal tau-function". In this section, let us assume that $\mathcal{L} = \mathbb{P}^1$, i.e. it has genus $\bar{g} = 0$. The higher genus case is discussed in section 4.6.6 below.

Definition 4.3 *The formal τ-function is defined as a formal series in a variable N:*

$$\ln \tau_N = \sum_{g=0}^{\infty} N^{2-2g} \, F_g. \tag{4-46}$$

Now, we shall explain why it makes sense to call it a τ-function. τ-functions are usually defined in the context of integrable systems, and they have several more or less equivalent definitions, see [14].

One possible definition of τ-functions relies on Hirota equations [72, 14], and another one relies on a free fermion representations, i.e. determinantal formulae [75, 86, 14].

4.6.1 Determinantal formulae

In the following of this section, most of the functions have an obvious formal N dependence. For the sake of brevity, we do not write it explicitly as long as it is not needed.

Out of the $\omega_n^{(g)}$'s, it is convenient to define the formal series:

$$\omega_n(z_1, \ldots, z_n) = -\frac{\delta_{n,2} \, dx(z_1) dx(z_2)}{(x(z_1) - x(z_2))^2} + \sum_g N^{2-2g-n} \omega_n^{(g)}(z_1, \ldots, z_n) \tag{4-47}$$

and also the "non-connected" correlators:

$$\overline{\omega}_n(J) = \sum_{k=1}^{n} \sum_{I_1 \cup \ldots \cup I_k = J} \prod_{i=1}^{k} \omega_{|I_i|}(I_i). \tag{4-48}$$

For example:

$$\overline{\omega}_2(z_1, z_2) = \omega_2(z_1, z_2) + \omega_1(z_1)\omega_1(z_2), \tag{4-49}$$

$$\begin{aligned}\overline{\omega}_3(z_1, z_2, z_3) &= \omega_3(z_1, z_2, z_3) + \omega_1(z_1)\omega_2(z_2, z_3) + \omega_1(z_2)\omega_2(z_1, z_3) \\ &\quad + \omega_1(z_3)\omega_2(z_1, z_2) + \omega_1(z_1)\omega_1(z_2)\omega_1(z_3).\end{aligned} \tag{4-50}$$

In other words, the ω_n are the cumulants of the $\overline{\omega}_n$'s.

The following proposition is proved in some cases (hyperelliptical spectral curves [15]), and in all matrix models, however, it is expected to hold for any spectral curve:

Proposition 4.1 *There exists a (formal) kernel $H(z_1, z_2)$, such that:*

$$\omega_1(z) = \lim_{z' \to z} \left(H(z, z') - \frac{\sqrt{dz \, dz'}}{z - z'} \right), \tag{4-51}$$

$$\omega_2(z_1, z_2) = -H(z_1, z_2)H(z_2, z_1) - \frac{dx(z_1)\, dx(z_2)}{(x(z_1) - x(z_2))^2} \qquad (4\text{-}52)$$

and if $n \geq 3$:

$$\overline{\omega}_n(z_1, \ldots, z_n) = \text{"} \det \text{"}_{i,j=1,\ldots,n}(H(z_i, z_j)) \qquad (4\text{-}53)$$

where the quotation mark "det" means the following: write the determinant as a sum over permutations of products of H's: $\det(H(z_i, z_j)) = \sum_\sigma (-1)^\sigma \prod_i H(z_i, z_{\sigma(i)})$. Then, every time a permutation has a fixed point $\sigma(i) = i$ we must replace $H(z_i, z_i)$ by $\omega_1(z_i)$, and every time a permutation has a length 2 cycle $\sigma(i) = j, \sigma(j) = i$ we must replace the factor $H(z_i, z_j)H(z_j, z_i)$ by $-\omega_2(z_i, z_j)$.

This is equivalent to saying that for $n \geq 3$, the cumulants are given by:

$$\omega_n(z_1, \ldots, z_n) = \sum_{\text{cyclic } \sigma} (-1)^\sigma \prod_{i=1}^{n} H(z_i, z_{\sigma(i)}). \qquad (4\text{-}54)$$

Example:

$$\omega_3(z_1, z_2, z_3) = H(z_1, z_2)H(z_2, z_3)H(z_3, z_1) + H(z_1, z_3)H(z_3, z_2)H(z_2, z_1). \qquad (4\text{-}55)$$

The determinantal formulae for correlation functions were first found by Dyson and Mehta [46, 95] in the context of random matrix theory, and have led to a huge number of applications.

Moreover the kernel H can be written rather explicitly. In all matrix cases, the kernel H for the determinantal formulae above, coincides with the kernel $\hat{H}$ which we define below:

Definition 4.4 *We define the formal kernel $\hat{H}$ as a formal spinor in z_1 and z_2, given by an exponential formula*

$$\hat{H}(z_1, z_2) = \frac{\sqrt{dx(z_1)\, dx(z_2)}}{x(z_1) - x(z_2)}\, e^{\sum_{n=1}^{\infty} \frac{1}{n!} \int_{z_2}^{z_1} \cdots \int_{z_2}^{z_1} \omega_n}. \qquad (4\text{-}56)$$

This exponential formula for the kernel is to be understood order by order in powers of N, namely:

$$
\begin{aligned}
\hat{H}(z_1, z_2) &= \frac{e^{-N \int_{z_2}^{z_1} y\, dx}}{E(z_1, z_2)} \left[1 + N^{-1} \int_{z_2}^{z_1} \omega_1^{(1)} \right. \\
&\quad \left. + \frac{N^{-1}}{6} \int_{z_2}^{z_1} \int_{z_2}^{z_1} \int_{z_2}^{z_1} \omega_3^{(0)} + O(N^{-2}) \right],
\end{aligned}
\qquad (4\text{-}57)
$$

where $E(z_1, z_2)$ is the prime form:

$$E(z_1, z_2) = \frac{z_1 - z_2}{\sqrt{dz_1\, dz_2}}. \qquad (4\text{-}58)$$

For example, one of the terms contributing to $\hat{H}$ to order N^{-1} is:

$$\int_{z_2}^{z_1} \int_{z_2}^{z_1} \int_{z_2}^{z_1} \omega_3^{(0)} = \sum_i \frac{(z_1 - z_2)^3}{y'(a_i)\, x''(a_i)\, (a_i - z_1)^3\, (a_i - z_2)^3}. \tag{4-59}$$

In all matrix model cases, the kernel H can be written as a Sato-formula, and coincides with $\hat{H}$, but this is not proved in general.

4.6.2 Examples

For example, if we consider the Airy curve $y = \sqrt{x}$, i.e. $\mathcal{E} = (\mathbb{P}^1, z^2, z)$, we find that $\hat{H}$ is the Airy kernel:

$$\hat{H}(z_1, z_2) = \frac{Ai(z_1^2)Bi'(z_2^2) - Ai'(z_1^2)Bi(z_2^2)}{z_1^2 - z_2^2} \sqrt{z_1 dz_1\, z_2 dz_2}. \tag{4-60}$$

The corresponding Baker-Akhiezer function is the Airy function, and the correlators ω_n, are the correlators given by the determinantal Airy process.

4.6.3 Sato formula

The theorem 4.3, implies that under an infinitesimal change of spectral curve of the 3rd kind eq.(4-11): $\delta y dx = t\, dS_{z_1, z_2}$, we have:

$$\frac{\partial \omega_n^{(g)}(z_1', \ldots, z_n')}{\partial t} = \int_{z_2}^{z_1} \omega_{n+1}^{(g)}(z_1', \ldots, z_n', z) \tag{4-61}$$

and thus:

$$\frac{\partial^n F_g}{\partial t^n} = \int_{z_2}^{z_1} \ldots \int_{z_2}^{z_1} \omega_n^{(g)}(z_1', \ldots, z_n'). \tag{4-62}$$

The exponential formula of proposition 4.4, is nothing but the Taylor expansion of $F_g(t)$ computed at $t = N^{-1}$ in terms of derivatives taken at $t = 0$, i.e. $F_g(N^{-1}) = \sum_k \frac{N^{-k}}{k!} \partial_t^k F_g(0)$. In other words:

Theorem 4.10

$$\hat{H}(z_1, z_2) = \frac{\sqrt{dx(z_1)\, dx(z_2)}}{x(z_1) - x(z_2)} \frac{\tau_N\left(\mathcal{L}, x, y + \frac{1}{N} \frac{dS_{z_1,z_2}}{dx}\right)}{\tau_N(\mathcal{L}, x, y)}. \tag{4-63}$$

This theorem can be interpreted as Sato's formula [106] for integrable systems. In the context of random matrix theory, it can be interpreted as Heine's formula [109].

4.6.4 Baker-Akhiezer functions

Let $\alpha_1, \ldots, \alpha_m$ be the poles of the function $x(z)$, of respective degrees $d_1, \ldots, d_m$. Since x is a meromorphic form of degree $d = \sum_i d_i$, there are d sheets, i.e. d points on $\mathcal{L}$, $z^1(x), \ldots, z^d(x)$, such that $x(z^k) = x$. The following matrix:

$$\mathcal{H}(x_1, x_2) = \left(\hat{H}(z^j(x_1), z^i(x_2)) \right)_{i,j=1,\ldots,d} \tag{4-64}$$

is a square matrix of size $d \times d$.

The Ψ-function of the Lax system [14], is obtained by choosing $x_2 = \infty$, i.e. $z^i(x_2) = \alpha_i$. Since some poles α_i are multiple poles, in order to get an invertible matrix, we take linear combinations of rows, and we define:

$$i = 1, \ldots, m, \ \ j = 1, \ldots, d_i \qquad \psi_{i,j}(z) = \lim_{z_2 \to \alpha_i} \left[\left(\frac{d}{d\xi_i(z_2)} \right)^{j-1} \frac{e^{-N \int_o^{z_2} y dx} \ \hat{H}(z, z_2)}{\sqrt{d\xi_i(z_2)}} \right] \tag{4-65}$$

where o is an arbitrary basepoint, and $\xi_i(z_2) = x(z_2)^{-1/d_i}$ is the local parameter in the vicinity of α_i.

Those functions are the Baker-Akhiezer functions.

We also have d couples $I = (i, j)$ with $i = 1, \ldots, m, \ j = 1, \ldots, d_i$, and thus the following matrix is a square matrix:

$$\Psi(x) = \left(\psi_I(z^k(x)) \right)_{I=1,\ldots,d, \ k=1,\ldots,d}. \tag{4-66}$$

It is the Ψ-function of the corresponding Lax system [14].

4.6.5 Hirota formula

Notice that $\hat{H}(z_1, z_2)$ has a simple pole at $z_1 = z_2$ and behaves like:

$$\hat{H}(z_1, z_2) \sim \frac{\sqrt{dz_1 \ dz_2}}{z_1 - z_2} \tag{4-67}$$

near $z_1 = z_2$, and this holds for any ($\bar{g} = 0$) spectral curve $\mathcal{E} = (\mathcal{L}, x, y)$. In particular, we have, for any two such spectral curves $\mathcal{E}$ and $\tilde{\mathcal{E}}$:

Theorem 4.11

$$\operatorname*{Res}_{z \to z_2} \hat{H}(z_1, z; \mathcal{E}) \hat{H}(z, z_2; \tilde{\mathcal{E}}) = \hat{H}(z_1, z_2; \mathcal{E}). \tag{4-68}$$

If we consider that $\hat{H}$ is given by the Sato formula of theorem 4.10, this theorem is precisely the Hirota equation for the τ-function τ_N [14, 1].

This theorem justifies that we can call τ_N a Tau-function. By expanding locally the dS_{z_1, z_2} in the vicinity of poles of x, we can see that it is the multicomponent Kadamtsev-Petviashvili (KP) tau-function. There is one set of component for each pole α_i of x. In the case where $\mathcal{E}$ is an hyperelliptical curve, of type $y^2 = \mathrm{Pol}(x)$, the function x has two poles, which are symmetric of oneanother, and everything can be written in terms of the expansion near only one pole. In that case τ_N reduces to the Kortweg-de-Vries (KdV) tau-function [14].

4.6.6 Higher genus

So far, in this section, we were considering genus zero spectral curves, i.e. $z \in \mathbb{C}$, and $x(z)$ and $y(z)$ analytical functions of a complex variable.

The integrability relied on the Sato formula, which gives the kernel $\hat{H}$ as the τ-function of a shifted spectral curve, i.e. the exponential formula.

For higher genus $\bar{g} \geq 1$, the problem is that the exponential formula does not define a well-defined spinor on $\mathcal{L}$. Indeed, $\mathcal{L}$ is not simply connected, and the abelian integrals $\int_{z_2}^{z_1} \dots \int_{z_2}^{z_1} \omega_n$ are multivalued functions of z_1 and z_2 because there is not a unique integration path between z_1 and z_2. The exponential formula has to be modified. It was proposed to modify it with some theta functions (see section 4.5).

Definition 4.5 *Given a characteristics (μ, ν), the "tau-function" is defined by the non-perturbative partition function of section 4.5:*

$$\tau_N(\mu, \nu, \mathcal{E}) = Z_{\mathcal{E}}(\mu, \nu). \tag{4-69}$$

Then define the spinor kernel $\hat{H}_{(\mu,\nu)}$ through the Sato formula:

Definition 4.6

$$\hat{H}_{(\mu,\nu)}(z_1, z_2) = \frac{\sqrt{dx(z_1)dx(z_2)}}{x(z_1) - x(z_2)} \frac{\tau_N(\mu, \nu, \mathcal{L}, x, y + \frac{1}{N} \frac{dS_{z_1,z_2}}{dx})}{\tau_N(\mu, \nu, \mathcal{L}, x, y)}. \tag{4-70}$$

With this definition, $\hat{H}_{(\mu,\nu)}$ is closely related to the Szegö kernel [109].

Theorem 4.12 $\hat{H}_{(\mu,\nu)}(z_1, z_2)$ *is well defined for* $z_1, z_2 \in \mathcal{L}$.

proof:
Integrals of $\omega_n^{(g)}$'s are in principle defined only on the universal covering of $\mathcal{L}$, and one needs to check that after going around an $\mathcal{A}$-cycle or $\mathcal{B}$-cycle, $\hat{H}_{(\mu,\nu)}(z_1, z_2)$ takes the same value.

Notice that, if z_1 goes around an $\mathcal{A}$-cycle, then dS_{z_1,z_2} is unchanged, and if z_1 goes around the cycle $\mathcal{B}_i$, then dS_{z_1,z_2} is shifted by a holomorphic differential:

$$dS_{z_1+\mathcal{B}_i,z_2} = dS_{z_1,z_2} + 2i\pi \, du_i. \tag{4-71}$$

However, it was proved in [55] that the τ function above is background independent, which exactly means that, for any λ:

$$\tau_N(\mu, \nu, \mathcal{L}, x, y + \lambda du_i/dx) = \tau_N(\mu, \nu, \mathcal{L}, x, y) \tag{4-72}$$

and therefore, we see that $\hat{H}_{(\mu,\nu)}(z_1, z_2)$ is unchanged if z_1 goes around a $\mathcal{B}$-cycle. $\square$

Then, we see that [64]

Theorem 4.13 $\hat{H}_{(\mu,\nu)}(z_1, z_2)$ *obeys the Hirota equation:*

$$\operatorname*{Res}_{z \to z_2} \hat{H}_{(\mu,\nu)}(z_1, z; \mathcal{E}) \hat{H}_{(\mu,\nu)}(z, z_2; \tilde{\mathcal{E}}) = \hat{H}_{(\mu,\nu)}(z_1, z_2; \mathcal{E}). \tag{4-73}$$

If we consider that $\hat{H}_{(\mu,\nu)}$ is given by the Sato formula of theorem 4-70, this theorem is precisely the Hirota equation for the Tau-function τ_N.

4.7 Virasoro constraints

Its has been understood for a long time that the random matrix integrals are fundamentally linked to Virasoro and $\mathcal{W}$-algebras through differentials equations on their moduli called Virasoro or $\mathcal{W}$-constraints. The definition of the symplectic invariants and of the correlation functions themselves were inspired by these constraints since they mimic the solution of the loop equations of random matrix models, the latter being considered as equivalent to the Virasoro constraints.

In a series of papers [7, 8, 9], Alexandrov, Mironov and Morozov go even further and propose to generalize the notion of random matrix integrals by defining a general string partition function interpolating between different matrix models. This partition function is characterized as a "D-module" solution of some Virasoro constraints.

It is natural to see the symplectic invariants and the τ-function built from them as a good candidate for this string partition function. It is thus interesting to clarify the arising of Virasoro constraints in the theory of symplectic invariants by looking at the variations of the latter wrt the moduli of the spectral curve.

4.7.1 Virasoro at the branch points

One can slightly rewrite the recursive relations defining the correlation functions eq.(2-44) by moving all the terms to the same side of the equation. One gets:

$$
\begin{aligned}
0 \quad = \quad & \sum_i \operatorname*{Res}_{z \to a_i} K(z_0, z) \Big[\omega_{n+2}^{(g-1)}(z, \bar{z}, J) + \sum_{h=0}^{g} \sum_{I \subset J} \omega_{1+|I|}^{(h)}(z, I) \omega_{1+n-|I|}^{(g-h)}(\bar{z}, J \backslash I) \\
& + y dx(z) \omega_{n+1}^{(g)}(\bar{z}, J) + y dx(\bar{z}) \omega_{n+1}^{(g)}(z, J) \Big].
\end{aligned}
$$

$$(4-74)$$

By summing over the genus g and interpreting the correlation function as the result of the loop insertion operator on the symplectic invariants, this equation can be rephrased as a Virasoro constraint:

Theorem 4.14 *For any point z on the spectral curve, the partition function is a zero mode of the* global Virasoro operator *$\widehat{\mathcal{V}}(z)$*

$$\widehat{\mathcal{V}}(z) \tau_N = 0 \qquad (4\text{-}75)$$

with

$$\widehat{\mathcal{V}}(z) = \sum_i \oint_{a_i} K(z, z') : \mathcal{J}(z') \mathcal{J}\left(\overline{z'}\right) : \qquad (4\text{-}76)$$

and the global current *is defined by:*

$$\mathcal{J}(z) = N y dx(z) + \frac{1}{N} D_z \qquad (4\text{-}77)$$

for any point z of the spectral curve.

This means that the recursive definition of the correlation functions is nothing but a Virasoro constraint on the τ-function defined **globally** on the spectral curve.

Let us now approach a particular branch point a_i and blow up the spectral curve around this point (see section 4.8). A rational parametrization of the blown up curve can read

$$\begin{cases} \tilde{x}(z) = z^2 \\ \tilde{y}(z) = \displaystyle\sum_{k=0}^{\infty} T_k^{(i)} z^k \end{cases} \tag{4-78}$$

where the $T_k^{(i)}$'s are the coefficients of the Taylor expansion of ydx around the branch points a_i

$$ydx(z) = \sum_{k=0}^{\infty} T_k^{(i)} \hat{\xi}_i^{k+1}(z) d\hat{\xi}_i(z) \tag{4-79}$$

with the local coordinates

$$\hat{\xi}_i(z) = \sqrt{x(z) - x(a_i)}. \tag{4-80}$$

From section 4.8, one knows that the projection of the correlation functions in the local patch around a_i built from the local parameter $\hat{\xi}_i$ is given by the correlation functions of the blown up curve, i.e. the correlation functions of the Kontsevich integral with times $T_k^{(i)}$, $k = 0, \ldots, \infty$. Since the recursive definition of these correlation functions is equivalent to Kontsevich Virasoro constraints [9], this means that the global Virasoro operator projects to the continuous Virasoro operator in this local patch of coordinates around a_i:

Theorem 4.15 *For any branch point a_i, the partition function is a zero mode of a set local Kontsevich Virasoro operator $\widehat{\mathcal{V}}_i(\hat{\xi}_i(z))$ for any point z in a neighborhood of a_i:*

$$\forall i, \quad \widehat{\mathcal{V}}_i(\hat{\xi}_i(z)) \tau_N = 0 \tag{4-81}$$

where $\hat{\xi}_i(z) = \sqrt{x(z) - x(a_i)}$ and the Virasoro operator annihilates the Kontsevich τ-function:

$$\widehat{\mathcal{V}}_i(\hat{\xi}) Z_K(T_k^{(i)}) = 0 \tag{4-82}$$

where

$$Z_K(T_k^{(i)}) := \int_{formal} e^{-N\,\mathrm{Tr}\left(\frac{M^3}{3} - \Lambda^2 M\right)} \tag{4-83}$$

with Kontsevich times

$$T_k^{(i)} := \frac{1}{N} \mathrm{Tr}\, \Lambda^{-k}. \tag{4-84}$$

Indeed, as it is exhibited in section 10.1, the corresponding spectral curve has only one branch point and all the moduli of the integral are summed up in the Taylor expansion of the differential form xdy at this branch point.

---

[9]See [8, 30] for detail on these continuous Virasoro contraints.

4.7.2 Loop equations and Virasoro at the poles

In the preceding section, we have translated the recursive definition of the correlation functions into a set of Virasoro operators related to the moduli of the spectral curve at the branch points. One can proceed in a similar way for the moduli at the poles of the one form ydx by building a set of equations solved by the correlation functions called "loop equations" since they mimic the loop equations of random matrix theory.

Theorem 4.16 *The correlation functions ω_n are solutions of the loop equations:*

$$\sum_{l=0}^{k} \omega_{l+1}(z, \mathbf{z_L})\omega_{k-l+1}(z, \mathbf{z_{K\backslash L}}) + \frac{1}{N^2}\omega_{k+2}(z, z, \mathbf{z_K}) = P_{1,k}(z, \mathbf{z_K})dx(z)^2 \tag{4-85}$$

where the function

$$P_{1,k}(z, \mathbf{z_K}) := \sum_i \oint_{\alpha_i} \frac{\sum\limits_{l=0}^{k} \omega_{l+1}(z', \mathbf{z_L})\omega_{k-l+1}(z', \mathbf{z_{K\backslash L}}) + \frac{1}{N^2}\omega_{k+2}(z', z', \mathbf{z_K})}{(z_i(z) - z_i(z'))dx(z')} \tag{4-86}$$

is a function of z with poles only at the poles of ydx.

In the matrix model case, these loop equations are often referred to as Virasoro constraints. They indeed encode a set of Virasoro constraints build from the poles of ydx. Let us make this assertion clear in the general framework of the symplectic invariants:

Theorem 4.17 *For any point $z \in \mathcal{L}$, the τ-function satisfies*

$$\mathcal{V}(z)\tau_N = 0 \tag{4-87}$$

where one defines the global Virasoro operator

$$\mathcal{V}(z) = \frac{1}{N^2} : \mathcal{J}^2(z) : + \sum_i \oint_{\alpha_i} \frac{: \mathcal{J}^2(z') :}{(\xi_i(z') - \xi_i(z))dx(z')} \tag{4-88}$$

and $\xi_i(z) = \frac{1}{\xi(z)}$ is a local parameter in the neighborhood of α_i (see eq.(4-10) for the definition of $\xi(z)$).

The τ-function can thus be seen as the zero mode of another Virasoro operator globally defined on the spectral curve. This new operator, equivalent to the loop equations, can be easily projected to a set of local Virasoro operators in the vicinity of the poles of ydx instead of the branch points for the first one. In order to follow this procedure one has to restrict to ydx which are holomorphic forms with poles α_i such that $ydx(z) \sim_{z\to a_i} \sum_k kt_{i,k}\xi_i^k(z)d\xi_i(z)$.

Theorem 4.18 *For any point z in the neighborhood of a pole α_i of ydx:*

$$\mathcal{V}_-^{(i)}(z)\tau_N = 0 \tag{4-89}$$

where the local Virasoro operator *is defined as the loop operator*

$$\mathcal{V}_-^{(i)}(z) := \oint_{\alpha_i} \frac{d\xi_i(z')}{\xi_i(z') - \xi_i(z)} : J^{(i)}(z')^2 : \tag{4-90}$$

with the local current

$$J^{(i)}(z) := \sum_{k\geq 0} \left[kt_{i,k}\xi_i(z)^{k-1}d\xi_i(z) + \frac{d\xi_i(z)}{\xi_i(z)^{k+1}}\frac{\partial}{\partial t_{i,k}} \right]. \tag{4-91}$$

Remark that these local Virasoro operators are indeed Laurent series in $\xi_i(z)$ with only negative powers whose coefficients are differential operators satisfying Virasoro commutation relations:

$$\mathcal{V}_-^{(i)}(z) = \sum_{k>0} \mathcal{V}_k^{(i)} \xi_i(z)^{-k}(d\xi_i(z))^2 \tag{4-92}$$

and

$$\left[\mathcal{V}_j^{(i)}, \mathcal{V}_l^{(k)} \right] = (j-l)\mathcal{V}_{j+l}\delta_{i,k}. \tag{4-93}$$

These local operators around the poles have also a natural solution: the one hermitian matrix integral

$$Z_{1MM} := \int_{formal} e^{-\frac{N}{t}\operatorname{Tr} V(M)} dM \tag{4-94}$$

with a polynomial potential

$$V(x) := \frac{x^2}{2} - \sum_{k=3}^{d} t_k x^k \tag{4-95}$$

whose coefficients t_k are identified with the moduli at the poles $t_{i,k}$ (see section 5.1 for more details).

4.7.3 Givental decomposition formulae

Let us suppose in this short section that the spectral curve has genus 0, i.e. $\mathcal{L}$=Riemann sphere. In this case, the only moduli of the curve are:

- either the position of the poles and moduli $t_{i,k}$ at these poles;

- either the position of the branch point and the moduli $T_k^{(i)}$.

Let us first focus on the branch points of the spectral curve. The dependence of τ_N on the moduli at the branch points is constrained by the local Virasoro equations eq.(4-81). Thus, this τ-function can be decomposed as a product of the zero modes of the different local operators at the branch points, i.e. a product of Kontsevich integrals, up to a conjugation operator mixing the moduli at the different branch points.

Theorem 4.19 τ_N can be decomposed into a product of Kontsevich integrals associated to the branch points a_i:

$$\boxed{\tau_N(\mathbf{T^{(1)}}, \mathbf{T^{(2)}}, \ldots) = e^{\widehat{\mathcal{U}}} \prod_i \mathcal{Z}_K(\mathbf{T^{(i)}}).}$$

(4-96)

where the symbol $\mathbf{T^{(i)}}$ stands for the infinite family $\left\{T_k^{(i)}\right\}_{k=0}^{\infty}$, with the intertwining operator $\widehat{\mathcal{U}}$ defined by

$$\widehat{\mathcal{U}} := \sum_{i,j} \oint_{a_j} \oint_{a_i} \widehat{A}^{(i,j)}(z, z') \widehat{\Omega}_j(z') \widehat{\Omega}_i(z)$$

(4-97)

where

$$\widehat{A}^{(i,j)}(z, z') := B(z, z') - \frac{d\hat{\xi}_i(z) d\hat{\xi}_j(z')}{(\hat{\xi}_i(z) - \hat{\xi}_j(z'))^2}$$

(4-98)

and

$$\widehat{\Omega}_i(z) := N \sum_k T_k^{(i)} \hat{\xi}_i^k(z) d\hat{\xi}_i(z) - \frac{1}{N} \frac{d\hat{\xi}_i(z)}{k \hat{\xi}_i^k(z)} \frac{\partial}{\partial T_k^{(i)}}.$$

(4-99)

One can proceed exactly in the same way by looking at the moduli at the poles: this time the decomposition is expressed as a product of 1-hermitian matrix integrals.

Theorem 4.20 τ_N can be decomposed into a product of one hermitian matrix integrals associated to the poles α_i of the meromorphic form ydx

$$\boxed{\tau_N(\mathbf{t_1}, \mathbf{t_2}, \ldots) = e^{\mathcal{U}} \prod_i \mathcal{Z}_{1MM}(\mathbf{t_i}).}$$

(4-100)

where $\mathbf{t_i}$ stands for the infinite set $\{t_{i,k}\}_{k=0}^{\infty}$, with the intertwining operator $\mathcal{U}$ defined by

$$\mathcal{U} := \sum_{i,j} \oint_{\alpha_j} \oint_{\alpha_i} A^{(i,j)}(z, z') \Omega_j(z') \Omega_i(z)$$

(4-101)

where

$$A^{(i,j)}(z, z') = B(z, z') - \frac{d\xi_i(z) d\xi_j(z')}{(\xi_i(z) - \xi_j(z'))^2}$$

(4-102)

and

$$\Omega_i(z) := N \sum_k t_{i,k} \xi_i^k(z) d\xi_i(z) - \frac{1}{N} \frac{d\xi_i(z)}{k \xi_i^k(z)} \frac{\partial}{\partial t_{i,k}}.$$

(4-103)

Remark that, in both cases, these decomposition formulae consist in writing a KP tau function as a product of KdV tau-functions. Indeed, these formula were already derived by Givental in the study of KP tau functions [67, 68].

4.7.4 Vertex operator and integrability

In this paragraph, we do not consider the Baker-Akhiezer functions as defined in section 4.6. On the contrary, we define them as the images of the partition function under the action of some global operator on the spectral curve.

Let us define the equivalent of the Baker-Akhiezer (BA) functions:

Definition 4.7 *One defines the x-type global BA and dual BA functions as*

$$\boldsymbol{\Psi}(z) := \exp\left(\Omega_x(z)\right)\mathcal{Z} \qquad and \qquad \boldsymbol{\Psi}^*(z) := \exp\left(-\Omega_x(z)\right)\mathcal{Z} \tag{4-104}$$

where

$$\Omega_x(q) := \int^q \mathcal{J}_x = N \int^q y\,dx(z) - \frac{1}{N}\int^q D_z. \tag{4-105}$$

One also defines the x-type local BA functions as:

$$\boldsymbol{\Psi}_{i,0}(z) := \exp\left(\Omega_i(z)\right)\mathcal{Z} \qquad and \qquad \boldsymbol{\Psi}^*_{i,0}(z) := \exp\left(-\Omega_i(z)\right)\mathcal{Z} \tag{4-106}$$

where the operator Ω_i was defined in eq.(4-103).

We finally define the corresponding y-type BA functions:

$$\widetilde{\boldsymbol{\Psi}}(z) := \exp\left(\Omega_y(z)\right)\mathcal{Z} \qquad and \qquad \widetilde{\boldsymbol{\Psi}}^*(z) := \exp\left(-\Omega_y(z)\right)\mathcal{Z} \tag{4-107}$$

where

$$\Omega_y(q) := \int^q \mathcal{J}_y = N \int^q x\,dy(z) - \frac{1}{N}\int^q D_z. \tag{4-108}$$

These functions correspond to deformations of the spectral curve and thus coincide with the Baker-Akhiezer functions of eq.(4-65).

Lemma 4.1 *The BA functions can be written in terms of the partition function as*

$$\boldsymbol{\Psi}(z) := \exp\left(\Omega_x(z)\right)\mathcal{Z}(\mathbf{t_i}) = \frac{\mathcal{Z}(\mathbf{t_i} + [z_i^{-1}])}{\mathcal{Z}(\mathbf{t_i})}, \tag{4-109}$$

$$\boldsymbol{\Psi}_{i,0}(z) := \exp\left(\Omega_i(z)\right)\mathcal{Z}(t_i) = \frac{\mathcal{Z}(t_i + [z_i^{-1}])}{\mathcal{Z}(t_i)}, \tag{4-110}$$

$$\widetilde{\boldsymbol{\Psi}}(z) := \exp\left(\Omega_y(z)\right)\mathcal{Z}(\tilde{\mathbf{t}}_\mathbf{i}) = \frac{\mathcal{Z}(\tilde{\mathbf{t}}_\mathbf{i} + [z_i^{-1}])}{\mathcal{Z}(\tilde{\mathbf{t}}_\mathbf{i})} \tag{4-111}$$

and

$$\widetilde{\boldsymbol{\Psi}}_{i,0}(z) := \exp\left(\Omega_{y,i}(z)\right)\mathcal{Z}(\tilde{t}_i) = \frac{\mathcal{Z}(\tilde{t}_i + [z_i^{-1}])}{\mathcal{Z}(\tilde{t}_i)} \tag{4-112}$$

where the $\tilde{t}_i$ are the coefficients of the Taylor expansion of $x\,dy(z)$ as $z \to \alpha_i$ and $[z^{-1}]$ is the usual Hirota symbol [1].

On the other hand, thanks to the pole structure of the BA functions, one gets

Theorem 4.21 *The Baker-Akhizer functions satisfy the bilinear Hirota equation*

$$\sum_i \oint_{\alpha_i} \boldsymbol{\Psi}(p|\mathbf{t}) e^{-N \int^p y dx + N \int^p y' dx'} \boldsymbol{\Psi}^*(p|\mathbf{t}') = \sum_i \oint_{\alpha_i} \widetilde{\boldsymbol{\Psi}}(p|\mathbf{t}) e^{-N \int^p x dy + N \int^p x' dy'} \widetilde{\boldsymbol{\Psi}}^*(p|\mathbf{t}')$$

$$(4\text{-}113)$$

where x' and y' are functions on $\mathcal{L}$ satisfying another algebraic equation

$$\mathcal{E}'(x'(z), y'(z)) = 0 \qquad (4\text{-}114)$$

compared to

$$\mathcal{E}(x(z), y(z)) = 0. \qquad (4\text{-}115)$$

proof:

The proof relies on the simple observation that

$$\Omega_x - N \int^z y dx = \Omega_y - N \int^z x dy = D_z. \qquad (4\text{-}116)$$

$\square$

Corollary 4.1 *If x has q poles and y has p poles, the partition function $\mathcal{Z}(\mathbf{t})$ is τ-function of the multi-component $p + q$-KP hierarchy since it satisfies the Hirota equations:*

$$\sum_i \oint_{\alpha_i} \frac{\mathcal{Z}(t_i + [z_i^{-1}])}{\mathcal{Z}(t_i)} \frac{\mathcal{Z}(t_i' - [z_i^{-1}])}{\mathcal{Z}(t_i')} e^{N \sum_k (t_{i,k} - t_{i,k}') z_i^{k-1}(p)} =$$
$$= \oint_{\alpha_i} \frac{\mathcal{Z}(\tilde{t}_i + [\tilde{z}_i^{-1}])}{\mathcal{Z}(\tilde{t}_i)} \frac{\mathcal{Z}(\tilde{t}_i' - [\tilde{z}_i^{-1}])}{\mathcal{Z}(\tilde{t}_i')} e^{N \sum_k (\tilde{t}_{i,k} - \tilde{t}_{i,k}') \tilde{z}_i^{k-1}(p)}.$$

$$(4\text{-}117)$$

4.8 Singular limits

The F_g's and $\omega_n^{(g)}$'s can be computed for any regular spectral curve, i.e. as long as the branchpoints are simple. When the spectral curve is singular, the F_g's are not defined.

Nevertheless, consider a 1-parameter family of spectral curves $\mathcal{E}(t)$, such that $\mathcal{E}(t_c)$ is singular, we prove below, that $F_g(t)$ diverges as $t \to t_c$, in the following form:

$$F_g(t) \sim (t - t_c)^{(2-2g)\mu} \tilde{F}_g. \qquad (4\text{-}118)$$

The goal of this section is to prove this divergent behavior, and compute the exponent μ and the prefactor $\tilde{F}_g$. These aymptotics are very important in many applications in mathematics and physics, for instance Witten's conjecture relates the asymptotics of large discrete surfaces, to integrals over moduli spaces of continuous Riemann surfaces. Asymptotic formulae play also a key role in the universal limits of random matrix eigenvalues statistics, or in the study of universality in the statistics of non-intersecting Brownian motions (see section 6).

In the context of matrix models quantum gravity, the prefactor $\tilde{F}_g$ is called the "double scaling limit" of F_g, and the exponent $2 - 2\mu$ is called

$$2 - 2\mu = \gamma_{\text{string}} = \text{"string susceptibility exponent"}. \qquad (4\text{-}119)$$

It is such that F_0'' formally diverges with the exponent $-\gamma_{\text{string}}$:

$$\frac{d^2 F_0}{dt^2} \sim (t - t_c)^{-\gamma_{\text{string}}}. \qquad (4\text{-}120)$$

4.8.1 Blow up of a spectral curve

Consider a one parameter family of spectral curves $\mathcal{E}(t) = (\mathcal{L}(t), x(z, t), y(z, t))$, such that $\mathcal{E}(t)$ is regular in an interval $]t_c, t_0]$. For the moment we do not assume that $\mathcal{E}(t_c)$ is singular, i.e. it may be either regular or singular. In a small vicinity of t_c, we can, to leading orders, parameterize $\mathcal{E}(t)$ in terms of $\mathcal{L} = \mathcal{L}(t_c)$.

Moreover, let a be a branchpoint, and let us study the correlators $\omega_n^{(g)}(z_1, \ldots, z_n)$ in the vicinity of a. We choose a rescaled local coordinate ζ in the vicinity of a. Let us write:

$$z = a + (t - t_c)^\nu \zeta + o((t - t_c)^\nu). \qquad (4\text{-}121)$$

We want to compute the asymptotic behavior of $\omega_n^{(g)}(a + (t - t_c)^\nu \zeta_1, \ldots, a + (t - t_c)^\nu \zeta_n; t)$ in the limit $t \to t_c$.

First, let us study the behavior of x and y, by Taylor expansion. Let q be the first non-trivial power in the Taylor expansion of x, i.e.:

$$x(a + (t - t_c)^\nu \zeta; t) = x(a; t_c) + (t - t_c)^{q\nu} \tilde{x}(\zeta) + o((t - t_c)^{q\nu}) \qquad (4\text{-}122)$$

and similarly, there is an exponent p such that:

$$y(a + (t - t_c)^\nu \zeta; t) = y(a; t_c) + (t - t_c)^{p\nu} \tilde{y}(\zeta) + o((t - t_c)^{p\nu}). \qquad (4\text{-}123)$$

This means that at $t = t_c$ the curve $\mathcal{E}(t_c)$ behaves like $y \sim y(a) + (x - x(a))^{p/q}$. It is regular if $\frac{p}{q} = \frac{1}{2}$ and singular otherwise.

The rescaled curve $\tilde{\mathcal{E}} = (\tilde{\mathcal{L}}, \tilde{x}, \tilde{y})$ is called the blow up of the spectral curve near the branchpoint a, in the limit $t \to t_c$.

The choice of the exponent ν, must be such that $\tilde{\mathcal{E}}$ is a regular spectral curve. We cannot give a general formula for ν, since it depends on the explicit choice of a 1-parameter family of spectral curves $\mathcal{E}(t)$, and how it is parametrized. Also, here we consider only algebraic singularities of type $y \sim x^{p/q}$, but the method could certainly be extended to other types of singularities.

Examples:

● The following spectral curve arises in the enumeration of quadrangulated surfaces (see section 7.4):

$$\mathcal{L} = \mathbb{P}^1 \quad , \quad x(z) = \gamma\left(z + \frac{1}{z}\right) \quad , \quad y(z) = \frac{-1}{\gamma z} + \frac{t_4 \gamma^3}{z^3} \qquad (4\text{-}124)$$

where
$$\gamma^2 = \frac{1 - \sqrt{1 - 12t_4}}{6t_4}. \tag{4-125}$$

The branchpoints $x'(a) = 0$ are $a = \pm 1$. Consider the branchpoint $a = 1$, and introduce an auxillary scaling variable t:
$$z = 1 + t\zeta, \tag{4-126}$$

such that x and y are independent of t, and let us study the vicinity of $t \to t_c = 0$.

We whish to study the behaviour of $\omega_n^{(g)}(1 + t\zeta_1, \dots, 1 + t\zeta_n)$ in the vicinity of $t \to 0$, i.e. the behaviour of $\omega_n^{(g)}$ in the vicinity of the branchpoint $a = 1$.

In the limit $t \to 0$, we Taylor expand x and y:

$$x(z) = 2\gamma + \gamma t^2 \zeta^2 + O(t^3), \tag{4-127}$$

$$y(z) = \frac{-1}{\gamma} + t_4 \gamma^3 + t\zeta(\frac{1}{\gamma} - 3t_4\gamma^3) + O(t^2). \tag{4-128}$$

Notice that we have $q = 2$ and $p = 1$, which means that our curve is not singular at $t = 0$ (which was expected since it is actually independent of t).

The blow up is:

$$\tilde{x}(\zeta) = \gamma \zeta^2 \quad , \quad \tilde{y}(\zeta) = \left(\frac{1}{\gamma} - 3t_4\gamma^3\right)\zeta. \tag{4-129}$$

Up to a rescaling, it is the Airy spectral curve (see section 8 and example eq.(2-4)).

• In the previous example, at $t_4 = \frac{1}{12}$, we have $(1 - 3t_4\gamma^4) = 0$, and thus, one needs to go further in the Taylor expansion. Let us now choose $t = 1 - 12t_4$, $t_c = 0$, and $a = 1$. We have in the limit $t \to t_c$:

$$x\left(1 + \frac{1}{\sqrt{2}}t^{\frac{1}{4}}\zeta\right) = 2\sqrt{2} + \frac{\sqrt{t}}{\sqrt{2}}(\zeta^2 - 2) + o(\sqrt{t}), \tag{4-130}$$

$$y\left(1 + \frac{1}{\sqrt{2}}t^{\frac{1}{4}}\zeta\right) = -\frac{\sqrt{2}}{3} + \frac{t^{\frac{1}{2}}}{2\sqrt{2}}(\zeta^2 - 2) - \frac{t^{\frac{3}{4}}}{12}(7\zeta^3 - 12\zeta) + o(t^{\frac{3}{4}}), \tag{4-131}$$

and, in fact, what we really need is the asymptotic behavior of $y(z) - y(\bar{z})$, i.e.:

$$y\left(1 + \frac{1}{\sqrt{2}}(t_c - t)^{\frac{1}{4}}\zeta\right) - y\left(\frac{1}{(1 + \frac{1}{\sqrt{2}}(t_c - t)^{\frac{1}{4}}\zeta)}\right) = -\frac{2\,t^{\frac{3}{4}}}{3}(\zeta^3 - 3\zeta) + o(t^{\frac{3}{4}}). \tag{4-132}$$

The blow up of the spectral curve in this limit is thus:

$$\tilde{x}(\zeta) = \frac{1}{\sqrt{2}}(\zeta^2 - 2) \quad , \quad \tilde{y}(\zeta) = -\frac{1}{3}(\zeta^3 - 3\zeta). \tag{4-133}$$

Notice that it is proportional to the "pure gravity" spectral curve $(p, q) = (3, 2)$, see section 8, and see the first example in section 2.1.1. It is the spectral curve which arises everytime we have a $y \sim x^{3/2}$ cusp singularity.

4.8.2 Asymptotics

In order to study the asymptotics of the $\omega_n^{(g)}$'s, we need to study the asymptotics of the kernels B and K, in the limit $t \to t_c$.

We have:

$$B(z_0, z) \sim \begin{array}{|c|c|c|} \hline & z \text{ near } a & z \text{ far from } a \\ \hline z_0 \text{ near } a & \tilde{B}(\zeta_0, \zeta) & O((t-t_c)^\nu) \\ \hline z_0 \text{ far from } a & O((t-t_c)^\nu) & O(1) \\ \hline \end{array} \times (1 + O((t-t_c)^\nu)), \quad (4\text{-}134)$$

where $\tilde{B}(\zeta_0, \zeta)$ is the Bergmann kernel of the blown up spectral curve $(\tilde{\mathcal{L}}, \tilde{x}, \tilde{y})$:

$$\tilde{B}(\zeta_0, \zeta) = \frac{d\zeta_0 \, d\zeta}{(\zeta - \zeta_0)^2}. \tag{4-135}$$

Similarly, the kernel K behaves like:

$$K(z_0, z) \sim \begin{array}{|c|c|c|} \hline & z \text{ near } a & z \text{ far from } a \\ \hline z_0 \text{ near } a & (t-t_c)^{-(p+q)\nu} \tilde{K}(\zeta_0, \zeta) & O(1) \\ \hline z_0 \text{ far from } a & O((t-t_c)^{-(p+q-1)\nu}) & O(1) \\ \hline \end{array} \times (1 + O((t-t_c)^\nu)),$$

$$(4\text{-}136)$$

where $\tilde{K}(\zeta_0, \zeta)$ is the recursion kernel of the blown up spectral curve $(\tilde{\mathcal{L}}, \tilde{x}, \tilde{y})$:

$$\tilde{K}(\zeta_0, \zeta) = \frac{1}{2} \left(\frac{1}{\zeta_0 - \zeta} - \frac{1}{\zeta_0 + \zeta} \right) \frac{1}{2\tilde{y}(\zeta) \, \tilde{x}'(\zeta)}. \tag{4-137}$$

Therefore, we see that the leading contribution to $\omega_n^{(g)}(1 + \delta\zeta_0, \dots, 1 + \delta\zeta_n)$ is given by the terms where all residues are taken near a, and the leading contribution can be computed only in terms of $\tilde{B}$ and $\tilde{K}$. By an easy recursion, on gets:

Theorem 4.22 *Singular limit of $\omega_n^{(g)}$ with $2 - 2g - n < 0$ and $n \geq 1$.*
If a is a branchpoint, the asymptotics of $\omega_n^{(g)}$ in the vicinity of $t \to t_c$ and $z_i \to a$, are given by:

$$\omega_n^{(g)}(a + (t - t_c)^\nu \zeta_1, \dots, a + (t - t_c)^\nu \zeta_n) \sim (t - t_c)^{(2-2g-n)(p+q)\nu} \, \tilde{\omega}_n^{(g)}(\zeta_1, \dots, \zeta_n) \tag{4-138}$$

where $\tilde{\omega}_n^{(g)}$ are the correlators of the blown up spectral curve $\tilde{\mathcal{E}} = (\tilde{\mathcal{L}}, \tilde{x}, \tilde{y})$.

In this theorem, the exponent ν is unspecified, it depends on our choice of 1-parameter family of curves, i.e. it depends on each example. We recall that it must be chosen such that the blown up curve $\tilde{\mathcal{E}}$ is regular. We see examples in section 8.

This theorem implies in particular, that all correlation functions in the vicinity of a regular branchpoint, are, to leading order, the same as the Airy process correlation functions, we recover the universals Airy law near regular branchpoints. This is related to the universal Tracy-Widom law [111].

One may extend this theorem to $\omega_0^{(g)} = F_g$'s:

Theorem 4.23 *Singular limit of F_g.*

If at $t = t_c$, the branchpoint a becomes singular, and no other branchpoint becomes singular, then the asymptotics of $F_g, g \geq 2$ in the vicinity of $t \to t_c$ are given by:

$$F_g \sim (t - t_c)^{(2-2g)(p+q)\nu} \; \widetilde{F_g} \quad + o((t - t_c)^{(2-2g)(p+q)\nu}) \tag{4-139}$$

where $\widetilde{F_g}$ are the symplectic invariants of the blown up spectral curve $\tilde{\mathcal{E}} = (\tilde{\mathcal{L}}, \tilde{x}, \tilde{y})$.

In fact this theorem holds also in the case where the branchpoint a is not singular, but it becomes useless. Indeed, if a is not singular, the Blown up spectral curve is the Airy curve, and the $\tilde{F}_g$'s of the Airy spectral curve vanish for all $g \geq 1$, and therefore all what the theorem says in this case, is that F_g does not diverge as $(t - t_c)^{(2-2g)(p+q)\nu}$. And this is obvious since F_g is not divergent at all.

Also, the condition that a is the only singular branchpoint, is not so necessary. In fact, if several branchpoints become singular simultaneously at t_c, with the same exponents ν and (p, q), then the asymptotics of F_g is the sum of contributions of most singular branchpoints. The most basic example is a symmetric spectral curve $y(x) = y(-x)$, for which branchpoints come by pairs. The leading order of F_g then gets a factor 2: $F_g \sim 2 \; (t - t_c)^{(2-2g)(p+q)\nu} \; \widetilde{F_g} \quad + o((t - t_c)^{(2-2g)(p+q)\nu})$.

5 Application to matrix models

The recursion relations defining the symplectic invariants and their correlation functions were originally found in the study of the one-Hermitian random matrix model [47, 32] where they appeared as the solution to the so-called loop equations. It is thus interesting to remind to which extent the symplectic invariants give a solution to the computation of the free energies and correlation functions' topological expansions in different matrix models. It is also interesting to emphasize the special properties of the spectral curves obtained from matrix models to remind that they represent only a particular subcase in the whole framework for symplectic invariants.

5.1 1-matrix model

The formal 1-matrix integral is defined as a formal power series in a variable t.

Consider a polynomial $V(x)$ (called "potential") of degree $d + 1 > 2$ as well as its d stationary points ξ_i:

$$\forall i = 1, \ldots, d \, , \quad V'(\xi_i) = 0 \tag{5-1}$$

and the non-quadratic part of its Taylor expansion around these points

$$\delta V_i(x) = V(x) - V(\xi_i) - \frac{V''(\xi_i)}{2}(x - \xi_i)^2. \tag{5-2}$$

Let us also consider a d-partition of N, i.e. a set of d integers n_i satisfying

$$\sum_{i=1}^{d} n_i = N. \tag{5-3}$$

Definition 5.1 *Formal 1-hermitian matrix integral.*
   The formal 1-matrix integral is defined as a formal power series in a variable t:

$$
\begin{aligned}
Z_{1\mathrm{MM}} \quad &= \quad \mathrm{e}^{-\frac{N}{t}\sum_i n_i V(\xi_i)} \sum_{k=0}^{\infty} \frac{(-1)^k N^k}{t^k\, k!} \int_{H_{n_1}} \cdots \int_{H_{n_d}} \prod_{i=1}^{d} dM_i \left(\sum_i \mathrm{Tr}\, \delta V_i(M_i) \right)^k \\
&\quad\ \ \mathrm{e}^{-\frac{N}{2t}\sum_i V''(x_i)\,\mathrm{Tr}\left(M_i - \xi_i \mathbf{1}_{n_i} \right)^2} \prod_{i>j} \det \left(M_i \otimes \mathbf{1}_{n_j} - \mathbf{1}_{n_i} \otimes M_j \right)^2 \\
&= \quad 1 + \sum_{k=1}^{\infty} t^k A_k
\end{aligned}
$$

$$(5-4)$$

where each A_k is a (well defined) polynomial moment of a Gaussian integral.
   It is denoted by:

$$
Z_{1\mathrm{MM}} = \int_{\mathrm{formal}} \mathrm{e}^{-\frac{N}{t}\,\mathrm{Tr}\, V(M)}\, dM. \tag{5-5}
$$

This last notation comes from the exchange of the Taylor expansion of $\exp -\frac{N}{t}\sum_i \mathrm{Tr}\, \delta V_i(M_i)$ and the integral. Once this commutation is performed, the integral obtained corresponds to the formal expansion of the integral $\int \mathrm{e}^{-\frac{N}{t}\,\mathrm{Tr}\, V(M)}\, dM$ around a saddle point $\tilde{M}$ solution of $V'(\tilde{M}) = 0$, which we choose as:

$$
\tilde{M} = \mathrm{diag}\left(\overbrace{\xi_1, \ldots, \xi_1}^{n_1}, \ldots, \overbrace{\xi_i, \ldots, \xi_i}^{n_i}, \ldots, \overbrace{\xi_d, \ldots, \xi_d}^{n_d} \right). \tag{5-6}
$$

The integers n_i thus correspond to choosing the number of eigenvalues of the saddle matrix located at a particular solution of the saddle-point equation $V'(\xi_i) = 0$.

However, in general, the Taylor expansion and the integral do not commute, and the formal matrix integral is different from the usual convergent matrix integral:

$$
\int_{\mathrm{formal}} \mathrm{e}^{-\frac{N}{t}\,\mathrm{Tr}\, V(M)}\, dM \neq \int \mathrm{e}^{-\frac{N}{t}\,\mathrm{Tr}\, V(M)}\, dM. \tag{5-7}
$$

In fact, typically, convergent matrix integrals are obtained for $V > 0$, whereas formal matrix integrals have combinatorical interpretations for $V < 0$.

Remark 5.1 The definition of the matrix integral does not depend only on the potential (i.e. the coefficients of this polynomial) but also on the filling fractions

$$
\epsilon_i := \frac{tn_i}{N} \quad , \quad i = 1, \ldots, d \quad , \quad \sum_{i=1}^{d} \epsilon_i = t. \tag{5-8}
$$

The formal logarithm of Z_{1MM} is also a formal power series in t of the form:

$$\ln Z_{1MM} = \sum_{k=0}^{\infty} t^k \, \widetilde{A}_k \tag{5-9}$$

and it can be seen from general properties of polynomial moments of Gaussian matrix integrals (observation first made by 't Hooft [110]), that each coefficient $N^{-2}\widetilde{A}_k$ is a **polynomial** in $1/N^2$:

$$\widetilde{A}_k = N^2 \sum_{g=0}^{g_{\max}(k)} \widetilde{A}_k^{(g)} \, N^{-2g}. \tag{5-10}$$

Therefore, we may collect together the coefficients of given powers of N, and define a formal power series:

$$F_g = t^{2-2g} \sum_{k=1}^{\infty} \tilde{A}_k^{(g)} \, t^k. \tag{5-11}$$

We have:

Theorem 5.1

$$\ln Z_{1MM} = \sum_{g=0}^{\infty} (N/t)^{2-2g} \, F_g. \tag{5-12}$$

This theorem is an equality between formal power series of t. This means the coefficients in the small t power series expansions of both sides are the same. And for a given power of t, the sum in the RHS is in fact a finite sum.

5.1.1   Loop equations

We may also define the following formal correlation functions:

$$W_n(x_1, \ldots, x_n) = \left\langle \mathrm{Tr} \, \frac{1}{x_1 - M} \ldots \mathrm{Tr} \, \frac{1}{x_n - M} \right\rangle_c \tag{5-13}$$

where the subscript c means the cumulant and the notation $\mathrm{Tr} \, \frac{1}{x-M}$ stands for the formal series:

$$\mathrm{Tr} \, \frac{1}{x - M} \equiv \sum_{i=1}^{d} \sum_{k=0}^{\infty} \, \mathrm{Tr} \, \frac{(M_i - \xi_i \mathbf{1}_{n_i})^k}{(x - \xi_i)^{k+1}} \tag{5-14}$$

to be inserted in the integrand of eq.(5-4). Again, $W_n(x_1, \ldots, x_n)$ is defined as a formal power series in t, whose coefficients are polynomial moments of Gaussian integrals. Moreover, one may notice that the coefficient of t^k is a polynomial in $1/N$, and is a rational fraction of $x_1, \ldots, x_n$ with poles at the $\xi_i's$.

Remark 5.2 Each coefficient of $W_n(x_1, \ldots, x_n)$ is a rational fraction of the x_j's with poles at the ξ_i's, and from eq.(5-14), one sees that simple poles can appear only when $k = 0$ in

eq.(5-14), i.e. terms independent of M_i. This implies that the cumulants $W_n(x_1, \ldots, x_n)$ can have no simple poles when $n > 1$, and for W_1, the only residue is:

$$\operatorname*{Res}_{x \to \xi_i} W_1(x)dx = n_i \tag{5-15}$$

and when $n > 1$:

$$\operatorname*{Res}_{x_1 \to \xi_i} W_n(x_1, x_2, \ldots x_n)dx_1 = 0. \tag{5-16}$$

Again those two equalities are equalities between the coefficients of formal series of t.

We may collect together coefficients with the same power of N. That allows to write:

$$W_n(x_1, \ldots, x_n) = \sum_{g \geq 0} \left(\frac{N}{t}\right)^{2-2g-n} W_n^{(g)}(x_1, \ldots, x_n) \tag{5-17}$$

where each $W_n^{(g)}$ is a formal power series in t, whose coefficients are rational fractions of $x_1, \ldots, x_n$ with poles at the $\xi_i's$. Eq.(5-17) is an equality of formal power series of t.

For further convenience we also define in a similar manner:

$$P_n(x_1; x_2, \ldots, x_n) = \left\langle \operatorname{Tr} \frac{V'(x_1) - V'(M)}{x_1 - M} \operatorname{Tr} \frac{1}{x_2 - M} \ldots \operatorname{Tr} \frac{1}{x_n - M} \right\rangle_c \tag{5-18}$$

which is a polynomial in the variable x_1. Again, we may collect together coefficients with the same power of N^{-1}, and define $P_n^{(g)}$ such that:

$$P_n(x_1; x_2, \ldots, x_n) = \sum_{g \geq 0} \left(\frac{N}{t}\right)^{2-2g-n} P_n^{(g)}(x_1; x_2, \ldots, x_n). \tag{5-19}$$

Then, the Schwinger-Dyson equations imply:

Theorem 5.2 *We have the loop equations, $\forall n, g$:*

$$
\begin{aligned}
W_{n+2}^{(g-1)}(x, x, J) &+ \sum_{h=0}^{g} \sum_{I \subset J} W_{1+|I|}^{(h)}(x, I) W_{1+n-|I|}^{(g-h)}(x, J\backslash I) \\
&+ \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{W_n^{(g)}(x, J\backslash\{x_j\}) - W_n^{(g)}(J)}{x - x_j} \\
&= V'(x) W_{n+1}^{(g)}(x, J) - P_{n+1}^{(g)}(x; J)
\end{aligned}
\tag{5-20}
$$

where $J = \{x_1, \ldots, x_n\}$.

proof:

This theorem can be proved by integrating by parts the gaussian integrals for each power of t, see [47]. It is called Schwinger-Dyson equations, or loop equations, or sometimes Ward identities... cf [96, 80, 36, 38]. □

Remark also that this theorem corresponds to the global Virasoro constraints of theorem 4.16.

Loop equations were initially used to find the topoloical expansion of the 1-matrix model in the special case of 1-cut [11] and also to the first orders 2-cuts [5].

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5.1.2 Spectral curve

For $n = 0$ and $g = 0$ the loop equation eq.(5-20) reduces to an algebraic equation for $W_1^{(0)}(x)$ sometimes known as the master loop equation:

$$(W_1^{(0)}(x))^2 = V'(x) W_1^{(0)}(x) - P_1^{(0)}(x) \tag{5-21}$$

where $P_1^{(0)}(x)$ is a polynomial of x of degree at most $d - 1$. The one point function is thus given by

$$W_1^{(0)}(x) = \frac{V'(x)}{2} - \sqrt{\frac{V'(x)^2}{4} - P_1^{(0)}(x)}. \tag{5-22}$$

We define:

$$y = \frac{V'(x)}{2} - W_1^{(0)}(x) = \sqrt{\frac{V'(x)^2}{4} - P_1^{(0)}(x)}, \tag{5-23}$$

and the master loop equation implies that the function y is solution of the algebraic equation (hyperelliptical) $H_{1MM}(x, y) = 0$ where

$$H_{1MM}(x, y) := y^2 - \left(\frac{V'(x)}{2}\right)^2 + P_1^{(0)}(x) \tag{5-24}$$

which is called the spectral curve associated to the one hermitian matrix model.

From eq.(5-15), we see that, if one chooses the $\mathcal{A}_i$-cycle to be a circle, independent of t, around ξ_i, then we have (order by order in t):

$$\frac{1}{2i\pi} \oint_{\mathcal{A}_i} W_1^{(0)}(x)dx = \frac{t\,n_i}{N} = \epsilon_i \quad , \quad i = 1, \ldots, d. \tag{5-25}$$

This last equation gives d constraints, i.e. the same number as the coefficients of the polynomial $P_1^{(0)}$ (which is of degree $d - 1$), therefore it determines $P_1^{(0)}(x)$. Just by looking at the first terms in the small t expansion, one has:

$$P_1^{(0)}(x) = \sum_{i=1}^{d} \epsilon_i \frac{V'(x)}{x - \xi_i} + O(t^2). \tag{5-26}$$

The data n_i/N are thus equivalent to the data of $P_1^{(0)}$. Notice that $\epsilon_i = tn_i/N$ is of order $O(t)$ in the small t expansion.

In case some ϵ_i's are vanishing, we define $\bar{g} + 1 =$ the number of non-vanishing ϵ_i's, and we assume that $\epsilon_1, \ldots, \epsilon_{\bar{g}+1}$ are non-vanishing and $\epsilon_{\bar{g}+2}, \ldots, \epsilon_d$ are zero.

Order by order in t we have:

$$y = \prod_{i=\bar{g}+2}^{d} (x - \xi_i - A_i(t)) \sqrt{\prod_{i=1}^{\bar{g}+1}((x - \xi_i - B_i(t))^2 - 4C_i(t))} \tag{5-27}$$

where $A_i(t), B_i(t), C_i(t)$ are formal power series of t. To the first orders:

$$A_i = \frac{4}{V''(\xi_i)} \sum_{j=1}^{\bar{g}+1} \frac{\epsilon_j}{\xi_i - \xi_j} + O(t^2), \tag{5-28}$$

$$B_i = \frac{1}{2V''(\xi_i)} \sum_{j \neq i} \frac{\epsilon_j}{\xi_i - \xi_j} \quad - \frac{\epsilon_i}{4} \frac{V'''(\xi_i)}{V''^2(\xi_i)} + O(t^2), \tag{5-29}$$

$$C_i = \frac{\epsilon_i}{V''(\xi_i)} + O(t^2). \tag{5-30}$$

Let us study the specificities of the 1-matrix model spectral curve $(\mathcal{L}_{1MM}, x, y)$.

Genus of $\mathcal{L}_{1MM}$

The Riemann surface $\mathcal{L}_{1MM}$ has genus $\bar{g}$ lower than $d - 1$:

$$\bar{g} \leq d - 1. \tag{5-31}$$

Sheeted structure

The polynomial $H_{1MM}(x, y)$ has degree 2 in y. This means that the embedding of $\mathcal{L}_{1MM}$ is composed by 2 copies of the Riemann sphere, called sheets, glued by $\bar{g} + 1$ cuts so that the resulting Riemann surface $\mathcal{L}_{1MM}$ has genus $\bar{g}$. Each copy of the Riemann sphere corresponds to one particular branch of the solutions of the equation $H_{1MM}(x, y) = 0$. Since there are only two sheets in involution, this spectral curve is said to be hyperelliptic. It also means that the application $z \to \bar{z}$ is globally defined since it is the map which exchanges both sheets:

$$y(\bar{z}) = -y(z). \tag{5-32}$$

Pole structure

The function $x(z)$ on the Riemann surface $\mathcal{L}_{1MM}$ has two simple poles (call them α_+ and α_-), one in each sheet. Near $\alpha_\pm$, $y(z)$ behaves like:

$$y(z) \underset{z \to \alpha_\pm}{\sim} \pm \frac{1}{2} V'(x(z)) \mp \frac{t}{x(z)} + O(1/x(z)^2). \tag{5-33}$$

5.1.3 The 2-point function

For $n = 1$ and $g = 0$, the loop equation eq.(5-20) reads:

$$\frac{\partial}{\partial x_1} \frac{W_1^{(0)}(x) - W_1^{(0)}(x_1)}{x - x_1} = (V'(x) - 2W_1^{(0)}(x))W_2^{(0)}(x, x_1) - P_2^{(0)}(x; x_1) \tag{5-34}$$

i.e.:

$$W_2^{(0)}(x, x_1) = \frac{\frac{\partial}{\partial x_1} \frac{W_1^{(0)}(x) - W_1^{(0)}(x_1)}{x - x_1} + P_2^{(0)}(x; x_1)}{2y(x)}$$

$$= -\frac{1}{2(x - x_1)^2} + \frac{\frac{1}{2} \frac{\partial}{\partial x_1} \frac{V'(x) - V'(x_1) + 2y(x_1)}{x - x_1} + P_2^{(0)}(x; x_1)}{2y(x)}.$$

$$(5 - 35)$$

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This equation shows that $W_2^{(0)}(x, x_1)$ is a meromorphic function of z and z_1 on the spectral curve $\mathcal{L}_{1MM}$. It is a multivalued function of x, x_1, but it is a monovalued function in the variables z, z_1. Therefore let us write:

$$W_2^{(0)}(x(z), x(z_1)) \, dx(z)dx(z_1) = \bar{\omega}_2^{(0)}(z, z_1) \tag{5-36}$$

where $\bar{\omega}_2^{(0)}(z, z_1)$ is a meromorphic form of z and z_1. It is clear from the first line of eq.(5-35), that $\bar{\omega}_2^{(0)}(z, z_1)$ has no pole at $z = z_1$, but it has a pole at $z = \bar{z}_1$. Moreover, since $dx(z)/y(z)$ has no pole at branchpoints, we see that $\bar{\omega}_2^{(0)}(z, z_1)$ has no pole when z approaches a branchpoint. By looking at the behavior at large x, we see, from $\deg P_2^{(0)} \leq d - 2$, that $\bar{\omega}_2^{(0)}(z, z_1)$ has no pole at the two infinities $\alpha_\pm$.

From eq.(5-35), it may seem that $\bar{\omega}_2^{(0)}(z, z_1)$ could have simple poles at the zeroes of $y(z)$, but the residues are computed by eq.(5-16) and they vanish. Thus, we see that the only possible pole of $\bar{\omega}_2^{(0)}(z, z_1)$ can be at $z = \bar{z}_1$.

Then, notice that the second line of eq.(5-35) is the sum of a term which is even under $z \to \bar{z}$, and a term which is odd under $z \to \bar{z}$. Since the sum of those two terms must have no pole at $z = z_1$, we see that the pole at $z = \bar{z}_1$ must be twice the pole of the even part. Therefore we find that $\bar{\omega}_2^{(0)}(z, z_1)$ has a double pole at $z = \bar{z}_1$, with no residue, and no other pole. Moreover, eq.(5-16) implies that on every $\mathcal{A}$-cycle we have:

$$\oint_{z \in \mathcal{A}_i} \bar{\omega}_2^{(0)}(z, z_1) = 0. \tag{5-37}$$

The only meromorphic differential having all those properties is the Bergmann kernel:

$$\bar{\omega}_2^{(0)}(z, z_1) = -B(z, \bar{z}_1) = B(z, z_1) - \frac{dx(z) \, dx(z_1)}{(x(z) - x(z_1))^2}. \tag{5-38}$$

(we choose $\kappa = 0$).

5.1.4 Higher correlators

Similarly to what we just did with $W_2^{(0)}$, we are going to compute every $W_n^{(g)}$ and relate it to the symplectic invariants of the curve $y(x)$.

First, notice that the loop equations eq.(5-20) imply recursively, that each $W_n^{(g)}$ is in fact a meromorphic function on the spectral curve, and thus we prefer to rewrite:

Definition 5.2

$$
\begin{aligned}
\omega_n^{(g)}(z_1, \ldots, z_n) &= W_n^{(g)}(x(z_1), \ldots, x(z_n)) \, dx(z_1) \ldots dx(z_n) \\
&+ \delta_{n,2}\delta_{g,0} \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2}.
\end{aligned} \tag{5-39}
$$

Indeed, correlation functions $W_n^{(g)}$ are multivalued functions of the complex variable x whereas the $\omega_n^{(g)}$ are monovalued n-forms on the spectral curve $\mathcal{L}$. Somehow, these forms are built to choose one particular branch of solution of the master loop equation eq.(5-21).

Structure of the form $\omega_n^{(g)}$:

One clearly sees from loop equation eq.(5-20), that $\omega_{n+1}^{(g)}(z, z_1, \ldots, z_n)$ can possibly have poles only at branchpoints, or at coinciding points $x(z) = x(z_j)$, i.e. at $z = z_j$ or $z = \bar{z}_j$, or also at the zeroes of $y(z)$.

From the degrees $\deg P_n^{(g)} \leq d-2$, one can see that there is no poles at the infinities $\alpha_{\pm}$. Also, there is manifestly no pole at $z = z_j$, and one may notice that $\omega_{n+1}^{(g)}$ is an odd function when $z \to \bar{z}$, thus it can also have no pole when $z = \bar{z}_j$.

The zeroes of $y(z)$ are either the branchpoints, or double points, which are, order by order in t, of the form (see eq.(5-27)):

$$\xi_i + A_i(t) + O(t^2) \quad , \quad A_i(t) = O(t). \tag{5-40}$$

Let us assume by recursion on $2g + n$, that each $\omega_n^{(g)}$ has no poles at those double points. This is true for $\omega_1^{(0)}$ and $\omega_2^{(0)}$. Assume that it is true for every $\omega_{n'}^{(g')}$ with $2g' + n' \leq 2g + n$, let us prove it for $2g + n + 1$. From the loop equation eq.(5-20), one sees that $\omega_{n+1}^{(g)}(z, z_1, \ldots, z_n)$ could have at most a simple pole at such double points. But because of eq.(5-16) and eq.(5-15), the residue must vanish, and thus there is no pole.

Therefore we obtain the following structure for the $\omega_n^{(g)}$'s:

Lemma 5.1 $\omega_n^{(g)}$ *can have poles only at branchpoints when $2g + n \geq 3$.*

Moreover, we see from eq.(5-16) and eq.(5-15), that if $2g + n \geq 3$ we have:

$$\oint_{x_1 \in \mathcal{A}_i} \omega_n^{(g)}(x_1, \ldots, x_n) = 0. \tag{5-41}$$

5.1.5 Symplectic invariants

Let:

$$dS_{z_1, z_2}(z) = \int_{z_2}^{z_1} B(z, z') \tag{5-42}$$

be the 3rd kind differential in z, having a simple pole at $z = z_1$ with residue $+1$, and a simple pole of residue -1 at $z = z_2$, and no other poles, and normalized on $\mathcal{A}$-cycles:

$$\oint_{\mathcal{A}_i} dS_{z_1, z_2} = 0. \tag{5-43}$$

The fact that it has only simple poles with residue $+1$ at z_1 allows to write the Cauchy formula on the spectral curve (o being an arbitrary base point on $\mathcal{L}_{1MM}$):

$$\omega_{n+1}^{(g)}(z, z_1, \ldots, z_n) = \operatorname*{Res}_{z' \to z} dS_{z', o}(z) \, \omega_{n+1}^{(g)}(z', z_1, \ldots, z_n). \tag{5-44}$$

On the other hand, the differential form $\omega_{n+1}^{(g)}(z, z_1, \ldots, z_n)$ has poles only at the branch points $z \to a_i$, and thus the Riemann bilinear identity tells us that:

$$\operatorname*{Res}_{z' \to z} dS_{z', o}(z) \, \omega_{n+1}^{(g)}(z', z_1, \ldots, z_n) + \sum_i \operatorname*{Res}_{z' \to a_i} dS_{z', o}(z) \, \omega_{n+1}^{(g)}(z', z_1, \ldots, z_n)$$

$$
= \sum_i \oint_{z' \in \mathcal{A}_i} B(z, z') \oint_{z' \in \mathcal{B}_i} \omega_{n+1}^{(g)}(z', z_1, \ldots, z_n)
$$

$$
- \sum_i \oint_{z' \in \mathcal{B}_i} B(z, z') \oint_{z' \in \mathcal{A}_i} \omega_{n+1}^{(g)}(z', z_1, \ldots, z_n).
$$

$(5-45)$

Due to eq.(5-43) and eq.(5-41), the right hand side vanishes and thus:

$$
\operatorname*{Res}_{z' \to z} dS_{z',o}(z) \, \omega_{n+1}^{(g)}(z', z_1, \ldots, z_n) = - \sum_i \operatorname*{Res}_{z' \to a_i} dS_{z',o}(z) \, \omega_{n+1}^{(g)}(z', z_1, \ldots, z_n). \quad (5\text{-}46)
$$

Finally, one can plug in the loop equation eq.(5-20) and remind that the polynomial $P_1^{(g)}$ has no pole at the branch points, and thus we find:

$$
\begin{aligned}
& \omega_{n+1}^{(g)}(z, z_1, \ldots, z_n) \\
= & \sum_i \operatorname*{Res}_{z' \to a_i} K(z, z') \Big[\omega_{n+2}^{(g-1)}(z', \bar{z}', z1, \ldots, z_n) \\
& + \sum_{h=0}^{g} \sideset{}{'}\sum_{I \subset \{z_1, \ldots, z_n\}} \omega_{1+|I|}^{(h)}(z', I) \omega_{1+n-|I|}^{(g-h)}(\bar{z}', \{z_1, \ldots, z_n\} \setminus I) \Big].
\end{aligned}
$$

$(5-47)$

where $K(z, z')$ is the kernel:

$$
K(z, z') = \frac{dS_{\bar{z}', z'}(z)}{2(y(z') - y(\bar{z}')) \, dx(z')}. \tag{5-48}
$$

In other words:

Theorem 5.3 $\omega_n^{(g)}(z_1, \ldots, z_n)$ *and* $F^{(g)}$ *are the correlators and symplectic invariants of the spectral curve* $\mathcal{E}_{1\mathrm{MM}}$ *of equation* $H_{1\mathrm{MM}}(x, y) = 0$.

To get this theorem, we also recover the F_g's from theorem 4.3, or just by homogeneity, see [32] for details. For the 1-matrix model, F_0 has been known from the origin of random matrices, F_1 was first found in [11] for the 1-cut case, and in [31] for the multicut case. The other F_g's were first found in [32].

5.2 2-matrix model

The method of loop equations can also be used to solve the formal 2-matrix model. One of the main applications and reasons for introducing the 2-matrix model, was the problem of counting Ising model configurations on random discrete surfaces, or in other words bi-colored maps (see section 7), it was first introduced and solved by V.Kazakov [81]. It corresponds to a formal 2-matrix integral. It can be rephrased in terms of symplectic invariants too.

For this purpose, one generalizes the notion of formal matrix integral to integrals over two normal matrices.

Definition 5.3 *Let N be an integer and V_1 and V_2 two polynomial potentials:*

$$V_1(x) = -\sum_{k=2}^{d_1+1} \frac{t_k}{k} x^k \qquad , \qquad V_2(y) = -\sum_{k=2}^{d_2+1} \frac{\tilde{t}_k}{k} y^k. \tag{5-49}$$

Let $d = d_1 d_2$, and let $\vec{n}$ be a d-partition of N:

$$\vec{n} := \{n_1, n_2, \ldots, n_d\} \qquad \text{such that} \qquad \sum_{i=1}^{d} n_i = N. \tag{5-50}$$

Let $\{(\xi_i, \eta_i)\}_{i=1}^{d_1 d_2}$ be the $d = d_1 d_2$ solutions of the system of equations

$$\begin{cases} V_1'(\xi_i) = \eta_i \\ V_2'(\eta_i) = \xi_i \end{cases}. \tag{5-51}$$

One defines the non-quadratic part of the Taylor expansions of the potentials around these saddle points

$$\delta V_{1,i}(x) = V_1(x) - V_1(\xi_i) - \frac{V_1''(\xi_i)}{2}(x - \xi_i)^2 \tag{5-52}$$

and

$$\delta V_{2,i}(y) = V_2(y) - V_2(\eta_i) - \frac{V_2''(\eta_i)}{2}(y - \eta_i)^2. \tag{5-53}$$

For all l, one defines the polynomial in t:

$$\begin{aligned}
\sum_{k=l/2}^{ld} A_{k,l} t^k &= \\
&= \frac{(-1)^l N^l}{l!\, t^l} \int dM_1 \ldots dM_d d\tilde{M}_1 \ldots d\tilde{M}_d (\sum_i \text{Tr } \delta V_{1,i}(M_i) + \delta V_{2,i}(\tilde{M}_i))^l \\
&\quad \prod_{i=1}^{d} e^{-\frac{N}{t}\left(\text{Tr } \frac{V_1''(\xi_i)}{2}(M_i - \xi_i \mathbf{1}_{n_i})^2 + \frac{V_2''(\eta_i)}{2}(\tilde{M}_i - \eta_i \mathbf{1}_{n_i})^2 - (M_i - \xi_i \mathbf{1}_{n_i})(\tilde{M}_i - \eta_i \mathbf{1}_{n_i}) \right)} \\
&\quad \prod_{i>j} \det(M_i \otimes \mathbf{1}_{n_j} - \mathbf{1}_{n_i} \otimes M_j) \prod_{i>j} \det(\tilde{M}_i \otimes \mathbf{1}_{n_j} - \mathbf{1}_{n_i} \otimes \tilde{M}_j)
\end{aligned} \tag{5-54}$$

as a gaussian integral over hermitian matrices M_i and $\widetilde{M_i}$ of size $n_i \times n_i$.

The formal 2-matrix model partition function is then defined as a formal power series in t (cf [53, 54]):

$$Z_{2\text{MM}} := \sum_{k=0}^{\infty} t^k (\sum_{j=0}^{2k} A_{k,j}). \tag{5-55}$$

As in the 1-matrix model, one uses the notation

$$Z_{2\text{MM}} = \int_{\text{formal}} e^{-\frac{N}{t} \text{Tr } V_1(M_1) + V_2(M_2) - M_1 M_2} \, dM_1 \, dM_2. \tag{5-56}$$

One is also interested in the formal logarithm of the partition function: the free energy

$$F_{2MM} := \ln Z_{2MM} \tag{5-57}$$

which has a topological expansion (due again to 't Hooft's observation [110])

$$F_{2MM} = \sum_{g \geq 0} \left(\frac{N}{t}\right)^{2-2g} F_g, \tag{5-58}$$

where each F_g is a formal power series of t.

5.2.1 Loop equations and spectral curve

As in the 1-matrix model (cf eq.(5-14)), one also defines correlation functions by

$$W_{k,l}(x_1, \ldots, x_k, y_1, \ldots, y_l) := \left\langle \prod_{i=1}^{k} \mathrm{Tr} \, \frac{1}{x_i - M_1} \prod_{i=1}^{l} \mathrm{Tr} \, \frac{1}{y_i - M_2} \right\rangle_c \tag{5-59}$$

denoted as the non-mixed correlation functions[10]. These correlation functions also admit a topological expansion

$$W_{k,l}(x_1, \ldots, x_k, y_1, \ldots, y_l) = \sum_{g \geq 0} \left(\frac{N}{t}\right)^{2-2g-k} W_{k,l}^{(g)}(x_1, \ldots, x_k, y_1, \ldots, y_l). \tag{5-60}$$

One also needs the polynomials in x_1 and y

$$P_n(x_1, y; x_2, \ldots, x_n) = \left\langle \mathrm{Tr} \left(\frac{V_1'(x_1) - V_1'(M_1)}{x_1 - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right) \prod_{i=2}^{n} \mathrm{Tr} \, \frac{1}{x_i - M_1} \right\rangle_c \tag{5-61}$$

as well as

$$U_n(x_1, y; x_2, \ldots, x_n) = \left\langle \mathrm{Tr} \left(\frac{1}{x_1 - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \right) \prod_{i=2}^{n} \mathrm{Tr} \, \frac{1}{x_i - M_1} \right\rangle_c \tag{5-62}$$

which are polynomials in y only.

5.2.2 Loop equations

Loop equations proceed from integration by parts in the formal matrix integral (i.e. integration by parts in each gaussian integral for each power of t), or by writing invariance under changes of variables, i.e. they are Schwinger-Dyson equations.

---

[10]There exists more general correlation functions mixing the two types of matrices M_1 and M_2 inside the same trace, for example $< \mathrm{Tr} \, (M_1^k M_2^l) >$, but their study is too far from the main topic of this review to be treated here. It is studied in [58, 59].

The loop equations for the 2-matrix model were first studied by M. Staudacher [108], and then written in a more concise form in [48, 51]. The loop equations for the 2-matrix model are (where $J = \{x_1, \ldots, x_n\}$):

$$
\frac{N}{t}(y - V_1'(x))\,U_{n+1}(x, y; J) + U_{n+2}(x, y; x, J)
$$
$$
+ \sum_{I \subset J} W_{1+|I|,0}(x, I)\,U_{1+n-|I|}(x, y; J/I)
$$
$$
+ \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{U_n(x, y; J/\{x_j\}) - U_n(x_j, y; J/\{x_j\})}{x - x_j}
$$
$$
= -\frac{N}{t}P_{n+1}(x, y; J) + \frac{N^2}{t^2}. \tag{5-63}
$$

And then, identifying the coefficients of polynomials of $1/N^2$ for each power of t, we have:

$$
(y - V_1'(x) + W_{1,0}^{(0)}(x))\,U_1^{(0)}(x, y) = (V_2'(y) - x)\,W_{1,0}^{(0)}(x) - P_1^{(0)}(x, y) + 1 \tag{5-64}
$$

and for $(g, n) \neq (0, 0)$:

$$
(y - V_1'(x) + W_{1,0}^{(0)}(x))\,U_{n+1}^{(g)}(x, y; J) + W_{n+1,0}^{(g)}(x, J)\,U_1^{(0)}(x, y)
$$
$$
+ U_{n+2}^{(g-1)}(x, y; x, J) + \sum_{h=0}^{g} \sum_{I \subset J}{}' W_{1+|I|,0}^{(h)}(x, I)\,U_{1+n-|I|}^{(g-h)}(x, y; J/I)
$$
$$
+ \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{U_n^{(g)}(x, y; J/\{x_j\}) - U_n^{(g)}(x_j, y; J/\{x_j\})}{x - x_j}
$$
$$
= -P_{n+1}^{(g)}(x, y; J). \tag{5-65}
$$

where $\sum_h \sum_I'$ means that we exclude the terms $(h, I) = (0, \emptyset)$ and (g, J).

5.2.3 Spectral curve

Consider the first loop equation:

$$
(y - V_1'(x) + W_{1,0}^{(0)}(x))\,U_1^{(0)}(x, y) = (V_2'(y) - x)\,W_{1,0}^{(0)}(x) - P_1^{(0)}(x, y) + 1. \tag{5-66}
$$

It is valid for any x and y, and in particular we may choose:

$$
y = y(x) = V_1'(x) - W_{1,0}^{(0)}(x). \tag{5-67}
$$

Since $U_1^{(0)}(x, y)$ is a polynomial in y, it cannot have a pole at this value of $y = y(x)$, and thus, for this value of $y = V_1'(x) - W_{1,0}^{(0)}(x)$, we have the algebraic equation:

$$
H_{2MM}(x, y(x)) = (V_2'(y(x)) - x)\,(V_1'(x) - y(x)) - P_1^{(0)}(x, y(x)) + 1 = 0 \tag{5-68}
$$

known as the 2-matrix model spectral curve.

Let us study the specificities of the 2-matrix model spectral curve $(\mathcal{L}_{2MM}, x, y)$ (see [82]).

Genus of $\mathcal{L}_{2MM}$

The Riemann surface $\mathcal{L}_{2MM}$ has genus $\bar{g}$ lower than $d_1 d_2 - 1$:

$$\bar{g} \leq d_1 d_2 - 1. \tag{5-69}$$

Sheeted structure

The polynomial $H_{2MM}(x, y)$ has degree $d_2 + 1$ (resp. $d_1 + 1$) in y (resp. x). This means that the embedding of $\mathcal{L}_{2MM}$ as a branch-covering of the $\mathbb{P}^1$ x-plane (resp. y-plane) is composed by $d_2 + 1$ (resp. $d_1 + 1$) copies of the Riemann sphere $\mathbb{P}^1$, called x-sheets (resp. y-sheets), glued by cuts, so that the resulting Riemann surface $\mathcal{L}_{2MM}$ has genus $\bar{g}$. Each copy of the Riemann sphere corresponds to one particular branch of the solutions of the equation $H_{2MM}(x, y)$ in y (resp. x).

Since there are $d_2 + 1$ x-sheets, this means that there are $d_2 + 1$ points in $\mathcal{L}_{2MM}$ corresponding to the same value of x. We write:

$$x(z^i) = x(z) , \quad i = 0, \dots, d_2 \tag{5-70}$$

We will take the convention that $z^0 = z$.

Branchpoints are zeroes of dx, and they are also places where 2 sheets merge $z \to z^i$ for some i. By convention, we call this other point z^i, and thus $\bar{z}$ is one of the z^i's. In general, $\bar{z}$ is not globally defined but only defined locally around the branch points.

Pole structure of the functions $x(z)$ and $y(z)$

The function $x(z)$ (resp. $y(z)$) on the Riemann surface $\mathcal{L}_{2MM}$ has two poles: one of degree 1 (resp. degree d_1) at a pole called ∞_x and one of degree d_2 (resp. degree 1) at a pole called ∞_y. It means that d_2 x-sheets merge at ∞_y, and only one x-sheet contains ∞_x alone.

Near ∞_x, a local parameter is $1/x$, and we have:

$$y(z) \underset{z \to \infty_x}{\sim} V_1'(x(z)) - \frac{t}{x(z)} + O(1/x(z)^2). \tag{5-71}$$

And near ∞_y, a local parameter is $1/y$, and we have:

$$x(z) \underset{z \to \infty_y}{\sim} V_2'(y(z)) - \frac{t}{y(z)} + O(1/y(z)^2). \tag{5-72}$$

According to section 4.7, the fact that we have two poles, means that the tau-function built from the symplectic invariants of this curve is the tau-function of the $1 + 1$-KP hierarchy[11].

---

[11] Actually, one should take d_1 and d_2 arbitrary large to obtain the 1+1-KP tau-function. The times of the hierarchy being given by the coefficients of the Laurent expansions of ydx and xdy around ∞_x and ∞_y respectively.

5.2.4 Preliminaries to the solution of loop equations

As in the preceding section, we promote the correlation functions to differential forms on the spectral curve to make them monovalued:

Definition 5.4

$$
\begin{aligned}
\omega_n^{(g)}(z_1,\ldots,z_n) &= W_{n,0}^{(g)}(x(z_1),\ldots,x(z_n))\,dx(z_1)\ldots dx(z_n) \\
&\quad + \delta_{n,2}\delta_{g,0}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}.
\end{aligned}
\tag{5-73}
$$

Exactly like in the 1-matrix formal model, the very definition of the model as a formal series in t, implies that each $\omega_n^{(g)}$ with $2g+n \geq 3$ is a meromorphic n-form with poles only at the branchpoints (i.e. the zeroes of dx), and with vanishing $\mathcal{A}$-cycle integrals. The only exception is $\omega_2^{(0)}(z_1,z_2)$ which can have a pole only at $z_1 = z_2$, and which is found to be the Bergmann kernel (see [34]):

$$
\omega_2^{(0)}(z_1,z_2) = B(z_1,z_2).
\tag{5-74}
$$

Before proving that the solution of loop equations for the $\omega_n^{(g)}$'s are the symplectic invariant's correlators, we need a small lemma. Consider the "full spectral curve":

$$
E(x,y) = (V_1'(x)-y)(V_2'(y)-x) - \frac{t}{N}P_1(x,y) + 1
\tag{5-75}
$$

where $P_1(x,y) = \sum_g (N/t)^{1-2g}P_1^{(g)}(x,y)$. We also consider its descendants:

$$
E_{n+1}(x,y;z_1,\ldots,z_n) = \delta_{n,0}(V_1'(x)-y)(V_2'(y)-x) - \frac{t}{N}P_{n+1}(x,y;z_1,\ldots,z_n) + \delta_{n,0}.
\tag{5-76}
$$

We have:

Lemma 5.2

$$
\begin{aligned}
&E_{n+1}(x(z),y;z_1,\ldots,z_n) \\
&= -\tilde{t}_{d_2+1}\,{}''\!\left\langle \prod_{i=0}^{d_2}\left(y-V_1'(x(z^i))+\frac{t}{N}\operatorname{Tr}\frac{1}{x(z^i)-M_1}\right)\right. \\
&\qquad\left. \prod_{j=1}^{n}\operatorname{Tr}\frac{1}{x(z_j)-M_1}\right\rangle_{c,\{x_1,\ldots,x_n\}}{}''\,, \\
&(5-77)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{t}{N}U_{n+1}(x(z),y) - \delta_{n,0}(V_2'(y)-x(z)) \\
&= -\tilde{t}_{d_2+1}\,{}''\!\left\langle \prod_{i=1}^{d_2}\left(y-V_1'(x(z^i))+\frac{t}{N}\operatorname{Tr}\frac{1}{x(z^i)-M_1}\right)\right.
\end{aligned}
$$

$$\prod_{j=1}^{n} \text{Tr} \frac{1}{x(z_j) - M_1} \Bigg\rangle_{c,\{x_1,\dots,x_n\}} \text{''}.$$

$(5-78)$

where $\tilde{t}_{d_2+1}$ is the leading coefficient of $V_2'(y)$, z^i are the preimages of $x(z)$ (see eq.(5-70)), the subscript $_{c,\{x_1,\dots,x_n\}}$ means that we take the connected part with respect to the $\text{Tr} \frac{1}{x(z_j)-M_1}$ terms, but not the $\text{Tr} \frac{1}{x(z^i)-M_1}$ terms, and the inverted comas " $\langle . \rangle$ " mean that, every time one encounters a two-point function in the cumulant expansion, one replaces it by[12]

$$W_{2,0}(x, x') := \left\langle \text{Tr} \frac{1}{x - M_1} \text{Tr} \frac{1}{x' - M_1} \right\rangle + \frac{1}{(x - x')^2}. \qquad (5\text{-}79)$$

For example formula eq.(5-77), for $n = 1$, reads to the first subleading order in t/N:

$$
\begin{aligned}
P_1^{(1)}(x, y; z_1) \quad = \quad & \tilde{t}_{d_2+1} \sum_{j=0}^{d_2} W_{2,0}^{(1)}(z^j, z_1) \prod_{i \neq j, i=0}^{d_2} (y - y(z^j)) \\
& + \tilde{t}_{d_2+1} \sum_{j \neq k=0}^{d_2} W_{2,0}^{(0)}(z^j, z_1) W_{1,0}^{(1)}(z^k) \prod_{i \neq j, k, i=0}^{d_2} (y - y(z^j)) \\
& + \tilde{t}_{d_2+1} \sum_{j \neq k=0}^{d_2} W_{3,0}^{(0)}(z^j, z^k, z_1) \prod_{i \neq j, k, i=0}^{d_2} (y - y(z^j)) \\
& + \tilde{t}_{d_2+1} \sum_{j \neq k \neq l=0}^{d_2} W_{2,0}^{(0)}(z^j, z_1) W_{2,0}^{(0)}(z^k, z^l) \prod_{i \neq j, k, l, i=0}^{d_2} (y - y(z^j)).
\end{aligned}
$$

$(5-80)$

Formula eq.(5-78) would be almost the same, but with the indices $i, j, k, l \geq 1$ instead of ≥ 0.

This lemma was proved in [33] and relies on the fact that the loop equation eq.(5-65) has a unique solution admitting a topological expansion. The way to prove this lemma mostly follows from Lagrange interpolation formula for polynomials, as well as Cauchy residue formula on $\mathcal{L}_{2MM}$.

Since $E_{n+1}(x, y; z_1, \dots, z_n)$ is a polynomial of y of degree $\geq d_2 + 1$, by expanding $E_{n+1}(x, y; z_1, \dots, z_n)$ in powers of y, this lemma gives $d_2 + 1$ equations. In particular, the term in y^{d_2} gives (if $2g + n \geq 2$):

$$\sum_{i=0}^{d_2} \omega_{n+1,0}^{(g)}(z^i, z_1, \dots, z_n) = 0. \qquad (5\text{-}81)$$

And the term in y^{d_2-1} gives a bilinear equation in the correlation functions:

$$\sum_{i \neq j}^{d_2} \omega_{n+2,0}^{(g-1)}(z^i, z^j, J) + \sum_{h=0}^{g} \sideset{}{'}\sum_{I \subset J} \omega_{1+|I|,0}^{(h)}(z^i, I) \omega_{1+n-|I|,0}^{(g-h)}(z^j, J/I)$$

[12]Remark that this notation reminds the notation "det" in th.4.1.

$$= \sum_{i \neq j} y(z^i) \omega^{(g)}_{n+1,0}(z^j, J) + y(z^j) \omega^{(g)}_{n+1,0}(z^i, J) - f^{(g)}_n(x(z), J) \, dx(z)^2 \quad (5\text{-}82)$$

where $f^{(g)}_n(x(z), J)$ is a rational function of $x(z)$ with no pole when z approaches a branchpoint.

5.2.5 Solution of loop equations and symplectic invariants

Let us write Cauchy formula, exactly like for the 1-matrix model eq.(5-44):

$$\omega^{(g)}_{n+1}(z_0, J) \quad = \quad - \operatorname*{Res}_{z \to z_0} dS_{z,0}(z_0) \, \omega^{(g)}_{n+1}(z, J)$$
$$= \quad \sum_i \operatorname*{Res}_{z \to a_i} dS_{z,0}(z_0) \, \omega^{(g)}_{n+1}(z, J)$$
$$(5-83)$$

where we have moved the integration contours using Riemann bilinear identity like for the 1-matrix model eq.(5-45).

Now, notice that near a branchpoint a_i, $\omega^{(g)}_{n+1,0}(z, J)$ and $\omega^{(g)}_{n+1,0}(\bar{z}, J)$ have a pole at $z = a_i$ and all the $\omega^{(g)}_{n+1,0}(z^i, J)$ such that $z^i \neq z, \bar{z}$ have no pole at $z \to a_i$. According to eq.(5-81), we have:

$$\omega^{(g)}_{n+1,0}(z, J) + \omega^{(g)}_{n+1,0}(\bar{z}, J) = \text{regular} \quad (5\text{-}84)$$

and from eq.(5-82), we have near a_i:

$$(y(z) - y(\bar{z})) \omega^{(g)}_{n+1,0}(z, J) \quad = \quad \omega^{(g-1)}_{n+2,0}(z, \bar{z}, J) + \sum_{h=0}^{g} \sideset{}{'}\sum_{I \subset J} \omega^{(h)}_{1+|I|,0}(z, I) \omega^{(g-h)}_{1+n-|I|,0}(\bar{z}, J/I)$$
$$+ \text{regular}. \quad (5\text{-}85)$$

Inserting this last equation into the Cauchy formula eq.(5-83), we find:

$$\omega^{(g)}_{1+n}(z_0, J) = \sum_i \operatorname*{Res}_{z \to a_i} K(z_0, z) \left[\omega^{(g-1)}_{n+2,0}(z, \bar{z}, J) + \sum_{h=0}^{g} \sideset{}{'}\sum_{I \subset J} \omega^{(h)}_{1+|I|,0}(z, I) \omega^{(g-h)}_{1+n-|I|,0}(\bar{z}, J/I) \right]$$
$$(5-86)$$

where

$$K(z_0, z) = \frac{dS_{\bar{z},z}(z_0)}{2(y(z) - y(\bar{z})) \, dx(z)}. \quad (5\text{-}87)$$

This gives the theorem:

Theorem 5.4 $\omega^{(g)}_n(z_1, \ldots, z_n)$ and F_g are the correlators and symplectic invariants of the spectral curve $(\mathcal{L}_{2\mathrm{MM}}, x, y)$.

Here we have only briefly sketched the proof of [33], and we refer the reader to details there, in particular for finding the F_g's.

F_0 was found for example in [88, 20], F_1 was found in [48, 49, 50], and the other F_g's in [33].

Remark 5.3 The correlation functions $\omega_n^{(g)}$ which we consider here, are expectation values of traces of only the matrix M_1 (in terms of combinatorics of maps of section 7, they are generating functions for bicolored maps, whose boundaries are of color 1 only). There exists a generalization of symplectic invariants for all other possible expectation values, i.e. all possible boundary conditions, but this is largely outside of the scope of this review. We refer the reader to [59] for further details.

An obvious remark, is that color 1 and 2, i.e. functions x and y play similar roles. We can obtain generating functions for bicolored maps, whose boundaries are of color 2 only, by just exchanging the roles of x and y, i.e. by computing residues at the zeroes of dy.

In particular we may compute generating functions for bicolored maps with no boundaries (i.e. the F_g's), with either x or y. In other words the F_g's are unchanged if we exchange x and y. This is a special case of the symplectic invariance property $F_g(\mathcal{L}, x, y) = F_g(\mathcal{L}, y, x)$.

In fact, the general proof of symplectic invariance consists in defining some mixed generating functions for bicolored maps, whose boundaries are bicolored, it was done in [61].

5.3 Chain of matrices in an external field

Another matrix model which can be solved with the same technics is the chain of matrices matrix model.

Consider the model of an arbitrary long open chain of matrices in an external field, which includes the one and two matrix models as particular cases.

Consider m potentials $V_k(x) = -\sum_{j=2}^{d_k+1} \frac{t_{k,j}}{j} x^j$, $k = 1, \ldots, m$. The formal chain of matrices matrix integral is:

$$
Z_{\text{chain}} = \int_{\text{formal}} e^{-\frac{N}{t} \text{Tr} \left(\sum_{k=1}^{m} V_k(M_k) - \sum_{k=1}^{m} c_{k,k+1} M_k M_{k+1} \right)} dM_1 \ldots dM_m = e^{\sum_g (N/t)^{2-2g} F_g}
$$
(5-88)

where the integral is a formal integral in the sense of the preceding sections and M_{m+1} is a constant given diagonal matrix $M_{m+1} = \Lambda$ with s distinct eigenvalues λ_i with multiplicities l_i:

$$
M_{m+1} = \Lambda = \text{diag} \left(\overbrace{\lambda_1, \ldots, \lambda_1}^{l_1}, \ldots, \overbrace{\lambda_i, \ldots, \lambda_i}^{l_i}, \ldots, \overbrace{\lambda_s, \ldots, \lambda_s}^{l_s} \right)
$$
(5-89)

with $\sum_i l_i = N$.

Note also that we may choose $c_{m,m+1} = 1$ since it can be reabsorbed as a rescaling of Λ.

Once again, in the definition of the formal integral, one has to choose around which saddle point one expands. Saddle points are solutions of:

$$
\forall k = 1, \ldots, m, \qquad V_k'(\xi_k) = c_{k-1,k}\xi_{k-1} + c_{k,k+1}\xi_{k+1} \qquad , \qquad \exists j, \; \xi_{m+1} = \lambda_j. \quad (5\text{-}90)
$$

This system is an algebraic equation with $D = sd_1 d_2 \ldots d_m$ solutions.

Therefore the choice of a saddle point is encoded in the choice of a set of filling fractions

$$\epsilon_i = T \frac{n_i}{N} \tag{5-91}$$

for $i = 1, \ldots, D$ with $D = d_1 d_2 \ldots d_m s$ and n_i arbitrary integers satisfying

$$\sum_i n_i = N. \tag{5-92}$$

5.3.1 Definition of the correlation functions

The loop equations of the chain of matrices were derived in [51, 52], and they require the definition of several quantities, as follows:

For convenience, we introduce in the sense of eq.(5-14):

$$G_i(x_i) := \frac{1}{x_i - M_i}. \tag{5-93}$$

We also consider the minimal polynomial of Λ, such that $S(\Lambda) = 0$, i.e.:

$$S(z) = \prod_{i=1}^{s} (z - \lambda_i) \tag{5-94}$$

and we introduce the following polynomial in z

$$Q(z) = \frac{1}{c_{n,n+1}} \frac{S(z) - S(\Lambda)}{z - \Lambda}. \tag{5-95}$$

We also define the polynomials $f_{i,j}(x_i, \ldots, x_j)$ by $f_{i,j} = 0$ if $j < i - 1$, $f_{i,i-1} = 1$, and

$$f_{i,j}(x_i, \ldots, x_j) = \det \begin{pmatrix} V_i'(x_i) & -c_{i,i+1}x_{i+1} & & 0 \\ -c_{i,i+1}x_i & V_{i+1}'(x_{i+1}) & \ddots & \\ & \ddots & \ddots & -c_{j-1,j}x_j \\ 0 & & -c_{j-1,j}x_{j-1} & V_j'(x_j) \end{pmatrix} \tag{5-96}$$

if $j \geq i$. They satisfy the recursion

$$c_{i-1,i}f_{i,j}(x_i, \ldots, x_j) = V_i'(x_i)f_{i+1,j}(x_{i+1}, \ldots, x_j) - c_{i,i+1} x_i x_{i+1} f_{i+2}(x_{i+2}, \ldots, x_j). \tag{5-97}$$

Let us then define the correlation functions and auxiliary functions:

$$W_0(x) = \langle \operatorname{Tr} G_1(x) \rangle. \tag{5-98}$$

For $i = 2, \ldots, m$, we define:

$$W_i(x_1, x_i, \ldots, x_m, z) = \operatorname{Pol}_{x_i, \ldots, x_m} f_{i,m}(x_i, \ldots, x_m) \langle \operatorname{Tr} (G_1(x_1)G_i(x_i) \ldots G_m(x_m)Q(z)) \rangle, \tag{5-99}$$

which is a polynomial in variables $x_i, \ldots, x_m, z$, but not in x_1. And for $i = 1$, we define:

$$W_1(x_1, x_2, \ldots, x_m, z) = \text{Pol}_{x_1,\ldots,x_m} f_{1,m}(x_1, \ldots, x_m) \langle \text{Tr} \ (G_1(x_1) G_2(x_2) \ldots G_m(x_m) Q(z)) \rangle . \tag{5-100}$$

which is a polynomial in all variables.

We also define:

$$
\begin{aligned}
&W_{i;1}(x_1, x_i, \ldots, x_m, z; x_1') \\
=\ & \text{Pol}_{x_i,\ldots,x_m} f_{i,m}(x_i, \ldots, x_m) \langle \text{Tr} \ (G_1(x_1')) \ \text{Tr} \ (G_1(x_1) G_i(x_i) \ldots G_m(x_m) Q(z)) \rangle_c .
\end{aligned}
$$
$(5-101)$

All these functions admit a topological expansion, for example:

$$W_0 = \sum_g (N/t)^{1-2g} W_0^{(g)} \quad , \quad W_1 = \sum_g (N/t)^{1-2g} W_1^{(g)} \quad , \quad W_i = \sum_g (N/t)^{1-2g} W_i^{(g)}$$
$$\tag{5-102}$$

and

$$W_{i;1} = \sum_g (N/t)^{-2g} W_{i;1}^{(g)}. \tag{5-103}$$

5.3.2 Loop equations and spectral curve

In this model, the master loop equation reads [51, 52]:

$$
\begin{aligned}
&W_{2;1}(x_1, \ldots, x_{m+1}; x_1) \\
&+ (c_{1,2} x_2 - V_1'(x_1) + \frac{t}{N} W_0(x_1)) \left(\frac{t}{N} W_2(x_1, \ldots, x_{m+1}) - S(x_{m+1}) \right) \\
&= -\frac{t}{N} W_1(x_1, \ldots, x_{m+1}) + (V_1'(x_1) - c_{1,2} x_2) \, S(x_{m+1}) + \\
&\qquad + \frac{t}{N} \sum_{i=2}^{m} (V_i'(x_i) - c_{i-1,i} x_{i-1} - c_{i,i+1} x_{i+1}) \, W_{i+1}(x_1, x_i, \ldots, x_{m+1}).
\end{aligned}
$$
$(5-104)$

This equation is valid for any set of variables $x_1, x_2, \ldots, x_{m+1}$, however, it can be simplified by choosing special values for those variables, in particular values for which the last terms in the RHS vanishes. For this purpose, one defines some $\hat{x}_i(x_1, x_2)$ as functions of the two first variables x_1 and x_2, as follows:

$$\hat{x}_1(x_1, x_2) = x_1 \quad , \quad \hat{x}_2(x_1, x_2) = x_2, \tag{5-105}$$

and for $i = 2, \ldots, m$:

$$c_{i,i+1} \hat{x}_{i+1}(x_1, x_2) = V_i'(\hat{x}_i(x_1, x_2)) - c_{i-1,i} \hat{x}_{i-1}(x_1, x_2). \tag{5-106}$$

Choosing $x_i = \hat{x}_i(x_1, x_2)$, reduces the master loop equation to an equation in x_1 and x_2:

$$\widehat{W}_{2;1}(x_1, x_2; x_1) + \frac{t}{N} (c_{1,2} x_2 - Y(x_1)) \, \widehat{U}(x_1, x_2) = \widehat{E}(x_1, x_2) \tag{5-107}$$

where

$$Y(x) = V_1'(x) - \frac{t}{N}W_0(x) \quad , \quad \widehat{U}(x_1, x_2) = W_2(x_1, x_2, \hat{x}_3, \ldots, \hat{x}_{m+1}) - \frac{N}{t}S(\hat{x}_{m+1}),$$
(5-108)

$$\widehat{W}_{2;1}(x_1, x_2; x_1) = W_{2;1}(x_1, x_2, \hat{x}_3, \ldots, \hat{x}_{m+1}; x_1)$$
(5-109)

and

$$\widehat{E}(x_1, x_2) = -\frac{t}{N}\widehat{W}_1(x_1, x_2) + (V_1'(x_1) - c_{1,2}x_2)\,\widehat{S}(x_1, x_2)$$
(5-110)

with

$$\widehat{S}(x_1, x_2) = S(\hat{x}_{m+1}) \quad , \quad \widehat{W}_1(x_1, x_2) = W_1(x_1, x_2, \hat{x}_3, \ldots, \hat{x}_{m+1}).$$
(5-111)

Notice that $\widehat{W}_1(x_1, x_2)$, and thus $\widehat{E}(x_1, x_2)$ is a polynomial in both x_1 and x_2.

Finally, the leading order in the topological expansion gives

$$\widehat{E}^{(0)}(x_1, x_2) = \left(c_{1,2}x_2 - Y^{(0)}(x_1)\right)\widehat{U}^{(0)}(x_1, x_2).$$
(5-112)

We may notice that this equation is more or less the same as in the 2-matrix model, and it is solved in the same way.

Again, this equation is valid for any x_1 and x_2, and if we choose x_2 such that $c_{1,2}x_2 = Y^{(0)}(x_1)$, we get:

$$H_{chain}(x_1, x_2) := \widehat{E}^{(0)}(x_1, x_2) = 0.$$
(5-113)

This algebraic equation is the spectral curve of our model.

Study of the spectral curve

The algebraic plane curve $H_{chain}(x_1, x_2) = 0$, can be parameterized by a variable z living on a compact Riemann surface $\mathcal{L}_{chain}$ of some genus $\bar{g}$, and two meromorphic functions $x_1(z)$ and $x_2(z)$ on it. Let us study it in greater details.

Genus of $\mathcal{L}_{chain}$

The Riemann surface $\mathcal{L}_{chain}$ has genus $\bar{g}$ lower than $D - s$:

$$\bar{g} \leq D - s,$$
(5-114)

where $D = s\,d_1 \ldots d_m$.

Sheeted structure

The polynomial $H_{2MM}(x_1, x_2)$ has degree $1 + \frac{D}{d_1}$ (resp. $d_1 + \frac{D}{d_1 d_2}$) in x_2 (resp. x_1). This means that the embedding of $\mathcal{L}_{chain}$ is composed by $1 + \frac{D}{d_1}$ (resp. $d_1 + \frac{D}{d_1 d_2}$) copies of the Riemann sphere, called x_1-sheets (resp. x_2-sheets), glued by cuts so that the resulting Riemann surface $\mathcal{L}_{chain}$ has genus $\bar{g}$. Each copy of the Riemann sphere corresponds to one particular branch of the solutions of the equation $H_{chain}(x_1, x_2) = 0$ in x_2 (resp. x_1).

Pole structure

In the preceding cases (1 and 2-matrix models), one was interested in the pole structure of only two functions x and y on the spectral curve. In the case of the chain of matrices, the problem is slightly richer since one can consider, not only the meromorphic functions x_1 and x_2, but also all the $x_i(p) := \hat{x}_i(x_1(p), x_2(p))$ as meromorphic functions on $\mathcal{L}_{chain}$. Their negative divisors are given by

$$[x_k(p)]_- = -r_k\infty - s_k \sum_{i=1}^{s} \hat{\lambda}_i \tag{5-115}$$

where ∞ is the only point of $\mathcal{L}_{chain}$ where x_1 has a simple pole, the $\hat{\lambda}_i$ are the preimages of λ_i under the map $x_{m+1}(p)$:

$$x_{m+1}(\hat{\lambda}_i) = \lambda_i \tag{5-116}$$

and the degrees r_k and s_k are integers given by

$$r_1 := 1, \ r_k := d_1 d_2 \ldots d_{k-1}, \ s_{m+1} := 0, \ s_m := 1 \quad \text{and} \quad s_k := d_{k+1} d_{k+2} \ldots d_m \, s. \tag{5-117}$$

Note that the presence of an external matrix creates as many poles as the number of distinct eigenvalues of this external matrix $M_{m+1} = \Lambda$[13].

Remark 5.4 This matrix model has also a combinatorics interpretation in terms of counting colored surfaces. This interpretation is discussed in chapter 7.

5.3.3 Solution of the loop equations

The loop equations have been solved in [52] by the same method as the 2-matrix model. It proceeds in three steps. One first shows that the loop equations eq.(5-104) have a unique solution admitting a topological expansion. One then propose an Ansatz of solution and prove that it is indeed right. This gives

Theorem 5.5

$$E(x(z), y) = -\tilde{t}_{d_2+1} \, " \left\langle \prod_{i=0}^{d_2} (y - V_1'(x(z^i)) + \frac{t}{N} \operatorname{Tr} \frac{1}{x(z^i) - M_1}) \right\rangle ". \tag{5-118}$$

One finally develops this expression as a polynomial in y to get the bilinear relation

$$\omega_1^{(g)}(z_0) = \sum_i \operatorname*{Res}_{z \to a_i} K(z_0, z) \left[\omega_2^{(g+1)}(z, \bar{z}) + \sum_{h=0}^{g} \omega_1^{(h)}(z) \omega_1^{(g-h)}(\bar{z}) \right] \tag{5-119}$$

where, as in the preceding section,

$$\begin{aligned} \omega_n^{(g)}(z_1, \ldots, z_n) &= W_n^{(g)}(x(z_1), \ldots, x(z_n)) \, dx(z_1) \ldots dx(z_n) \\ &+ \delta_{n,2} \delta_{g,0} \frac{dx(z_1) dx(z_2)}{(x(z_1) - x(z_2))^2}. \end{aligned} \tag{5-120}$$

This allows to obtain the theorem:

---

[13]The cases of matrix models without external field correspond to a totally degenerate external matrix $\Lambda = c \operatorname{Id}$ with only 1-eigenvalue. There are thus two poles as in the 1 or 2 matrix models studied earlier.

Theorem 5.6 $\omega_n^{(g)}(z_1, \ldots, z_n)$ *and* F_g *are the correlators and symplectic invariants of the spectral curve* $(\mathcal{L}_{chain}, c_{1,2}x_1, x_2)$.

Since $c_{1,2}x_1 + c_{2,3}x_3 = V_2'(x_2)$, we may use the symplectic invariance theorem 4.1, and homogeneity theorem eq.(4-4), which allows also to write:

$$
\begin{aligned}
F_g &= & F_g(\mathcal{L}_{chain}, c_{1,2}x_1, x_2) \\
&= & F_g(\mathcal{L}_{chain}, -c_{2,3}x_3, x_2) \\
&= & F_g(\mathcal{L}_{chain}, c_{2,3}x_3, x_2) \\
&= & F_g(\mathcal{L}_{chain}, x_2, c_{2,3}x_3) \\
&= & F_g(\mathcal{L}_{chain}, c_{2,3}x_2, x_3).
\end{aligned}
$$
$$(5-121)$$

And by an easy recursion, for any $k = 1, \ldots, m$:

$$F_g = F_g(\mathcal{L}_{chain}, c_{k,k+1}\, x_k, x_{k+1}) = F_g(\mathcal{L}_{chain}, x_k, c_{k,k+1}\, x_{k+1}). \tag{5-122}$$

In other words, the F_g's can be computed by choosing the spectral curve of any two consecutive x_k's. It means it doesnot depends on k, it doesnot depend on where we are in the chain.

5.3.4   Matrix quantum mechanics

Matrix quantum mechanics is the limit of an infinitely long chain of matrices $m \to \infty$. This model is very useful in string theory [10].

In this limit, the index k of the matrix M_k, becomes a continuous time variable t. The coefficients are scaled in a way such that the coupling term $\mathrm{Tr}\,(M_k - M_{k+1})^2$ becomes a kinetic energy $\mathrm{Tr}\,(dM/dt)^2$.

More explicitly, consider the chain of m matrices

$$Z = \int dM_1 \ldots dM_m \; \mathrm{e}^{-N\,\mathrm{Tr}\left[\sum_{k=1}^{m}\eta V_k(M_k)+\frac{\mu}{2\eta}\sum_{k=1}^{m-1}(M_k-M_{k+1})^2\right]} \tag{5-123}$$

and take the $\eta \to 0$ limit, and $T = m\eta$ of order 1. The index k becomes a time $t = k\eta$, and in the $\eta \to 0$ limit we have:

$$Z = \int D[M(t)] \; \mathrm{e}^{-N\,\mathrm{Tr}\int_0^T[V(M(t),t)+\frac{\mu}{2}\,(dM/dt)^2]\,dt} \tag{5-124}$$

The spectral curve is characterized as before:

find a compact Riemann surface $\mathcal{L}$, and some time-dependent function $x(z,t)$ analytical in the variable z on some domain of $\mathcal{L}$, which satisfy eq.(5-106) which become Newton's equations of motion:

$$\mu\,\ddot{x}(z,t) = -V'(x(z,t),t) \tag{5-125}$$

and such that the initial and final impulsions

$$p(x(z,0),0) = \mu\dot{x}(z,0) \quad , \quad p(x(z,T),T) = \mu\dot{x}(z,T) \tag{5-126}$$

are analytical outside some cuts.

Therefore, theorem 5.6 gives $\forall t \in [0, T]$:

$$\ln Z = \sum_g N^{2-2g} \; F_g(\mathcal{L}, x(z, t), \mu \dot{x}(z, t)). \tag{5-127}$$

In other words, the spectral curve is here the realtionship between impulsion and position for the classical equation of motion. Although the spectral curve depends on time t, the F_g's are independent of t.

In the case where the potential $V(x, t) = V(x)$ does not depend on time t, the equations of motion eq.(5-125) can be integrated and give the energy conservation:

$$E(z) = \frac{\mu}{2} \dot{x}^2(z, t) + V(x(z, t)) \tag{5-128}$$

i.e.

$$\mu \dot{x}(z, t) = \sqrt{2\mu(E(z) - V(x(z, t)))} \tag{5-129}$$

and we have:

$$\ln Z = \sum_g N^{2-2g} \; F_g(\mathcal{L}, x(z, t), \sqrt{2\mu(E(z) - V(x(z, t)))}). \tag{5-130}$$

5.4   1-Matrix model in an external field

As a special example of the chain of matrices above, let us consider the special case $m = 1$, i.e. 1-matrix model with an external field.

The formal 1-matrix model in an external field $\hat{\Lambda}$ is defined as [118]:

$$Z_{M.\text{ext}}(\hat{\Lambda}) = \int_{\text{formal}} e^{-\frac{N}{t} \operatorname{Tr} (V(M) - \hat{\Lambda} M)} \, dM \quad , \quad \hat{\Lambda} = \operatorname{diag}\left(\overbrace{\hat{\Lambda}_1, \ldots, \hat{\Lambda}_1}^{m_1}, \ldots, \overbrace{\hat{\Lambda}_s, \ldots, \hat{\Lambda}_s}^{m_s}\right) \tag{5-131}$$

where *formal*, as usual means that we Taylor expand near a critical value and then exchange the order of Taylor expansion and gaussian integral. A critical point is a matrix M_0 solution of $V'(M_0) = \hat{\Lambda}$. Let us assume that $\hat{\Lambda}$ has s distinct eigenvalues $\hat{\Lambda}_i$ of multiplicities m_i. Its minimal polynomial is:

$$S(y) = \prod_{i=1}^s (y - \hat{\Lambda}_i). \tag{5-132}$$

For each $\hat{\Lambda}_i$, let $\xi_{i,j}, j = 1, \ldots, \deg V'$ be the $d = \deg V'$ solutions of $V'(\xi_{i,j}) = \hat{\Lambda}_i$. A critical point M_0 is characterized by s partitions of the m_i's into at most $d = \deg V'$ parts

$$m_i = \sum_{j=1}^d n_{i,j}. \tag{5-133}$$

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It is of the form:

$$M_0 = \mathrm{diag}\left(\overbrace{\xi_{i,j}, \ldots, \xi_{i,j}}^{n_{i,j}}\right).$$ (5-134)

The parameters:

$$\epsilon_{i,j} = \frac{t\, n_{i,j}}{N}$$ (5-135)

are called the filling fractions, and they parametrize which formal integral we are considering.

The 1-matrix model in an external field is of course a special case of the chain of matrices described in the previous section 5.3, and therefore one finds that it has a topological expansion given by the symplectic invariants of a spectral curve:

$$\ln\left(Z_{M\,.\mathrm{ext}}(\hat{\Lambda})\right) = \sum_{g=0}^{\infty}(N/T)^{2-2g}\, F_g(\mathcal{E}_{M\,.\mathrm{ext}})$$ (5-136)

and

$$\left\langle \mathrm{Tr}\, \frac{dx(z_1)}{x(z_1) - M} \cdots \mathrm{Tr}\, \frac{dx(z_n)}{x(z_n) - M} \right\rangle_c = \sum_g (N/t)^{2-2g-n}\, \omega_n^{(g)}(z_1, \ldots, z_n).$$ (5-137)

However, let us make the results of section 5.3 a little bit more explicit in that case.

5.4.1   Spectral curve

The spectral curve $\mathcal{E}_{M\,.\mathrm{ext}}$, obeys the equation:

$$0 = \mathcal{E}_{M\,.\mathrm{ext}}(x, y) = (V'(x) - y) - \frac{t}{N}\sum_{j=1}^{s} \frac{P_j(x)}{y - \hat{\Lambda}_j}$$ (5-138)

where $P_j(x)$ is a polynomial of degree at most $d - 1$. Such a spectral curve is typically of genus $\bar{g} \leq sd - s$.

All the coefficients of all the P_j's are fixed by the filling fractions requirement that (order by order in t):

$$\frac{1}{2i\pi}\oint_{\mathcal{A}_I} y\,dx = \epsilon_I$$ (5-139)

where $I = (i, j)$ runs through the values $i = 1, \ldots, s$, $j = 1, \ldots, d$, and $\mathcal{A}_I$ is a small circle around $\xi_{i,j}$. We remind that the $\epsilon_{i,j}$ are not independent, only $sd - s$ of them are independent:

$$\sum_{j=1}^{d} \epsilon_{i,j} = \frac{t\, m_i}{N}.$$ (5-140)

The two functions $x(z)$ and $y(z)$ are characterized by the fact that $x(z)$ has a simple pole at ∞, simple poles at some λ_i such that $y(\lambda_i) = \hat{\Lambda}_i$, and $y(z)$ has a pole of degree d at ∞. And we have:

$$V'(x(z)) - y(z) \underset{z\to\infty}{\sim} \frac{1}{x(z)} + O(1/x(z)^2)$$ (5-141)

and near λ_i, the residues of x are such that:

$$\operatorname*{Res}_{z \to \lambda_i} x dy = -\frac{t\, m_i}{N} \tag{5-142}$$

and the cycle integrals:

$$\frac{1}{2i\pi} \oint_{\mathcal{A}_{i,j}} y dx = \epsilon_{i,j} = \frac{t\, n_{i,j}}{N}. \tag{5-143}$$

5.4.2 Rational case

It is interesting to study the case of a rational spectral curve. The two rational functions $x(z)$ and $y(z)$ are of the form:

$$\mathcal{E}_{M\,\mathrm{ext}} = \begin{cases} x(z) = z - \frac{t}{N} \operatorname{Tr} \frac{1}{Q'(\Lambda)(z-\Lambda)} \\ y(z) = Q(z) \end{cases} \tag{5-144}$$

where Λ is a diagonal matrix determined by:

$$Q(\Lambda) = \hat{\Lambda} \tag{5-145}$$

and Q is a polynomial of degree $d = \deg V'$, determined by:

$$V'(x(z)) = Q(z) + \frac{1}{z} + O(z^{-2}). \tag{5-146}$$

5.5 Convergent matrix integrals

So far, we have been discussing formal matrix integrals, which consist in exchanging integration and the small t Taylor series of $e^{-\frac{N}{t} \operatorname{Tr} V(M)}$. Formal matrix integrals always have a "topological expansion" of the type:

$$\ln Z = \sum_g (N/t)^{2-2g} F_g(t). \tag{5-147}$$

Now, let us consider a "convergent" matrix integral:

$$Z = \int_{H_N(\gamma)} dM\; e^{-\frac{N}{t} \operatorname{Tr} V(M)} \tag{5-148}$$

where the integration domain $H_N(\gamma)$ is the set of normal matrices with eigenvalues on a path γ:

$$H_N(\gamma) = \{M \; / \; M = U\Lambda U^\dagger, U \in U(N)\,, \Lambda = \mathrm{diag}(\Lambda_1, \dots, \Lambda_N)\,, \Lambda_i \in \gamma\} \tag{5-149}$$

equipped with the complex $U(N)$ invariant measure

$$dM = \prod_{i>j} (\Lambda_i - \Lambda_j)^2 \; dU \prod_i d\Lambda_i \tag{5-150}$$

where dU is the Haar measure on $U(N)$, and $d\Lambda_i$ is the curviline measure along γ.

- For example $H_N(\mathbb{R}) = H_N$ is the set of hermitian matrices.
- For example $H_N(S_1) = U(N)$ is the set of unitary matrices (S_1 is the unit circle).

In eigenvalues, the convergent matrix integral is:

$$Z(\gamma) = \frac{1}{N!} \int_{\gamma^N} dx_1 \ldots dx_N \prod_{i>j}(x_i - x_j)^2 \prod_{i=1}^{N} e^{-\frac{N}{t}V(x_i)}. \qquad (5\text{-}151)$$

Imagine that V is a polynomial of degree $d+1$ (the present discussion can be extended easily to a 2-matrix model, or a chain of matrices, and to all cases where V' is a rational fraction [18]). There are d homologically independent paths on which the integral $\int e^{-V(x)}\, dx$ is convergent, let us call $\gamma_1, \ldots, \gamma_d$, a basis of such paths (a choice of basis is not unique). See the figure for the example of a quartic potential ($d=3$):
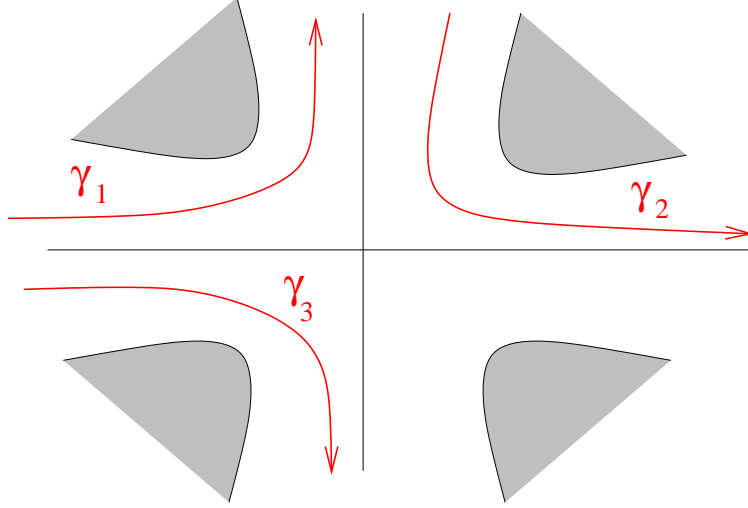
The path γ in the matrix integral eq.(5-148) is a linear combination of such paths:

$$\gamma = \sum_{i=1}^{d} c_i \gamma_i \qquad (5\text{-}152)$$

and the convergent matrix integral eq.(5-151) can be written:

$$Z(\gamma) = \sum_{n_1+\ldots+n_d=N} \frac{c_1^{n_1} \ldots c_d^{n_d}}{n_1! \ldots n_d!} \int_{\gamma_1^{n_1} \times \ldots \times \gamma_d^{n_d}} dx_1 \ldots dx_N \prod_{i>j}(x_i - x_j)^2 \prod_{i=1}^{N} e^{-\frac{N}{t}V(x_i)}. \qquad (5\text{-}153)$$

This leads us to define the convergent matrix integral with **fixed filling fractions** n_i, as:

$$\hat{Z}_{n_1,\ldots,n_d} \overset{\text{def}}{=} \frac{1}{n_1! \ldots n_d!} \int_{\gamma_1^{n_1} \times \ldots \times \gamma_d^{n_d}} dx_1 \ldots dx_N \prod_{i>j}(x_i - x_j)^2 \prod_{i=1}^{N} e^{-\frac{N}{t}V(x_i)}, \qquad (5\text{-}154)$$

and thus we have:
$$Z(\gamma) = \sum_{n_1+\ldots+n_d=N} c_1^{n_1} \ldots c_d^{n_d} \; \hat{Z}_{n_1,\ldots,n_d}. \tag{5-155}$$

This holds for any choice of basis $\gamma_1, \ldots, \gamma_d$.

There is a conjecture[14] that there exists a "good" basis of paths $\gamma_1, \ldots, \gamma_d$ [37], such that $\hat{Z}_{n_1,\ldots,n_d}$ is a formal matrix integral (and thus it has a topological expansion in powers of t/N)!

The "good" paths $\gamma_1, \ldots, \gamma_d$ can be seen as "steepest descent" paths, they should be such that the effective potential:
$$V_{\text{eff}}(x) = V(x) - \frac{t}{N} \left\langle \ln\left(\det(x - M)\right)\right\rangle \tag{5-156}$$

is such that along each γ_i, the real part of the large N leading order of V_{eff} is decreasing, then constant and finally increasing, and at the same time, the imaginary part of V_{eff} is constant, then increasing, then constant:
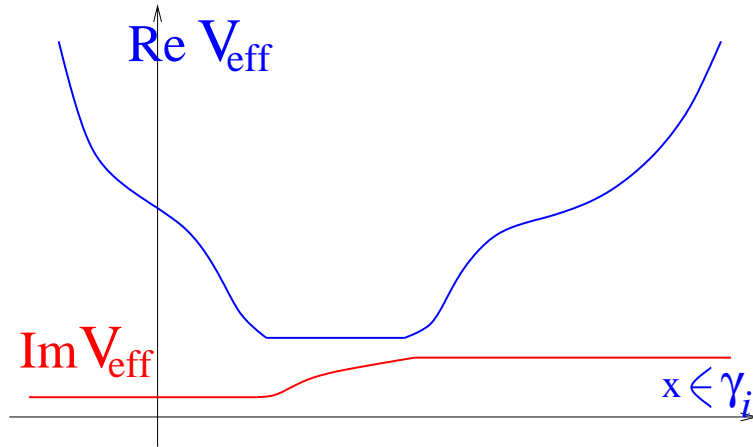
If such paths exist, then we can write:
$$\hat{Z}_{n_1,\ldots,n_d} = e^{\displaystyle\sum_g (N/t)^{2-2g}\, F_g(\epsilon_i)} \tag{5-157}$$

where ϵ_i are the filling fractions:
$$\epsilon_i = \frac{t\, n_i}{N} \tag{5-158}$$

and the coefficients F_g in the expansion, are the symplectic invariants F_g of the corresponding formal matrix integral. Since the F_g's are analytical functions of the filling fractions, we have, for any "background" filling fraction $\eta = (\eta_1, \ldots, \eta_{d-1})$
$$F_g(\epsilon_i) = \sum_{k=0}^{\infty} \frac{(\epsilon - \eta)^k}{k!} \; \partial_\eta^k \; F_g(\eta) \tag{5-159}$$

---

[14]This conjecture is proved in some cases, and in particular proved for the 1-matrix model with arbitrary γ and arbitrary polynomial V (the proof follows from M. Bertola's work [19]), but, at the time this article is being written, it is not proved in more general cases, for instance not proved for the general 2-matrix model.

where we assume tensorial notations and sums over indices. For simplicity, the unfamiliar reader may assume $d = 2$, i.e. η and ϵ are scalar.

Eq.(5-157) thus becomes:

$$
\begin{aligned}
\hat{Z}_{n_1,\ldots,n_d} \quad &= \quad \frac{e^{\sum_g (N/t)^{2-2g} F_g(\eta)}}{e^{\sum_g \sum_{k \geq 2-2g} \frac{(N/t)^{2-2g-k}}{k!} (n-N\eta/t)^k F_g^{(k)}(\eta)}} e^{(n-N\eta/t)NF_0'/t} e^{\frac{1}{2}(n-N\eta/t)^2 F_0''} \\[2ex]
&= \quad e^{\sum_g (N/t)^{2-2g} F_g(\eta)} \; e^{(n-N\eta/t)NF_0'/t} e^{\frac{1}{2}(n-N\eta/t)^2 F_0''} \\[1ex]
&\qquad \sum_l \frac{1}{l!} \sum_{g_1,\ldots,g_l} {\sum_{k_1,\ldots,k_l}}' \frac{(N/t)^{\sum_i (2-2g_i-k_i)}}{k_1!\ldots k_l!} \prod_i F_{g_i}^{(k_i)}(\eta) \; (n-N\eta/t)^{\sum k_i}
\end{aligned}
$$

$$(5-160)$$

where we have separated the terms with a positive power of N from those with a negative power of N ($\sum'$ means that we consider only terms with $k_i > 0$ and $2g_i + k_i - 2 > 0$). Then, writing:

$$c_i = e^{2i\pi\nu_i}, \tag{5-161}$$

we perform the sum over filling fractions in eq.(5-155), and we get [21, 55]:

$$
\begin{aligned}
Z(\gamma) \quad &\sim \quad e^{\sum_g (N/t)^{2-2g} F_g(\eta)} \; \sum_{n\in\mathbb{Z}^{d-1}} e^{2i\pi n\nu}\, e^{(n-N\eta/t)NF_0'/t} e^{\frac{1}{2}(n-N\eta/t)^2 F_0''} \\[1ex]
&\qquad \sum_l \frac{1}{l!} \sum_{g_1,\ldots,g_l} {\sum_{k_1,\ldots,k_l}}' \frac{(N/t)^{\sum_i (2-2g_i-k_i)}}{k_1!\ldots k_l!} \prod_i F_{g_i}^{(k_i)}(\eta) \; (n-N\eta/t)^{\sum k_i}.
\end{aligned}
$$

$$(5-162)$$

That is, we find the non-perturbative partition function of section 4.5 (see [55, 64]):

$$
\begin{aligned}
Z(\gamma) \quad &\sim \quad e^{\sum_g (N/t)^{2-2g} F_g(\eta)} \\[1ex]
&\qquad \sum_l \frac{1}{l!} \sum_{g_1,\ldots,g_l} {\sum_{k_1,\ldots,k_l}}' \frac{(N/t)^{\sum_i (2-2g_i-k_i)}}{k_1!\ldots k_l!} \prod_i F_{g_i}^{(k_i)}(\eta) \; \Theta_{(0,\nu)}^{(\sum k_i)}(NF_0'/t, F_0'')
\end{aligned}
$$

$$(5-163)$$

where

$$\Theta_{(\mu,\nu)}(u, F_0'') = \sum_{n\in\mathbb{Z}^{d-1}} e^{2i\pi n\nu}\, e^{2i\pi(n-N\eta/t+\mu)u}\, e^{\frac{1}{2}(n-N\eta/t)^2 F_0''}. \tag{5-164}$$

This formula is expected to give the large N expansion of convergent matrix models. It was proved in several cases, but a general proof is still missing.

To the first orders, eq.(5-163) reads:

$$Z(\gamma) \sim e^{\frac{N^2}{t^2} F_0} e^{F_1} \left[\Theta + \frac{t}{N} \left(\Theta' F_1' + \frac{\Theta''' F_0'''}{6} \right) + \dots \right]. \tag{5-165}$$

We have to make several remarks:

• background independence:

Formula eq.(5-163) is independent of the background η. This was discussed in section 4.5, and it is related to the fact that the non-perturbative partition function is modular.

This also implies that eq.(5-163) cannot be a good large N asymptotic expansion for all values of η. The conjecture is that one should choose η as a real minimum of $\mathrm{Re} F_0$. Notice that if η is real, then

$$\mathrm{Re} F_0'' = 2\pi \, \mathrm{Im} \tau > 0 \tag{5-166}$$

where τ is the Riemann matrix of periods of the spectral curve, and thus $\mathrm{Re} F_0$ is a convex function of η, within each cell of the moduli space of spectral curves corresponding to the potential V. Unfortunately, this moduli space is not very well known, and it is not known how to find such η in general.

If such η can be found we have $\frac{1}{2i\pi} \oint_{\mathcal{A}} y dx = \eta \in \mathbb{R}$, and $F_0' = \oint_{\mathcal{B}} y dx$, and thus, for any contour $\mathcal{C}$ we have:

$$\mathrm{Re} \oint_{\mathcal{C}} y dx = 0. \tag{5-167}$$

A spectral curve with that property is called a "Boutroux" curve (See [19]).

• Characteristics (μ, ν):

We see here, that the characteristics (μ, ν) of the Θ-function, is related to the path γ chosen at the beginning to define the convergent integral. It is not fully understood how to associate a path to a characteristic and vice-versa.

6 Non-intersecting Brownian motions

6.1 Dyson motions and integrability: introduction

Let us consider N Brownian motions on the real line whose positions at time t are denoted by $x_i(t)$ for $i = 1, \dots, N$. Let us constrain them not to intersect and fix their starting and ending points: the particle i goes from a_i at $t = 0$ to b_i at $t = 1$:

$$x_i(0) = a_i \qquad , \qquad x_i(1) = b_i \tag{6-1}$$

with

$$a_1 \leq a_2 \leq \dots \leq a_N \qquad \text{and} \qquad b_1 \leq b_2 \leq \dots \leq b_N. \tag{6-2}$$

Once these parameters are fixed, one is interested in the statistic of these Brownian movers at a given time $t \in (0, 1)$. They are given by the correlation functions:

$$R_k(\lambda_1, \lambda_2, \dots, \lambda_k | t) = \frac{1}{N^k} \left\langle \prod_{i=1}^{k} \mathrm{Tr} \, \delta(\lambda_i - M) \right\rangle_t \tag{6-3}$$

where we denote $M = \mathrm{diag}(x_1, \ldots, x_N)$.

These correlation functions can be written under a determinantal form [45, 94]

$$R_k(x_1, \ldots, x_k|t) = \det\left[H_{N,t}(x_i, x_j)\right]_{i,j=1}^k \tag{6-4}$$

for some kernel $H_{N,t}(x_i, x_j)$ depending on time t. In particular, the density of Brownian movers at time t is given by

$$R_1(x) = H_{N,t}(x, x). \tag{6-5}$$

Let us now consider a particular case where some of the starting and ending points merge in groups:

$$(a_1, \ldots, a_N) = (\overbrace{\alpha_1, \ldots, \alpha_1}^{n_1}, \overbrace{\alpha_2, \ldots, \alpha_2}^{n_2}, \ldots, \overbrace{\alpha_2, \ldots, \alpha_2}^{n_2}, \ldots, \overbrace{\alpha_p, \ldots, \alpha_p}^{n_p}) \tag{6-6}$$

and

$$(b_1, \ldots, b_N) = (\overbrace{\beta_1, \ldots, \beta_1}^{\tilde{n}_1}, \overbrace{\beta_2, \ldots, \beta_2}^{\tilde{n}_2}, \ldots, \overbrace{\beta_2, \ldots, \beta_2}^{\tilde{n}_2}, \ldots, \overbrace{\beta_q, \ldots, \beta_q}^{\tilde{n}_q}) \tag{6-7}$$

with

$$\sum_{i=1}^p n_i = \sum_{i=1}^q \tilde{n}_i = N. \tag{6-8}$$

It means that one considers p groups of n_i particles starting from distinct points α_i at $t = 0$, merging for intermediate times and splitting into q groups of $\tilde{n}_i$ movers reaching β_i at $t = 1$. Remark [1] that the kernel reduces to the kernel of the $p + q$ mutli-component KP integrable hierarchy:

$$H_{N,t}(x, x') = H_{N,t}^{(p,q)}(x, x') \tag{6-9}$$

since it satisfies the corresponding Hirota equation.

In the following one studies the behavior of this phenomenon as the number of particles goes to infinity while the ratios

$$\epsilon_i = \frac{n_i}{N} \qquad \text{and} \qquad \tilde{\epsilon}_i = \frac{\tilde{n}_i}{N} \tag{6-10}$$

are kept fixed and finite. In this case, the Brownian movers form clouds which fill a connected region of the complex plane describing the space time. For example, for 1 starting point and two ending points one typically gets a configuration of the type depicted in figure 1: all movers leave the origin and begin to flee from one another. It creates a larger and larger segment of the space filled by the Brownian movers. As the time grows, because the movers want to reach different points, they split into two groups heading towards these two end-points.

As often in the study of such integrable system, the kernel exhibits universal behaviors : in any point of the space time, one can rescale the kernel so that one obtains a universal kernel independent of the position of the considered point; typically, one recovers the Sine, Airy and Pearcy kernels. The purpose of this part is to emphasize the role played by the spectral curve and algebraic geometry in the study of these universality properties.

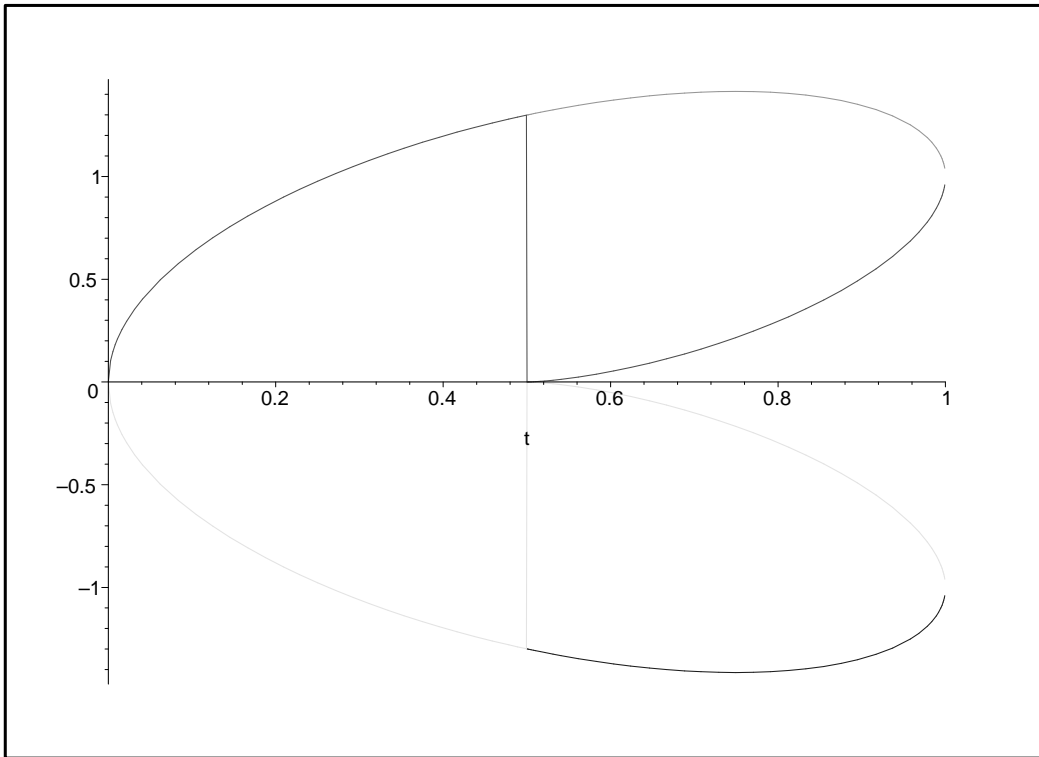Figure 1: Example of one starting point at 0 and two ending points at +1 and -1 half of the movers going to +1 and the other half to -1(the particles go from the left to the right). In the large N limit, the Brownian movers fill the sector of the space time delimited by the figure. One sees that before $t = \frac{1}{2}$, they are distributed along a unique segment which splits into two disjoint segments for $t > \frac{1}{2}$.

6.2 Particles starting from one point: a matrix model representation

Let us consider the particular case of one starting point $p = 1$. This means that all the Brownian movers start from the same point which we may assume to be located at the origin 0 of the real axis. It was proved [46] that this case is equivalent to a matrix model. More precisely, the statistic of the Brownian movers at a given time t is the same as the statistic of the rescaled eigenvalues of an hermitian random matrix M of size $N \times N$ submitted to an external field $A(t)$. This matrix model is given by the partition function

$$Z\left(A(t)\right) = \int dM e^{-N\,\mathrm{Tr}\,\left(M^2 + A(t)M\right)} \tag{6-11}$$

and

$$A(t) = \mathrm{diag}\left(\overbrace{A_1(t), \ldots, A_1(t)}^{\tilde{n}_1}, \overbrace{A_2(t), \ldots, A_2(t)}^{\tilde{n}_2}, \ldots, \overbrace{A_q(t), \ldots, A_q(t)}^{\tilde{n}_q}\right) \tag{6-12}$$

with the time dependent elements

$$A_i(t) = \sqrt{\frac{2t}{t(t-1)}}\beta_i. \tag{6-13}$$

The limit of a large number of particles corresponds to the large matrix limit.

This model can also be expressed in terms of the eigenvalues $(x_1, \ldots, x_N)$ of the random matrix M using the sell known HCIZ integral formual [70, 74]. They are submitted to a probability measure

$$d\mu(x_1, \ldots, x_N) = \prod_{i=1}^{N} dx_i \Delta(x)^2 e^{-N\sum_i \left(\frac{x_i^2}{2} - x_i a_i\right)}. \tag{6-14}$$

After the rescaling

$$x_i \rightarrow x_i \sqrt{t(1-t)}, \tag{6-15}$$

they have the same statistic as N Brownian movers starting from the origin.

The correlation functions

$$R_k(x_1, x_2, \ldots, x_k) = \frac{1}{N^k}\left\langle \prod_{i=1}^{k} \mathrm{Tr}\,\delta(x_i - M)\right\rangle, \tag{6-16}$$

have also a determinantal expression

$$R_k(x_1, \ldots, x_k) = \det\left[H_{N,t}(x_i, x_j)\right]_{i,j=1}^{k}, \tag{6-17}$$

in terms of a kernel $H_{N,t}$. As N goes to infinity, these eigenvalues merge into dense intervals of the real axis and the density $R_1(x)$ is supported by a finite number of

segments $[z_{2i-1}, z_{2i}]$. A classical result of the study of random matrices states that these segments correspond to the discontinuity of the resolvent $W(x) = \left\langle \sum_i \frac{1}{x-x_i} \right\rangle$:

$$W_+(x) - W_-(x) = R_1(x) \tag{6-18}$$

for $x \in [z_{2i-1}, z_{2i}]$.

6.2.1   Gaussian matrix model in an external field

Let us now quickly remind this matrix model's loop equations and large N solution. This model is a special case of the matrix model in an external field studied in section 5.4.

The resolvent $W(x)$ is thus the solution of the algebraic equation eq.(5-138) with $V'(x) = x$, i.e.:

$$E(x, Y(x)) = Y(x) - x + \sum_{i=1}^{q} \frac{\epsilon_i}{Y(x) - a_i(t)} = 0 \tag{6-19}$$

where

$$Y(x) = W(x) - x. \tag{6-20}$$

This equation can be seen as the embedding of a Riemann surface $\mathcal{L}$ into $\mathbb{CP}^1 \times \mathbb{CP}^1$.

First of all, one can see that this spectral curve has always genus 0: it admits a simple rational parametrization

$$\begin{cases} x(z) = z + \sum_{i=1}^{q} \frac{n_i}{N(z-a_i(t))} \\ y(z) = z \end{cases} . \tag{6-21}$$

It is composed of $q+1$ sheets, one of which contains the pole $z = \infty$. This sheet is called the physical sheet.

For a fixed number q of distinct eigenvalues of the external matrix, it may have up to q cuts linking the physical sheet to the q others. These cuts correspond to the discontinuity of the resolvent giving rise to the density of eigenvalues eq.(6-18). The cuts $[z_{2i-1}, z_{2i}]$ are thus the support of the eigenvalues. Each z_i is solution of $x'(z_i) = 0$. To be precise, using the notations of section 2, the density of eigenvalues on the cut i is given by

$$\rho(x(p)) = y(p^{(0)}) - y(p^{(i)}) \quad , \quad \text{for} \quad x(p) \in [z_{2i-1}, z_{2i}] \tag{6-22}$$

where one labels the physical sheet by 0 and the label i corresponds to the sheet linked to the physical one by the cut $[z_{2i-1}, z_{2i}]$.

Let us now study the evolution of the structure of this spectral curve as the time evolves from 0 to 1: one is particularly interested in the time evolution of the position of the branch points $z_i(t)$.
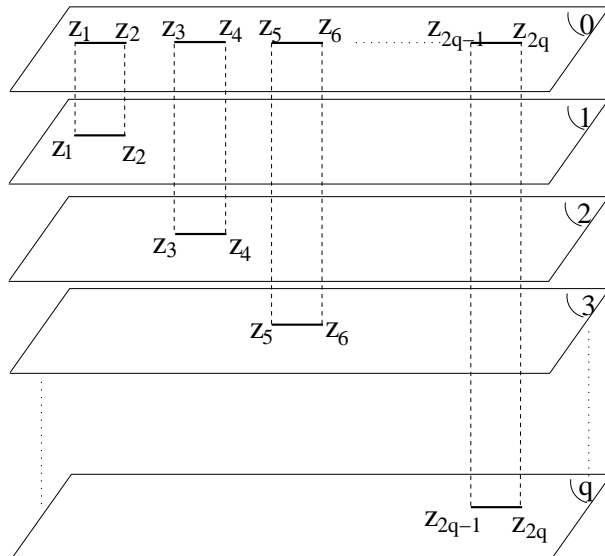
Figure 2: For large times, the spectral curve is composed of $q + 1$ sheets linked by q real cuts $[z_{2i-1}, z_{2i}]$. These cuts can be seen as the section of the support of the Brownian movers at fixed time. In the example depicted in figure 1, it corresponds to times greater than $\frac{1}{2}$.

For a given time t, these branch points are the simple real roots of the equation $\partial_y E(x, y)\big|_{y=y(x)} = 0$:

$$x'(z_j) = 1 - \sum_{i=1}^{k} \frac{n_i}{N(z_j - a_i(t))^2} = 0. \tag{6-23}$$

Let us first consider large times close to 1. In this case, the eigenvalues $a_i(t)$ become large and are far apart from one another. Thus, the equation has $2k$ distinct real roots in z: the spectral curve has k distinct cuts $[z_{2i-1}, z_{2i}]$ (see figure 2).

Then, as the time decreases, the branch points come closer from one another and merge for some critical time before becoming complex conjugated with an increasing imaginary part. It means that two real cuts merge into one and an imaginary cut linking two non physical sheets appear (see figure 3).

Finally, as the time becomes small and approaches zero, all the branch point are coupled complex conjugated numbers except two of them: the ones with the smallest and the largest real parts: there is only one real cut left.

This time evolution of the spectral curve has a simple interpretation in terms of statistic of the eigenvalues. Indeed, thanks to eq.(6-18), the real cuts are the support of the random matrix eigenvalues (whereas imaginary cuts just follow from the interaction between the different groups of eigenvalues). In terms of non intersecting Brownian motions, these real cuts are the segments filled by the Brownian movers at a given time t: it is the constant time section of the region of space-time filled by the Brownian movers.

The time evolution of the spectral curve can thus be interpreted as follows. For
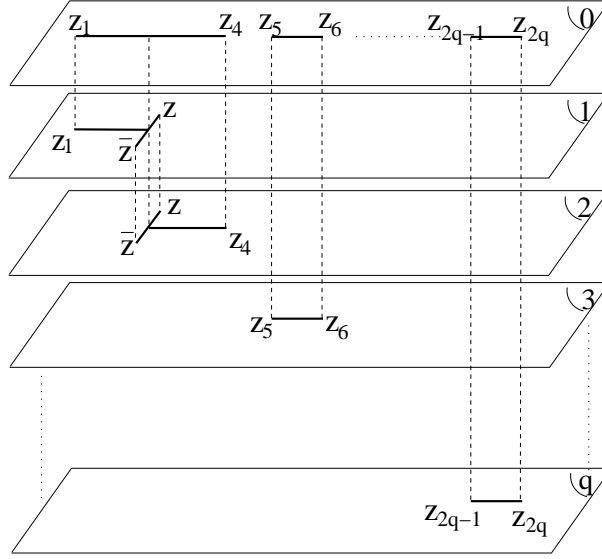
Figure 3: For intermediate times some of the real cuts have merged and created imaginary cuts linking two non-physical sheets. In this example, z_2 and z_3 have collapsed and given rise to the imaginary cut $[z, \overline{z}]$.

times close to one, the movers form q groups lying on segments centered around the q end points and whose extremities are the branch points of the spectral curve. As the time decreases, the branch points come closer to each other and finally some of them merge, i.e. two of the disjoints segments supporting the Brownian movers merge into one and the sector filled by the Brownian movers exhibits a cusp. Then the different segments keep on merging as time decreases until they give a simply connected support for $t \to 0$. This follows the intuition that all the particles leave 0 in one group which step by step splits into smaller groups to end up with q groups reaching the end points at $t = 1$ (see figure 1 for $q = 2$).

Remark 6.1 For the following, it is interesting to note that the critical times when two disjoint segments merge correspond to singular spectral curves in the sense of definition 2.3. Indeed, at this time, the spectral curve has a double branch point at the location where the two simple branch points merge.

6.2.2 Replica formula and spectral curve

Let us now follow another approach, exact for finite N, exhibiting the role played by the spectral curve directly in the formulation of the kernel H_N. For this purpose, we sketch the derivation of a double integral representation of the kernel using the replica method developed in this context by Brézin and Hikami [24, 25, 26, 27].

Let us first consider the "Fourier" transforms of the correlation functions

$$U_l(t_1, t_2, \ldots, t_l) = \left\langle \prod_{i=1}^{l} \mathrm{Tr}\, e^{iNt_i M} \right\rangle \tag{6-24}$$

and, in particular, one gets the Fourier transform of the two points correlation function:

$$U_2(t_1, t_2) = \frac{1}{Z(A)N^2} \sum_{\alpha_1, \alpha_2 = 1}^{N} \int \left(\prod_{j=1}^{N} dx_j \right) \frac{\Delta(x)}{\Delta(a)} e^{-N \sum_{j=1}^{N} \left[\frac{x_j^2}{2} - x_j \left(a_j + it_1 \delta_{j,\alpha_1} + it_2 \delta_{j,\alpha_2} \right) \right]}.$$

$$(6\text{-}25)$$

One can now integrate the variables x_j by noting that

$$\int \left(\prod_{j=1}^{N} dx_j \right) \Delta(x) e^{-N \sum_{j=1}^{N} \left[\frac{x_j^2}{2} + x_j b_j \right]} = \Delta(b) e^{\frac{N}{2} \sum_{j=1}^{N} b_j^2} \qquad (6\text{-}26)$$

and using the expansion $\Delta(x) = \prod_{i \neq j} (x_i - x_j)$:

$$U_2(t_1, t_2) = \sum_{\alpha_1, \alpha_2 = 1}^{N} e^{N \left(it_1 a_{\alpha_1} + it_2 a_{\alpha_2} - \frac{t_1^2 + t_2^2}{2} - t_1 t_2 \delta_{\alpha_1, \alpha_2} \right)} \times$$

$$\times \frac{\prod_{1 \leq l < m \leq N} (a_l - a_m + it_1 (\delta_{l,\alpha_1} - \delta_{m,\alpha_1}) + it_2 (\delta_{l,\alpha_2} - \delta_{m,\alpha_2}))}{\prod_{1 \leq l < m \leq N} (a_l - a_m)}.$$

$$(6-27)$$

One can see that this can be written as a double contour integral

$$U_2(t_1, t_2) = \frac{e^{-N \frac{t_1^2 + t_2^2}{2}}}{t_1 t_2} \oint \oint \frac{dudv}{(2i\pi)^2} e^{Ni(t_1 u + t_2 v)} \frac{(u - v + it_1 - it_2)(u - v)}{(u - v + it_1)(u - v - it_2)} \times$$

$$\times \prod_k \left(1 + \frac{it_1}{u - a_k} \right) \left(1 + \frac{it_2}{v - a_k} \right)$$

$$(6-28)$$

or

$$U_2(t_1, t_2) = \frac{e^{-N \frac{t_1^2 + t_2^2}{2}}}{t_1 t_2} \oint \oint \frac{dudv}{(2i\pi)^2} e^{Ni(t_1 u + t_2 v)} \left(1 - \frac{t_1 t_2}{(u - v + it_1)(u - v - it_2)} \right) \times$$

$$\times \prod_k \left(1 + \frac{it_1}{u - a_k} \right) \left(1 + \frac{it_2}{v - a_k} \right)$$

$$(6-29)$$

where the integration contours encircle all the eigenvalues a_k and the pole $v = u - it_1$[15].

We can now go back to the correlation function

$$R_2(\lambda, \mu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dt_1 dt_2}{4\pi^2} e^{-iN(t_1 \lambda + t_2 \mu)} U(t_1, t_2).$$

$$(6\text{-}30)$$

---

[15]see for instance [24] for more details around eq.(2-20) and eq.(4-40).

By first integrating on t_1 and t_2 with the shifts $t_1 \to t_1 - iu$ and $t_2 \to t_2 - iu$, we can show that:

$$R_2(\lambda, \mu) = K_N(\lambda, \lambda)K_N(\mu, \mu) - K_N(\mu, \lambda)K_N(\lambda, \mu) \tag{6-31}$$

where the kernel is defined by

$$K_N(\lambda, \mu) = \int \frac{dt}{2\pi} \oint \frac{dv}{2i\pi} \prod_{k=1}^{N} \left(\frac{it - a_k}{v - a_k}\right) \frac{1}{v - it} e^{-N\left(\frac{v^2 + t^2}{2} + it\lambda - v\mu\right)} \tag{6-32}$$

where the integration contour for v goes around all the points a_k and the and the integration for t is parallel to the real axis and avoids the v contour. Moreover, it is straightforwardly proven that any k-point function can be written as the Fredholm determinant:

$$R_k(x_1, \ldots, x_k) = \det \left[K_N(x_i, x_j)\right]_{i,j=1}^{k}. \tag{6-33}$$

By Wick rotating the integration variable $t \to it$, one gets

$$K_N(\lambda, \mu) = \int \frac{dt}{2i\pi} \oint \frac{dv}{2i\pi} \prod_{k=1}^{N} \left(\frac{t - a_k}{v - a_k}\right) \frac{1}{v - t} e^{-N\left(\frac{v^2 - t^2}{2} + t\lambda - v\mu\right)} \tag{6-34}$$

where the integration contour for t is now parallel to the imaginary axis. One can then rewrite it under a more factorized form:

$$K_N(\lambda, \mu) = \int \frac{dt}{2i\pi} \oint \frac{dv}{2i\pi} e^{-N(S(\mu,v) - S(\lambda,t))} \frac{1}{v - t} \tag{6-35}$$

where

$$S(x, y) = \frac{y^2}{2} - xy + \sum_{i=1}^{q} \epsilon_i \ln(y - a_i) \tag{6-36}$$

with the "filling fractions" given by

$$\epsilon_i := \frac{n_i}{N}. \tag{6-37}$$

How to compute such an integral ? Let us use the saddle point method for both integrals. The saddle points of the first exponential are given by the y solutions of

$$\partial_y S(x, y) = y - x + \sum_{i=1}^{k} \frac{\epsilon_i}{y - a_i} = 0 \tag{6-38}$$

which is nothing but the equation of the spectral curve eq.(6-19) !

In this setup, the spectral curve can thus be seen as the location of the saddle points of the action $S(x, y)$ in this formulation of the kernel. Remark also that this formulation of the kernel is very similar to the formulation of th.4.4 in terms of symplectic invariants.

6.2.3   From the replica formula to the symplectic invariants formalism

Let us now start from the expression of th.4.4 for the kernel. For this purpose, one has to compute the one form $y\,dx$. Using the global parameterization

$$\begin{cases} x(z) = z + \sum_i \frac{\epsilon_i}{z - a_i(t)} \\ y(z) = z \end{cases}, \tag{6-39}$$

one gets

$$y\,dx(z) = z\,dz + \sum_i \frac{\epsilon_i\,dz}{z - a_i(t)}. \tag{6-40}$$

Integrating by parts, one can see that

$$\int_{z_2}^{z_1} y\,dx = -\int_{z_2}^{z_1} x(z)\,dy(z) + [x(z)y(z)]_{z_2}^{z_1}. \tag{6-41}$$

Moreover, the function $y(z)$ can also be considered as a global coordinate on the spectral curve since it coincides with the z coordinate. The previous equation can hence be written

$$\int_{y_2}^{y_1} y\,dx(y) = -\int_{y_2}^{y_1} x(y)\,dy + [x(y)y]_{y_2}^{y_1}. \tag{6-42}$$

On the other hand, the spectral curve has a particular form: it has only one x-sheet and takes the form

$$H(x, y) = x - x(y) = 0 \tag{6-43}$$

with the function

$$x(y) = y + \sum_i \frac{\epsilon_i}{y - a_i(t)}. \tag{6-44}$$

Thus the action, i.e. the integral of the spectral curve wrt y, reads

$$S(x, y) = xy - \int x(y)\,dy \tag{6-45}$$

and further

$$\begin{aligned} S(x_1, y_1) - S(x_2, y_2) \quad &= \quad x_1 y_1 - x_2 y_2 - \int_{y_1}^{y_2} x(y)\,dy \\ &= \quad \int_{y_2}^{y_1} y\,dx(y) + y_1\left[x_1 - x(y_1)\right] + y_2\left[x_2 - x(y_2)\right]. \end{aligned}$$
$$(6-46)$$

It is now possible to compare the kernel built from the symplectic invariants and the spectral curve

$$H_N(x_1, x_2) = \frac{1}{x_1 - x_2} e^{N \int_{y(x_2)}^{y(x_1)} y\,dx(y)} \left[1 + O\left(\frac{1}{N}\right)\right] \tag{6-47}$$

and the kernel following the replica formula

$$\widetilde{H}_N(x_1, x_2) = \oint \oint dy_1 dy_2 \frac{1}{y_1 - y_2} e^{N[S(x_1, y_1) - S(x_2, y_2)]}. \tag{6-48}$$

Indeed, the later reads

$$\widetilde{H}_N(x_1, x_2) = \oint \oint dy_1 dy_2 H_N(x(y_1), x(y_2)) e^{N\{y_1[x_1 - x(y_1)] + y_2[x_2 - x(y_2)]\}}. \qquad (6\text{-}49)$$

The saddle point equation directly then states the equality of both kernels in the large N limit if the integration contours are good steepest descent contours.

Remark 6.2 Note that this particular representation of the kernel under the form of a double integral on the complex plane (or more precisely the Riemann sphere) is a direct consequence of the specific parameterization of the spectral curve $x = x(y)$.

6.2.4 Critical behaviors and singular spectral curves

Since both kernels coincide, one can use the singular limits derived in the symplectic invariant setup to obtain some critical universal behaviors of the exclusion process.

● **Universality on the Edge: the Airy kernel**

Let us study the statistic of Brownian movers around the edge of their limiting support. It means that one is interested in a point (x, t) of the space time such that $(x, y(x))$ is close to a branch point $(x_c, y_c) = (x(z_c), y(z_c))$ of the spectral curve at time t_c. Following the study of section 4.8, let us rescale the global coordinate $z = z_c + \frac{1}{N^{\frac{1}{3}}} Z$ and one gets the blown up spectral curve

$$\begin{cases} x(z) = x(z_c) + \frac{Z}{N^{\frac{1}{3}}} x'(z_c) + \frac{Z^2}{2N^{\frac{2}{3}}} x''(z_c) + O\left(\frac{1}{N}\right) \\ y(z) = y(z_c) + \frac{Z}{N^{\frac{1}{3}}} y'(z_c) + O\left(\frac{1}{N^{\frac{2}{3}}}\right) \end{cases}. \qquad (6\text{-}50)$$

Since the point z_c is a branch point, one has $x'(z_c) = 0$, and the blown up curve reduces to

$$\begin{cases} x_{airy}(Z) = \frac{1}{N^{\frac{2}{3}}} \frac{x''(z_c)}{2} Z^2 \\ y_{airy}(Z) = \frac{1}{N^{\frac{1}{3}}} y'(z_c) Z \end{cases} \qquad (6\text{-}51)$$

to leading order as $N \to \infty$. Remark that the scaling is such that $ydx(Z) = O\left(\frac{1}{N}\right)$. This curve is the Airy curve described in the second example of section 2.1.1. Theorem 4.22 states that, in terms of the rescaled variable Z, i.e. the distance from the considered critical point, the kernel and the correlation functions reduce to the one of the Airy curve, independently of the position of the critical point.

Note that this Airy curve has also the form $x = x(y)$ where the function $x(y) = \frac{x''(z_c)}{2y'(z_c)^2} y^2$. The kernel can thus also be written under a double integral form

$$H_{Airy}(x_1, x_2) = \oint \oint \frac{dy_1 dy_2}{y_1 - y_2} e^{S_{Airy}(x_1, y_1) - S_{Airy}(x_1, y_1)} \qquad (6\text{-}52)$$

with $S_{Airy}(x, y) = xy - \frac{x''(z_c)}{y'(z_c)^2} \frac{y^3}{6}$, i.e. the Airy kernel.

● **Universality at the cusp: the Pearcy kernel**

Let us finally consider a point where two groups of Brownian movers merge. This is obtained when two edges merge or, in the spectral curve formalism, when two branch points merge as time decreases. At this critical time t_c, the corresponding spectral curve is singular since the merging of two simple branch points gives rise to a double branch point. Let us be more specific: one considers a cusp at the position (x_c, t_c) in the space time. Let us blow up the space time around this point using the rescaling

$$
\begin{cases}
t := t_c + \frac{\alpha_t T}{N^{\frac{1}{2}}} \\
z := z_c + \frac{\alpha_y Z}{N^{\frac{1}{4}}}
\end{cases}
\tag{6-53}
$$

where z is the global parameter of the spectral curve. The rational parameterization of the spectral curve thus reads

$$
\begin{cases}
x(z,t) = x(z_c, t_c) + \frac{\alpha_y Z}{N^{\frac{1}{4}}} \partial_z x(z_c, t_c) + \frac{\alpha_y^2 Z^2}{2N^{\frac{1}{2}}} (\partial_z)^2 x(z_c, t_c) + \frac{\alpha_y^3 Z^3}{6N^{\frac{3}{4}}} (\partial_z)^3 x(z_c, t_c) \\
\quad + \frac{\alpha_y \alpha_t Z T}{N^{\frac{3}{4}}} \partial_z \partial_t x(z_c, t_c) + O\left(\frac{1}{N}\right) \\
y(z) = y(z_c) + \frac{\alpha_y Z}{N^{\frac{1}{4}}} y'(z_c) + O\left(\frac{1}{N^{\frac{2}{3}}}\right)
\end{cases}
\tag{6-54}
$$

Since the critical point is a double branch point, the blown up curve reduces to

$$
\begin{cases}
x(Z,T) = \frac{1}{N^{\frac{3}{4}}} \left[\frac{\alpha_y^3 Z^3}{6} (\partial_z)^3 x(z_c, t_c) + \alpha_y \alpha_t Z T \partial_z \partial_t x(z_c, t_c) \right] \\
y(Z) = \frac{\alpha_y Z}{N^{\frac{1}{4}}} \partial_z y(z_c)
\end{cases}
\tag{6-55}
$$

which could be called the Pearcy curve. Indeed, since this curve also has the form $x = x(y)$, the associated kernel has the double contour integral representation

$$
H_{Pearcy}(x_1, x_2) = \oint \oint \frac{dy_1 dy_2}{y_1 - y_2} e^{S_{Pearcy}(x_1, y_1) - S_{Pearcy}(x_1, y_1)}
\tag{6-56}
$$

with $S_{Pearcy}(x, y) = xy - \left(\frac{\alpha_y^2 (\partial_z)^3 x(z_c, t_c)}{(\partial_z y(z_c))^3} \right) \frac{y^4}{24} - \left(\frac{\alpha_t \partial_z \partial_t x(z_c, t_c)}{2 \partial_z y(z_c)} \right) y^2 T$. This is noting but the Pearcy kernel once the right integration contour is found and the rescaling coefficients are fixed by:

$$
\frac{\alpha_y^2 (\partial_z)^3 x(z_c, t_c)}{(\partial_z y(z_c))^3} = 6
\tag{6-57}
$$

and

$$
\frac{\alpha_t \partial_z \partial_t x(z_c, t_c)}{2 \partial_z y(z_c)} = 1.
\tag{6-58}
$$

7 Enumeration of discrete surfaces or maps

The symplectic invariants provide a solution to Tutte's equations for counting discrete surfaces (also called maps), for arbitrary topologies. Indeed, it was found by Brezin-Itzykson-Parisi-Zuber [28], and further developed by [12, 35, 79], that generating functions for discrete surfaces can be written as formal matrix models. Let us review how to enumerate various ensembles of discrete surfaces.

7.1 Introduction

Definition 7.1 *Let $\mathbb{M}_n^{(g)}$ be the set of connected orientable discrete surfaces of genus g obtained by gluing together polygonal faces, namely n_3 triangles, n_4 quadrangles, ... n_k k-angles, as well as n marked polygonal faces of perimeters $l_1, \ldots, l_n$, each of the marked faces having one marked edge on its boundary. Let us call v the number of vertices of a discrete surface, and let $\mathbb{M}_n^{(g)}(v)$ be the set of discrete surfaces in $\mathbb{M}_n^{(g)}$, with v vertices.*

We require that unmarked faces have perimeter ≥ 3, whereas marked faces are only required to have perimeter $l_i \geq 1$.

Marked faces are also called "boundaries".

Notice that nothing in our definition prevents from gluing a side of a polygon, to another side of the same polygon.

Theorem 7.1 $\mathbb{M}_n^{(g)}(v)$ *is a finite set.*

proof:

Let e be the number of edges of a discrete surface in $\mathbb{M}_n^{(g)}(v)$. The total number of half edges is:

$$2e = \sum_{j \geq 3} j n_j + \sum_{i=1}^n l_i. \tag{7-1}$$

The Euler characteristics is:

$$\chi = 2 - 2g = v - e + n + \sum_{j \geq 3} n_j = v + n - \frac{1}{2}\sum_{j \geq 3}(j-2)n_j - \frac{1}{2}\sum_{i=1}^n l_i. \tag{7-2}$$

This implies:

$$\frac{1}{2}\sum_{j \geq 3}(j-2)n_j + \frac{1}{2}\sum_{i=1}^n l_i = 2g - 2 + n + v \tag{7-3}$$

and therefore the n_j's are bounded, and the l_i's are bounded. There is then a finite number of possible discrete surfaces having a finite number of faces, edges and vertices. $\square$

In order to enumerate discrete surfaces, we define the generating functions:

Definition 7.2 *The generating function is the formal power series in t:*

$$W_n^{(g)}(x_1, \ldots, x_n; t_3, \ldots, t_d; t) = \frac{t}{x_1}\delta_{n,1}\delta_{g,0}$$

$$+ \sum_{v=1}^{\infty} t^v \sum_{S \in \mathbb{M}_n^{(g)}(v)} \frac{1}{\#\mathrm{Aut}(S)} \frac{t_3^{n_3(S)} \ldots t_d^{n_d(S)}}{x_1^{l_1(S)} \ldots x_n^{l_n(S)}} \prod_{i=1}^n \frac{1}{x_i}.$$

$$(7-4)$$

Most often, we will write only the dependance in the x_i's explicitly, and write:

$$W_n^{(g)}(x_1, \ldots, x_n; t_3, \ldots, t_d; t) = W_n^{(g)}(x_1, \ldots, x_n). \tag{7-5}$$

The generating functions counting surfaces with marked faces of given perimeters $l_1, \ldots, l_n$ are by definition:

$$T^{(g)}_{l_1,\ldots,l_n} = (-1)^n \operatorname*{Res}_{x_1 \to \infty} \ldots \operatorname*{Res}_{x_n \to \infty} x_1^{l_1} \ldots x_n^{l_n} \, W_n^{(g)}(x_1, \ldots, x_n) \, dx_1 \ldots dx_n. \qquad (7\text{-}6)$$

Notice that rooted discrete surfaces, i.e. surfaces with only 1 marked edge, have no non-trivial automorphisms, and thus $\#\mathrm{Aut} = 1$ when $n = 1$.

7.2 Tutte's recursion equations

Tutte's equations are recursions on the number of edges [112, 113]. If one erases the marked edge on the 1st marked face whose perimeter is l_1+1, several mutually exclusive possibilities may occur:

- the marked edge separates the marked face with some unmarked face (let us say a j-gon with $j \geq 3$), and removing that edge is equivalent to removing a j-gon (with weight t_j). We thus get a discrete surface of genus g with the same number of boundaries, and the length of the first boundary is now $l_1 + j - 1$.

- the marked edge separates two distinct marked faces (face 1 and face m with $2 \leq m \leq n$,), thus the marked edge of the first boundary is one of the l_m edges of the mth boundary. We thus get a discrete surface of genus g with $n-1$ boundaries. The other $n-2$ boundaries remain unchanged, and there is now one boundary of length $l_1 + l_m - 1$.

- the same marked face lies on both sides of the marked edge, therefore by removing it, we disconnect the boundary. Two cases can occur: either the discrete surface itself gets disconnected into two discrete surfaces of genus h and $g-h$, one having $|J| + 1$ boundaries of lengths j, J, where J is a subset of $K = \{l_2, \ldots, l_n\}$, and the other discrete surface having $k - |J|$ boundaries of lengths $l_1 - 1 - j, K/J$, or the discrete surface remains connected because there was a handle connecting the two sides, and thus by removing the marked edge, we get a discrete surface of genus $g - 1$, with $n + 1$ boundaries of lengths $j, l_1 - j - 1, K$.

This procedure is (up to the symmetry factors) bijective, and all those possibilities correspond to the following recursive equation:

$$\sum_{j=0}^{l_1-1} \left[\sum_{h=0}^{g} \sum_{J \subset K} \mathcal{T}^{(h)}_{j,J} \mathcal{T}^{(g-h)}_{l_1-1-j,K/J} + \mathcal{T}^{(g-1)}_{j,l_1-1-j,K} \right] + \sum_{m=2}^{n} l_m \, \mathcal{T}^{(g)}_{l_m+l_1-1,K/\{l_m\}}$$

$$= \mathcal{T}^{(g)}_{l_1+1,K} - \sum_{j=3}^{d} t_j \mathcal{T}^{(g)}_{l_1+j-1,K}. \qquad (7\text{-}7)$$

This equation is illustrated as follows (where the 1st marked face is the "exterior face"):
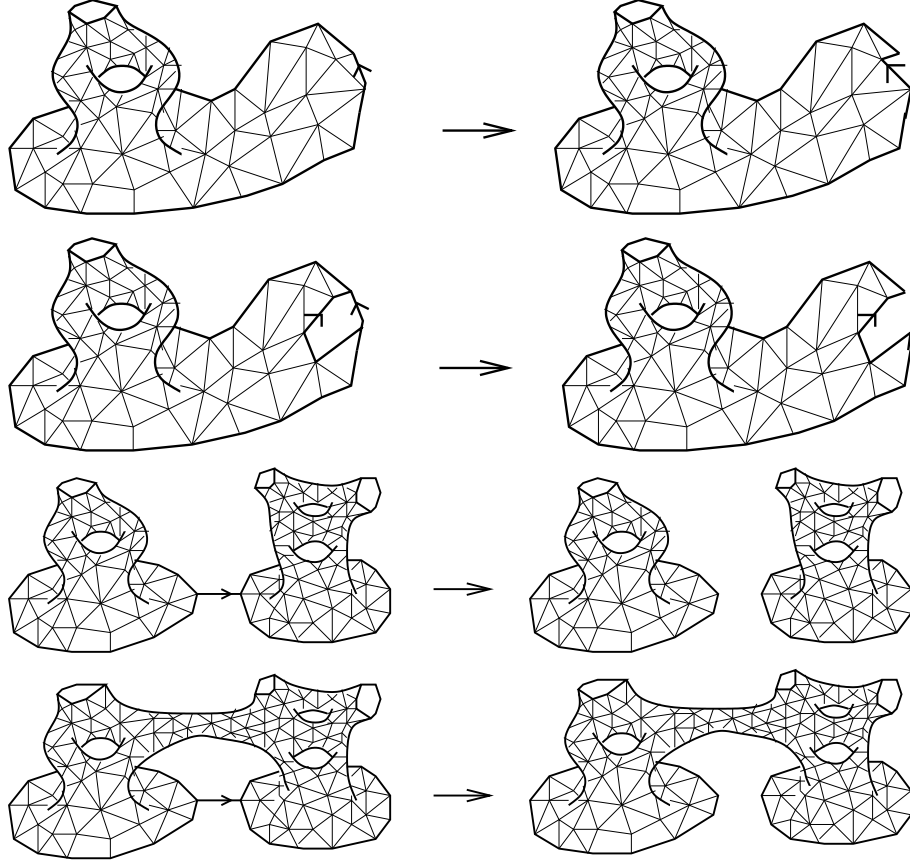
7.3 Loop equations

Rewritten in terms of the $W_n^{(g)}$'s, Tutte's equations eq.(7-7) read:

Theorem 7.2 *Loop equations. For any n and g, and $L = \{x_2, \ldots, x_n\}$, we have:*

$$
\sum_{h=0}^{g} \sum_{J \subset L} W_{1+|J|}^{(h)}(x_1, J) W_{n-|J|}^{(g-h)}(x_1, L/J) + W_{n+1}^{(g-1)}(x_1, x_1, L)
$$
$$
+ \sum_{j=2}^{n} \frac{\partial}{\partial x_j} \frac{W_{n-1}^{(g)}(x_1, L/\{j\}) - W_{n-1}^{(g)}(L)}{x_1 - x_j}
$$
$$
= V'(x_1) W_n^{(g)}(x_1, L) - P_n^{(g)}(x_1, L) \tag{7-8}
$$

where

$$
V'(x) = x - \sum_{j \geq 3} t_j x^{j-1} \tag{7-9}
$$

where $P_n^{(g)}(x_1, L)$ is a polynomial in x_1, of degree $d - 3$ (except $P_1^{(0)}$ which is of degree $d - 2$):

$$
P_n^{(g)}(x_1, x_2, \ldots, x_n) = - \sum_{j=2}^{d-1} t_{j+1} \sum_{i=0}^{j-1} x_1^i \sum_{l_2,\ldots,l_n=1}^{\infty} \frac{\mathcal{T}_{j-1-i,l_2,\ldots,l_n}^{(g)}}{x_2^{l_2+1} \ldots x_n^{l_n+1}} + t\, \delta_{g,0} \delta_{n,1}. \tag{7-10}
$$

proof:

Indeed, if we expand both sides of eq.(7-8) in powers of $x_1 \to \infty$, and identify the coefficients on both side, we find that the negative powers of the x_i's give precisely the loop equations eq.(7-7), whereas the coefficients of positive powers of x_1 cancel due to the definition of $P_n^{(g)}$, which is exactely the positive part of $V'(x_1) W_n^{(g)}$:

$$P_n^{(g)}(x_1, x_2, \ldots, x_n) = \operatorname*{Pol}_{x_1 \to \infty} \left(V'(x_1) W_n^{(g)}(x_1, x_2, \ldots, x_n) \right) \tag{7-11}$$

where Pol means that we keep only the polynomial part, i.e. the positive part of the Laurent series at $x_1 \to \infty$. $\square$

We see that Tutte's equations eq.(7-8) are identical to the matrix model loop equations eq.(5-20). The solution is thus the same, and it is expressed in terms of symplectic invariants. One only has to find the corresponding spectral curve.

7.3.1 Spectral curve and disc amplitude

The spectral curve is given by the function $W_1^{(0)}(x)$.

With $n = 1$ and $g = 0$, the loop equation eq.(7-8) reads:

$$(W_1^{(0)}(x))^2 = V'(x) W_1^{(0)}(x) - P_1^{(0)}(x) \tag{7-12}$$

which implies:

$$W_1^{(0)}(x) = \frac{1}{2} \left(V'(x) - \sqrt{V'(x)^2 - 4P_1^{(0)}(x)} \right) \tag{7-13}$$

where $P_1^{(0)}(x)$ is a polynomial of degree $\deg V' - 1$ in x, namely:

$$P_1^{(0)}(x) = t - \sum_{j=2}^{d-1} t_{j+1} \sum_{i=0}^{j-1} x^i \, \mathcal{T}_{j-1-i}^{(0)}. \tag{7-14}$$

Notice that eq.(7-3) implies that discrete surfaces with $g = 0$ and $n = 1$ must have $v \geq 2$, and thus:

$$\mathcal{T}_j^{(0)} = t\delta_{j,0} + O(t^2). \tag{7-15}$$

Therefore $P_1^{(0)}(x)$ is a formal series in t such that:

$$P_1^{(0)}(x) = t\frac{V'(x)}{x} + O(t^2). \tag{7-16}$$

A general spectral curve with filling fractions $\epsilon_i = \frac{-1}{2i\pi} \oint_{\mathcal{A}_i} W_1^{(0)}(x)dx$, would correspond to (see eq.(5-26)):

$$P_1^{(0)}(x) = \sum_{i=1}^{d-1} \epsilon_i \, \frac{V'(x)}{x - X_i} + O(t^2) \tag{7-17}$$

where X_i, $i = 1, \ldots, d - 1$ are the zeroes of $V'(x)$.

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In other words, the spectral curve counting discrete surfaces, has only one non-vanishing filling fraction, it is a 1-cut spectral curve, or equivalently, it is a genus $\bar{g} = 0$ spectral curve.

More precisely, eq.(7-16) implies that the zeroes of $V'(x)^2 - 4P_1^{(0)}(x)$ have the following small t behaviour:

• two zeroes a, b are of the form:

$$a \sim 2\sqrt{t} + O(t) \quad , \quad b \sim -2\sqrt{t} + O(t); \tag{7-18}$$

• there are $d - 2$ double zeroes of the form $X_i \pm O(t)$, where $V'(X_i) = 0$, and $X_i \neq 0$.

Therefore there exists a, b and a polynomial $M(x)$ such that:

$$V'(x)^2 - 4P_1^{(0)}(x) = (x - a)(x - b) M(x)^2 \tag{7-19}$$

and thus:

$$W_1^{(0)}(x) = \frac{1}{2} \left(V'(x) - M(x)\sqrt{(x - a)(x - b)} \right) \tag{7-20}$$

where $a = 2\sqrt{t} + O(t)$, $b = -2\sqrt{t} + O(t)$, $M(x) = \frac{V'(x)}{x} + O(t)$.

7.3.2 Rational parametrization

Since the spectral curve has only one cut $[a, b]$, it has genus $\bar{g} = 0$, and thus it is a rational spectral curve, and it can be parameterized by rational functions of a complex variable. Here, this can be done very explicitly.

We parameterize x as:

$$x(z) = \frac{a + b}{2} + \frac{a - b}{4} \left(z + \frac{1}{z} \right) = \alpha + \gamma \left(z + \frac{1}{z} \right) \quad , \quad \alpha = \frac{a + b}{2}, \quad \gamma = \frac{a - b}{4}. \tag{7-21}$$

This parametrization is convenient because we have:

$$\sqrt{(x - a)(x - b)} = \gamma \left(z - \frac{1}{z} \right) \tag{7-22}$$

and therefore, from eq.(7-20), we see that $W_1^{(0)}(x(z))$ is a rational fraction of z.

Since $x(z) = x(1/z)$, we can find some complex numbers $u_0, u_1, \ldots, u_{d-1}$ such that:

$$V'(x(z)) = \sum_{k=0}^{d-1} u_k(z^k + z^{-k}) \tag{7-23}$$

and similarly:

$$M(x(z)) \sqrt{(x(z) - a)(x(z) - b)} = \sum_{k=1}^{d-1} \tilde{u}_k(z^k - z^{-k}). \tag{7-24}$$

Thus we have:

$$W_1^{(0)}(x(z)) = u_0 + \frac{1}{2} \sum_{k \geq 1} (u_k - \tilde{u}_k) z^k + \frac{1}{2} \sum_{k \geq 1} (u_k + \tilde{u}_k) z^{-k}. \tag{7-25}$$

Since, by definition, $W_1^{(0)}(x(z))$ contains only negative powers of x, $W_1^{(0)}(x) \sim \frac{t}{x} + O(1/x^2)$, it must contain only negative powers of z, and therefore we must have $u_0 = 0$ and $\tilde{u}_k = u_k$, i.e. $W_1^{(0)}(x(z))$ is a polynomial in $1/z$:

$$W_1^{(0)}(x(z)) = \sum_{k=1}^{d-1} u_k z^{-k}. \tag{7-26}$$

Since $W_1^{(0)}(x) \sim \frac{t}{x}$ at large x, the coefficient of $1/z$ must be $u_1 = t/\gamma$. Therefore, a and b are determined by the two equations:

$$u_0 = 0 \quad , \quad u_1 = \frac{t}{\gamma}. \tag{7-27}$$

All this can be summarized by the theorem:

Theorem 7.3 *Let $V'(x) = x - \sum_{k=2}^{d-1} t_{k+1} x^k$ where t_k is the Boltzmann weight for k-gons. For an arbitrary α and γ, we write:*

$$V'(\alpha + \gamma(z + 1/z)) = \sum_{k=0}^{d-1} u_k(z^k + z^{-k}). \tag{7-28}$$

The coefficients u_k are thus polynomials of α and γ. We determine α and γ by:

$$u_0 = 0 \quad , \quad u_1 = \frac{t}{\gamma} \tag{7-29}$$

and by the conditions that $\alpha = O(t)$ and $\gamma^2 = t + O(t^2)$ at small t.
Then the spectral curve $\mathcal{E} = (\mathbb{P}^1, x, y)$ is:

$$x(z) = \alpha + \gamma(z + 1/z) \quad , \quad y(z) = -W_1^{(0)}(x(z)) = -\sum_k u_k z^{-k}. \tag{7-30}$$

7.3.3 Generating function of the cylinder, annulus

For $n = 2$ and $g = 0$, Tutte's equations give:

$$2W_1^{(0)}(x_1)W_2^{(0)}(x_1, x_2) + \frac{\partial}{\partial x_2} \frac{W_1^{(0)}(x_1) - W_1^{(0)}(x_2)}{x_1 - x_2} = V'(x_1)W_2^{(0)}(x_1, x_2) - P_2^{(0)}(x_1, x_2) \tag{7-31}$$

and one can prove that:

$$W_2^{(0)}(x(z_1), x(z_2)) = \frac{1}{(z_1 - z_2)^2 \, x'(z_1)x'(z_2)} - \frac{1}{(x(z_1) - x(z_2))^2} \tag{7-32}$$

i.e.

$$W_2^{(0)}(x(z_1), x(z_2))dx(z_1)dx(z_2) + \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2} = B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} \tag{7-33}$$

which is the Bergmann kernel on $\mathbb{P}^1$, i.e. the Bergmann kernel of the spectral curve $\mathcal{E} = (\mathbb{P}^1, x, y)$.

7.3.4 Generating functions of discrete surfaces of higher topologies

For arbitrary n and g, the generating functions $W_n^{(g)}$ counting discrete surfaces of genus g with n boundaries, are obtained from the $\omega_n^{(g)}$'s of the spectral curve $\mathcal{E} = (\mathbb{P}^1, x, y)$, by:

$$
\begin{aligned}
\omega_n^{(g)}(z_1, \ldots, z_n) &= W_n^{(g)}(x(z_1), \ldots, x(z_n)) \, dx(z_1) \ldots dx(z_n) \\
&\quad + \delta_{n,2}\delta_{g,0} \, \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2}.
\end{aligned}
\tag{7-34}
$$

In particular, with $n = 0$, the generating function for counting surfaces of genus g and no boundary, is given by the symplectic invariants of $\mathcal{E}$:

$$
\sum_{v=1}^{\infty} t^v \sum_{S \in \mathbb{M}_0^{(g)}(v)} \frac{t_3^{n_3(S)} \ldots t_d^{n_d(S)}}{\#\mathrm{Aut}(S)} = F_g(\mathcal{E}).
\tag{7-35}
$$

7.4 Example quadrangulations

If we count only quadrangulations, we choose $t_4 \neq 0$, and all other $t_k = 0$, i.e.:

$$
V'(x) = x - t_4 x^3.
\tag{7-36}
$$

The spectral curve is found from theorem eq.(7.3). We write:

$$
x(z) = \alpha + \gamma \left(z + \frac{1}{z} \right)
\tag{7-37}
$$

and

$$
\begin{aligned}
&\quad V'(x(z)) \\
&= x(z) - t_4 x^3(z) \\
&= \alpha + \gamma(z + z^{-1}) - t_4 \Big(\alpha^3 + 3\alpha^2\gamma(z + z^{-1}) \\
&\quad + 3\alpha\gamma^2(z^2 + 2 + z^{-2}) + \gamma^3(z^3 + 3z + 3z^{-1} + z^{-3}) \Big).
\end{aligned}
\tag{7-38}
$$

In other words:

$$
2u_0 = \alpha - t_4(\alpha^3 + 6\alpha\gamma^2) \quad , \quad u_1 = \gamma - 3t_4(\alpha^2\gamma + \gamma^3)
\tag{7-39}
$$

$$
u_2 = -3t_4\alpha\gamma^2 \quad , \quad u_3 = -t_4\gamma^3.
\tag{7-40}
$$

The condition $u_0 = 0$ implies:

$$
0 = \alpha(1 - t_4(\alpha^2 + 6\gamma^2)).
\tag{7-41}
$$

Since we must choose a solution where $\alpha = O(t)$ and $\gamma^2 = O(t)$ at small t, we must choose $\alpha = 0$. Then the condition $u_1 = t/\gamma$ gives:

$$
t = \gamma^2 - 3t_4\gamma^4
\tag{7-42}
$$

i.e.

$$\gamma^2 = \frac{1 - \sqrt{1 - 12tt_4}}{6t_4}. \tag{7-43}$$

The spectral curve is then:

$$x(z) = \gamma\left(z + \frac{1}{z}\right) \quad , \quad y(z) = -\frac{t}{\gamma z} + \frac{t_4\gamma^3}{z^3}. \tag{7-44}$$

7.4.1 Rooted planar quadrangulations

The number of planar $(g = 0)$ quadrangulations with only 1 boundary of length $2l$ is $\mathcal{T}_{2l}^{(0)} = \operatorname*{Res}_{\infty} x(z)^{2l}\, y(z)dx(z)$ and thus:

$$\mathcal{T}_{2l}^{(0)} = \gamma^{2l}\, \frac{(2l)!}{l!\,(l+2)!}\, (2(l+1)t - l\gamma^2). \tag{7-45}$$

In particular, if we require all faces, including the marked face of the boundary, to be quadrangles, we choose $2l = 4$, and we find the generating function counting planar quadrangulations with one marked edge (rooted quadrangulations)[16]:

$$\begin{aligned}
\mathcal{T}_4^{(0)} &= \sum_v t^v \sum_{S \in \mathbb{M}_1^{(0)}(v), l(S)=4} t_4^{n_4(S)} \\
&= \gamma^4\, (3t - \gamma^2) \\
&= t^3 \sum_n \frac{2\,\, 3^n\, (2n)!}{n!\,(n+2)!}\, (tt_4)^{n-1}.
\end{aligned} \tag{7-46}$$

Thus we recover the famous result of Tutte [113] that the number of rooted planar quadrangulations with $n = n_4 + 1$ faces is:

$$\frac{2\,\, 3^n\, (2n)!}{n!\,(n+2)!}. \tag{7-47}$$

7.4.2 Quadrangulations of the annulus

The generating function counting quadrangulations of the annulus $g = 0, n = 2$ is given by the Bergmann kernel:

$$\begin{aligned}
W_2^{(0)}(x_1, x_2) &= \sum_v t^v \sum_{S \in \mathbb{M}_2^{(0)}(v)} \frac{t_4^{n_4(S)}}{x_1^{l_1(s)}\, x_2^{l_2(s)}\, \#\mathrm{Aut}(S)} \\
&= \frac{1}{(z_1 - z_2)^2\, x'(z_1)x'(z_2)} - \frac{1}{(x_1 - x_2)^2}
\end{aligned} \tag{7-48}$$

and thus, if we fix the perimeter lengths of the 2 boundaries as $2l_1$ and $2l_2$, we have:

$$\mathcal{T}_{2l_1,2l_2}^{(0)} = \sum_v t^v \sum_{S \in \mathbb{M}_2^{(0)}(v), l_1(S)=2l_1, l_2(S)=2l_2} \frac{t_4^{n_4(S)}}{\#\mathrm{Aut}(S)}$$

---

[16]Notice that discrete surfaces with $n = 1$ marked face and marked edge can have no non-trivial symmetry conserving the marked edge, and thus $\#\mathrm{Aut} = 1$.

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$$= \operatorname*{Res}_{z_1 \to \infty} \operatorname*{Res}_{z_2 \to \infty} \frac{x(z_1)^{2l_1} \, x(z_2)^{2l_2}}{(z_1 - z_2)^2} \, dz_1 dz_2$$

$$= \gamma^{2(l_1+l_2)} \sum_{j=1}^{\min(l_1,l_2)} 2j \, \frac{(2l_1)! \, (2l_2)!}{(l_1+j)!(l_1-j)!(l_2+j)!(l_2-j)!}.$$

$$(7-49)$$

If we require the marked boundary faces to be quadrangles, we choose $2l_1 = 2l_2 = 4$ and thus:

$$
\begin{aligned}
\mathcal{T}_{4,4}^{(0)} &= \sum_v t^v \sum_{S \in \mathbb{M}_2^{(0)}(v), l_1(S)=4, l_2(S)=4} \frac{t_4^{n_4(S)}}{\#\mathrm{Aut}(S)} \\
&= 36 \, \gamma^8 \\
&= 36 \, t^4 \sum_{n \geq 2} \frac{(2n+2)!}{n! \, (n+2)!} \, (3tt_4)^{n-2}
\end{aligned}
\qquad (7\text{-}50)
$$

i.e. the number of annulus quadrangulations with $n = n_4 + 2$ faces, where all faces including the 2 marked faces are quadrangles, is $4 \times 3^n \frac{(2n+2)!}{n!\,(n+2)!}$.

7.4.3 Quadrangulations on a pair of pants

The generating function for quadrangulations on the pair of pants $g = 0, n = 3$, is given by $\omega_3^{(0)}$:

$$\omega_3^{(0)}(z_1, z_2, z_3) = \frac{1}{2\gamma y'(1)} \left(\frac{dz_1}{(z_1-1)^2} \frac{dz_2}{(z_2-1)^2} \frac{dz_3}{(z_3-1)^2} - \frac{dz_1}{(z_1+1)^2} \frac{dz_2}{(z_2+1)^2} \frac{dz_3}{(z_3+1)^2} \right)$$

$$(7\text{-}51)$$

and for instance we find:

$$\mathcal{T}_{4,4,4}^{(0)} = (12)^3 \frac{\gamma^{12}}{2t} \left(1 + \frac{1}{\sqrt{1-12tt_4}} \right) = t^5 \sum_n 2^6 \, 3^n \frac{(2n+1)!}{(n+2)!\,(n-1)!} \, (tt_4)^{n-3} \quad (7\text{-}52)$$

i.e. the number of quadrangulations on a pair of pants, where all n faces, including the 3 marked faces, are quadrangles, is $2^6 \, 3^n \frac{(2n+1)!}{(n+2)!\,(n-1)!}$.

7.4.4 Quadrangulations on a genus 1 disc

The generating function for quadrangulations on the genus 1 disc $g = 1, n = 1$, is given by $\omega_1^{(1)}$:

$$\omega_1^{(1)}(z) = \frac{-z + 8z^3 - z^5 + \gamma^2(z - 5z^3 + z^5)}{3(\gamma^2 - 2)^2(z^2 - 1)^4} \qquad (7\text{-}53)$$

and:

$$\mathcal{T}_4^{(1)} = \frac{\gamma^6}{(\gamma^2 - 2)^2} = \frac{1}{6t_4} \left(\frac{1}{1 - 12t_4} - \frac{1}{\sqrt{1 - 12t_4}} \right) \qquad (7\text{-}54)$$

$$\mathcal{T}_4^{(1)} = 2 \left(1 + \binom{-1/2}{n} (-1)^{n-1} \right) (12t_4)^{n-1} = 2 \left(1 - \frac{(2n-1)!!}{n! \, 2^n} \right) (12t_4)^{n-1} \quad (7\text{-}55)$$

i.e. the number of rooted quandrangulations of genus 1 with n faces is:

$$\frac{1}{6} \left(1 - \frac{(2n-1)!!}{n! \, 2^n} \right) (12)^n = \frac{3^n}{6} \left(2^{2n} - \frac{(2n)!}{n! \, n!} \right). \qquad (7\text{-}56)$$

7.5 Colored surfaces

Exactly like the 1-matrix integral is related to the enumeration of discrete surfaces, the 2-matrix integral and the chain of matrices are also related to enumeration of discrete surfaces carrying colors (1 color per matrix).

7.5.1 The Ising model on a discrete surface

The Ising model is a problem of enumeration of bicolored discrete surfaces, and it is related to the 2-matrix model. It was introduced by Kazakov [81].

Consider $\mathbb{M}_n^{(g)} =$ the set of connected orientable discrete surfaces of genus g obtained by gluing together polygonal faces of two possible colors (or spin) $\pm$, namely n_{3+} triangles of color $+$, n_{4+} quadrangles of color $+$, ... n_{k+} k-angles of color $+$, and also n_{3-} triangles of color $-$, n_{4-} quadrangles of color $-$, ... n_{k-} k-angles of color $-$, as well as n marked polygonal faces of color $+$ of perimeters $l_1, \ldots, l_n$, each of the marked faces having one marked edge on its boundary. Let us call v the number of vertices of a discrete surface, and let $\mathbb{M}_n^{(g)}(v)$ be the set of discrete surfaces in $\mathbb{M}_n^{(g)}$, with v vertices. We call n_{++} the number of edges with $+$ on both sides, n_{--} the number of edges with $-$ on both sides, and n_{+-} the number of edges separating faces of different colors.

We require that unmarked faces have perimeter ≥ 3, whereas marked faces are only required to have perimeter $l_i \geq 1$. Moreover, we recall that marked faces are required to have color $+$.

We define their generating functions as follows:
the generating function is the formal power series in t:

$$
\begin{aligned}
& W_n^{(g)}(x_1, \ldots, x_n; t_2, t_3, \ldots, t_d; \tilde{t}_2, \tilde{t}_3, \ldots, \tilde{t}_{\tilde{d}}; t) \\
= \quad & \frac{t}{x_1} \delta_{n,1} \delta_{g,0} \\
& + \sum_{v=1}^{\infty} t^v \sum_{S \in \mathbb{M}_n^{(g)}(v)} \frac{1}{\#\mathrm{Aut}(S)} \\
& \frac{t_3^{n_{3+}(S)} \ldots t_d^{n_{d+}(S)} \, \tilde{t}_3^{n_{3-}(S)} \ldots \tilde{t}_d^{n_{\tilde{d}-}(S)}}{x_1^{1+l_1(S)} \ldots x_n^{1+l_n(S)}} \, \frac{t_2^{n_{--}(S)} \, \tilde{t}_2^{n_{++}(S)}}{(t_2 \tilde{t}_2 - 1)^{n_{++}(S) + n_{++}(S) + n_{+-}(S)}}.
\end{aligned}
$$
$(7-57)$

For instance if we want to consider the Ising model on a random triangulation, we choose $t_k = 0$ for $k \geq 4$, and $\tilde{t}_k = 0$ for $k \geq 4$.

One may write Tutte-like recursion equations on the number of edges, which are identical to the loop equations of the 2-matrix model [108, 48], and for which we may use the results of section 5.2. Like in the 1-matrix model, the spectral curve is found by the requirement that $W_1^{(0}(x)$ be a 1-cut solution of some algebraic equation, and have a good small t expansion. The recipe for finding the correct spectral curve is summarized in the following theorem:

Theorem 7.4 *We define the potentials:*

$$V_1'(x) = t_2 x - \sum_{j=2}^{d-1} t_{j+1} x^j \quad , \quad V_2'(y) = \tilde{t}_2 y - \sum_{j=2}^{\tilde{d}-1} \tilde{t}_{j+1} y^j. \tag{7-58}$$

For arbitrary coefficients $\gamma, \alpha_i, \beta_i$, we define the two rational functions:

$$x(z) = \gamma z + \sum_{i=0}^{\tilde{d}-1} \alpha_i z^{-i} \quad , \quad y(z) = \gamma z^{-1} + \sum_{i=0}^{d-1} \beta_i z^i. \tag{7-59}$$

The coefficients $\gamma, \alpha_i, \beta_i$ are uniquely determined by the conditions:

$$V_1'(x(z)) - y(z) \sim \frac{t}{\gamma z} + O(z^{-2}) \quad , \quad V_2'(y(z)) - x(z) \sim \frac{tz}{\gamma} + O(z^2) \tag{7-60}$$

and such that γ^2 as well as α_i, β_i are power series in t which behave like $O(t)$ for $t \to 0$.

Then, the spectral curve is:

$$\mathcal{E}_{\text{Ising}} = (\mathbb{C}P^1, x, y), \tag{7-61}$$

the function $W_1^{(0)}(x)$ is given by:

$$W_1^{(0)}(x(z)) = V_1'(x(z)) - y(z), \tag{7-62}$$

the function $W_2^{(0)}(x_1, x_2)$ is given by:

$$W_2^{(0)}(x(z_1), x(z_2)) \, x'(z_1) x'(z_2) = \frac{1}{(z_1 - z_2)^2} - \frac{x'(z_1) x'(z_2)}{(x(z_1) - x(z_2))^2} \tag{7-63}$$

and all the stable $W_n^{(g)}$'s are given by the $\omega_n^{(g)}$'s of the spectral curve $\mathcal{E}_{\text{Ising}}$ by:

$$W_n^{(g)}(x(z_1), \dots, x(z_n)) \, dx(z_1) \dots dx(z_n) = \omega_n^{(g)}(z_1, \dots, z_n). \tag{7-64}$$

In particular, the generating function for counting bicolored maps with no boundaries are the symplectic invariants $F_g(\mathcal{E}_{\text{Ising}})$.

Example: Ising model on quadrangulations.

We have
$$V_1'(x) = t_2 x - t_4 x^3 \quad , \quad V_2'(y) = \tilde{t}_2 y - \tilde{t}_4 y^3. \tag{7-65}$$

We find:
$$x(z) = \gamma z + \alpha_1 z^{-1} + \alpha_3 z^{-3} \quad , \quad y(z) = \gamma z^{-1} + \beta_1 z + \beta_3 z^3 \tag{7-66}$$

with the equation:
$$\beta_3 = -t_4 \gamma^3 \quad , \quad \beta_1 = t_2 \gamma - 3 t_4 \gamma^2 \alpha_1 \quad , \quad \alpha_3 = -\tilde{t}_4 \gamma^3, \tag{7-67}$$
$$\alpha_1 = \tilde{t}_2 \gamma - 3 \tilde{t}_4 \gamma^2 \beta_1 \quad , \quad \alpha_1 \beta_1 + 3 \alpha_3 \beta_3 = \gamma^2 + t. \tag{7-68}$$

That gives an algebraic equation for γ^2:

$$3 t_4 \tilde{t}_4 \gamma^4 + \frac{(t_2 - 3\tilde{t}_2 t_4 \gamma^2)(\tilde{t}_2 - 3 t_2 \tilde{t}_4 \gamma^2)}{(1 - 9\tilde{t}_4 t_4 \gamma^4)^2} = 1 + \frac{t}{\gamma^2} \tag{7-69}$$

and we choose the unique solution such that:

$$\gamma^2 \sim \frac{t}{t_2 \tilde{t}_2 - 1} + O(t^2). \tag{7-70}$$

7.5.2 The chain of matrices discrete surfaces

A chain of matrices (see section 5.3), with m matrices $M_1, \ldots, M_m$ can also be interpreted as a generating function for enumerating discrete surfaces with m possible colors. The "colors" are labeled $1, \ldots, m$.

We are going to consider discrete surfaces, whose unmarked faces can have any color, and are at least triangles, and marked faces have color 1.

Consider $\mathbb{M}_n^{(g)}$ be the set of connected orientable discrete surfaces of genus g obtained by gluing together polygonal faces of m possible colors $k = 1, \ldots, m$ (let $n_{j,k}$ be the number of faces of size j and color k, we assume $j \geq 3$), and n marked faces of color 1, with a marked edge, and of size $l_i \geq 1$, $i = 1, \ldots, n$.

Let $n_{<i,j>}$ be the number of edges such that the two sides are faces of color i and j.

Let C be the following Toeplitz matrix of color couplings:

$$
C^{-1} = \begin{pmatrix}
t_{2,1} & -1 & 0 & \cdots & & 0 \\
-1 & t_{2,2} & -1 & & & \\
0 & \ddots & \ddots & \ddots & & 0 \\
& & -1 & t_{2,m-1} & -1 \\
0 & & 0 & -1 & t_{2,m}
\end{pmatrix}. \tag{7-71}
$$

We define their generating functions as follows. The generating function is the formal power series in t:

$$
\begin{aligned}
& W_n^{(g)}(x_1, \ldots, x_n; t_{i,j}; t) \\
={} & \frac{t}{x_1} \delta_{n,1} \delta_{g,0} \\
& + \sum_{v=1}^{\infty} t^v \sum_{S \in \mathbb{M}_n^{(g)}(v)} \frac{\prod_{i \geq 3} \prod_{j=1}^m t_{i,j}^{n_{i,j}(S)}}{x_1^{1+l_1(S)} \ldots x_n^{1+l_n(S)}} \frac{\prod_{i,j=1}^m C_{i,j}^{n_{<i,j>}(S)}}{\#\mathrm{Aut}(S)}.
\end{aligned}
$$
$$(7-72)$$

Again, one may write Tutte-like recursion equations on the number of edges, which are identical to the loop equations of the chain of matrices model [51], and we may apply the results of section 5.3. Like in the 1-matrix model, the spectral curve is found by the requirement that $W_1^{(0)}(x)$ be a 1-cut solution of some algebraic equation, and have a good small t expansion. The recipe for finding the correct spectral curve is summarized in the following theorem:

Theorem 7.5 *We define the potentials $V_1, \ldots, V_m$ by:*

$$
V_k'(x) = t_{2,k} x - \sum_{j=2}^{d_k-1} t_{j+1,k} x^j, \qquad k = 1, \ldots, m. \tag{7-73}
$$

For arbitrary coefficients $\alpha_{i,k}$, we define the rational functions:

$$x_k(z) = \sum_{j=-s_k}^{r_k} \alpha_{j,k} z^j \tag{7-74}$$

with:

$$r_1 = 1, \qquad r_{k+1} = r_k.(d_k - 1), \tag{7-75}$$

$$s_m = 1, \qquad s_{k-1} = s_k.(d_k - 1). \tag{7-76}$$

The coefficients $\gamma, \alpha_{i,k}$ are uniquely determined by the conditions:

$$\alpha_{1,1} = \alpha_{-1,m} = \gamma, \tag{7-77}$$

$$\forall\, k = 2, \ldots, m-1\,, \qquad V_k'(x_k(z)) = x_{k-1}(z) + x_{k+1}(z), \tag{7-78}$$

$$\forall\, k = 1, \ldots, m-1\,, \qquad \sum_j j\,\alpha_{j,k+1}\alpha_{-j,k} = t \tag{7-79}$$

and such that γ^2 as well as $\alpha_{i,k}$ are power series in t which behave like $O(t)$ for small t.

   Then, the spectral curve is:

$$\mathcal{E}_{\text{ch.mat}} = (\mathbb{C}P^1, x_1, x_2), \tag{7-80}$$

the function $W_1^{(0)}(x)$ is given by:

$$W_1^{(0)}(x_1(z)) = V_1'(x_1(z)) - x_2(z), \tag{7-81}$$

the function $W_2^{(0)}(x_1(z_1), x_1(z_2))$ is given by:

$$W_2^{(0)}(x_1(z_1), x_1(z_2))\, x_1'(z_1) x_2'(z_2) = \frac{1}{(z_1 - z_2)^2} - \frac{x_1'(z_1) x_1'(z_2)}{(x_1(z_1) - x_1(z_2))^2} \tag{7-82}$$

and all the stable $W_n^{(g)}$'s are given by the $\omega_n^{(g)}$'s of the spectral curve $\mathcal{E}_{\text{ch.mat}}$ by:

$$W_n^{(g)}(x_1(z_1), \ldots, x_1(z_n))\, dx_1(z_1) \ldots dx_1(z_n) = \omega_n^{(g)}(z_1, \ldots, z_n), \tag{7-83}$$

i.e. F_g is independent of k.

Remark 7.1 Because of eq.(7-78), we see that the spectral curves $(\mathbb{C}P^1, x_k, x_{k+1})$ are all symplectically equivalent for any $k = 1, \ldots, m-1$, and thus, we have:

$$\forall\, k = 1, \ldots, m-1\,, \qquad F_g = F_g((\mathbb{C}P^1, x_k, x_{k+1})) = F_g((\mathbb{C}P^1, x_{k+1}, x_k)). \tag{7-84}$$

8 Double scaling limits and large maps

8.1 Minimal models and continuous surfaces

In the preceding section, we explained how the symplectic invariants can be used to count discrete surfaces. The theorem 4.23, allows to find various limits of symplectic invariants, and here, it can be used to find the asymptotics of generating functions of large discrete surfaces [78].

The conjecture [116] was that large discrete surfaces tend towards continuous surfaces weighted with the Liouville theory action, possibly coupled to some conformal matter fields.

The idea is to count discrete surfaces made of large numbers of polygons, and send the size of polygons to 0, so that the total area remains finite [29, 42, 69].

In section 7 we found the generating function for counting maps of genus g as the symplectic invariants $F_g(\mathcal{E})$ of some rational spectral curve $\mathcal{E}$. In this enumeration of maps, the expectation value of the number of k-gons in the considered maps is given by

$$\langle n_k \rangle = t_k \frac{\partial \ln F_g}{\partial t_k} \tag{8-1}$$

and the expectation value of the number of vertices is:

$$\langle v \rangle = t \frac{\partial \ln F_g}{\partial t}. \tag{8-2}$$

Large discrete surfaces are obtained when those numbers diverge, i.e. when the parameters t_k (or t) approach a singularity, for which the spectral curve is singular (in the sense of eq.(2.3)).

Let us now study the blow up of the matrix models' spectral curves around these singularities.

8.2 Minimal model (p, q) and KP hierarchy

For discrete surfaces (formal 1-matrix model), or bicolored discrete surfaces with an Ising model (2-matrix model), or for the formal chain of matrices, the spectral curve depends on the parameter t in an algebraic way. We choose potentials V_k as well as a critical $t = t_c$, such that at $t = t_c$ the spectral curve has a cusp singularity of the type $y(z) \sim (x(z) - x(a))^{p/q}$ (see [34] for a list of critical potentials of minimal degrees, for the 2-matrix model).

We expand $\mathcal{E}$ in the vicinity of the critical branchpoint $z \to a$ and $t \to t_c$:

$$z = a + (t - t_c)^\nu \zeta \tag{8-3}$$

$$\mathcal{E} \sim \begin{cases} x(z) = x(a) + (t - t_c)^{q\nu}\, Q(\zeta) + o((t - t_c)^{q\nu}) \\ y(z) = y(a) + (t - t_c)^{p\nu}\, P(\zeta) + o((t - t_c)^{p\nu}) \end{cases} \tag{8-4}$$

where Q and P are polynomials of the complex variable ζ, of respective degrees q and p, and where the exponent ν is a scaling exponent such that the blown up spectral curve is regular.

We define the double scaling limit [78] spectral curve as the blow-up of the singularity:

$$\mathcal{E}_{(p,q)} = \begin{cases} x(\zeta) = Q(\zeta) \\ y(\zeta) = P(\zeta) \end{cases}.$$ (8-5)

In order to find the exponent ν, one may notice that for the chain of matrices (and thus also 1-matrix and 2-matrix model), the derivative with respect to t of the form ydx is:

$$\frac{d}{dt}(ydx(z)) = \frac{dz}{z} \sim (t - t_c)^\nu \frac{1}{a} d\zeta.$$ (8-6)

Comparing this with eq.(8-4), it is easy to see that this implies:

$$\nu = \frac{1}{p+q-1}$$ (8-7)

and:

$$pP(\zeta)Q'(\zeta) - qQ(\zeta)P'(\zeta) = \frac{1}{a}.$$ (8-8)

Theorem 4.23 of section 4.8, imply that the symplectic invariants defined in eq.2-46 give the double scaling limit:

$$F_g(\mathcal{E}) \sim (t - t_c)^{(2-2g)\frac{p+q}{p+q-1}} F_g(\mathcal{E}_{(p,q)}) (1 + o(1)).$$ (8-9)

The spectral curve eq.(8-5), is the spectral curve of the (p, q) minimal model in conformal field theory [39], coupled to gravity [84]. It corresponds to a finite dimensional irreducible representation of the group of conformal transformations, it has a central charge:

$$c = 1 - 6\frac{(p-q)^2}{pq}.$$ (8-10)

The exponent $\nu = \frac{1}{p+q-1}$ or more precisely the exponent "$\gamma-$string":

$$\gamma = 2 - 2(p+q)\nu$$ (8-11)

is given by the famous KPZ formula [39, 84]. Notice that the symplectic invariance of the F_g's under $x \leftrightarrow y$, is related to the $(p,q) \leftrightarrow (q,p)$ duality [83].

The corresponding tau function is:

$$\tau(N) = \exp\left(\sum_{g=0}^{\infty} N^{2-2g} F_g\left(\mathcal{E}_{(p,q)}\right)\right).$$ (8-12)

It is a tau function of the (p,q) reduction of the integrable hierarchy of Kadamtsev-Petviashvili (KP) [14]. The function

$$F(t) = \sum_{g=0}^{\infty} t^{(2-2g)\frac{p+q}{p+q-1}} F_g\left(\mathcal{E}_{(p,q)}\right) = \ln\left(\tau(t^{\frac{p+q-1}{p+q}})\right)$$ (8-13)

and its second derivative

$$u(t) = F'''(t) \qquad (8\text{-}14)$$

can be found as follows: find two differential operators $\hat{P} = d^p + pud^{p-2} + \dots$ and $\hat{Q} = d^q + ud^{q-2} + \dots$ (where $d = d/dt$) satisfying the string equation:

$$[\hat{P}, \hat{Q}] = 1. \qquad (8\text{-}15)$$

This equation implies the differential equation of KP for the function $u(t)$.

For example, for pure gravity $(p, q) = (3, 2)$ (with central charge $c = 0$), we have:

$$\hat{Q} = d^2 - 2u \quad , \quad \hat{P} = d^3 - 3ud - \frac{3}{2}u' \qquad (8\text{-}16)$$

and $[\hat{P}, \hat{Q}] = 1$ imply the Painlevé equation for $u = F''$:

$$3u^2 - \frac{1}{2}u'' = t. \qquad (8\text{-}17)$$

8.3 Minimal model $(p, 2)$ and KdV

As a special case, we consider the 1-matrix model. In that case, the spectral curve is always hyperelliptical $y^2 = \mathrm{Pol}(x)$, and thus the only cusp singularities must be half integers $y \sim x^{p/2}$, i.e. $q = 2$ and $p = 2k + 1$.

The operators $\hat{Q}$ and $\hat{P}$ are such that:

$$\hat{Q} = d^2 - 2u(t), \qquad (8\text{-}18)$$

$$\hat{P} = d^{2k+1} - (2k+1)u(t)d^{2k-1} + \sum_{j=0}^{2k-2} v_j(t)d^j. \qquad (8\text{-}19)$$

The string equation $[\hat{P}, \hat{Q}] = 1$ implies a differential equation for $u(t)$:

$$R_{k+1}(u) = t \qquad (8\text{-}20)$$

where R_k is the kth Gelfand-Dikii differential polynomial (see [39, 38, 14]). They obey the recursion:

$$R_0 = 2 \quad , \quad R'_{j+1} = -2u\,R'_j - u'\,R_j + \frac{1}{4}\,R'''_j. \qquad (8\text{-}21)$$

The first few of them are:

$$
\begin{aligned}
R_0 &= 2 \\
R_1 &= -2u \\
R_2 &= 3u^3 - \frac{1}{2}u'' \\
R_3 &= -5u^3 + \frac{5}{2}uu'' - \frac{1}{4}u'^2 - \frac{1}{8}u''' \\
&\vdots \\
&(8-22)
\end{aligned}
$$

We have seen in the previous section, that the spectral curve eq.(8-5) of the $(2k + 1, 2)$ scaling limit of the 1-matrix model, is:

$$\mathcal{E}_{(2k+1,2)} = \begin{cases} \tilde{x}(z) = z^2 - 2u_0 \\ \tilde{y}(z) = \sum_{j=0}^{k} \bar{t}_k\, u_0^{k-j}\, z^{2j+1} \end{cases} . \tag{8-23}$$

It is the classical limit of $(\hat{P}, \hat{Q})$, where $d = d/dt$ becomes a complex number z:

$$\begin{array}{ccc} \hat{Q} = d^2 - 2u & \rightarrow & x = z^2 - 2u_0 \\ \hat{P} = d^{2k+1} - (2k+1)ud^{2k-1} + \ldots & \rightarrow & y = z^{2k+1} - (2k+1)u_0 z^{2k-1} + \ldots \end{array}$$
$$(8-24)$$

The symplectic invariants defined in eq.2-46 give the double scaling limit:

$$F_g(\mathcal{E}_{1\mathrm{MM}}) \sim (t - t_c)^{\frac{(2-2g)(2k+3)}{(2k+2)}} F_g(\mathcal{E}_{(2k+1,2)}) + o\left((t - t_c)^{\frac{(2-2g)(2k+3)}{(2k+2)}}\right) \tag{8-25}$$

and, if $z_1, \ldots, z_n$ all lie in the vicinity of $1 + O((t - t_c)^{\frac{1}{2k+2}})$, we have:

$$\omega_n^{(g)}(\mathcal{E}_{1\mathrm{MM}})(z_1, \ldots, z_n) \sim (t - t_c)^{\frac{(2-2g-n)(2k+3)}{(2k+2)}} \omega_n^{(g)}(\mathcal{E}_{(p,2)})(\zeta_1, \ldots, \zeta_n) + o\left((t - t_c)^{\frac{(2-2g-n)(2k+3)}{(2k+2)}}\right).$$
$$(8-26)$$

where $z_i = 1 + (t - t_c)^{\frac{1}{2k+2}} \zeta_i$

The formal function:

$$F(\xi) = \sum_g \xi^{\frac{(2-2g)(2k+3)}{(2k+2)}} F_g(\mathcal{E}_{(2k+1,2)}) \tag{8-27}$$

is such that its second derivative

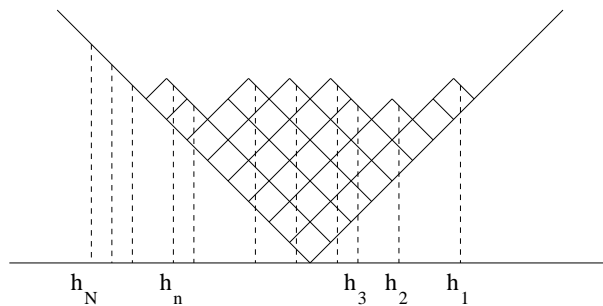$$u(\xi) = F''(\xi) \tag{8-28}$$

satisfies the $k + 1^{\text{st}}$ Gelfand-Dikii equation:

$$R_{k+1}(u) = \xi. \tag{8-29}$$

9 Partitions and Plancherel measure

In many different problems of mathematics or physics, one needs to count partitions with the Plancherel Weight.

Given a partition $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_N \geq 0$, we define:
- its weight $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_N$.
- its length $n(\lambda) = \#\{i, \lambda_i \neq 0\}$, we have $n(\lambda) \leq N$.
- its Plancherel weight:

$$\mathcal{P}(\lambda) = \left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2 = \frac{\prod_{i>j}(h_i - h_j)^2}{\prod_i (h_i!)^2} \tag{9-1}$$

where the h_i's correspond to the down-right edges of the partition rotated by $\pi/4$:

$$h_i = \lambda_i - i + N \quad , \quad h_1 > h_2 > \ldots > h_N \geq 0. \tag{9-2}$$

- The Casimirs:

$$C_k(\lambda) = \sum_{i=1}^N \left(h_i - \frac{N-1}{2}\right)^k - \sum_{i=1}^N \left(\frac{1}{2} - i\right)^k + (1 - 2^{-k})\,\zeta(-k). \tag{9-3}$$

For example

$$C_1(\lambda) = |\lambda| - \frac{1}{24} \quad , \quad C_2(\lambda) = \sum_{i=1}^N \lambda_i(\lambda_i - 2i + 1). \tag{9-4}$$

9.1 The partition function

The partition function one would like to compute is:

$$Z_N(Q) = \sum_{n(\lambda) \leq N} \left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2 Q^{2|\lambda|} \tag{9-5}$$

and particularly its large Q expansion:

$$\ln Z_N(Q) = \sum_g Q^{2-2g}\, F_g. \tag{9-6}$$

Indeed, since Q is coupled to the weight $|\lambda|$ of partitions, the large Q limit corresponds to the limit of large partitions.

More generally, one is also interested in expectation values of moments of Casimirs, and one whishes to compute the partition function:

$$Z_N(Q, t_k) = \sum_{n(\lambda) \leq N} \left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2 Q^{2|\lambda|}\, \mathrm{e}^{\sum_{k=1}^d \frac{t_k\, Q^{1-k}}{k} C_k(\lambda)} \tag{9-7}$$

where we have taken into account the scaling of the Casimirs in the large Q limit. Again, we are interested in the large size expansion:

$$\ln Z_N(Q, t_k) \sim \sum_g Q^{2-2g}\, F_g(t_k, N). \tag{9-8}$$

In the case where $t_k = 0$ for all k, the answer has been known for a long time, and we have the following identity:

$$\sum_\lambda \left(\frac{\dim(\lambda)}{|\lambda|!}\right)^2 Q^{2|\lambda|} = e^{Q^2} \tag{9-9}$$

and therefore:

$$F_g = \delta_{g,0}. \tag{9-10}$$

When some of the t_k's are turned on, the answer can be written in terms of a matrix model and symplectic invariants of an appropriate spectral curve.

Some applications of this model include the statistics of the longest increasing subsequence of a random sequence, which is equivalent to the statistical physics of growing 2D crystals, indeed the Plancherel measure $\mathcal{P}(\lambda)$ is precisely the probability to obtain shape λ by dropping square boxes from the sky at random times and random positions, see [103, 105].

Another application concerns algebraic geometry and topological string theory. The partition function $Z_N(Q, t_k)$ is also the generating function for counting ramified branched coverings of $\mathbb{P}^1$, i.e. the Hurwitz numbers, see [92, 99, 100, 102, 103].

It has been observed in many works [13, 77, 105], that locally, in the large Q limit, the Plancherel statistics of partitions, shows many similarities with universal statistical laws observed in various limits of matrix models. In fact, it was found in [56], that the similarity is much stronger, and in fact $Z_N(Q, t_k)$ **is a matrix model** for all Q (not only large Q). In particular this shows that the F_g's are again symplectic invariants.

9.2   Matrix model for counting partitions

Consider the contour $\mathcal{C}$ which surrounds all positive integer points.
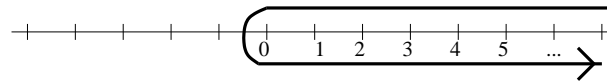
The following matrix integral[17]:

$$Z = \int_{H_N(\mathcal{C})} dM \, e^{-Q \operatorname{Tr} V(M)} \tag{9-11}$$

can be written in eigenvalues:

$$Z = \frac{1}{N!} \int_{\mathcal{C}^N} dx_1 \ldots dx_N \prod_{i>j}(x_i - x_j)^2 \prod_i e^{-QV(x_i)} \tag{9-12}$$

where we choose:

$$e^{-QV(x)} = \frac{e^{i\pi Qx}}{\sin(\pi Qx)} \frac{Q^{2Qx}}{\Gamma(Qx+1)^2} e^{-Q\sum_k \frac{t_k}{k}(x-(N-\frac{1}{2})/Q)^k}. \tag{9-13}$$

---

[17]$H_N(\mathcal{C})$ stands for the set of normal matrices whose eigenvalues lie on a contour $\mathcal{C}$.

The integration over $\mathcal{C}$ picks up residues at the poles of $\mathrm{e}^{-QV(x)}$, i.e. at the poles of $\frac{1}{\sin \pi Q x}$, and thus forces $Q x_i \in \mathbb{N}$, and the factor $\prod_{i>j}(x_i - x_j)^2$ forces $x_i \neq x_j$, thus we have $Q x_i = h_i \in \mathbb{N}$, and since they are dum variables, we can always reorder them so that $h_1 > h_2 > \ldots > h_N \geq 0$. Therefore we have:

$$
\begin{aligned}
Z &= \int_{H_N(\mathcal{C})} dM \ \mathrm{e}^{-Q \operatorname{Tr} V(M)} \\
&= \sum_{h_1 > \ldots > h_N \geq 0} \frac{\prod_{i>j}(h_i - h_j)^2}{\prod_i (h_i!)^2} \ Q^{2 \sum_i h_i} \ \mathrm{e}^{\frac{1}{Q} \sum_k \sum_i \frac{t_k}{k} Q^{-k} (h_i - N + \frac{1}{2})^k} \\
&\propto Z_N(Q, t_k),
\end{aligned}
\tag{9-14}
$$

i.e. we recover the Plancherel generating function $Z_N(Q, t_k)$ for partitions eq.(9-7). Therefore, $Z_N(Q, t_k)$ can be written as a normal matrix integral with eigenvalues supported on $\mathcal{C}$. Since the loop equations are independent of the integration contour (provided there is no boundary term when integrating by parts), we have the same loop equations as for any other matrix model, and thus the same solution in terms of symplectic invariants.

The spectral curve, i.e. the equilibrium density of eigenvalues of the matrix, is computed like for any other matrix model, like in section 5. Here, the potential $V(M)$ may look quite complicated, and the corresponding spectral curve also looks at first sight quite complicated. However, due to the special properties of the Γ functions, it simplifies considerably, and reduces to a rather simple expression, which coincides with a "naive" large Q limit. The naive large Q limit has been computed in many works [93, 102, 103, 114], and has been known for some time. It has the property that it is nearly independent of N. Let us just state the final result, and refer the reader to [56] for details.

The spectral curve is obtained from the following recipe:

Theorem 9.1 *Define:*

$$
U(x) = \sum_{k=1}^{d-1} t_{k+1} x^k.
\tag{9-15}
$$

Then define the coefficients $u_0, u_1, \ldots, u_{d-1}$ by the equation:

$$
U(\mathrm{e}^{-u_0}(z + 1/z - \alpha)) = \sum_{k=0}^{d-1} u_k (z^k + z^{-k}).
\tag{9-16}
$$

Then the spectral curve is:

$$
\mathcal{E}_{\text{Plancherel}} =
\begin{cases}
x(z) = \dfrac{N - \frac{1}{2}}{Q} + \mathrm{e}^{-u_0}(z + 1/z - u_1) \\[2mm]
y(z) = \ln z + \dfrac{1}{2} \sum_{k=1}^{d-1} u_k (z^k - z^{-k})
\end{cases}
\tag{9-17}
$$

and the large Q expansion of $Z_N(Q, t_k)$ is given by:

$$\ln Z_N(Q, t_k) \sim \sum_{g=0}^{\infty} Q^{2-2g} \, F_g(\mathcal{E}_{\text{Plancherel}}) \tag{9-18}$$

where F_g are the symplectic invariants defined in section 2.

Remark 9.1 Since $x \to x - \frac{N-\frac{1}{2}}{Q}$ is a symplectic transformation, F_g is independent of N, and therefore eq.(9-18) seems to be independent of N. In fact, the N dependence is smaller than any power of Q, it is exponentially small. This phenomenon is known as the arctic circle property. All partitions with size $n(\lambda) > Q$ seem to be "frozen" and contribute only exponentially to the partition function.

9.2.1 Example no Casimir

When all $t_k = 0$ we have $U = 0$, i.e. $u_0 = 0$ and $u_1 = 0$. The spectral curve is thus:

$$\begin{cases} x(z) = \frac{N-\frac{1}{2}}{Q} + z + 1/z \\ y(z) = \ln z \qquad = \text{Arcosh}((x - \frac{N-\frac{1}{2}}{Q})/2) \end{cases} \tag{9-19}$$

For that spectral curve one finds:

$$F_0 = 1, F_1 = F_2 = F_3 = \ldots = 0 \tag{9-20}$$

which is in agreement with

$$Z(Q) = e^{Q^2}. \tag{9-21}$$

9.2.2 Example: Plancherel measure with the 2nd Casimir

If we choose $t_2 \neq 0$ and all other $t_k = 0$, we have from eq.(9-16):

$$t_2 e^{-u_0} \left(z + \frac{1}{z} - u_1 \right) = 2u_0 + u_1 \left(z + \frac{1}{z} \right) \tag{9-22}$$

and thus:

$$t_2^2 = -2u_0 e^{2u_0} \qquad , \qquad u_1 = \sqrt{-2u_0}. \tag{9-23}$$

That gives:

$$2u_0 = L(-t_2^2) \tag{9-24}$$

where $L(x)$ is the Lambert function, solution of $Le^L = x$.

The spectral curve is thus:

$$\begin{cases} x(z) = \frac{N-\frac{1}{2}}{Q} + e^{-u_0}(z + 1/z - u_1) \\ y(z) = \ln z + \frac{1}{2}u_1(z - \frac{1}{z}) \end{cases} \tag{9-25}$$

For that spectral curve one finds:

$$F_0 = \frac{e^{-2u_0}}{2}(1 + u_0)(2 - u_0), \tag{9-26}$$

$$F_1 = \frac{1}{24} \ln \left(e^{-2u_0} (1 + 2u_0) \right), \tag{9-27}$$

$$F_2 = \frac{e^{2u_0}}{180} \frac{u_0^3 (1 - 12u_0)}{(1 + 2u_0)^5}, \tag{9-28}$$

and so on...

9.3 q-deformed partitions

The q-deformed Plancherel weight is obtained by replacing integer numbers $h_i \in \mathbb{N}$ by the q−numbers $[h_i] = \frac{q^{\frac{h_i}{2}} - q^{\frac{-h_i}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$, and thus:

$$\mathcal{P}_q(\lambda) = \left(\frac{\dim_q(\lambda)}{[|\lambda|]!} \right)^2 = \left(\frac{\prod_{i>j}[h_i - h_j]}{\prod_{i=1}^{N} \prod_{j=1}^{h_i}[j]} \right)^2 = \frac{\prod_{i>j}(q^{\frac{h_i - h_j}{2}} - q^{\frac{h_j - h_i}{2}})^2}{\prod_{i=1}^{N} \prod_{j=1}^{h_i}(q^{\frac{j}{2}} - q^{\frac{-j}{2}})^2} \tag{9-29}$$

which can also be written:

$$\mathcal{P}_q(\lambda) = \prod_{i>j}(q^{h_i} - q^{h_j})^2 \prod_{i=1}^{N} q^{(1-N)h_i} \ q^{\frac{h_i(h_i+1)}{2}} \left(\frac{g(q^{-h_i})}{g(1)} \right)^2 \tag{9-30}$$

where $g(x)$ is the q-product:

$$g(x) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{x} q^n \right). \tag{9-31}$$

Again, our goal, for various applications in physics and mathematics, is to compute the following sum:

$$Z_N(q; t_k) = \sum_{n(\lambda) \leq N} \mathcal{P}_q(\lambda) \ e^{\frac{1}{\ln q} \sum_k \frac{t_k}{k} (\ln q)^k C_k(\lambda)} \tag{9-32}$$

and one would like to compute it in the $q \to 0$ limit, i.e.

$$\ln Z_N \sim \sum_{g=0}^{\infty} (\ln q)^{2-2g} F_g. \tag{9-33}$$

Again, we will find that the F_g's are the symplectic invariants of some spectral curve.

The main application concerns algebraic geometry and topological strings. $Z_N(q; t_k)$ is the partition function for the Gromov-Witten invariants of some family of Calabi-Yau 3-fold, see [92].

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9.3.1 Matrix model

Again, the idea is to represent the sum eq.(9-32) as a matrix integral.

Consider the contour $\mathcal{C}$ which surrounds all points of the form $1, q, q^2, q^3, \ldots$, i.e. $\mathcal{C}$ is a circle of radius $1 < r < |q^{-1}|$:
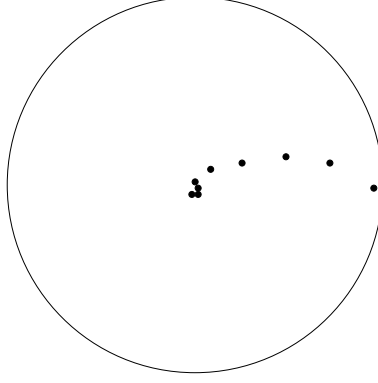
The following matrix integral:

$$Z = \int_{H_N(\mathcal{C})} dM \; e^{\frac{1}{\ln q} \operatorname{Tr} V(M)} \tag{9-34}$$

can be written in eigenvalues:

$$Z = \frac{1}{N!} \int_{\mathcal{C}^N} dx_1 \ldots dx_N \; \prod_{i>j} (x_i - x_j)^2 \; \prod_i e^{\frac{1}{\ln q} V(x_i)} \tag{9-35}$$

where we choose:

$$e^{\frac{1}{\ln q} V(x)} = e^{\frac{1}{\ln q} \sum_k \frac{t_k}{k} (\ln x - (N-\frac{1}{2}) \ln q)^k} \; e^{(1-N)\ln x} \frac{\sqrt{x}\, g(1/x)^2}{g(1)^2} \; e^{\frac{(\ln x)^2}{2\ln q}} \; f(x) \tag{9-36}$$

with

$$f(x) = - \frac{e^{i\pi \frac{\ln x}{\ln q}} \, g(1)^2 \; e^{\frac{(\ln x)^2}{2\ln q}}}{\sqrt{x}(1-x)\, g(x)\, g(1/x)}. \tag{9-37}$$

The integration over $\mathcal{C}$ picks up residues at the poles of $e^{\frac{1}{\ln q} V(x)}$, i.e. at the poles of $f(x)$, and thus forces $\frac{\ln x_i}{\ln q} \in \mathbb{N}$, and the factor $\prod_{i>j}(x_i - x_j)^2$ forces $x_i \neq x_j$, thus we have $x_i = q^{h_i}$ where $h_i \in \mathbb{N}$, and since they are dum variables, we can always reorder them so that $h_1 > h_2 > \ldots > h_N \geq 0$. The residues are such that we have:

$$Z_N(q; t_k) \propto \int_{H_N(\mathcal{C})} dM \; e^{\frac{1}{\ln q} \operatorname{Tr} V(M)}. \tag{9-38}$$

Therefore, as in the preceding case, $Z_N(q, t_k)$ can be written as a normal matrix integral with eigenvalues supported on $\mathcal{C}$. Once again, the small q expansion of the free energy is thus given by the corresponding symplectic invariants.

9.3.2 Spectral curve

It remains to compute the spectral curve, i.e. the equilibrium density of eigenvalues for the matrix potential:

$$
\begin{aligned}
V(x) \;=\; & \sum_k \frac{t_k}{k} \left(\ln x - (N - \tfrac{1}{2}) \ln q \right)^k + (\ln x)^2 + i\pi \ln x \\
& + (1 - N) \ln q \, \ln x + \ln q \, \ln \frac{g(1/x)}{g(x)} - \ln q \ln (1 - x).
\end{aligned}
\tag{9-39}
$$

Again, the spectral curve, i.e. the equilibrium density of eigenvalues of the matrix, is computed like for any other matrix model, like in section 5. Here, the potential $V(M)$ may look quite complicated, and the corresponding spectral curve also looks at first sight quite complicated. However, due to the special properties of the q-product $g(x)$, it simplifies considerably, and reduces to a rather simple expression, which coincides with a "naive" large $\ln q$ limit. The naive large $\ln q$ limit has been computed in many works [92], and has been known for some time. It has the property that it is nearly independent of N. Let us just state the final result, and refer the reader to [56] for details.

In principle, the spectral curve could be found for all t_k's, but for simplicity, we compute it explicitly for $t_3 = t_4 = \ldots = 0$, i.e. for only t_1 and t_2. We write:

$$
t_1 = t \quad , \quad t_2 = p - 1 \quad , \quad t_3 = t_4 = \ldots = 0.
\tag{9-40}
$$

In that case the spectral curve is:

$$
\mathcal{E} =
\begin{cases}
x(z) = \frac{(1 - \frac{z}{z_0})(1 - \frac{1}{z z_0})}{(1 + \frac{1}{z_0})^2} \\[2ex]
y(z) = \frac{1}{x(z)} \left(- \ln z + \frac{p}{2} \ln \left(\frac{1 - z/z_0}{1 - 1/z z_0} \right) \right)
\end{cases}
\quad , \quad
e^{-t} = \frac{1}{z_0^2} \left(1 - \frac{1}{z_0^2} \right)^{p(p-2)}
\tag{9-41}
$$

and thus we have:

$$
Z = \sum_{n(\lambda) \leq N} \left(\frac{\dim_q(\lambda)}{[|\lambda|]!} \right)^2 e^{-t|\lambda|} \, q^{\frac{p-1}{2} C_2(\lambda)} \sim e^{\sum_g (\ln q)^{2-2g} F_g(\mathcal{E})}.
\tag{9-42}
$$

The large $\ln q$ expansion of $\ln Z$ is given by the symplectic invariants of curve $\mathcal{E}$, and this expansion is independent of N, provided that $N > \overline{n}$, where:

$$
\overline{n} = \frac{1}{2} + \ln q \, \left(p \ln (1 - 1/z_0) + (p - 2) \ln (1 + 1/z_0) \right).
\tag{9-43}
$$

In fact, this means that the N dependence is in non-perturbative terms, smaller than any power of $\ln q$. This is again the arctic circle phenomenon, the partitions of size $> \overline{n}$ have an exponentially small probability, the size of the system seems to be frozen to $\overline{n}$.

9.3.3 Mirror curve

In topological strings (see [3, 71, 90, 22, 115] and section 11), it is known that Gromov-Witten invariants of some Toric-Calabi-Yau manifolds, can be written as sums over partitions, this is called the topological vertex method.

In particular, for the Calabi-Yau manifold $X_p = \mathcal{O}(-p) \oplus \mathcal{O}(p-2) \to \mathbb{P}^1$, which is a rank 2 bundle over $\mathbb{P}^1$, the Gromov Witten invariants $\mathcal{N}_{g,d}(X_p)$ which count the number of Riemann surfaces of genus g and degree d which can be embedded in X_p and going through given points, are given by:

$$\sum_g (\ln q)^{2-2g} \sum_d e^{-td} \mathcal{N}_{g,d}(X_p) = \ln Z \tag{9-44}$$

where

$$Z = \sum_\lambda \left(\frac{\dim_q(\lambda)}{[|\lambda|]!} \right)^2 e^{-t|\lambda|} q^{\frac{p-1}{2} C_2(\lambda)}. \tag{9-45}$$

Therefore, we have found in the previous section that the Gromov-Witten invariants of X_p are given by the symplectic invariants of $\mathcal{E}$:

$$\sum_d e^{-td} \mathcal{N}_{g,d}(X_p) = F_g(\mathcal{E}). \tag{9-46}$$

This result is interesting by itself, since it already gives a practical way of computing the Gromov-Witten invariants of X_p.

But we can go further. Notice that $x(z) = u(z)$ is a rational function of z, and $y(z) = \frac{1}{u(z)} \ln v(z)$ where $v(z)$ is also a rational function of z, and notice that:

$$dx \wedge dy = d \ln u \wedge d \ln v \tag{9-47}$$

thus, $\mathcal{E}$ is symplectically equivalent to the following spectral curve:

$$\tilde{\mathcal{E}} = \begin{cases} \tilde{x}(z) = \ln\left((1 - \frac{z}{z_0})(1 - \frac{1}{zz_0}) \right) \\ \\ \tilde{y}(z) = \ln\left(\frac{1}{z} \left(\frac{1-z/z_0}{1-1/zz_0} \right)^{\frac{p}{2}} \right) \end{cases} , \quad e^{-t} = \frac{1}{z_0^2} (1 - \frac{1}{z_0^2})^{p(p-2)} \tag{9-48}$$

and thus we have:

$$F_g(\mathcal{E}) = F_g(\tilde{\mathcal{E}}). \tag{9-49}$$

Notice that $u = e^{\tilde{x}}$ and $v = e^{\tilde{y}}$ are both rational fractions of z, and thus, by eliminating z, there exists an algebraic relationship between them, i.e. there exists a polynomial H such that

$$H(u, v) = 0. \tag{9-50}$$

This curve is known in the context of topological strings [71]:

$$H(e^{\tilde{x}}, e^{\tilde{y}}) = \omega_+ \omega_- \tag{9-51}$$

is a Calabi-Yau 3-fold $\tilde{X}_p$, which is mirror of X_p under mirror symmetry, and $H(e^x, e^y) = 0$ is the singular locus of $\tilde{X}_p$. Therefore, we have obtained that, in agreement with the "remodelling of the B-model proposal" (see [22] and section 11), we have:

the Gromov-Witten invariants of X_p, are the symplectic invariants of the singular curve of its mirror $\tilde{X}_p$.

10 Intersection numbers and volumes of moduli spaces

10.1 Kontsevich integral and intersection numbers

10.1.1 Matrix integral

Let Λ be a diagonal matrix $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N)$.

Kontsevich's integral [85] is the following formal matrix integral (defined as a formal power series in large Λ)

$$Z_{\text{Kontsevich}}(\Lambda) = \int dM \, e^{-N \text{Tr} \frac{M^3}{3} + \Lambda M^2} \tag{10-1}$$

and we consider its topological expansion:

$$\ln Z_{\text{Kontsevich}}(\Lambda) = \sum_{g=0}^{\infty} N^{2-2g} \, F_g. \tag{10-2}$$

Notice that, upon shifting $M \to M - \Lambda$ we have

$$Z_{\text{Kontsevich}}(\Lambda) = e^{\frac{N}{3} \text{Tr} \Lambda^3} \int dM \, e^{-N \text{Tr} \left[\frac{M^3}{3} - \Lambda^2 M \right]} \tag{10-3}$$

and thus, this integral is a special case of 1-matrix integral with an external field (see section 5.4), which implies that the coefficients F_g are the symplectic invariants of the corresponding spectral curve.

10.1.2 Kontsevich's Spectral curve

We have seen in section 5.4, that the topological expansion of a matrix integral with external field, of the type:

$$Z = \int dM \, e^{-N \text{Tr} V(M) - \Lambda^2 M} \tag{10-4}$$

is given by the symplectic invariants of its spectral curve:

$$\ln Z = \sum_g N^{2-2g} F_g(\mathcal{E}) \tag{10-5}$$

where the spectral curve $\mathcal{E}$ is characterized by the algebraic equation:

$$V'(x) - y = \frac{1}{N} \sum_i \frac{P_i(x)}{y - \lambda_i^2} \tag{10-6}$$

where $P_i(x)$ is a polynomial of x of degree at most $\deg V''$, which behaves at large x like $V'(x)/x$. Here we have $V(x) = \frac{x^3}{3}$, hence $V'(x) = x^2$, and thus $P_i(x) = x + P_i(0)$.

The fact that Z is to be understood as a formal power series at large Λ, means that we look for a rational spectral curve, and this determines all the polynomials $P_i(x)$.

Here we find that the rational spectral curve of the form eq.(10-6) is:

$$
\mathcal{E}_K = \begin{cases} x(z) = z - \frac{1}{N}\operatorname{Tr}\frac{1}{2\hat{\Lambda}(\hat{\Lambda}-z)} \\ y(z) = z^2 + t_1 \end{cases} \tag{10-7}
$$

where $t_1 = \frac{1}{N}\operatorname{Tr}\frac{1}{\hat{\Lambda}}$ and

$$
\Lambda^2 = \hat{\Lambda}^2 + t_1. \tag{10-8}
$$

From now on, for simplicity, we shall assume that $t_1 = 0$:

$$
t_1 = 0 = \frac{1}{N}\operatorname{Tr}\frac{1}{\hat{\Lambda}} \tag{10-9}
$$

and therefore we have

$$
\hat{\Lambda} = \Lambda \tag{10-10}
$$

and the spectral curve is:

$$
\mathcal{E}_K = \begin{cases} x(z) = z - \frac{1}{N}\operatorname{Tr}\frac{1}{2\Lambda(\Lambda-z)} \\ y(z) = z^2 \end{cases} . \tag{10-11}
$$

10.1.3   Symplectic invariants

To compute the F_g's of the spectral curve $\mathcal{E}_K$, we need to consider the branchpoints, i.e. the zeroes of $x'(z)$, and they are quite complicated.

However, we may use symplectic invariance, and compute the F_g's after exchanging the roles of x and y, and thus, consider the spectral curve:

$$
\tilde{\mathcal{E}}_K = \begin{cases} x(z) = z^2 \\ y(z) = z - \frac{1}{N}\operatorname{Tr}\frac{1}{2\Lambda(\Lambda-z)} \end{cases} . \tag{10-12}
$$

This spectral curve has now only one branchpoint solution of $x'(z) = 0$, which is located at $z = 0$. Since the F_g's are obtained by computing residues near $z = 0$, we may Taylor expand $y(z)$ near $z = 0$, and we have:

$$
\tilde{\mathcal{E}}_K = \begin{cases} x(z) = z^2 \\ y(z) = z - \frac{1}{2}\sum_{k=0}^{\infty} t_{k+2}\, z^k \end{cases} \tag{10-13}
$$

Now, it is rather easy to compute the first few symplectic invariants:

$$
\omega_1^{(1)}(z) = -\frac{dz}{8(2-t_3)}\left(\frac{1}{z^4} + \frac{t_5}{(2-t_3)z^2}\right), \tag{10-14}
$$

$$
\omega_3^{(0)}(z_1, z_2, z_3) = -\frac{1}{2-t_3}\frac{dz_1\, dz_2\, dz_3}{z_1^2 z_2^2 z_3^2}, \tag{10-15}
$$

$$
\begin{aligned}
\omega_2^{(1)}(z_1, z_2) \quad = \quad &\frac{dz_1\, dz_2}{8(2-t_3)^4 z_1^6 z_2^6}\Big[(2-t_3)^2(5z_1^4 + 5z_2^4 + 3z_1^2 z_2^2) \\
&+ 6t_5^2 z_1^4 z_2^4 + (2-t_3)(6t_5 z_1^4 z_2^2 + 6t_5 z_1^2 z_2^4 + 5t_7 z_1^4 z_2^4)\Big],
\end{aligned}
$$

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$$(10 - 16)$$

$$
\begin{aligned}
\omega_1^{(2)}(z) \quad = \quad & -\frac{dz}{128(2-t_3)^7 z^{10}} \Big[252\, t_5^4 z^8 + 12\, t_5^2 z^6 (2-t_3)(50\, t_7 z^2 + 21\, t_5) \\
& + z^4 (2-t_3)^2 (252\, t_5^2 + 348\, t_5 t_7 z^2 + 145\, t_7^2 z^4 + 308\, t_5 t_9 z^4) \\
& + z^2 (2-t_3)(203\, t_5 + 145\, z^2 t_7 + 105\, z^4 t_9 + 105\, z^6 t_{11}) \\
& + 105\, (2-t_3)^4 \Big].
\end{aligned}
$$

$$(10 - 17)$$

And so on...

For example, the first and second order free energies are:

$$
F_{\text{Kontsevitch}}^{(1)} = -\frac{1}{24} \ln \left(1 - \frac{t_3}{2} \right) \tag{10-18}
$$

and

$$
F_{\text{Kontsevitch}}^{(2)} = \frac{1}{1920} \frac{252\, t_5^3 + 435\, t_5 t_7 (2-t_3) + 175\, t_9 (2-t_3)^2}{(2-t_3)^5}. \tag{10-19}
$$

Remark 10.1 The fact that the symplectic invariants depend only on odd t_k's can be understood in terms of symplectic invariance: indeed, adding to $y(z)$ any rational function of $x(z)$ (i.e. any rational function of z^2) is a symplectic transformation. It does not change the F_g's, and therefore the F_g's depend only on the odd part of $y(z)$, i.e. only on the odd t_k's.

10.1.4 Intersection numbers

The Kontsevich integral is important because it computes intersection numbers of Chern classes of line bundles over the moduli spaces of Riemann surfaces [116, 85].

Let $\mathcal{M}_{g,n}$ be the moduli space of Riemann surfaces Σ of genus g, with n marked points $p_1, \ldots, p_n$. This moduli space is a complex manifold of dimension:

$$
\dim \mathcal{M}_{g,n} = d_{g,n} = 3g - 3 + n. \tag{10-20}
$$

This moduli space can be compactified into a compact space $\overline{\mathcal{M}}_{g,n}$ by adding all stable nodal surfaces (stable means that each component has a Euler characteristics < 0).

The cotangent bundle $\mathcal{L}_i$ is the bundle over $\mathcal{M}_{g,n}$, whose fiber is the cotangent space of Σ at the point p_i. Let

$$
\psi_i = c_1(\mathcal{L}_i) \tag{10-21}
$$

be its first Chern class. $\mathcal{L}_i$ and ψ_i can be extended to $\overline{\mathcal{M}}_{g,n}$.

Chern classes ψ_i provide useful information on the topology of $\overline{\mathcal{M}}_{g,n}$. Indeed if one computes the integral of ψ_i over a cycle in $\overline{\mathcal{M}}_{g,n}$, this integral tells how many times the cotangent space rotates. The intersection numbers are defined as:

$$
< \tau_{d_1} \ldots \tau_{d_n} > = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \psi_2^{d_2} \ldots \psi_n^{d_n} \tag{10-22}
$$

and are non-zero only if $\sum_i d_i = d_{g,n} = 3g - 3 + n$.

The Kontsevich integral is a generating function for those numbers:

$$
\begin{aligned}
&\ln Z_{\text{Kontsevich}}(\Lambda) \\
&= \sum_g (N/2)^{2-2g} \sum_n \sum_{d_1 + \ldots + d_n = d_{g,n}} \prod_i (2d_i - 1)!! \frac{t_{2d_i+1}}{2} < \tau_{d_1} \ldots \tau_{d_n} >
\end{aligned}
$$
$$(10-23)$$

where the sum is restricted to $d_i > 0$ because we have assumed $t_1 = 0$.

For example we have:

$$
F_2 = \sum_k (t_3/2)^k \left(\frac{105\, t_9}{2} < \tau_1^k \tau_4 > + \frac{45\, t_5 t_7}{4} < \tau_1^k \tau_2 \tau_3 > + \frac{27\, t_5^3}{8} < \tau_1^k \tau_2^3 > \right).
$$
$$(10-24)$$

10.1.5 Correlators and unmarked faces

The symplectic invariants of the spectral curve eq.(10-13) are the F_g's. They generate intersection numbers of ψ-classes, i.e. Chern classes of cotangent line bundles over marked points, i.e. they generate the intersection numbers of the type:

$$< \psi_1^{d_1} \ldots \psi_n^{d_n} > . \tag{10-25}$$

The correlators $\omega_n^{(g)}$s of that spectral curve, also have some interpretation in terms of integrals of some classes over moduli-spaces.

Notice that Kontsevich integral eq.(10-23) contains a summation over n, i.e. over the number of marked points. One may whish to distinguish some of those marked points, and fix marked faces around them, and perform the sum over the other marked points, in some sense forget the other marked points.

The forgetful map, is the map from $\overline{\mathcal{M}}_{g,n+m}$ to $\overline{\mathcal{M}}_{g,n}$, which consists in forgetting m marked points. Under this map, ψ classes project to the Mumford κ-classes:

$$
\int_{\overline{\mathcal{M}}_{g,n+m}} \psi_1^{d_1} \ldots \psi_n^{d_n} \prod_{k=1}^m \psi_{n+k}^{a_k+1} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} \sum_{\sigma \in \Sigma_m} \prod_{c=\text{cycles of } \sigma} \kappa_{(\sum_{i \in c} a_i)}. \tag{10-26}
$$

For examples with $m = 1$ and $m = 2$:

$$
\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \ldots \psi_n^{d_n} \psi_{n+1}^{a+1} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} \kappa_a, \tag{10-27}
$$

$$
\int_{\overline{\mathcal{M}}_{g,n+2}} \psi_1^{d_1} \ldots \psi_n^{d_n} \psi_{n+1}^{a_1+1} \psi_{n+2}^{a_2+1} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \ldots \psi_n^{d_n} (\kappa_{a_1+a_2} + \kappa_{a_1} \kappa_{a_2}). \tag{10-28}
$$

One finds [57], that the correlators $\omega_n^{(g)}(z_1, \ldots, z_n)$, are the generating functions for κ classes, coupled to n ψ-classes:

$$
\omega_n^{(g)}(z_1, \ldots, z_n) = 2^{-d_{g,n}} (t_3 - 2)^{2-2g-n} \sum_{d_0+d_1+\ldots+d_n=d_{g,n}} \sum_{k=1}^{d_0} \frac{1}{k!} \sum_{b_1+\ldots+b_k=d_0, b_i>0}
$$

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$$\prod_{i=1}^{n} \frac{2d_i + 1!}{d_i!} \frac{dz_i}{z_i^{2d_i+2}} \prod_{l=1}^{k} \tilde{t}_{b_l} < \prod_{l=1}^{k} \kappa_{b_l} \prod_{i=1}^{n} \psi_i^{d_i} >_{g,n}$$

$(10-29)$

where the coefficients $\tilde{t}_b$ are the Schur transform of the t_k's:

$$\tilde{t}_b = \sum_{l=1}^{b} \frac{(-1)^l}{l} \sum_{a_1+...+a_l=b, a_i>0} \prod_j \frac{2a_j + 1!}{a_j!} \frac{t_{2a_j+3}}{t_3 - 2}. \qquad (10\text{-}30)$$

Their generating function is obtained by:

$$f(z) = \sum_{a=1}^{\infty} \frac{2a + 1!}{a!} \frac{t_{2a+3}}{2 - t_3} z^a \quad , \quad -\ln(1 - f(z)) = \sum_{b=1}^{\infty} \tilde{t}_b \, z^b. \qquad (10\text{-}31)$$

Example:

$$\tilde{t}_1 = -6 \frac{t_5}{t_3 - 2} \quad , \quad \tilde{t}_2 = -60 \frac{t_7}{t_3 - 2} + 18 \frac{t_5^2}{(t_3 - 2)^2} \, , \, ... \qquad (10\text{-}32)$$

$\omega_n^{(g)}$ is the Laplace transform of:

$$2^{d_{g,n}} (t_3 - 2)^{2g-2+n} V_{g,n}(L_1, \ldots, L_n)$$
$$= \sum_{d_0+d_1+...+d_n=d_{g,n}} \prod_{j=1}^{n} \frac{L_j^{2d_j}}{d_j!} \sum_k \frac{1}{k!} \sum_{b_1+b_2+...+b_k=d_0, b_i>0} \prod_{i=1}^{k} \tilde{t}_{b_i} < \prod_{i=1}^{k} \kappa_{b_i} \prod_j \psi_F^{d_j} > .$$

$(10-33)$

$V_{g,n}$ can be interpreted as the generating function for counting intersection numbers of κ-classes, on the moduli-space of Riemann surfaces with n boundaries (discs removed from the surface), of perimeters $L_1, \ldots, L_n$.

10.2 Application: Weil-Petersson volumes

Consider the stable moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n boundaries (stability means $2 - 2g - n < 0$). Every surface in $\mathcal{M}_{g,n}$ has a negative Euler-characteristics, and thus a negative average curvature. It can be equipped with a unique constant negative curvature metric, called Poincaré metric, such that the boundaries are geodesics. Let $L_1, \ldots, L_n$ be the geodesic lengths of the boundaries.

Every Riemann surface in $\mathcal{M}_{g,n}$ can be decomposed into $2g - 2 + n$ pairs of pants, whose boundaries are geodesics (such a decomposition is not unique). Conversely, $2g - 2 + n$ pairs of pants with fixed given boundary perimeters can be glued together to form a Riemann surface of $\mathcal{M}_{g,n}$ provided that the lengths of boundaries which are glued together match. The gluing is not unique, because 2 circles of the same perimeter

can be glued in many ways, twisted by an arbitrary angle.
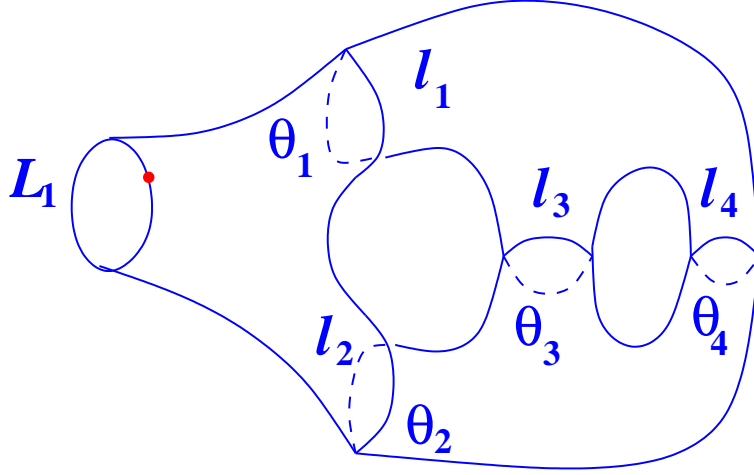
The $3g - 3 + n$ geodesic lengths of the glued boundaries $l_1, \ldots, l_{3g-3+n}$, together with the $3g - 3 + n$ twisting angles $\theta_1, \ldots, \theta_{3g-3+n}$, provide a local set of coordinates parameterizing $\mathcal{M}_{g,n}$. It turns out that, although we don't have uniqueness of the decomposition, the corresponding symplectic form, called Weil-Petersson symplectic metric is well defined on $\mathcal{M}_{g,n}$:

$$\Omega = \prod_{i=1}^{3g-3+n} dl_i \wedge d\theta_i \tag{10-34}$$

and it can be extended to the compactified $\overline{\mathcal{M}}_{g,n}$. The Weil-Petersson volume of $\overline{\mathcal{M}}_{g,n}$ is then defined as:

$$\mathrm{Vol}_{g,n}(L_1, \ldots, L_n) = \int_{\overline{\mathcal{M}}_{g,n}} \Omega \tag{10-35}$$

where the n external boundaries are restricted to have fixed geodesic lengths L_i's and fixed angles (i.e. some marked points).

It can be proved [117] that the Weil-Petersson metrics comes from the Kähler metrics:

$$2\pi^2 \kappa_1 \tag{10-36}$$

i.e. we have:

$$\mathrm{Vol}_{g,n}(L_1, \ldots, L_n) \quad = \quad \left\langle \left(2\pi^2 \kappa_1 + \frac{1}{2}\sum_{i=1}^n L_i^2 \psi_i\right)^{d_{g,n}} \right\rangle$$

$$= \quad 2^{-d_{g,n}} \sum_{d_0 + \ldots + d_n = 3g-3+n} \frac{2^{d_0}}{d_0!} \prod_{i=1}^n \frac{L_i^{2d_i}}{d_i!} \left\langle (2\pi^2 \kappa_1)^{d_0}\, \psi_1^{d_1} \ldots \psi_n^{d_n} \right\rangle.$$

$$(10-37)$$

It can be made to coincide with eq.(10-33), provided that we choose $\tilde{t}_1 = 4\pi^2$ and $t_3 = 3$. Doing the reverse transform using eq.(10-31), it corresponds to $-\ln(1 - f(z)) = 4\pi^2 z$, i.e. $f(z) = 1 - e^{-4\pi^2 z}$ and thus:

$$t_{2d+3} = \frac{(2i\pi)^{2d}}{(2d+1)!} + 2\delta_{d,0} \tag{10-38}$$

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i.e. it corresponds to the spectral curve

$$\mathcal{E}_{WP} = \begin{cases} x(z) = z^2 \\ y(z) = \frac{1}{2\pi} \sin(2\pi z) \end{cases} .$$

(10-39)

In other words, the Laplace transforms of the Weil-Petersson volumes $\mathrm{Vol}_{g,n}(L_1, \ldots, L_n)$, are the $W_n^{(g)}$'s of the spectral curve $\mathcal{E}_{WP}$.

Taking the inverse Laplace transform of the recursion relations eq.(2-44) which define the $W_n^{(g)}$'s for the spectral curve eq.(10-39), we recover Mirzakhani's recursion [97, 98] relations for the Weil-Petersson volumes (see [62, 57] for the proof).

10.3 Application: Witten-Kontsevich theorem

Witten's conjecture [116] was the assertion that the limit of the generating function of large discrete surfaces was indeed the generating function of intersection numbers for continuous Riemann surfaces. In other words the double scaling limit of the F_g's of maps, should coincide with the F_g's of the Kontsevich integral.

We have seen that the F_g's of maps are given by symplectic invariants, and thus their limits as $t \to t_c$, are given by the symplectic invariants of the blown up curve. Thus the F_g's of the double scaling limit of maps, are the F_g's of the minimal $(p, 2)$ model, i.e. the F_g's of the spectral curve:

$$F_g(\mathcal{E}_{\mathrm{maps}}) \sim (t - t_c)^{(2-2g)\mu} F_g(\mathcal{E}_{(p,2)})$$

(10-40)

where $\mu = \frac{p+2}{p+1}$, and:

$$\mathcal{E}_{(p,2)} = \begin{cases} x(z) = z^2 - 2u \\ y(z) = \sum_{k=0}^{p} \overline{t}_k z^k \end{cases} .$$

(10-41)

On the other hand, the F_g's of the Kontsevich integral, are also given by the symplectic invariants of a spectral curve, which is the equilibrium density of eigenvalues in the Kontsevich integral's Matrix Airy function. The spectral curve (see section 10.1) is (again we assume for simplicity that $t_1 = 0$):

$$\mathcal{E}_K = \begin{cases} x(z) = z - \frac{1}{N} \mathrm{Tr} \frac{1}{2\Lambda(\Lambda-z)} \\ y(z) = z^2 \end{cases} .$$

(10-42)

We have seen (theorem 4.1), that if two spectral curve are equivalent under symplectic transformations, then they have the same F_g's. In particular the F_g's don't change if we change $x \to y$ and $y \to -x$, thus:

$$\mathcal{E}_k \sim \tilde{\mathcal{E}}_K = \begin{cases} x(z) = z^2 \\ y(z) = -z + \frac{1}{N} \mathrm{Tr} \frac{1}{2\Lambda(\Lambda-z)} \end{cases} .$$

(10-43)

In that new formulation, there is only one branchpoint located at $z = 0$, and since all quantities computed are residues at this branchpoint, we may Taylor expand $y(z)$ in

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the vicinity of $z = 0$ and thus:

$$\mathcal{E}_K \sim \begin{cases} x(z) = z^2 \\ y(z) = -z + \frac{1}{2} \sum_{k=0}^{\infty} z^k \frac{1}{N} \operatorname{Tr} \Lambda^{-k-2} \end{cases} . \tag{10-44}$$

If we choose the diagonal matrix Λ such that:

$$\bar{t}_k = \frac{1}{2N} \operatorname{Tr} \Lambda^{-k-2} - \delta_{k,1}, \tag{10-45}$$

we have:

$$\mathcal{E}_K \sim \mathcal{E}_{(p,2)} \tag{10-46}$$

and therefore:

$$\boxed{F_g(\mathcal{E}_K) = F_g(\mathcal{E}_{(p,2)})}$$

$$\tag{10-47}$$

which proves Witten's conjecture.

In fact this conjecture was first proved by M. Kontsevich [85]. Kontsevich's method consisted of two parts: first prove that the Airy matrix integral now known as Kontsevich integral was the generating function of intersection numbers, and then prove that both $F_{g,K}$ and $F_{g,(p,2)}$ obeyed the same set of differential equations, namely KdV hierarchy. Witten's conjecture has received several proof since then, in particular Loijeenga's [89], or Okounkov-Pandaripande [104].

Here we see that the spectral curve of the Kontsevich model and the spectral curve of the $(p,2)$ model are obtained from one another by the symplectic transformation $x \to y, y \to -x$. That symplectic transformation (the $\pi/2$ rotation in the (x,y) plane), leaves F_g invariant, but it changes the $\omega_n^{(g)}$'s.

11 Application: topological strings

The counting of surfaces with particular weights, or enumerative geometry, has been very important in physics since the arising of string theories. Indeed, those theories consist in replacing the point-like particles of usual theories such as classical mechanics (0 dimensional objects) by strings (1 dimensional objects): a state is now given by a string state instead of a point state. A string evolving in time describes a surface in the space-time, the world-sheet, instead of a line for a point evolving in space-time. Hence the usual path integral corresponding to sum over all possible stories from one initial state to one final state is now a "sum" over all possible surfaces in space-time (target space), linking the initial strings to the final ones. This could explain why these theories are related to the symplectic invariants since the latter already appear in many problems of "surface enumeration". So far, no proof is available but there exist many hints that the symplectic invariants of some specific spectral curve should be the partition functions of some particular string theories: type IIB toplogical strings on some special target space. Many checks have been made that this is indeed the case [92], so that Bouchard, Klemm, Mariño and Pasquetti [22] proposed to define the topological string partition function and observables as the symplectic invariants and

correlation functions, considering the spectral curve as the target space of the string theory.

Even if this conjecture is not proved yet, some new clues were recently given by Dijkgraaf and Vafa [41] who already conjectured that some matrix models are dual to some particular topological string theories [40]. Those new hints rely on the study of an effective field theory for the string theory: the Kodaira-Spencer theory. We review those advances in the second part of the present section.

11.1 Topological string theory

11.1.1 Introduction

Type IIB topological string theories are obtained by twisting $N = 2$ superconformal sigma model in 2 dimensions. The precise description would lead us too far away from the main topic of this review, and the interested reader is invited to consult [90, 91, 115, 71] for details as well as [23] for the particular topic of toric geometries. There exist two ways of twisting this theory leading to two different models referred to as A and B models. We will be particularly interested in this paper in the B-model and won't develop the A-model too far. Nevertheless, since the special geometries of the target space of the B-model we are interested in, are inherited from the A-model, we will say a few words about the latter and especially the possible geometries of its target space and of the objects one wants to compute in the next section.

First of all, let us mention that the topological string theories can be thought of, as theories of the maps from a Riemann surface Σ (the world-sheet) to a Calabi-Yau manifold M (the target space) of arbitrary dimension. More precisely, in the A model side, the amplitudes to be computed are related to the Gromov-Witten invariants as follows. Let us consider a worldsheet $\Sigma_k^{(g)}$ of genus g with k holes (or boundaries). One wants to count maps which map the boundaries to a Lagrangian submanifold of M denoted as the brane[18] L. Such maps are characterized by two additional parameters prescribing how the boundaries are mapped to the brane L: a bulk class $\beta \in H_2(M, L)$[19] and winding numbers $\omega_i \in \mathbb{Z}$, $i = 1, \ldots, k$, telling how many times the boundaries wrap around the brane L. One sums up those "number of maps" called Gromov-Witten invariants $N_{\beta,\omega}^{(g)}$ in generating functions:

$$F_\omega^{(g)} := \sum_{\beta \in H_2(M,L)} N_{\beta,\omega}^{(g)} e^{-\beta t} \tag{11-1}$$

and one also considers the open string amplitudes

$$A_k^{(h)}(z_1, \ldots, z_k) := \sum_{\omega \in \mathbb{Z}^k} F_\omega^{(g)} \prod_{i=1}^k z_i^{\omega_i} \tag{11-2}$$

where the open string parameters z_i are parameters of the moduli space of the brane L as well as the closed string amplitudes:

$$\mathcal{F}^{(g)} := F_0^{(g)}. \tag{11-3}$$

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[18]The geometry of such submanifolds is studied in the next section.
[19]One assumes for simplicity that $b_1(L) = 1$.

Let us also precise the moduli spaces of both models, since they play an important role in the link between topological string theories and the symplectic invariants. In the A model side, the moduli are the Kälher parameters of the target space M whereas the B model depends on the complex structure of M.

11.1.2   Mirror symmetry, branes and toric geometries

One of the most fascinating features of topological string theories is the existence of a duality linking the A and B models: the mirror symmetry. This symmetry states the equivalence of the A model on a target space M and the B model on a mirror target space $\widetilde{M}$ obtained from M by a mirror map which exchanges the Kälher structure of M with the complex structure of $\widetilde{M}$[20].

In the following, we will only be concerned with a special but interesting class of target spaces M: Toric Calabi-Yau threefolds for the A model and their image under mirror symmetry. We now precise the structure of those geometries as well as the geometry of their Lagrangian submanifolds.

Let us start with a Toric Calabi-Yau on the A model side. A Toric Calabi-Yau threefold M can be built as a submanifold of $\mathbb{C}^{k+3}$ as follows. Consider $k+3$ complex scalars $X_i = |X_i| e^{i\theta_i}$, $i = 1, \ldots, k+3$, transforming under the action of $U(1)^k$ as

$$X_i \to e^{iQ_i^\alpha \epsilon_\alpha} X_i \tag{11-4}$$

for some integers Q_i^α, $\alpha = 1, \ldots, k$. One then considers the 3-dimensional submanifold of $\mathbb{R}_+^{k+3}$ obtained by constraining the $|X_i|^2$'s to satisfy

$$\sum_{i=1}^{k+3} Q_i^\alpha |X_i|^2 = r_\alpha \quad , \quad \alpha = 1, \ldots, k. \tag{11-5}$$

The CY 3-fold M is the bundle of Tori generated by the θ_i's modulo the action of $U(1)^k$, over this real submanifold.

The parameters r_α are the Kähler moduli of the Toric threefold M. The Calabi-Yau condition is then:

$$\forall \alpha = 1, \ldots, k \quad , \quad \sum_{i=1}^{k+3} Q_i^\alpha = 0. \tag{11-6}$$

Remark that one can see the coordinates X_i as a S_1-fibration (coordinates $e^{i\theta_i}$) over $\mathbb{R}^+$ (coordinates $|X_i|$) giving to M a structure of T^3 fibration over the subspace of $\mathbb{R}_+^3$ defined by the constraints eq.(11-5). It is then interesting to note that this fibration has singular loci when one or several $|X_i|$ vanish. Indeed, the S^1 fiber defined by the corresponding θ_i shrinks. These loci will be important in the folowing study of the type A branes and they are encoded in the so-called toric graph of the threefold M [92, 22, 23].

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[20]The works on the subject of mirror symmetry are numerous in physics and mathematics. Entering this subject would quickly lead us too far away from our main topic. For a nice review of the subject, one can refer to the excellent book [71].

The boundaries of the worldsheet must be mapped to special Lagrangian submanifolds of X called branes. For this purpose, let us remind the notation

$$X_i = |X_i| e^{i\theta_i}. \tag{11-7}$$

The canonical symplectic form on M is then given by

$$\omega = \frac{1}{2} \sum_{i=1}^{3} d |X_i|^2 \wedge d\theta_i. \tag{11-8}$$

One can cancel this form and obtain a special Lagrangian submanifold L by fixing the θ_i's with the equation

$$\sum_{i=1}^{3} \theta_i = 0 \quad [\pi] \tag{11-9}$$

as well as constraining the moduli of X_i by

$$\sum_{i=1}^{k+3} q_i^{\alpha} |X_i|^2 = c^{\alpha} \tag{11-10}$$

for $\alpha = 1, \ldots, r$, where the q_i^{α} satisfy

$$\sum_{i=1}^{k+3} q_i^{\alpha} = 0. \tag{11-11}$$

A special submanifold in M is then given by the set of complex numbers q_i^{α} and c^{α}.

Moreover, one can consider such manifolds L passing through some singular locus of the manifold M. In this case, it splits into two submanifolds L^+ and L^-. This is one of the latter submanifolds that we consider as A brane, e.g. L^+[21].

Let us now describe the mirror geometry of this target space and of the branes in the B model.

The mirror map transforming M into $\widetilde{M}$ can be built as follows. $\widetilde{M}$ has homogenous coordinates $\widetilde{X}_i := e^{x_i} \in \mathbb{C}^*$ with $i = 1, \ldots, k+3$ whose moduli are constrained by

$$\left| \widetilde{X}_i \right| = e^{-|X_i|^2}. \tag{11-12}$$

The mirror geometry $\widetilde{M}$ of M is then given by

$$\omega^+ \omega^- = \sum_{i=1}^{k+3} \widetilde{X}_i \tag{11-13}$$

for two complex scalars $(\omega^+, \omega^-) \in \mathbb{C}^2$ and non vanishing complex homogenous coordinates $\widetilde{X}_i \in \mathbb{C}^*$ satisfying

$$\prod_{i=1}^{k+3} \widetilde{X}_i^{Q_i^{\alpha}} = e^{-t_\alpha} = q_\alpha \tag{11-14}$$

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[21]One can chose L^+ or L^- as this brane without changing anything in the following.

for any $\alpha = 1, \ldots, k$, where

$$t_\alpha := r_\alpha + i\theta_\alpha \tag{11-15}$$

are complexified Kähler parameters of the threefold M. The equation of the mirror geometry $\widetilde{M}$ reduces to

$$\widetilde{H}(\widetilde{X}, \widetilde{Y}|t_\alpha) = \omega^+ \omega^- = H(e^x, e^y|t_\alpha) \tag{11-16}$$

where $\widetilde{X} = e^x$ and $\widetilde{Y} = e^y$ are two non-vanishing coordinates chosen among the $\widetilde{X}_i$'s[22]. The holomorphic volume form on $\widetilde{M}$ is then given by

$$\Omega = \frac{d\omega \, d\widetilde{X} \, d\widetilde{Y}}{\omega \widetilde{X} \widetilde{Y}} = \frac{d\omega}{\omega} dx dy. \tag{11-17}$$

Under this mirror map, one can easily characterize the B model branes, i.e. the image of the special Lagrangian submanifolds of M under the mirror map [4]. The constraints eq.(11-10) are translated into constraints on the $\widetilde{X}_i$:

$$\prod_{i=1}^{k+3} \widetilde{X}_i^{q_i^\alpha} = e^{-c_\alpha} \tag{11-18}$$

for $\alpha = 1, \ldots, r$. Moreover, for $r = 2$, if one considers the singular brane L^+, its image is a one dimensional complex submanifold described by the algebraic equation

$$H(e^x, e^y) = 0 = \widetilde{H}(\widetilde{X}, \widetilde{Y}) \tag{11-19}$$

on $\mathbb{C}^*$ and can thus be obtained by fixing $\omega^- = 0$ and considering ω^+ as a parameter of this brane. In the following, we will consider this equation as the spectral curve.

11.1.3   Embedding of the spectral curve and open string parameters

In the preceding section, we showed that the spectral curve corresponds to the moduli space of the open string boundaries, i.e. of the B-branes. In particular, it was shown that the description of this moduli space depends on a choice of local coordinates. Let us make this statement more precise by studying the whole target space of the B model and its projection to local patches.

Remember that the moduli space of the B-branes is a Riemann surface $\mathcal{L}$ given by a set of coordinates $\widetilde{X}_i$ and $\omega_\pm$ constrained by

$$\sum_{i=1}^{k+3} \widetilde{X}_i = \omega_- = 0 \tag{11-20}$$

as well as the constraints

$$\prod_{i=1}^{k+3} \widetilde{X}_i^{q_i^\alpha} = e^{-t_\alpha} \quad , \quad \alpha = 1, \ldots, k \tag{11-21}$$

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[22]The choice of such coordinates $\widetilde{X}_i$ and $\widetilde{X}_j$ depends on the sector of the moduli space that we are studying. We explicit this choice in the next section.

and

$$\prod_{i=1}^{k+3} \widetilde{X}_i^{q_i^\alpha} = e^{-c_\alpha} \qquad , \qquad \alpha = 1, \dots, r. \tag{11-22}$$

In order to describe this moduli space, one has to choose as set of coordinates characterizing it. Indeed, one can describe it as an embedding of the Riemann surface $\mathcal{L}$ into $\mathbb{C}^* \times \mathbb{C}^*$ (resp. to $\mathbb{C} \times \mathbb{C}$) through the coordinates $\widetilde{X}_i$ (resp. the coordinates x_i), e.g. the choice of two coordinates $\widetilde{X}_i$ and $\widetilde{X}_j$ (resp. x_i and x_j) among the set $\left\{ \widetilde{X}_a \right\}_{a=1}^{k+3}$ (resp. $\{x_a\}_{a=1}^{k+3}$) allowing to describe $\mathcal{L}$ by an equation

$$\widetilde{H}_{i,j}(\widetilde{X}_i, \widetilde{X}_j) = H_{i,j}(e^{x_i}, e^{x_j}) = 0 \tag{11-23}$$

obtained by elimination of all the other coordinates $\widetilde{X}_a$ in eq.(11-20), eq.(11-21) and eq.(11-22). In the following, one generically denotes this embedding of the spectral curve by the equation

$$H(e^x, e^y) = 0 = \widetilde{H}(\widetilde{X}, \widetilde{Y}). \tag{11-24}$$

If all these equations represent the embedding of the same surface, they correspond to different description of the branes, i.e. different types of boundary conditions for the worldsheet. The choice of such an embedding is not random: depending on the regime we are considering, i.e. the sector of the moduli space we are studying, some embeddings are more appropriate (see for example the discussion in section 2.2 of [22]).

Remark that the reparameterization group of the spectral curve $\mathcal{L}$ is

$$G_\mathcal{L} = SL(2, \mathbb{Z}) \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{11-25}$$

in terms of the variables $(\widetilde{X}, \widetilde{Y})$ of $\widetilde{H}(\widetilde{X}, \widetilde{Y}) = 0$.

Remark 11.1 Note that these transformations preserve the symplectic form $|dx \wedge dy| = \left| \frac{d\widetilde{X}}{\widetilde{X}} \wedge \frac{d\widetilde{Y}}{\widetilde{Y}} \right|$. This reminds of the symplectic transformations section 4.2 which do not change the symplectic invariants. In this topological string setup, these transformations acting on the open string parameters should preserve the closed string amplitudes.

These transformations, i.e. changing the open string parameters, are important in the study of the open string amplitudes. Indeed, the whole open string moduli space, or moduli space of branes, exhibits different phases. In each of these particular regimes, one can use a specific embedding to describe the brane moduli space in appropriate coordinates (see [71, 22] for a review on the subject). The usual methods of computation allow to know the open string amplitudes in some very particular regime. This means that one can compute these amplitudes in a specific patch of $\widetilde{M}$ and not on the others. It is thus interesting to be able to go from one patch to the others. These "phase transitions", corresponding for example to blow ups of $\widetilde{M}$, are elements of $G_\mathcal{L}$ encoding the transition from the open string parameters of one embedding to the others.

The choice of an embedding, i.e. the choice of a coordinate x, does not only fix the location of a brane but also the last remaining ambiguity known as the framing[23]. Roughly speaking, the framing consists in a discrete ambiguity and can be fixed by choosing an integer f. This ambiguity correspond to the elements of $G_{\mathcal{L}}$:

$$(\widetilde{X}, \widetilde{Y}) \to (\widetilde{X}\widetilde{Y}^f, \widetilde{Y}) \tag{11-26}$$

or, in x and y coordinates,

$$(x, y) \to (x + fy, y) \tag{11-27}$$

for integer f. Note that this is also a symplectic transformation considered in section 2.

We have thus shown that fixing an embedding of the spectral curve $\mathcal{L}$ in $\mathbb{C} \times \mathbb{C}$ (or $\mathbb{C}^* \times \mathbb{C}^*$), one fixes the open string parameter space which can be seen as the coordinate x. Going from one patch in the parameter space to another is obtained by changing embedding thanks to an element of $G_{\mathcal{L}}$.

11.1.4 Open/closed flat coordinates

As it was already mentioned in the preceding sections, the moduli space of the B model (resp. A model) is given by the complex (resp. Kähler) parameters of the target space $\widetilde{M}$ (resp. M). Let us precise the flat coordinates describing these complex and Kähler structures which are mapped to each other by the closed string mirror map.

These flat coordinates $T^\alpha, \alpha = 1, \ldots, \bar{g}$ are given by the periods of the meromorphic one form $ydx = \ln \widetilde{Y} \frac{d\widetilde{X}}{\widetilde{X}}$ on the spectral curve $\mathcal{L}$:

$$T^\alpha = \frac{1}{2i\pi} \oint_{\mathcal{A}_\alpha} ydx \tag{11-28}$$

where $(\mathcal{A}_\alpha, \mathcal{B}_\alpha)$ is a canonical basis of one cycles on $\mathcal{L}$. This also ensures the existence of a holomorphic function $F(T_\alpha)$ such that the dual periods can be expressed as

$$\frac{\partial F}{\partial T_\alpha} = \frac{1}{2i\pi} \oint_{\mathcal{B}_\alpha} ydx. \tag{11-29}$$

What about the open flat coordinates? If the closed coordinates are given by closed integrals of λ over the cycles, the open flat coordinate is expected to be given by chain integrals

$$U = \frac{1}{2i\pi} \int_{\alpha_x} ydx \tag{11-30}$$

where α_x is an open path over which y jumps by $2i\pi$.

Moreover, it is interesting to note that the open string disc amplitude $A_1^{(0)}(u)$ can be computed explicitly: it is also a chain integral of the one form Θ

$$A_1^{(0)}(x) = \int_{[x^*, x]} ydx. \tag{11-31}$$

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[23]See [22] for explanations on this phenomenon.

Remark 11.2 Once again, it is interesting to note the similarities between the theory of symplectic invariants and B model. Let us summarize this correspondence in a short array:

Symplectic invariants	B model
Spectral curve	Brane moduli space
Symplectic transformations	phase transition
filling fraction ϵ_i	closed flat coordinates T_α
genus zero free energy $F^{(0)}$	prepotential F
variations $\frac{\partial F^{(0)}}{\partial \epsilon_i} = \frac{1}{2i\pi} \oint_{\mathcal{B}_i} y dx$	variations $\frac{\partial F}{\partial T_\alpha} = \frac{1}{2i\pi} \oint_{\mathcal{B}_\alpha} y dx$
Genus 0 one point function $y dx$	Disc amplitude $\int y dx$

11.1.5 Symplectic invariants formalism: a conjecture

In [22], following some checks of [92] and the seminal paper [40], Bouchard, Klemm, Mariño and Pasquetti proposed to define the open and closed string amplitudes of the A model as the symplectic invariants and correlation functions computed on the spectral curve of the mirror B model branes.

The conjecture of [22] simply states that the open string amplitudes $A_k^{(h)}(z_1, \ldots, z_k)$ and closed string amplitudes $\mathcal{F}^{(h)}$ of the A model whose mirror background gives rise to the spectral curve $H(e^x, e^y) = 0$ are given by correlation functions and symplectic invariants built from the equation $H(e^x, e^y) = 0$:

$$A_k^{(h)}(z_1, \ldots, z_k) = \int \omega_{k,string}^{(h)}(z_1, \ldots, z_k)$$

(11-32)

and

$$\mathcal{F}^{(h)} = F_h(t_\alpha).$$

(11-33)

Remark 11.3 In [22], the authors seem to slightly change the recursive rules defining the correlation functions and symplectic invariants. However, this apparent transformation results from their choice to work with the coordinates $\widetilde{X} = e^x$ and $\widetilde{Y} = e^y$ instead of x and y in order to start from an equation

$$\widetilde{H}(\widetilde{X}, \widetilde{Y}) = 0 \tag{11-34}$$

which is algebraic. It appears that the algebraicity of the spectral curve is not essential and that one can directly work with the coordinates x and y, avoiding this change of coordinates.

Let us emphasize a few important points. In order to get the A model amplitudes, one first has to compute the correlation functions and symplectic invariants from the B model spectral curve and then plug in the mirror map to obtain the result in terms of the A model parameters. It should be underlined that the choice of coordinates x and y out of the x_i's, i.e. a choice of embedding of $\mathcal{L}$ in $\mathbb{C} \times \mathbb{C}$, corresponds to a choice of brane in the A model[24]. Thus, it is interesting to study the behavior of

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[24]That is to say, the choice of the location of the Brane as well as a choice of framing. This topic is well developed in [22].

the amplitudes when one moves in the brane moduli space, which corresponds to the space of parameterizations of the spectral curve. In particular, the closed amplitudes should not depend on this choice of parametrization since only the boundaries of the worldsheet are sensitive to the definition of the branes. This independence of the closed amplitudes on the embedding of the spectral curve is indeed true and follows directly from the symplectic invariance of the $F^{(g)}$'s.

11.1.6 Checks of the conjecture

There have been many checks of this conjecture before the definition of the symplectic invariants and after.

First of all, most of this conjecture was inspired by the idea of Dijkgraaf and Vafa who conjectured that the partition function of the type B topological string on some special backgrounds is given by a random matrix integral [40]. Now that the symplectic invariants extend the notion of random matrix partition function to any spectral curve, it seemed natural to conjecture that these symplectic invariants do coincide with the partition function of B model topological string with more general backgrounds.

Another further property of the symplectic invariants points in the same direction. As it is reminded in section 4.4.2, using the modular properties of the symplectic invariants, it was proved that one can promote the latter to modular invariants whose non-holomorphic part is fixed by the same set of equations as the B model topological string partition function [63]: the holomorphic anomaly equations of BCOV [17]. This means that the non-holomorphic part of these functions coincide. To prove the conjecture, one thus have to prove it only for the holomorphic part.

Further studies were led by Mariño and collaborators [92, 22]: they checked for many explicit examples of possible backgrounds for the B model topological strings that the partition function and open string amplitudes indeed coincide with the correlation functions and symplectic invariants computed on the associated spectral curve. Every single check indeed works, giving more weight to this conjecture!

Another general check can be made by looking at the short summary made in the array in section 11.1.4. It is also interesting to note that the disc and cylinder amplitudes can be computed independently from this conjecture for any background in the B model: they satisfy the relation conjectured by [22].

Moreover, the computation of the sum over large partition with respect to the q-deformed Plancherel measure makes the link with the topological vertex approach and proves the conjecture in a particular family of target spaces. The extension of this method could lead to a direct proof of this conjecture.

Finally, a last clue has been added recently by Dijkgraaf and Vafa [41], using an effective field theory conjectured to be equivalent to B model: the Kodaira-Spencer theory. This check is the subject of the next section.

11.2 Kodaira-Spencer theory

11.2.1 Introduction: an effective field theory for the B-model

The six dimensional Kodaira-Spence theory is the string field theory for the B model on Calabi-Yau threefold. Consider the case of non-compact Calabi-Yau threefold $\widetilde{M}$ defined by

$$H(x,y) = \omega^+ \omega^- \tag{11-35}$$

whose holomorphic volume form is

$$\Omega = \frac{d\omega}{\omega} dx dy. \tag{11-36}$$

The Kodaira-Spencer theory is the quantization of the cohomologically trivial variations of the operator $\overline{\partial}$ on $\widetilde{M}$ with fixed complex structure.

The setup dual to the preceding section corresponds to the local surface

$$H(x,y) = 0. \tag{11-37}$$

Since one can see that the periods of Ω can be reduced on this local surface to the integrals of the one form ydx, the two dimensional reduction of the Kodaira-Spencer theory is defined by the pair $(\overline{\partial}, ydx)$ on the spectral curve: it means that it is the study of the deformations of $\overline{\partial}$ keeping the cohomology class of ydx fixed. For this purpose, one looks at the variations of the this operator under the form

$$\overline{\partial} \rightarrow \overline{\partial} - \frac{\overline{\partial}\phi}{ydx}\partial \tag{11-38}$$

where ϕ is a scalar field satisfying

$$\overline{\partial}\partial\phi = 0. \tag{11-39}$$

Indeed, under this variation, the cohomology class of ydx is not changed since it transforms as

$$ydx \rightarrow ydx + d\phi. \tag{11-40}$$

Finally, we are left with a field theory on the spectral curve $\mathcal{L}$ given by the action[25]:

$$S = \int_{\mathcal{L}} \partial\phi\overline{\partial}\phi + \frac{ydx}{\lambda}\overline{\partial}\phi + \frac{\lambda}{ydx}\overline{\partial}\phi\,(\partial\phi)^2 \tag{11-41}$$

where we rescale the differential ydx by the string coupling constant $\lambda = \frac{t}{N}$: $ydx \rightarrow \frac{ydx}{\lambda}$. Let us just explain the three different terms of this action. The first term is a simple kinetic term whereas the second term corresponds to the coupling to a holomorphic background gauge field $\frac{ydx}{\lambda}$. The most interesting term is the third one: this cubic interaction encodes the perturbative corrections and is the fundamental ingredient of this action.

Let us now move to the observables of this theory. First of all, the partition function can be written as

$$\mathcal{Z} = e^{-\mathcal{F}} \tag{11-42}$$

---

[25]For a very pedagogic construction of this action, see [41].

where the free energy $\mathcal{F}$ has a topological expansion in terms of the string coupling constant

$$\mathcal{F} = \sum_{g \geq 0} \lambda^{2g-2} F^{(g)}. \tag{11-43}$$

One also defines the correlation functions

$$W_k(z_1, \ldots, z_k | \lambda) := \langle \partial \phi(z_1) \ldots \partial \phi(z_k) \rangle_c \tag{11-44}$$

where the subscript c denotes the connected part. These correlation function also have a topological expansion

$$W_k(z_1, \ldots, z_k | \lambda) = \sum_{g \geq 0} \lambda^{2g+k-2} W_k^{(g)}(z_1, \ldots, z_k) \tag{11-45}$$

coming from the interaction term $e^{\lambda \int_{\mathcal{L}} \frac{\overline{\partial} \phi (\partial \phi)^2}{y dx}}$.

11.2.2 Recursive relations as Schwinger-Dyson equations

One can remark that the integrant in this interaction can be written as a total derivative and does not give any contribution except at the zeroes at the denominator, i.e. the zeroes of $y dx$. These zeroes are the zeroes of $y(z)$ and $dx(z)$. However, one can show that only the zeroes a_i of dx do give non-vanishing contributions (see [41]). Thus, the interaction term can be written as follows:

$$\int_{\mathcal{L}} \frac{\overline{\partial} \phi (\partial \phi)^2}{y dx} = \sum_i \oint_{a_i} \frac{\phi \, \partial \phi \, \partial \phi}{y dx}. \tag{11-46}$$

This means that the interaction vertex is localized at the branch points.

In order to compute the correlation functions, one thus has to compute terms of the form $\left\langle \partial \phi(z_1) \oint_{z \to a_i} \frac{\phi(z) \, \partial \phi(z) \, \partial \phi(z)}{y(z) dx(z)} \ldots \right\rangle$. A first step consists in the computation of the two point chiral operator $\langle \partial \phi(z) \partial \phi(z_1) \rangle$ which is known to be the Bergmann kernel. From this point, one can easily compute the contraction of $\partial \phi(z_1)$ with $\phi(z)$

$$\langle \phi(z) \partial \phi(z_1) \rangle_{twist} = \frac{1}{2} \int_{\xi(z')=-\xi(z)}^{\xi(z)} B(z', z_1) \tag{11-47}$$

where the subscript *twist* refers to the fact that one constrains the scalar field ϕ to be an odd function of a local variable $\xi(z)$ as z approaches a branch point:

$$\phi(-\xi(z)) = -\phi(\xi(z)). \tag{11-48}$$

In other terms, using the notations of section 2, it implies, thanks to De l'Hôpital's rule:

$$\lim_{z \to a_i} \frac{\langle \phi(z) \partial \phi(z_1) \rangle_{twist}}{y dx(z)} = \lim_{z \to a_i} \frac{1}{2} \frac{dE_z(z_1)}{(y(z) - y(\overline{z})) dx(z)} = - \lim_{z \to a_i} K(z_1, z). \tag{11-49}$$

Finally, taking into account the normal ordering of the cubic interaction term, the Schwinger-Dyson equations of this theory give the recursion relation for the correlation functions

$$W_{n+1}^{(g)}(z_0, J) = \sum_i \operatorname*{Res}_{z \to a_i} K(z_0, z) \left[W_{n+2}^{(g-1)}(z, \bar{z}, J) + \sum_{h=0}^{g} \sideset{}{'}\sum_{I \subset J} W_{1+|I|}^{(h)}(z, I) W_{1+n-|I|}^{(g-h)}(\bar{z}, J \backslash I) \right].$$
(11-50)

The correlators of the Kodaira-Spencer theory on $\mathcal{L}$ are thus the correlation functions computed from the latter curve.

What about the partition function? On the one hand, from the topological expansion, one easily finds that

$$\lambda \frac{\partial \mathcal{F}}{\partial \lambda} = \sum_{g \geq 0} (2g - 2) \lambda^{2g-2} F^{(g)}.$$
(11-51)

On the other hand, let us reexpress the LHS directly in terms of the correlators of the theory. Indeed, thanks to the expression of the action eq.(11-41), one gets

$$\lambda \frac{\partial \mathcal{Z}}{\partial \lambda} = -\frac{1}{\lambda} \left\langle \int_{\mathcal{L}} y dx \bar{\partial} \phi \right\rangle + \frac{\lambda^2}{y dx} \left\langle \int_{\mathcal{L}} \bar{\partial} \phi \, (\partial \phi)^2 \right\rangle.$$
(11-52)

In order to compute these terms, one proceed as in the case of the correlation functions by localizing these expressions around the branch points. One can first remark that the second term vanishes since it corresponds to the interaction operator with no field inserted. Let us thus compute the first term. Since the integrant can be written as a total derivative $y dx \, \bar{\partial} \phi = d \, (y dx \, \phi)$, this term can be localized around the poles of the integrant which are nothing but the branch points, i.e.

$$\lambda \frac{\partial \mathcal{Z}}{\partial \lambda} = \frac{1}{\lambda} \sum_i \left\langle \oint_{a_i} y dx \, \phi \right\rangle.$$
(11-53)

Consider now a primitive Φ of $y dx$

$$d\Phi = y dx.$$
(11-54)

Integrating by parts, it implies

$$\lambda \frac{\partial \mathcal{Z}}{\partial \lambda} = \frac{1}{\lambda} \sum_i \oint_{a_i} \Phi \, \langle \partial \phi \rangle.$$
(11-55)

In terms of the topological expansion, this equation coincides with the definition of the symplectic invariants

$$F^{(g)} = \frac{1}{2 - 2g} \sum_i \oint_{a_i} \Phi(z) W_1^{(g)}(z).$$
(11-56)

This means that the partition function of the Kodaira-Spencer theory is the tau function τ_N defined from the symplectic invariants in section 2.

12   Conclusion

In this review article, we have presented overview of the recent method introduced for solving matrix models loop equations, and its further extension to a more general context. We have defined the notion of symplectic invariants of a spectral curve, and we have studied its main applications, as in the present state of the art. In some sense, starting from the spectral curve of a classical integrable system, we have proposed a way to reconstruct the full quantum integrable system.

The study of applications to enumerative geometry and integrable systems of those notions is probably only at its beginning, and in particular the consequences for topological string theory are still mostly to be understood...

Acknowledgments

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