Wave propagation and spatial dispersion in random media

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The study of spatial dispersion in electromagnetic wave propagation in random media is approched via the quasi-crystalline approximation in the framework of multiple-scattering theory. The longitudinal and transverse permittivity kernels are obtained explicitely by using a simplified resonant model for the T-matrix of the scatterers. The transverse dispersion equation is solved numerically for all its frequency-dependent solutions in a given domain of the complex plane. The physical meaning of these solutions is discussed.

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I. INTRODUCTION

It is well known that standard effective-medium theories [1] are unable to account for the behaviour of random materials exposed to an electromagnetic wave, as soon as one wants to go beyond the quasi-static regime: for high frequencies, scattering effects on impurities become relevant and can not be correctly treated by such theories. Mathematically, this shows up as the appearance of new possible solutions of the dispersion equations: one equation for the tranverse fields, and another one for the longitudinal fields. We shall mainly focus here on the transverse equation. All standard effective-medium theories assume the transverse dispersion equation for the averaged fields to be of the form $k^2 - k_e^2 = 0$, where $k_e^2 = (\omega/c)^2\varepsilon_e\mu_e$. The frequency-dependent effective permittivity ε_e and permeability μ_e are obtained from the theory. This equation possesses only one (outgoing) solution with positive imaginary part. However, what distinguishes an inhomogeneous medium from an homogeneous one is the presence of (at least) one scale of length: say a, the radius of the impurities, for instance. In the space Fourier domain, this length generates a dimensionless group ak on which k_e^2 must depend. As a consequence, the transverse dispersion relation should read $k^2 - (\omega/c)^2\varepsilon_\perp(k,\omega) = 0$ and possesses in general more than one solution. The solutions which will effectively contribute as propagation channels in the averaged medium are those with least imaginary parts. On physical grounds, one expects that the more disordered the medium (that is: the higher the volumic concentration of impurities, and their dielectric mismatch with the background), the higher the number of relevant solutions there is, invalidating henceforth a description of the medium in terms of effective constitutive parameters. This picture is usually deliberatly ignored because of the lack of simple models allowing for a systematical study of these effects. The purpose of the present communication is to introduce such a model, and to describe some of its basic features [2]. This model describes a system of spherical resonant scatterers with relative dielectric permittivity ε_s embedded in some background medium with relative permittivity ε_m. Both the scatterers and the background medium are assumed non-magnetic. The model provides a simple non-local and frequency-dependent extension to the Maxwell-Garnett effective-medium theory.

II. NON-LOCAL MAXWELL-GARNETT THEORY

We consider first the multiple-scattering equations for the Green operators of the electric fields, expressed in terms of the T-matrices T_i of the scatterers, $i = 1 \ldots N$ [3]. Let G denote the full Green operator for the electric field in the disordered medium and G_0 its counterpart in the background bare propagation medium. Then,

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$$G = G_0 + G_0 \sum_i T_i G_i \tag{2.1a}$$

$$G_i = G_0 + G_0 \sum_{j \neq i} T_j G_j, \tag{2.1b}$$

where G_i is the Green operator for the *local field* impinging on the scatterer i. Let also the averaging operator $< . >$ denote statistical averaging over all possible configurations of scatterers, and $< . >_{i_1,\ldots,i_k}$ denote averaging over all possible configurations letting the scatterers $i_1, \ldots i_k$ at fixed positions. Then the average Green function reads $< G >= G_0 + NG_0 < T_i G_i >= G_0 + NG_0 << T_i G_i >_i>$, where $< T_i G_i >_i = T_i G_0 + (N-1) T_i G_0 < T_j G_j >_i = T_i G_0 + (N-1) T_i G_0 << T_j G_j >_{i,j} >_i$. A similar equation for $< T_j G_j >_{i,j}$ may be written, and so on with higher order correlations between scatterers. The first order mean-field closure approximation $< T_j G_j >_{i,j} \simeq < T_j G_j >_j$, known as the Quasi-Crystalline Approximation [4], permits to solve for $< T_i G_i >_i$, hence for $< G >$, in terms of the two-point correlation function of the scatterers.

We assume the scatterers to be spherical and identical, with radius a, and we represent them by a simplified momentum independent, resonant, and energy-conserving point-like model [2,5]

$$\mathsf{T}_i(\mathbf{k}_1|\mathbf{k}_2) = \frac{1}{(2\pi)^3}\frac{4\pi}{3} a^3 k_m^2\, e^{i(\mathbf{k}_1-\mathbf{k}_2).\mathbf{r}_i} \alpha_\varepsilon\, \mathsf{I}, \tag{2.2a}$$

$$\alpha_\varepsilon = \frac{\varepsilon_s - \varepsilon_m}{\varepsilon_m + L(\varepsilon_s - \varepsilon_m)}, \tag{2.2b}$$

$$L = (1/3) - (4/15)(ak_m)^2 - (2/9)i(ak_m)^3. \tag{2.2c}$$

The dyadic I is the 3×3 identity matrix. We choose the two-body normalized correlation function to be the unit step $g(\mathbf{r}_i - \mathbf{r}_j) = \theta(|\mathbf{r}_i - \mathbf{r}_j| - 2a)$, where θ denotes the Heaviside function. It implements volume exclusion between two scatterers. We also introduce the volumic concentration of scatterers $f = (4\pi/3)(N/V)$, where V is the volume of the system. After all calculations have been performed in the thermodynamic limit (N, $V \to \infty$ at f constant), we find the translation-invariant QCA-averaged dyadic Green function to be $< \mathsf{G} > (\mathbf{k}_1|\mathbf{k}_2) = \delta(\mathbf{k}_1 - \mathbf{k}_2)\mathsf{G}(\mathbf{k}_1)$, with ($\hat{\mathbf{k}} \equiv \mathbf{k}/k$):

$$\mathsf{G}(\mathbf{k}) = \frac{\mathsf{I} - \hat{\mathbf{k}}\hat{\mathbf{k}}}{k^2 - (\omega/c)^2 \varepsilon_\perp(k)} - \frac{\hat{\mathbf{k}}\hat{\mathbf{k}}}{(\omega/c)^2 \varepsilon_\parallel(k)}. \tag{2.3}$$

The transverse ($\perp$) and longitudinal ($\parallel$) permittivity kernels, which both tend to the same value when $k \to 0$ as implied by statistical isotropy [6], allow one to define non-local dielectric permittivity and magnetic permeability kernels $\varepsilon(k) \equiv \varepsilon_\parallel(k)$, $\mu^{-1}(k) \equiv 1 - (\omega/c)^2[\varepsilon_\perp(k) - \varepsilon_\parallel(k)]/k^2$, such that the statistical-averaged fields in the medium be linked by spatially non-local constitutive relations in direct space: $< \mathbf{D} > (\mathbf{r}) = \varepsilon_0 \int d^3r'\, \varepsilon(|\mathbf{r} - \mathbf{r}'|) < \mathbf{E} > (\mathbf{r}')$, and $< \mathbf{H} > (\mathbf{r}) = (1/\mu_0) \int d^3r'\, \mu^{-1}(|\mathbf{r} - \mathbf{r}'|) < \mathbf{B} > (\mathbf{r}')$. Thus spatial dispersion implies in principle the existence of an induced magnetic-like response (even in the medium is initially non-magnetic), whose physical origin lies in the presence of polarization currents due to the dielectric contrast between the impurities and the background. The two kernels read here:

$$\varepsilon_{\perp,\parallel}(k) = \varepsilon_m \frac{1 + (2/3) f \alpha_\varepsilon [1 + (1/2) Q_{\perp,\parallel}(k)]}{1 - (1/3) f \alpha_\varepsilon [1 - Q_{\perp,\parallel}(k)]} \tag{2.4}$$

$$Q_\perp(k) = 1 - 3(1 - 2iak_m) e^{i2ak_m} \frac{j_1(2ak)}{2ak} + 3k_m^2 \frac{1 - e^{i2ak_m}[\cos(2ak) - 2iak_m\, j_0(2ak)]}{k^2 - k_m^2} \tag{2.5a}$$

$$Q_\parallel(k) = -2 + 6(1 - 2iak_m) e^{i2ak_m} \frac{j_1(2ak)}{2ak}. \tag{2.5b}$$

The j_l, $l = 0, 1$, are spherical Bessel functions [7]. We check that, as expected: 1) $Q_\perp(k)$ is non-singular when $k \to k_m$; 2) $Q_\perp(k)$ and $Q_\parallel(k)$ both tend to the same expression when $k \to 0$; 3) these two quantities both tend to 0 when $a \to 0$, so that (2.4) reduces to the well-known quasi-static Maxwell-Garnett effective-medium formula in this limit. The transverse (resp. longitudinal) modes in the time-harmonic regime which contribute to the coherent part of the fields are the solutions $k(\omega)$ of the dispersion equation $k^2 - (\omega/c)^2 \varepsilon_\perp(k, \omega) = 0$ (resp. $\varepsilon_\parallel(k, \omega) = 0$).

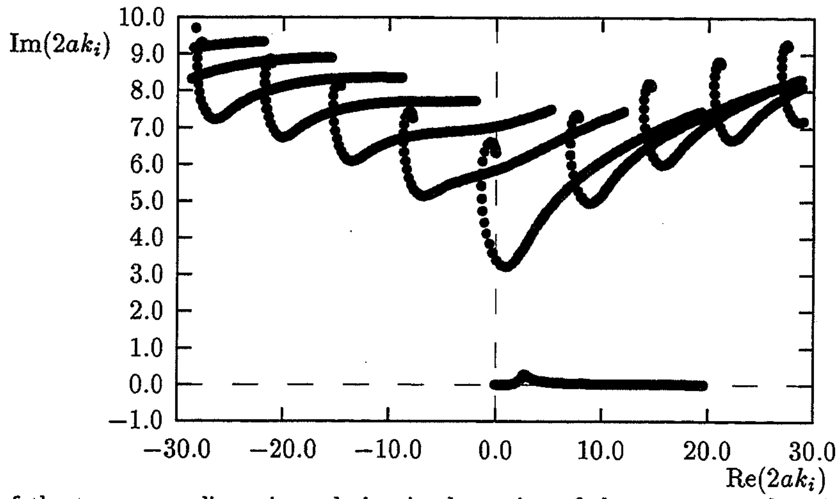

FIG. 1. Solutions of the transverse dispersion relation in the region of the upper complex plane $-30 \leq \mathrm{Re}(2ak) \leq 30$, $0 \leq \mathrm{Im}(2ak) \leq 10$, for reduced frequencies $0 \leq 2ak_m \leq 20$. The solutions move from left to right as the frequency increases, past the resonant regime. See text.

In Fig. 1 are displayed the paths followed by the all the solutions of the transverse dispersion equation in the displayed region of the upper complex plane (retarded solutions). The dispersion relation has an infinite number of solutions. Since it is an even function of k, advanced solutions $-k$ also exist in the lower complex plane. The solutions are computed numerically [8] for a moderate concentration $f = 0.10$ and dimensionless frequency variable $2ak_m$ ranging from 0 to 20. For testing purposes, we chose frequency-independent permittivities $\varepsilon_m = 1$ and $\varepsilon_s = 5$. The solutions in Fig. 1 are of two different types: 1) the solution we term as the perturbative one (for it could be perturbatively obtained as a series in powers of f), which starts from the origin at $\omega = 0$, an whose imaginary part displays a peak at the resonance freqency; 2) non-perturbative solutions, which are identified in the limit $\omega \to 0$ with the roots of the function $1 - f\alpha_\varepsilon j_1(2ak)/(2ak)$. In this limit, the infinite set $\{k_n\}_{n\in\mathbb{Z}}$ of these roots, may be computed asymptotically [9] as

$$k_n = u_n - 2i\ln(u_n) - u_n^{-1}\left[1 + 4\ln(u_n)\right] + O\left(u_n^{-2}\ln^2(u_n)\right) \tag{2.6a}$$

$$\text{with} \quad u_n = (2n+1)\pi + i\ln\left(\frac{2}{3}\frac{1}{f\alpha_\varepsilon}\right). \tag{2.6b}$$

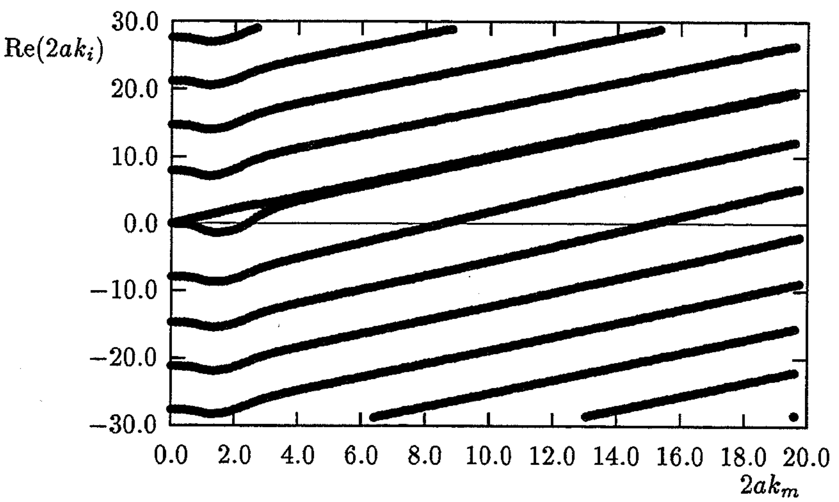

FIG. 2. Real parts Re($2ak$) of the solutions of the transverse dispersion relation, displayed vs. $2ak_m$.

The imaginary part of these solutions explodes as $-\ln(f)$ when $f \to 0$, while the solutions themselves become irrelevant, as they must. Pairing the solutions with equal imaginary part and opposite real part, we obtain standing waves to be identified as proper static modes for the electric field between particles. As the frequency increases, each pair decouples and leads finally to the opening of two different propagation channels (a new outgoing wave is allowed as soon as the real part of its propagation constant becomes positive). These solutions have a minimum imaginary part in the resonant regime, where they become most relevant, since closer from the perturbative solution.

Another light is shed on the solutions by Fig. 2, where their real parts are displayed as functions of $2ak_m$ (compare to Fig. 1). This plot emphasizes their regularity. Actually, such a regularity is a direct consequence of the quasi-crystalline approximation: being of the mean-field type, this approximation indeed amounts to consider the averaged polarizability $< T_i G_i >_{i,j}$ of the scatterer i to be independent of the locations of the other scatterers j. This implements an artificial crystalline structure in the system, for such an hypothesis can only be realized in truly periodic crystalline systems; hence the name. However, we expect our model to allow for a qualitatively correct understanding of the main features of spatial dispersion effects in random media.

A detailled analysis of the model will be presented elsewhere [8].

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