Universal scaling limits of matrix models, and (p,q) Liouville gravity

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Abstract:

We show that near a point where the equilibrium density of eigenvalues of a matrix model behaves like $y \sim x^{p/q}$, the correlation functions of a random matrix, are, to leading order in the appropriate scaling, given by determinants of the universal (p, q)-minimal models kernels. Those (p, q) kernels are written in terms of functions solutions of a linear equation of order q, with polynomial coefficients of degree $\leq p$. For example, near a regular edge $y \sim x^{1/2}$, the $(1, 2)$ kernel is the Airy kernel. Those kernels are associated to the (p, q) minimal model, i.e. the (p, q) reduction of the KP hierarchy solution of the string equation. Here we consider only the 1-matrix model, for which $q = 2$.

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1 Introduction

In this article, we shall consider "scaling limits" of matrix integrals.

We shall show, under certain assumptions, that scaling limits of matrix integrals are governed by some well known integrable systems. The fact that double scaling limits of matrix models are minimal models (p, q) of conformal field theories [26], has been well known in the physics literature for a long time (see [27, 20, 41] for review, and among others see [51, 46, 48, 47, 59, 53, 15, 29, 42, 23, 42]), and here we merely summarize some results scattered in the physics literature, we present the main features of those universal limit laws, and provide a mathematical proof.

The idea of the proof works backwards: we show that (p, q) minimal models determinantal correlation functions satisfy the same recursion as the scaling limits of matrix models.

We shall consider only the 1-matrix model, whose corresponding limit integrable systems are the $(p, 2)$ minimal models, reductions of KdV, and we hope to later generalize those results to multi-matrix models and general (p, q) limits, as claimed in many physics works [27, 48, 51].

The main result, theorem 2.3, is that limit correlation functions are given by determinantal formulae of the (p, q) kernel.

1.0.1 Example: $(1, 2)$ law: Airy kernel and Tracy-Widom law

The equilibrium density of eigenvalues of a $N \times N$ random hermitian matrix, generically behaves near the edge of the distribution, like:

$$\rho(x) \sim x^{\frac{1}{2}}. \tag{1-1}$$

It is well known that, after rescaling x by $N^{2/3}$, the $n-$points correlation functions in the vicinity of the edge, are given by determinants of the Airy kernel which appears in Tracy-Widom law [66] of extreme eigenvalues statistics:

$$\rho_n(N^{-2/3}x_1, \dots, N^{-2/3}x_n) \underset{N\to\infty}{\sim} N^{\frac{2n}{3}} \det(\hat{K}_{\text{Airy}}(x_i, x_j)) \ (1 + O(N^{-1/3})), \tag{1-2}$$

and the Airy kernel is the Christoffel-Darboux kernel of the Airy function:

$$\hat{K}_{\text{Airy}}(x_1, x_2) = \frac{Ai(x_1)Ai'(x_2) - Ai'(x_1)Ai(x_2)}{x_1 - x_2}. \tag{1-3}$$

Notice that the Airy function satisfies a 2nd order ODE, whose coefficients are polynomials of degree 1:

$$Ai''(x) = x\, Ai(x), \tag{1-4}$$

which can also be written as a 2×2 differential system:

$$\frac{d}{dx} \begin{pmatrix} Ai(x) \\ Ai'(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \begin{pmatrix} Ai(x) \\ Ai'(x) \end{pmatrix}. \tag{1-5}$$

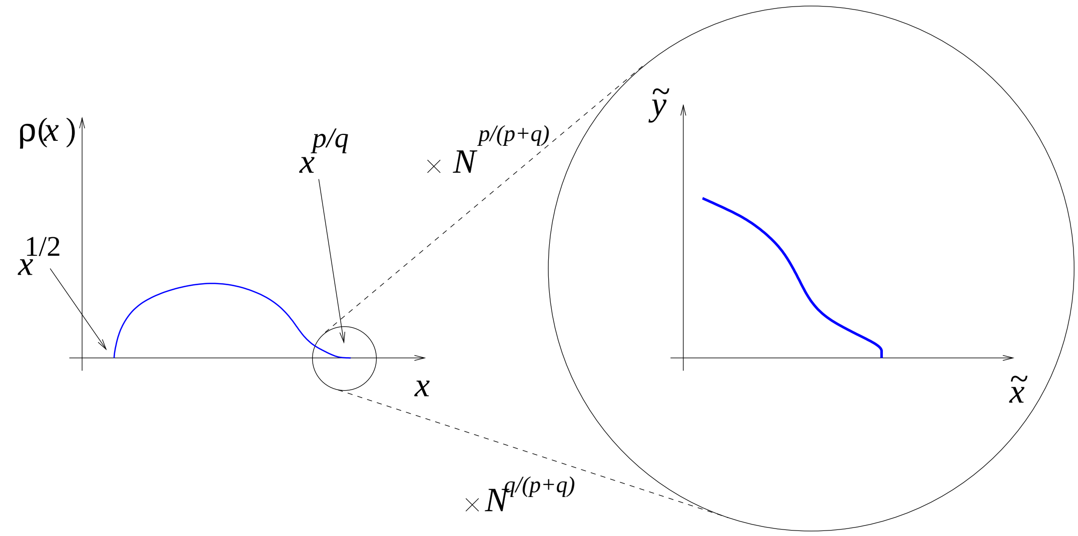

Figure 1: Generically, the equilibrium density of eigenvalues behaves like $x^{1/2}$ near endpoints. It may happen, after fine tuning the parameters, that it behaves like $x^{p/q}$ ($q = 2$ for 1-matrix model). The eigenvalue statistics at a scale $\tilde{x} = x\ N^{q/(p+q)}$ is governed by the universal (p,q) kernel law. The (p,q) kernel is an integrable kernel, associated to the (p,q) reduction of the KP hierarchy. For a regular endpoint $(p,q) = (1,2)$ it is the Airy kernel.

1.0.2 Higher $(p,2)$ laws

More generally consider a p/q singularity of the equilibrium density of eigenvalues:

$$\rho(x) \sim x^{p/q}. \tag{1-6}$$

We shall consider only $q = 2$ and $p = 2m+1$ in this article, but we recall that the physics literature claims that general $y \sim x^{p/q}$ case can be treated the same way, and should correspond to (p,q) minimal models.

We shall see, after rescaling x by $N^{q/(p+q)}$, that the correlation functions in the vicinity of the edge, are given by determinants of the (p,q) kernel which appears in (p,q) minimal models of conformal field theory [26].

$$\rho_n(N^{-\frac{q}{p+q}}x_1,\ldots,N^{-\frac{q}{p+q}}x_n) \underset{N\to\infty}{\sim} N^{n\frac{q}{p+q}} \det(\hat{K}_{(p,q)}(x_i,x_j))\ (1+O(N^{-1/(p+q)})), \tag{1-7}$$

and the $(p,2)$ kernel is the Christoffel-Darboux kernel of the $(p,2)$ Baker-Akhiezer function:

$$\hat{K}_{(p,2)}(x_1,x_2) = \frac{\psi(x_1)\tilde{\psi}(x_2) - \tilde{\psi}(x_1)\psi(x_2)}{x_1 - x_2}. \tag{1-8}$$

where the $(\psi, \tilde{\psi})$ functions satisfy a 2nd order ODE:

$$\frac{d}{dx} \begin{pmatrix} \psi(x) \\ \tilde{\psi}(x) \end{pmatrix} = \mathcal{D}_{(p,q)}(x) \begin{pmatrix} \psi(x) \\ \tilde{\psi}(x) \end{pmatrix} \tag{1-9}$$

where $\mathcal{D}_{(p,2)}(x)$ is a 2×2 matrix with polynomial coefficients, such that the degree of $\det \mathcal{D}$ is of degree at most p.

Moreover this differential system, is associated to the Lax matrix of the $(p, 2)$ reduction of the integrable KdV hierarchy, which means that the coefficients of the matrix $\mathcal{D}_{(p,q)}(x)$, are themselves solution of some non-linear integrable differential equations.

The coefficients of $\mathcal{D}_{(p,q)}(x)$ are differential polynomials of a function $u(t)$ which satisfies (for $p = 2m + 1$, $q = 2$), the $m + 1^{\text{th}}$ Gelfand-Dikii non linear equation [27], of the form:

$$u^{m+1} + u^{m-1}\ddot{u} + \ldots + u^{(2m)} = t. \tag{1-10}$$

The time t measures the distance to the critical point.

For example for pure gravity $(p, q) = (3, 2)$, $\mathcal{D}_{(3,2)}(x)$ is a 2×2 matrix, with polynomial coefficients such that the degree of $\det \mathcal{D}_{(3,2)}(x)$ is 3:

$$\mathcal{D}_{(3,2)}(x,t) = \begin{pmatrix} \dot{u} & 2x - 2u \\ (x + 2u)(2x - 2u) + \ddot{u} & -\dot{u} \end{pmatrix} \tag{1-11}$$

and where $u(t)$ is solution of the Painlevé I equation (1^{st} Gelfand-Dikii equation):

$$3u^2(t) - \frac{\ddot{u}(t)}{2} = t. \tag{1-12}$$

All this has been stated for a long time in the physics literature, and we shall just present it concisely and prove it.

1.1 Universality of eigenvalues statistics point of view

In this subsection, we summarize some well known facts about random matrices [57, 70, 33, 9, 58, 27, 13, 16, 45], and we fix the notations.

Consider a probability law of the form of the joint law of eigenvalues of a random hemitian type[3] matrix:

$$d\mu(\lambda_1, \ldots, \lambda_N) = \frac{1}{Z} \prod_{i<j} (\lambda_j - \lambda_i)^2 \prod_i \mathrm{e}^{-\frac{N}{s} V(\lambda_i)} \, d\lambda_i, \tag{1-13}$$

[3]Hermitian matrices correspond to real eigenvalues and positive measure $d\mu$, but it is customary to generalize random matrices to normal matrices having their eigenvalues on some contours in the complex plane, and the measure $d\mu$ can be complex. The loop equations are the same for all those models, they are independent of the integration contour, and thus, they can all be treated in the same framework.

where Z is the partition function

$$Z = \int d\mu = \int dM \, \mathrm{e}^{-\frac{N}{s} \mathrm{tr}\, V(M)}. \tag{1-14}$$

Here s is a parameter, often called the *temperature.* we shall be interested in the large N limit, and possibly a limit $s \to s_c$, where Z has a singularity at $s = s_c$. The name **double scaling limit** [48, 27, 46, 42, 41, 30, 29, 23, 11] means that we consider a regime where the limits $s \to s_c$ and $N \to \infty$ are related by a scaling relation

$$\boxed{(s - s_c)\, N^{-\alpha} = O(1)} \tag{1-15}$$

where α is some appropriate exponent ($\alpha = 0$ if Z is not singular).

We are interested in computing expectation values of resolvents:

$$\bar{\omega}_n(x_1, \dots, x_n) = \left\langle \mathrm{Tr}\, \frac{1}{x_1 - M} \dots \mathrm{Tr}\, \frac{1}{x_n - M} \right\rangle = \left\langle \sum_{i_1, \dots, i_n} \frac{1}{x_1 - \lambda_{i_1}} \dots \frac{1}{x_n - \lambda_{i_n}} \right\rangle \tag{1-16}$$

as well as in their cumulants

$$\hat{\omega}_n(x_1, \dots, x_n) = \left\langle \mathrm{Tr}\, \frac{1}{x_1 - M} \dots \mathrm{Tr}\, \frac{1}{x_n - M} \right\rangle_c. \tag{1-17}$$

The density correlation functions $\rho_n(x_1, \dots, x_n)$ can be easily deduced from them: densities are discontinuities of resolvents, and resolvents are Stieljes transforms of densities, for example for the 1-point function:

$$\hat{\omega}_1(x) = \int \frac{\rho_1(x')dx'}{x - x'} \qquad , \qquad \rho_1(x) = \frac{1}{2i\pi} \left(\hat{\omega}_1(x - i0) - \hat{\omega}_1(x + i0)\right). \tag{1-18}$$

Imagine, that, for $s < s_c$, the potential $V(x)$ is such that there is a large N expansion of the type:

$$\ln Z = \sum_{g=0}^{\infty} (N/s)^{2-2g} \, \hat{f}_g, \tag{1-19}$$

and similarly:

$$\hat{\omega}_n(x_1, \dots, x_n) = \sum_{g=0}^{\infty} (N/s)^{2-2g-n} \, \hat{\omega}_n^{(g)}(x_1, \dots, x_n). \tag{1-20}$$

First, let us emphasize that such an expansion does not exist for any potential V, it exists only if the integration contour for the λ_i's is a "steepest descent contour" for the potential V (i.e. a landpath and bridges path in the Riemann-Hilbert language of [6, 24]). For instance it was proved [40] that such a large N expansion holds for s sufficiently small.

From now on, let us assume that we are in a situation where such an expansion exists. In that case, the coefficients $\hat{\omega}_n^{(g)}$ and $\hat{f}_g$ were computed in [35, 19, 36], and they are the "spectral invariants" of some spectral curve associated to V/s. The spectral curve $\hat{y}(x)$, in that case, is the function $\hat{y}(x) = V'(x)/2 - \hat{\omega}_1^{(0)}(x)$, it is the large N density, also called equilibrium density $\hat{y}(x) = i\pi\rho_{\text{eq}}(x) = i\pi\rho_1^{(0)}(x)$:

Theorem 1.1 *(proved in [35, 18]) The coefficents $\hat{f}_g$ and $\hat{\omega}_n^{(g)}$ of the topological expansion of $\ln Z$ and $\hat{\omega}_n$, are the spectral invariants (in the sense of [36]) of the spectral curve:*

$$\hat{y}(x) = i\pi\, \rho_{\text{eq}}(x) = \frac{1}{2}\, V'(x) - \hat{\omega}_1^{(0)}(x). \tag{1-21}$$

Remark 1.1 We refer the reader to [36] to see how to compute the spectral invariants of an arbitrary plane curve $\hat{y}(x)$. We shall give an explicit example of computation of spectral invariants for formal matrix models below in section 1.2.1, see theorem 1.3.

Let us say that we shall not be really using any deep result of [36] in this article, except the theorem 8.1. of [36] (which is very easy to prove by recursion).

Here, as is well known in random matrix theory, $\hat{\omega}_1^{(0)}(x)$ is an algebraic curve (hyperelliptical for the 1-matrix model), with typical square-root branchpoints at the endpoints of the distribution of eigenvalues, we shall write it:

$$\hat{\omega}_1^{(0)}(x) = \frac{1}{2}V'(x) - \hat{y}(x) \tag{1-22}$$

where $\hat{y}(x)$ is the square root of some polynomial

$$\hat{y}^2 = \text{Polynomial}(x). \tag{1-23}$$

Generically, this s-dependent polynomial has only simple zeroes, and $\hat{y}(x)$ has square root singularities, but for some appropriate choices of $s = s_c$, the polynomial may have multiple zeroes, and we shall consider that, at $s = s_c$, there is a zero of order $2m+1$ at $x = 0$:

$$s = s_c \quad \longrightarrow \qquad \hat{y}(x) \sim x^{m+\frac{1}{2}}. \tag{1-24}$$

When s is close to s_c, we typically have:

$$\hat{y} \sim \sum_{k=0}^{m} x^{k+\frac{1}{2}}\, c_k\, (s - s_c)^{\frac{m-k}{m+1}} (1 + O((s - s_c)^{\frac{1}{m+1}})\ )\ , \tag{1-25}$$

which we write:

$$\hat{y}((s - s_c)^{\frac{1}{m+1}}\, x) \sim (s - s_c)^{\frac{2m+1}{2m+2}}\ y(x)\ (1 + O((s - s_c)^{\frac{1}{m+1}}) \qquad , \qquad y(x) = \sum_{k=0}^{m} c_k\, x^{k+\frac{1}{2}}. \tag{1-26}$$

At $s = s_c$ we have $\hat{y} \sim x^{m+\frac{1}{2}}$ and at $s \neq s_c$ we have $\hat{y} \sim \sqrt{x}$.

Notice that a regular endpoint corresponds to $m = 0$, and in that case, s_c can be chosen as any value of s.

1.1.1 Double scaling limit

In this article, we shall be interested in the behavior of $\hat{\omega}_n^{(g)}$ when s is close to s_c, and the x_i's are in the vicinity of a branchpoint. Theorem 8.1. of [36], implies that after rescaling, we have (when $2-2g-n<0$):

Theorem 1.2 *(theorem 8.1. of [36]): If $m>0$:*

$$\hat{f}_g \sim (s-s_c)^{(2-2g)\frac{2m+3}{2m+2}}\, f_g, \tag{1-27}$$

and if $m \geq 0$:

$$\begin{aligned} &\hat{\omega}_n^{(g)}((s-s_c)^{\frac{1}{m+1}}x_1,\dots,(s-s_c)^{\frac{1}{m+1}}x_n) \\ \sim\ &(s-s_c)^{(2-2g-n)\frac{2m+3}{2m+2}-\frac{n}{m+1}}\,\omega_n^{(g)}(x_1,\dots,x_n)\ \ (1+O((s-s_c)^{\frac{1}{m+1}})), \end{aligned} \tag{1-28}$$

where the f_g's and $\omega_n^{(g)}$'s are the spectral invariants of [36] for the spectral curve $y(x)$ appearing in eq.(1-26).

Remark 1.2 This theorem is very easy to prove by recursion on n and g from the definitions of spectral invariants in [36]. A more detailed proof is also given in section 4.8.2 of [38].

Our goal in this article, is to show that the coefficients $\omega_n^{(g)}$ and f_g can also be computed from determinantal formulae for the $(p,2)$ kernel appearing in the $(p,2)$ minimal model, this is our theorem 2.3.

1.1.2 Specific heat

If $m>0$, we consider the resummation to leading order:

$$\ln Z = \sum_g (N/s)^{2-2g}\,\hat{f}_g \sim \sum_g (N/s_c)^{(2-2g)}(s-s_c)^{(2-2g)\frac{2m+3}{2m+2}}\, f_g = F((s-s_c)N^{\frac{2m+2}{2m+3}}). \tag{1-29}$$

This shows that the double scaling limit is $N\to\infty$, $s\to s_c$ with a scaling:

$$\boxed{t=(s-s_c)\,N^{\frac{2m+2}{2m+3}}=O(1).} \tag{1-30}$$

This is a special case of the double scaling limit $(s-s_c)\sim N^{-(p+q-1)/(p+q)}$ for general (p,q).

We defined the function

$$F(t)=\sum_{g=0}^{\infty} t^{(2-2g)\frac{2m+3}{2m+2}}\, f_g. \tag{1-31}$$

Consider its second derivative, often called specific heat:

$$u(t) = \frac{d^2}{dt^2}\, F(t) \ = \sum_g u_g\, t^{\frac{1-g(2m+3)}{m+1}}. \tag{1-32}$$

i.e.

$$u_g = (1-g)\, f_g\ \frac{(2m+3)(m+2-g(2m+3))}{(m+1)^2}. \tag{1-33}$$

We shall prove in this article, that, as claimed in many physics articles, this function satisfies the $m+1^{\text{th}}$ Gelfand-Dikii non-linear equation. For instance if $m=1$, it satisfies the Painlevé I equation:

$$3u^2 - \frac{1}{2}\ddot{u} = t. \tag{1-34}$$

Moreover, it is well known from general considerations in statistical physics, that the free energy $-\ln Z$ should be convex, i.e. $u(t)$ should be negative for $t>0$:

$$u(t) \leq 0. \tag{1-35}$$

Remark 1.3 The case $m=0$, needs some care. The correlation functions ω_n's, indeed correspond to the $s \to s_c$ limits of $\hat{\omega}_n$'s, i.e. a zoom $x \to (s-s_c)x$ near a regular branch point, but the free energy $F = \ln Z$ is not divergent at $s = s_c$, and thus $F(t)$ cannot be seen as the $s \to s_c$ limit of $\ln Z$. In that case $m=0$, the 1st Gelfand-Dikii equation is not differential, it is simply

$$u(t) = -\frac{t}{2}, \tag{1-36}$$

and, if it made sense, it would correspond to a free energy diverging as $F \sim N^2$, but in fact the free energy F is not divergent, and one finds that $f_g = 0$ for $g \geq 1$.

1.2 Formal matrix models and combinatorics point of view

As it was discovered by Brezin-Itzykson-Parisi-Zuber [14], matrix integrals are (in the formal sense) generating functions for counting discrete surfaces of a given topology [22, 25, 27].

Consider the potential:

$$V(x) = \frac{x^2}{2s} - \frac{1}{s}\delta V(x) \qquad , \qquad \delta V(x) = \sum_{j=3}^{d+1} \frac{s_j}{j}\, x^j \tag{1-37}$$

Formal matrix integrals are defined as:

$$Z = \int_{\text{formal}} dM\, \mathrm{e}^{-\frac{N}{s}\operatorname{Tr} V(M)} \ \stackrel{\text{def}}{=}\ \sum_{k=0}^{\infty} \frac{N^k}{k!\, s^k} \int_{H_N} dM\ (\operatorname{tr} \delta V(M))^k\ \mathrm{e}^{-\frac{N}{s}\operatorname{Tr}\frac{M^2}{2}}. \tag{1-38}$$

where dM is the Lebesgue measure on H_N, normalized such that $\int dM\, \mathrm{e}^{-\frac{N}{s}\mathrm{Tr}\,\frac{M^2}{2}} = 1$. In other words, we have exchanged the integral and the summation over k. Z is a formal power series in s, where each coefficient is a finite sum of polynomial expectation values of a Gaussian integral:

$$Z = 1 + \sum_{j=1}^{\infty} s^j \tilde{A}_j. \tag{1-39}$$

After taking the Log, we also have a formal power series:

$$\ln Z = \sum_{j=1}^{\infty} s^j A_j. \tag{1-40}$$

It was noticed by t'Hooft [64] and then BIPZ[14], that, after dividing by N^2, each coefficient A_j is a polynomial in $1/N^2$, namely:

$$A_j = \sum_{g=0}^{g_{\max}(j)} N^{2-2g}\; A_{j,g}. \tag{1-41}$$

Then, one defines:

$$\hat{f}_g = \sum_{j=1}^{\infty} s^{j+2-2g}\; A_{j,g} \tag{1-42}$$

which is also a formal series in powers of s (one can easily prove that $A_{j,g} = 0$ if $j+2-2g < 0$). In that case, the following large N topological expansion holds as an equality between formal series of s, order by order in s:

$$\boxed{\ln Z = \sum_{g=0}^{\infty} (N/s)^{2-2g}\; \hat{f}_g.} \tag{1-43}$$

We emphasize that this equality is not a large N asymptotic expansion, it is a small s asymptotic expansion, and order by order in s the sum over g is finite.

It was proved by BIPZ in 1978 [14], by a mere application of Wick's theorem, that $\hat{f}_g$ is the generating function for maps of genus g:

$$\hat{f}_g = \sum_{\text{maps, genus } g} \frac{s^{\#\text{vertices}}}{\#\text{Aut}}\; s_3^{\#\text{triangles}}\; s_4^{\#\text{quadrangles}} \ldots s_{d+1}^{\#(d+1)-\text{gons}}. \tag{1-44}$$

Similarly one may compute formal expectation values:

$$\hat{\omega}_n(x_1,\ldots,x_n) \;=\; \sum_{l_1,\ldots,l_n} \frac{1}{x_1^{l_1+1}\ldots x_n^{l_n+1}} \left\langle \mathrm{tr}\, M^{l_1} \ldots \mathrm{tr}\, M^{l_n} \right\rangle_{c,\text{formal}}$$

$$= \left\langle \mathrm{tr}\,\frac{1}{x_1-M}\dots \mathrm{tr}\,\frac{1}{x_n-M}\right\rangle_{c,\mathrm{formal}} \tag{1-45}$$

where c means cumulant, and *formal* means that we compute the integral by exchanging the order of the Taylor expansion of $\mathrm{e}^{-\frac{N}{s}\mathrm{tr}\,\delta V(M)}$ and the integral, as in eq.(1-38). $\hat\omega_n(x_1,\dots,x_n)$ is thus a formal power series in s, whose coefficients are polynomial expectation values of a Gaussian integral. For each power of s, the coefficient in $\hat\omega_n$ is a polynomial in the $1/x_i$'s, and is a polynomial in $1/N$. We write:

$$\hat\omega_n(x_1,\dots,x_n) = \sum_{g=0}^{\infty}(N/s)^{2-2g-n}\,\hat\omega_n^{(g)}(x_1,\dots,x_n) \tag{1-46}$$

which is an equality between formal series of s (order by order the sum over g is finite).

It was also proved by BIPZ [14], that $\hat\omega_n^{(g)}$ is the generating function for maps of genus g, with n marked faces, and with 1 marked edge on each marked face:

$$\hat\omega_n^{(g)} = \sum_{\text{maps, genus } g}\frac{s^{\#\text{vertices}}}{\#\text{Aut}}\;\frac{s_3^{\#\text{triangles}}\,s_4^{\#\text{quadrangles}}\dots s_{d+1}^{\#(d+1)-\text{gons}}}{x_1^{l_1+1}\dots x_n^{l_n+1}} \tag{1-47}$$

where l_i is the length of the i^{th} marked face.

All this is now standard result in combinatorics of maps.

1.2.1 Spectral invariants and spectral curve

More recently, it was proved in [35, 36], that the functions $\hat\omega_n^{(g)}$ and $\hat f_g$ are the spectral invariants (in the sense of [36]) of the formal matrix model spectral curve $\mathcal{E}_{\text{formal MM}} = (\hat x,\hat y)$ defined parametrically by:

$$\mathcal{E}_{\text{formal MM}} = \begin{cases} \hat x(z) = \alpha+\gamma(z+\frac{1}{z}) \\ \hat y(z) = \sum_{k=1}^{d} u_k(z^k - z^{-k}) \end{cases} \tag{1-48}$$

where the coefficients α,γ,u_k are entirely determined by the following algebraic constraints:

$$\hat x(z) - \sum_{j=2}^{d} s_{j+1}\,\hat x(z)^j = \sum_{k=1}^{d} u_k(z^k+z^{-k}) \quad , \quad u_1=\frac{s}{\gamma} \quad , \quad u_0=0, \tag{1-49}$$

which give an algebraic equation for α and γ, whose solution we choose such that α and γ^2 are formal power series of s starting with:

$$\gamma^2 = s+O(s^2) \quad , \quad \alpha = O(s). \tag{1-50}$$

(We give the example of quadrangulations below)

Then, from [35, 18] we have:

Theorem 1.3 *(proved in [35, 18, 36]): $\hat\omega_n^{(g)}$ and $\hat f_g$ are the spectral invariants (in the sense of [36]) of the formal matrix model spectral curve $\mathcal{E}_{\mathrm{formal\,MM}} = (\hat x, \hat y)$.*

The spectral invariants of that curve are defined as follows (see [36]):

$$\hat\omega_2^{(0)}(\hat x(z_1), \hat x(z_2)) = \frac{1}{(z_1-z_2)^2\,\hat x'(z_1)\hat x'(z_2)} - \frac{1}{(\hat x(z_1)-\hat x(z_2))^2}, \tag{1-51}$$

and with $J = \{\hat x(z_1), \dots, \hat x(z_n)\}$, we have recursively:

$$\begin{aligned} & \hat\omega_{n+1}^{(g)}(J, \hat x(z_{n+1})) \\ = \quad & \frac{1}{2\hat x'(z_{n+1})} \mathop{\mathrm{Res}}_{z\to\pm1} \frac{z\,\hat x'(z)^2\,dz}{z_{n+1}\,(\hat x(z_{n+1})-\hat x(z))\hat y(z)} \Big(\hat\omega_{n+2}^{(g-1)}(J, \hat x(z), \hat x(z)) \\ & + \sum_{h=0}^{g} {\sum_{I\subset J}}' \hat\omega_{1+|I|}^{(h)}(I, \hat x(z))\hat\omega_{1+n-|I|}^{(g-h)}(J\setminus I, \hat x(z))\Big) \end{aligned} \tag{1-52}$$

and for $g \geq 2$:

$$\hat f_g = \frac{1}{2-2g} \mathop{\mathrm{Res}}_{z\to\pm1} \hat\omega_1^{(g)}(\hat x(z))\,\hat\Phi(z)\,dz \tag{1-53}$$

where $\hat\Phi'(z) = \hat y(z)\,\hat x'(z)$, and for $g = 1$:

$$\hat f_1 = \frac{1}{24}\,\ln\left(\gamma^2 \hat y'(1)\hat y'(-1)\right) \tag{1-54}$$

and for $g = 0$:

$$\begin{aligned} \hat f_0 \quad = \quad & \frac{1}{2}\Big(\sum_{j\geq1} \frac{\gamma^2}{j}(u_{j+1}-u_{j-1})^2 + \frac{2s\gamma}{j}(-1)^j(u_{2j-1}-u_{2j+1}) \\ & + \frac{3s^2}{2} + s^2\,\ln\left(\frac{\gamma^2}{s}\right)\Big). \end{aligned} \tag{1-55}$$

Example, for quadrangulations we have $s_4 \neq 0$ and all the other $s_k = 0$. That gives:

$$\mathcal{E}_{\mathrm{quadrangulations}} = \begin{cases} \hat x(z) = \gamma(z+\frac{1}{z}) \\ \hat y(z) = \frac{s}{\gamma}(z-\frac{1}{z}) - s_4\gamma^3(z^3-z^{-3}) \\ \\ \gamma^2 = \frac{1}{6s_4}\,(1-\sqrt{1-12ss_4}), \\ u_0 = u_2 = \alpha = 0, \\ u_1 = \frac{s}{\gamma} \qquad , \qquad u_3 = -s_4\,\gamma^3. \end{cases} \tag{1-56}$$

That gives:

$$\hat f_0 = \frac{1}{2}\Big(\frac{\gamma^2}{2}(u_3-u_1)^2 + \frac{\gamma^2}{4}(u_1)^2 - 2s\gamma(u_1-u_3) + \frac{2s\gamma}{2}(u_3) + \frac{3s^2}{2} + s^2\ln\frac{\gamma^2}{s}\Big), \tag{1-57}$$

$$\hat{f}_1 = \frac{1}{12} \ln (2(2s - \gamma^2)), \tag{1-58}$$

$$\hat{f}_2 = \frac{178s^3 - 465s^2\gamma^2 + 420s\gamma^4 - 130\gamma^6}{6! \; s_4^2 \; (\gamma^2 - 2s)^5}, \tag{1-59}$$

and so on... Notice that at $s = s_c = \frac{1}{12s_4}$, $\hat{y}$ has a singular branch point $\hat{y} \sim (\hat{x} - 2\gamma)^{3/2}$, and when $s \to s_c$, $\hat{f}_g$ diverges as $(s - s_c)^{\frac{5}{4}(2-2g)}$. Anticipating on what follows, we see that the exponent $5/4 = (p+q)/(p+q-1)$ indeed corresponds to the $(p,q) = (3,2)$ minimal model, of central charge $c = 0$, called pure gravity. In other words, the statistics of large quadrangulations is equivalent to the pure gravity $(3,2)$ conformal minimal model, i.e. Liouville field theory.

1.2.2 Limits of large maps

It is well known that the asymptotic large size behavior of a number of objects is related to the singularities of its generating series. Therefore, the number of maps with a large number of vertices (large maps), is governed by the singularities, i.e. the values of s_c, such that $\ln Z$ is not analytical at $s = s_c$. One should consider the singularity s_c closest to the origin, i.e. $|s_c|$ minimal, and see how the $\hat{f}_g$ and $\hat{\omega}_n^{(g)}$'s diverge at $s \to s_c$. Thus, the scaling limit $s \to s_c$ of a formal matrix integral near a singularity s_c, corresponds to the asymptotics of large discrete surfaces.

For instance, one easily sees from eq.(1-44), that the expectation value of the number of vertices for maps of genus g is:

$$< \#\text{vertices} >= s \, \frac{d}{ds} \, \ln \hat{f}_g \tag{1-60}$$

and thus large maps become dominant when $s \to s_c$ a singularity of $\hat{f}_g$. Typically, if we have an algebraic singularity of the type $\hat{f}_g \sim (s_c - s)^{-\alpha_g} \; f_g$, the expectation value of the number of vertices is:

$$< \#\text{vertices} >\sim \frac{\alpha_g \, s_c}{s_c - s}, \tag{1-61}$$

and we see that $(s_c - s)$, i.e. the distance to critical point, can be thought of as the "mesh size", so that the area (i.e. number of vertices times mesh size) remains finite in the limit.

Another way to say that, is imagine that $\hat{f}_g$ has an algebraic singularity of type

$$\hat{f}_g \sim (s_c - s)^{-\alpha_g} \; f_g \tag{1-62}$$

and notice that

$$(1 - s/s_c)^{-\alpha_g} = \sum_{v=0}^{\infty} \binom{-\alpha_g}{v} \; (-s/s_c)^v = \sum_{v=0}^{\infty} \frac{\Gamma(v + \alpha_g)}{v! \, \Gamma(\alpha_g)} \; (s/s_c)^v \tag{1-63}$$

This means that the (possibly weighted) number of maps of genus g with v vertices, behaves for large v as:

$$f_g\, s_c^{-\alpha_g - v}\, \frac{\Gamma(v+\alpha_g)}{v!\,\Gamma(\alpha_g)} \sim f_g\, s_c^{-\alpha_g - v}\, \frac{v^{\alpha_g - 1}}{\Gamma(\alpha_g)} \tag{1-64}$$

where we used the large v Stirling asymptotic formula for the Γ-function.

Similarly, the $s \to s_c$ asymptotics of $\hat{\omega}_n^{(g)}(x_1, \dots, x_n)$ give the enumeration of large maps with n marked faces, and if we also rescale $x_i \to (s_c - s)^{\alpha_{n,g}}\, \tilde{x}_i$, we ca also consider large maps with large marked faces.

Therefore, we see that the enumeration of large maps, is asymptotically given by the knowledge of:
- the exponents $\alpha_{n,g}$ (and $\alpha_g = \alpha_{0,g}$),
- the critical point s_c
- and the prefactor f_g.

• It turns out that the critical point s_c is independent of n and g, and it can be easily found from the resolvent $\omega_1^{(0)}$, it is not universal, it is strongly model dependent.

• The exponent $\alpha_{n,g}$ turns out to be proportional to $(2-2g-n)$:

$$\alpha_{n,g} = (2-2g-n)\,(1-\gamma_{\text{string}}/2) \tag{1-65}$$

where γ_{string} is a universal exponent, it depends only on (p,q), and it is one of the exponents computed by the famous KPZ formula. Here, we shall see that it is:

$$\gamma_{\text{string}} = \frac{-2}{p+q-1}. \tag{1-66}$$

• The last thing to compute, is the prefactor f_g, or $\omega_n^{(g)}$. Here, in this article, we prove in theorem 2.3 the long claim statement that this prefactor is the same as the one computed directly from conformal field theory technics, with the Liouville theory coupled to matter reprensented by a minimal model (p,q) of central charge $c = 1 - 6(p-q)^2/pq$ (notice that the $(3,2)$ model has $c=0$ and thus is called pure Liouville gravity). In particular, we show that the generating function of the coefficients f_g, satisfies the Gelfand-Dikii non linear equation, see eq.(1-31).

1.2.3 Singularities of spectral invariants

One can easily convince oneself that the algebraic equations eq.(1-49) obeyed by α and γ, are singular whenever $\hat{y}'(1) = 0$ or $\hat{y}'(-1) = 0$, and then from theorem 1.3, one can see that the $\hat{f}_g$'s and $\hat{\omega}_n^{(g)}$'s diverge whenever $\hat{y}'(1) = 0$ or $\hat{y}'(-1) = 0$, i.e. whenever $\hat{y}$ doesn't behave as a square root branchpoint.

Let us assume that we fix the parameters s_k and $s = s_c$ such that:

$$\hat{y}(z) \underset{z\to 1}{\sim} (\hat{x}(z) - \hat{x}(1))^{m+\frac{1}{2}}. \tag{1-67}$$

This can be obtained for instance if we choose:

$$V'(x) = (x - \alpha - 2)^m \; (T_{m+1}(x-\alpha) - T_m(x-\alpha))$$

$$\begin{aligned} s_c = (-\alpha - 2)^m \; (T'_{m+1}(-\alpha) - T'_m(-\alpha)) \\ T_{m+1}(-\alpha) = T_m(-\alpha) \end{aligned} \tag{1-68}$$

where $T_m(z + z^{-1}) = z^m + z^{-m}$ is the Tchebychev's polynomial of degree m. In that case we have at $s = s_c$:

$$\begin{cases} \hat{x}(z) = z + \frac{1}{z} \\ \hat{y}(z) = (z-1)^{2m+1} - (\frac{1}{z} - 1)^{2m+1} \end{cases} \tag{1-69}$$

When s is close to s_c but not exactly equal to s_c, we have like in eq.(1-26):

$$\hat{y}((s-s_c)^{\frac{1}{m+1}} \, x) \sim (s-s_c)^{\frac{2m+1}{2m+2}} \; y(x) \; (1 + O((s-s_c)^{\frac{1}{m+1}}) \qquad , \qquad y(x) = \sum_{k=0}^{m} c_k \, x^{k+\frac{1}{2}}. \tag{1-70}$$

At $s = s_c$ we have $\hat{y} \sim \hat{x}^{m+\frac{1}{2}}$ and at $s \neq s_c$ we have $\hat{y} \sim \sqrt{\hat{x} - \hat{x}(1)}$. The value of m and the coefficients c_k depend on which limit of large maps we are interested in. Indeed we may fine-tune the coefficients s_j, in order to favor one value of m or another.

Again, theorem 8.1. of [36] implies that:

Theorem 1.4 *(theorem 8.1. of [36]): If $m > 0$:*

$$\hat{f}_g \sim (s-s_c)^{(2-2g)\frac{2m+3}{2m+2}} \, f_g, \tag{1-71}$$

and if $m \geq 0$:

$$\begin{aligned} &\hat{\omega}_n^{(g)}((s-s_c)^{\frac{1}{m+1}} x_1, \dots, (s-s_c)^{\frac{1}{m+1}} x_n) \\ &\sim (s-s_c)^{(2-2g-n)\frac{2m+3}{2m+2} - \frac{n}{m+1}} \, \omega_n^{(g)}(x_1, \dots, x_n) \; (1 + O((s-s_c)^{\frac{1}{m+1}})), \end{aligned} \tag{1-72}$$

where the f_g's and $\omega_n^{(g)}$'s are the spectral invariants of [36] for the spectral curve $y(x)$ appearing in eq.(1-70).

It was argued and highly debated, that this limit should be equivalent to the Liouville gravity conformal field theory, coupled to some matter field given by a conformal minimal model (p, q) of central charge $c = 1 - 6\frac{(p-q)^2}{pq}$. Intuitively, discrete surfaces made of a very large number of small polygons, should give a good approximation of smooth Riemann surfaces...

It was indeed proved that the critical exponents $-\alpha_g = (2-2g)\frac{2m+3}{2m+2}$ are the same (given by KPZ formula [51, 31, 32]) as those of the Liouville conformal field theory, but it is only recently that it became possible to compute explicitly partition functions and correlation functions on both sides: on the matrix model side (in particular in the double scaling limit), and in the conformal theory side.

On the Liouville conformal theory side, recent progress was obtained following Zamolodchikov, Belavin, Hosomichi, Ribault, Teschner, ... [3, 4, 5, 63, 61, 43].

On the matrix model side, recent progress was obtained in [35], and formalized as a special case of the symplectic invariants of [36], which allow to compute all correlation functions of all genus.

From here, we can repeat all what was said in section 1.1, after theorem 1.1.

In this article, we shall show how to apply the spectral invariants method of [36], for the double scaling limit of matrix models which is expected to coincide with Liouville theory.

We prove that the scaling limits of the matrix model correlation functions, i.e. the generating functions counting discrete surfaces, is indeed the $(p, 2)$ reduction of KdV satisfying string equation, i.e. the minimal model $(p, 2)$.

2 Minimal models

There exists several equivalent definitions of minimal models coupled to gravity. They correspond to representations of the conformal group in 2 dimensions. They are classified by two integers (p, q), and their central charge is:

$$c = 1 - 6\frac{(p-q)^2}{pq} \tag{2-1}$$

Some of them have received special names:

- $(1, 2)$ = Airy, $c = -2$ (related to Tracy-Widom law [66])
- $(3, 2)$ = pure gravity, $c = 0$
- $(5, 2)$ = Lee-Yang edge singularity, $c = -\frac{22}{5}$
- $(4, 3)$ = Ising, $c = \frac{1}{2}$
- $(6, 5)$ = Potts-3, $c = \frac{4}{5}$

Minimal models can also be viewed as finite reductions of the Kadamtsev-Petviashvili (KP) integrable hierarchy of partial differential equations.

The case $q = 2$ is a little bit simpler to address, and is a reduction of the Korteweg de Vries (KdV) hierarchy.

The KdV hierarchy, and the minimal models $(p, 2)$ have generated a huge amount of works, and have been presented in many different (but equivalent) formulations. For instance in terms of a string equation for differential operators, in terms of a Lax pair, in terms of commuting hamiltonians, in terms of Schrödinger equation, in terms of Hirota equations, in terms of isomonodromic systems, in terms of Riemann Hilbert problems, in terms of tau functions, in terms of Grasman manifolds, in terms of Yang-Baxter equations, ...etc, see [2] for a comprehensive lecture.

All those formulations are equivalent, and let us recall some of the well known features of the $(p, 2)$ reduction of KdV (see [26, 2]), presented in a way convenient for our purposes.

2.1 String equation

The KdV minimal model $(p, 2)$ with $p = 2m + 1$ can be formulated in terms of two differential operators P, Q of respective orders p and 2, satisfying the string equation:

$$[P, Q] = \frac{1}{N}\,\mathrm{Id} \tag{2-2}$$

$$Q = d^2 - 2u(t) \qquad , \qquad P = d^p - p\,u\,d^{p-2} + \dots \qquad , \qquad d = \frac{1}{N}\frac{d}{dt} \tag{2-3}$$

$\frac{1}{N}$ is a scaling parameter, which we can send to zero to get the "classical limit".

The general solution of the string equation eq.(2-2) is of the form:

$$P = \sum_{j=0}^{m} t_j (Q^{j+1/2})_+ \qquad , \qquad t_m = 1 \tag{2-4}$$

where $(Q^{j+1/2})_+$ is the unique differential operator of order $2j + 1$, such that:

$$\mathrm{order}[((Q^{j+1/2})_+)^2 - Q^{2j+1}] \leq 2j. \tag{2-5}$$

For example:

$$(Q^{1/2})_+ = d \qquad , \qquad (Q^{3/2})_+ = d^3 - 3ud - \frac{3\dot{u}}{2}, \tag{2-6}$$

$$(Q^{5/2})_+ = d^5 - 5ud^3 - \frac{15\dot{u}}{2}\,d^2 - \frac{25\ddot{u}}{4}\,d - \frac{45u^2}{2}\,d - \frac{15}{8}\,\dddot{u} - \frac{45u\dot{u}}{2}. \tag{2-7}$$

It is a classical result (see [27]) that it satisfies:

$$[(Q^{j-1/2})_+, Q] = \frac{1}{N}\frac{d}{dt}(R_j(u(t))) \tag{2-8}$$

where the right hand side is a function (a differential operator of order 0), and the coefficients $R_j(u)$ are the Gelfand-Dikii differential polynomials [27]. They can be obtained by the recursion:

$$R_0 = 2 \qquad , \qquad \dot{R}_{j+1} = -2u\dot{R}_j - \dot{u}R_j + \frac{1}{4\,N^2}\dddot{R}_j. \tag{2-9}$$

The first few of them are:

$$\begin{aligned} R_0 &= 2 \\ R_1 &= -2u \\ R_2 &= 3\,u^2 - \tfrac{1}{2\,N^2}\ddot{u} \\ R_3 &= -5u^3 + \frac{5}{2N^2}u\ddot{u} + \frac{5}{4N^2}\dot{u}^2 - \frac{1}{8N^4}\ddddot{u} \\ &\vdots \end{aligned} \tag{2-10}$$

and in general:

$$R_j(u) = \frac{2\,(-1)^j\,(2j-1)!!}{j!}\,u^j + \quad \dots \quad - \frac{2}{(2N)^{2j-2}}\,u^{(2j-2)}. \tag{2-11}$$

After substitution of eq.(2-4) into the string equation eq.(2-2), the property eq.(2-8) gives a non-linear differential equation for the function $u(t)$:

$$\boxed{\sum_{j=0}^{m} t_j R_{j+1}(u) = t.} \tag{2-12}$$

Since $R_0 = 2$, we see that we can identify t with $t = -2t_{-1}$.

- For instance for Airy $p = 1$, this gives:

$$-2u = t. \tag{2-13}$$

- For instance for pure gravity $p = 3$, this is the Painlevé I equation:

$$3\,u^2 - \frac{1}{2N^2}\ddot{u} - 2t_0 u = t. \tag{2-14}$$

- For instance for Lee-Yang $p = 5$, we have:

$$-5u^3 + \frac{5}{2N^2}u\ddot{u} - \frac{1}{4N^2}\dot{u}^2 - \frac{1}{8N^4}\ddddot{u} + t_1(3\,u^2 - \frac{1}{2N^2}\ddot{u}) - 2t_0 u = t. \tag{2-15}$$

2.2 Tau function

We define the Tau-function $\tau(t, t_0, \dots, t_m)$ and its log, the free energy function $F(t, t_0, \dots, t_m) = \ln \tau(t, t_0, \dots, t_m)$ such that:

$$N^{-2}\,\ddot{F} = u. \tag{2-16}$$

The Tau-function has many other properties, which can be found in textbooks and classical works on the subject [2, 55, 56], but which are beyond the scope of the present article.

2.3 Lax pair

Consider the following matrices:

$$\mathcal{R}(x,t) = \begin{pmatrix} 0 & 1 \\ x+2u(t) & 0 \end{pmatrix}, \tag{2-17}$$

and for any integer k:

$$\mathcal{D}_k(x,t) = \begin{pmatrix} A_k & B_k \\ C_k & -A_k \end{pmatrix}, \tag{2-18}$$

where $A_k(x,t), B_k(x,t), C_k(x,t)$ are polynomials of respective degree $k-1, k, k+1$ in x, which are determined by:

$$B_k(x,t) = \sum_{j=0}^{k} x^{k-j}\ R_j(u) \qquad , \qquad A_k = -\frac{1}{2N}\dot{B}_k \qquad , \qquad C_k = (x+2u)\,B_k + \frac{1}{N}\dot{A}_k. \tag{2-19}$$

The recursion relation eq.(2-9) implies that B_k satisfies the equation:

$$2\dot{u}B_k + 2(x+2u)\dot{B}_k - \frac{1}{2N^2}\dddot{B}_k = -2\dot{R}_{k+1}(u) \tag{2-20}$$

and we see that the matrix $\mathcal{D}_k(x,t)$ satisfies:

$$\frac{1}{N}\frac{\partial}{\partial t}\mathcal{D}_k(x,t) + [\mathcal{D}_k(x,t), \mathcal{R}(x,t)] = -\frac{2}{N}\dot{R}_{k+1}(u)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2-21}$$

the right hand side is independent of x, and is proportional to $\frac{\partial}{\partial x}\mathcal{R}(x,t)$.

2.4 Lax equation

If we consider u solution of the string equation eq.(2-12), then, the matrix:

$$\mathcal{D}(x,t) = \sum_{j=0}^{m} t_j \mathcal{D}_j(x,t) \qquad , \qquad t_m = 1 \tag{2-22}$$

satisfies the Lax equation:

$$\frac{1}{N}\frac{\partial}{\partial t}\mathcal{D}(x,t) + [\mathcal{D}(x,t), \mathcal{R}(x,t)] = -\frac{2}{N}\frac{\partial}{\partial x}\mathcal{R}(x,t) \tag{2-23}$$

which can also be written as a commutation relation:

$$\left[\frac{2}{N}\frac{\partial}{\partial x} + \mathcal{D}(x,t), \mathcal{R}(x,t) - \frac{1}{N}\frac{\partial}{\partial t}\right] = 0 \tag{2-24}$$

This relation means that the operator $\frac{2}{N}\frac{\partial}{\partial x} + \mathcal{D}(x,t)$ is a Lax operator [2].

2.5 The differential system

The Lax equation eq.(2-24) is the compatibility condition, which says that the following two differential systems have a common solution $\Psi(x,t)$:

$$\frac{1}{N}\frac{d}{dx}\Psi(x,t) = -\frac{1}{2}\mathcal{D}(x,t)\,\Psi(x,t) \qquad , \qquad \frac{1}{N}\frac{d}{dt}\Psi(x,t) = \mathcal{R}(x,t)\,\Psi(x,t) \tag{2-25}$$

and $\Psi(x,t)$ is a matrix such that:

$$\Psi(x,t) = \begin{pmatrix} \psi & \phi \\ \tilde{\psi} & \tilde{\phi} \end{pmatrix} \qquad , \qquad \det\Psi = 1. \tag{2-26}$$

In particular we have the Schrödinger equation for ψ:

$$\frac{1}{N^2}\ddot{\psi}(x,t) = (x + 2u(t))\,\psi(x,t) \tag{2-27}$$

where t can be interpreted as the space variable, and x the energy. x is called the spectral parameter.

2.6 Correlators

Consider the Christoffel-Darboux kernel associated to the system $\mathcal{D}(x)$:

$$K(x_1,x_2) = \frac{\psi(x_1)\tilde{\phi}(x_2) - \tilde{\psi}(x_1)\phi(x_2)}{x_1 - x_2} \tag{2-28}$$

Definition 2.1 *We define the connected correlation functions by the "determinantal formulae":*

$$W_1(x) = \lim_{x'\to x} K(x,x') - \frac{1}{x-x'} = \psi'(x)\tilde{\phi}(x) - \tilde{\psi}'(x)\phi(x) \tag{2-29}$$

and for $n \geq 2$:

$$W_n(x_1,\dots,x_n) = -\frac{\delta_{n,2}}{(x_1-x_2)^2} - (-1)^n \sum_{\sigma=\text{cyles}} \prod_{i=1}^{n} K(x_{\sigma(i)}, x_{\sigma(i+1)}) \tag{2-30}$$

For example:

$$W_3(x_1,x_2,x_3) = K(x_1,x_2)K(x_2,x_3)K(x_3,x_1) + K(x_1,x_3)K(x_3,x_2)K(x_2,x_1). \tag{2-31}$$

Although we have not written it explicitly, the kernel K and the correlators W_n depend on t.

The *non-connected* correlation functions are defined by:

$$W_{n,n.c.}(x_1,\dots,x_n) = \sum_{k} \sum_{J_1\cup J_2\cup\ldots\cup J_k = J} \prod_{i=1}^{k} W_{|J_i|}(J_i), \tag{2-32}$$

where $J = \{x_1, \dots, x_n\}$ and the sum runs over all partitions of J into k non-empty disjoint subsets. In other words, the connected W_n's are the cumulants of the non-connected ones.

For instance:

$$W_{2,\,n.c.}(x_1,x_2) = W_2(x_1,x_2) + W_1(x_1)W_1(x_2), \tag{2-33}$$

$$\begin{aligned} W_{3,\,n.c.}(x_1,x_2,x_3) &= W_3(x_1,x_2,x_3) + W_1(x_1)W_2(x_2,x_3) + W_1(x_2)W_2(x_1,x_3) \\ &\quad + W_1(x_3)W_2(x_1,x_2) + W_1(x_1)W_1(x_2)W_1(x_3). \end{aligned} \tag{2-34}$$

The formula eq.(2-30) is called "determinantal formula", because for the non-connected correlation functions we have:

$$W_{n,\,n.c.}(x_1,\dots,x_n) = \det{}'(K(x_i,x_j)), \tag{2-35}$$

where $\det'$ means that when we compute the determinant as a sum over permutations of products $(-1)^\sigma \prod_i K(x_i, x_{\sigma(i)})$, then if $\sigma(i) = i$ we replace $K(x_i,x_i)$ by $W_1(x_i)$, and if $\sigma(i) = j$ and $\sigma(j) = i$, we replace $K(x_i,x_j)K(x_j,x_i)$ by $-W_2(x_i,x_j)$, see [7].

For instance $W_{3,\,n.c.}$ is the sum of 6 terms coming from the 6 permutations:

$$\begin{aligned} W_{3,\,n.c.}(x_1,x_2,x_3) &= \det{}' \begin{pmatrix} K(x_1,x_1) & K(x_1,x_2) & K(x_1,x_3) \\ K(x_2,x_1) & K(x_2,x_2) & K(x_2,x_3) \\ K(x_3,x_1) & K(x_3,x_2) & K(x_3,x_3) \end{pmatrix} \\ &= W_1(x_1)W_1(x_2)W_1(x_3) + W_1(x_1)W_2(x_2,x_3) + W_1(x_2)W_2(x_1,x_3) \\ &\quad + W_1(x_3)W_2(x_1,x_2) + K(x_1,x_2)K(x_2,x_3)K(x_3,x_1) \\ &\quad + K(x_1,x_3)K(x_3,x_2)K(x_2,x_1) \end{aligned} \tag{2-36}$$

It was proved in [8], that the correlators W_n satisfy an infinite set of equations, called loop equations, and equivalent to Virasoro constraints for the τ function. The loop equation simply states that the following quantity:

Theorem 2.1 *Loop equations (proved in [8]):*

$$\begin{aligned} & P_n(x; x_1,\dots,x_n) \\ = \quad & W_{n+2,\,n.c.}(x,x,x_1,\dots,x_n) \\ & + \sum_{j=1}^n \frac{\partial}{\partial x_j} \frac{W_n(x,x_1,\dots,x_{j-1},x_{j+1},\dots,x_n) - W_n(x_1,\dots,x_n)}{x - x_j} \end{aligned}$$

$(2-37)$

is a polynomial of the variable x.

For example, one can easily check that:

$$P_0(x) = W_2(x,x) + W_1(x)^2 = -\det \mathcal{D}(x,t). \tag{2-38}$$

Notice that

$$\frac{1}{N}\frac{\partial}{\partial t} \det \mathcal{D}(x,t) = 2B(x,t) = 2\sum_{j=0}^{m} t_j B_j(x,t). \tag{2-39}$$

2.7 Example: Airy kernel

Let us write the $(1,2)$ model, i.e. $m=0$. We have:

$$P = d \qquad , \qquad Q = d^2 - 2u \tag{2-40}$$

the string equation is:

$$[P,Q] = -\frac{2}{N}\,\dot{u} = \frac{1}{N} \tag{2-41}$$

i.e.

$$u(t) = -\frac{t}{2} = t_{-1} \tag{2-42}$$

The Lax pair is:

$$\mathcal{D}_0(x,t) = \begin{pmatrix} 0 & 2 \\ 2x+4u & 0 \end{pmatrix} \qquad , \qquad R(x,t) = \begin{pmatrix} 0 & 1 \\ x+2u & 0 \end{pmatrix} \tag{2-43}$$

The differential system is:

$$\frac{1}{N}\frac{d}{dx}\Psi(x,t) = -\begin{pmatrix} 0 & 1 \\ x-t & 0 \end{pmatrix} \Psi(x,t) \tag{2-44}$$

i.e.

$$\psi'' = N^2(x-t)\psi \tag{2-45}$$

whose solution is the Airy function [1]:

$$\psi(x,t) = Ai(N^{\frac{2}{3}}(x-t)) \qquad , \qquad \tilde{\psi}(x,t) = -Ai'(N^{\frac{2}{3}}(x-t)) \tag{2-46}$$

and the other independent solution is the "Bairy" function [1]:

$$\phi(x,t) = Bi(N^{\frac{2}{3}}(x-t)) \qquad , \qquad \tilde{\phi}(x,t) = -Bi'(N^{\frac{2}{3}}(x-t)) \tag{2-47}$$

and thus the kernel is the famous Airy kernel [65]:

$$K_{\text{Airy}}(t+N^{-2/3}x_1, t+N^{-2/3}x_2) = \frac{Ai'(x_1)Bi(x_2) - Ai(x_1)Bi'(x_2)}{x_1 - x_2} \tag{2-48}$$

The Airy kernel plays a very important role in many problems, in particular in the universal laws of extreme values, related to the Tracy-Widom law [66].

The τ function is simply:

$$\tau = \mathrm{e}^{-\frac{N^2 t^3}{12}}. \tag{2-49}$$

For the Airy system, the polynomial of theorem 2.1 is simply:

$$P_n(x) = 4(x+2u)\,\delta_{n,0}. \tag{2-50}$$

2.8 Classical limit

The classical limit is the large N limit, or equivalently, it is also the large t limit.

Intuitively, in the classical limit, P and Q commute, and they can be represented without differential operators. In this limit $d \to z$ can be represented as a number, and operators $Q = d^2 - 2u$ and P are replaced by functions of z and t. Therefore, in analogy with $Q = d^2 - 2u(t)$, and $P = d^p + \ldots$, let us define two functions $x(z,t)$ and $y(z,t)$:

$$x(z,t) = z^2 - 2u_0(t) \qquad , \qquad y(z,t) = z^p + \ldots \, . \tag{2-51}$$

In the classical limit, we replace the string equation $[P,Q] = N^{-1}$ with a Poisson bracket:

$$\{y,x\} = 1 = \frac{\partial y}{\partial z}\frac{\partial x}{\partial t} - \frac{\partial y}{\partial t}\frac{\partial x}{\partial z} \tag{2-52}$$

whose general solution is:

$$x(z,t) = z^2 - 2u_0(t) \qquad , \qquad y(z,t) = \sum_{j=0}^{m} t_j \left(z^{2j+1}\, (1 - \frac{2u_0(t)}{z^2})^{j+1/2} \right)_+ \, , \tag{2-53}$$

where $()_+$ means the positive part in the large z Laurent series expansion. Explicitly we get:

$$y(z,t) = \sum_{j=0}^{m}\sum_{l=0}^{j} t_j z^{2j+1-2l}\, (-u_0/2)^l\, \frac{(2j+1)!}{j!}\, \frac{(j-l)!}{l!\,(2j+1-2l)!}. \tag{2-54}$$

The string equation $\{y,x\} = 1$ reduces to:

$$\dot{u}_0\, y'(0) = \frac{-1}{2}, \tag{2-55}$$

i.e.

$$\sum_{j=0}^{m} t_j\, \dot{u}_0\, (-u_0/2)^j\, \frac{(2j+1)!}{(j!)^2} = -\frac{1}{2} \tag{2-56}$$

which can be integrated with respect to t and gives a polynomial equation for $u_0(t)$:

$$\mathcal{P}(u_0) = \sum_{j=0}^{m} t_j\, (-u_0/2)^{j+1}\, \frac{(2j+1)!}{j!\,(j+1)!} = \frac{t}{4} \tag{2-57}$$

which is clearly the classical limit of eq.(2-12). In other words, the non-linear differential equation eq.(2-12) for $u(t)$, becomes an algebraic equation for $u_0(t)$.

For example, for pure gravity $m = 1$ we have the classical limit of eq.(2-14):

$$4\mathcal{P}(u_0) = 3\, u_0^2 - 2t_0\, u_0 = t. \tag{2-58}$$

2.9 Topological expansion

We now have the polynomial equation eq.(2-57):

$$\mathcal{P}(u_0) = t/4 \tag{2-59}$$

which implies:

$$\dot{u}_0 = \frac{1}{4\mathcal{P}'(u_0)} \quad , \quad \ddot{u}_0 = \frac{-\mathcal{P}''(u_0)}{16\,(\mathcal{P}'(u_0))^3} \quad , \quad \dots \tag{2-60}$$

and in general, any derivative of u_0 with respect to t can be written as a rational function of u_0.

Since $u_0(t)$ satisfies the string equation eq.(2-12) at $N = \infty$, the full solution $u(t)$ to the string equation eq.(2-12), can be expanded as an N^{-2} power series:

$$u(t) = u_0 + \sum_k N^{-2k}\, u_k(t) \tag{2-61}$$

where all coefficients u_k are rational functions of u_0 (their denominator is a power of $\mathcal{P}'(u_0)$).

For example for pure gravity $m = 1$, the Painlevé equation eq.(2-14) implies that to the first few orders we have:

$$u(t) = u_0 - \frac{3}{N^2}\,(6u_0 - 2t_0)^{-4} + O(N^{-4}). \tag{2-62}$$

And the Free energy $F(t)$ such that $u = \frac{1}{N^2}\,\ddot{F}$, also has a $1/N^2$ expansion:

$$\ln\tau = F = \sum_{g=0}^{\infty} N^{2-2g} F_g(u_0) \quad , \quad \ddot{F}_g = u_g. \tag{2-63}$$

Also, since the coefficients of the differential system $\mathcal{D}(x,t)$ depend on $u(t)$, the matrix $\mathcal{D}(x,t)$ has a $1/N^2$ expansion:

$$\mathcal{D}(x,t) = \sum_g N^{-2g}\mathcal{D}^{(g)}(x,t) \tag{2-64}$$

To leading order we have:

$$\mathcal{D}^{(0)}(x,t) = \begin{pmatrix} 0 & \overline{B}(x,u_0) \\ (x+2u_0)\,\overline{B}(x,u_0) & 0 \end{pmatrix} \tag{2-65}$$

$$\overline{B}(x,u_0) = 2\sum_{j=0}^{m}\sum_{k=0}^{j} t_j x^{j-k}\, u_0^k\, \frac{(-1)^k\ (2k-1)!!}{k!} \tag{2-66}$$

Notice that:

$$z\,\overline{B}(z^2-2u_0,u_0)=y(z,t). \tag{2-67}$$

The classical spectral curve is given by the eigenvalues of $\mathcal{D}^{(0)}(x,t))$, i.e. the values of y such that $\det\left(y-\mathcal{D}^{(0)}(x,t)\right)=0$, i.e., if we parametrize x as $x=z^2-2u_0$, we have:

$$y=\pm\,y(z,t) \tag{2-68}$$

where $y(z,t)$ is the function defined in eq.(2-54). This explains why we call the function $y(z,t)$ the *classical spectral curve*.

Written in a parametric form where $u_0=u_0(t)$, the classical spectral curve is thus:

$$\boxed{\mathcal{E}_{(2m+1,2)}=\left\{\begin{array}{l} x(z)=z^2-2u_0 \\ y(z)=\sum_j\sum_l t_j z^{2j+1-2l}\,(-u_0/2)^l\ \frac{(2j+1)!}{j!}\,\frac{(j-l)!}{l!\,(2j+1-2l)!}\end{array}\right.} \tag{2-69}$$

It is important to notice that it is a genus 0 hyperelliptical curve, which is equivalent to saying that it can be parametrized by a complex variable z (higher genus would be parametrized by a variable z living on a Riemann surface), and which is equivalent to saying that the polynomial y^2, written as a polynomial in x, has only one simple zero, located at $x=-2u_0$, all the other zeroes are double zeroes.

2.10 BKW expansion

Similarly, we can look for a BKW asymptotic solution of the solutions $\psi(x,t)$ of the differential system. It takes the form:

$$\psi(x,t)\sim\ \frac{\mathrm{e}^{N\int_{-2u_0}^{x}ydx}}{\sqrt{2}\,(-x-2u_0)^{\frac14}}\ \left(1+\sum_k N^{-k}\psi_k(x,u_0)\right) \tag{2-70}$$

$$\tilde\psi(x,t)\sim\ \mathrm{e}^{N\int_{-2u_0}^{x}ydx}\ (x+2u_0)^{\frac14}\ \left(1+\sum_k N^{-k}\tilde\psi_k(x,u_0)\right) \tag{2-71}$$

and we recall that $z=(x+2u_0)^{\frac12}$. the BKW expansion of the other solutions ϕ and $\tilde\phi$, are obtained by changing the sign of the square root $z\to -z$.

We have the following Lemma:

Lemma 2.1 *Each $\psi_k(x,u_0)$ and $\tilde\psi_k(x,u_0)$ is a polynomial of $1/z$.*

proof:

The proof uses the Schrödinger equation eq.(2-27):

$$\frac{1}{N^2}\ddot{\psi}(x,t) = (x+2u(t))\,\psi(x,t). \tag{2-72}$$

Let us write:

$$\psi(x,t) = \sqrt{f(x,t)}\,\mathrm{e}^{\int^t \frac{dt'}{f(x,t')}}. \tag{2-73}$$

The Schrödinger equation implies that:

$$N^2(x+2u(t))\,f^2(x,t) = \frac{1}{2}f(x,t)\ddot{f}(x,t) - \frac{1}{4}\dot{f}(x,t)^2 + 1, \tag{2-74}$$

and after differentiating once more with respect to t, we obtain a third order linear equation for f:

$$(x+2u(t))\,\dot{f}(x,t) + \dot{u}(t)\,f(x,t) = \frac{1}{2\,N^2}\,\dddot{f}(x,t). \tag{2-75}$$

To leading order we have $u(t) = u_0(t)$, and recall that $u(t)$ has a $1/N^2$ expansion, therefore, one easily sees that:

$$f(x,t) = \frac{-1}{N\sqrt{x+2u_0(t)}}\,\Big(1+\sum_k N^{-2k}\,f_k(x,t)\Big), \tag{2-76}$$

and by an easy recursion, we see that each $f_k(x,t)$ is a polynomial in $1/z$ with $z = \sqrt{x+2u_0(t)}$.

Then, notice that the Poisson equation eq.(2-52) implies:

$$\left.\frac{\partial\,y}{\partial t}\right|_x = -\frac{1}{x'(z)} = -\frac{1}{2z} = -\frac{1}{2\sqrt{x+2u_0}} \tag{2-77}$$

And therefore:

$$\left.\frac{\partial\,\int^x y dx}{\partial t}\right|_x = -z. \tag{2-78}$$

This implies that:

$$\int^t \frac{1}{f(x,t)} = N\int^x y dx + \sum_{k\geq 1} N^{1-2k}\,g_k(x,t) \tag{2-79}$$

and where all coefficients $g_k(x,t)$ are polynomials of $1/z$.

Since $\psi(x,t) = \sqrt{f(x,t)}\,\mathrm{e}^{\int^t \frac{dt'}{f(x,t')}}$, we find that $\psi(x,t)$ is of the form:

$$\psi(x,t) \sim \frac{\mathrm{e}^{N\int^x y dx}}{(x+2u_0)^{\frac{1}{4}}}\,\left(1+\sum_k N^{-k}\psi_k(x,u_0)\right) \tag{2-80}$$

where each $\psi_k(x,t)$ is a polynomial in $1/z$.

The proof for $\tilde\psi(x,t)$ works in a similar manner. $\square$

This lemma implies that the kernel also have a $1/N$ expansion:

$$K(z_1,z_2) = \frac{\mathrm{e}^{N\int_{z_2}^{z_1} ydx}}{2\sqrt{z_1z_2}\,(z_1-z_2)}\Big(1+\sum_k N^{-k}K_k(z_1,z_2)\Big), \tag{2-81}$$

where each $K_k(z_1,z_2)$ is a polynomial in $1/z_1$ and in $1/z_2$.

This implies that the correlators also have a $1/N$ expansion:

Lemma 2.2

$$W_n(x_1,\dots,x_n) = \sum_g N^{2-2g-n}\,W_n^{(g)}(x_1,\dots,x_n) \tag{2-82}$$

where each $W_n^{(g)}$ is a rational function of the $z_i=\sqrt{x_i+2u_0}$, with poles only at $z_i=0$, except $W_2^{(0)}$ and $W_1^{(0)}$ which are:

$$W_1^{(0)} = y(z,t) \tag{2-83}$$

$$W_2^{(0)} = \frac{1}{4z_1z_2}\,\frac{1}{(z_1-z_2)^2} - \frac{1}{(z_1^2-z_2^2)^2} = \frac{1}{4z_1z_2(z_1+z_2)^2}. \tag{2-84}$$

The important point, is that each $W_n^{(g)}$ has no other pole than $z_i=0$, in particular, has no pole at the other zeroes of $y(z,t)$.

proof:

Notice that in the products $\prod_i K(z_{\sigma(i),\sigma(i+1)})$, all the exponentials cancel, and the result is, order by order in N^{-k}, a rational fraction of the z_i's having poles at $z_i=0$, or at $z_i=z_j$. Except for $W_1^{(0)}$ and $W_2^{(0)}$, the poles at $z_i=z_j$ are simple poles, and it is easy to see that in the sum over permutations, the residues cancel, therefore, each $W_n^{(g)}$ is a rational function of the z_i's having poles only at $z_i=0$. The cases of W_2 and W_1 need to be treated separately, and are easy.

The fact that W_n has a $1/N^2$ expansion instead of $1/N$ comes from a simple symmetry argument. In the expression of W_n, changing $\psi\to\phi$ and $\tilde\psi\to\tilde\phi$, can also be obtained as changing the order of the x_i's, and since we take a symmetric sum, only the terms which are invariant under the exchange $\psi\to\phi$ and $\tilde\psi\to\tilde\phi$ contribute to W_n. Exchanging the two solutions $\psi\to\phi$ and $\tilde\psi\to\tilde\phi$, is also equivalent to changing $N\to -N$, and therefore W_n has a given parity in N. $\square$

2.11 Symplectic invariants

It was found in [8], that the correlators obtained from the determinantal formulae eq.(2-29), eq.(2-30) of a Christoffel-Darboux kernel K of type eq.(2-28), do satisfy loop equations, i.e. for any n and g, and $J = \{x_1, \dots, x_n\}$, the following quantity:

$$\begin{aligned} P_n^{(g)}(x;J) &= \sum_{h=0}^{g} \sum_{I \subset J} W_{1+|I|}^{(h)}(x,I) W_{1+n-|I|}^{(g-h)}(x, J/I) \\ &\quad + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{W_n^{(g)}(x, J/\{x_j\}) - W_n^{(g)}(x_j, J/\{x_j\})}{x - x_j} \end{aligned} \tag{2-85}$$

is a polynomial in x. This property, as was proved in [8], is a direct consequence of eq.(2-28) and eq.(2-29), eq.(2-30).

Moreover we know from section 2.10, that $W_n^{(g)}(x(z_1), \dots, x(z_n))$ the following differential form:

$$\mathcal{W}_n^{(g)}(z_1, \dots, z_n) = W_n^{(g)}(x(z_1), \dots, x(z_n))\, x'(z_1) \dots x'(z_n) + \frac{\delta_{n,2}\delta_{g,0} x'(z_1) x'(z_2)}{(x(z_1) - x(z_2))^2} \tag{2-86}$$

is a symmetric rational function of all its variables, and if $2g+n-2 > 0$, due to lemma 2.2, it has poles only at $z_i = 0$, and

$$\mathcal{W}_2^{(0)}(z_1, z_2) = \frac{1}{(z_1 - z_2)^2} \tag{2-87}$$

It was found in [35], that the unique solution of loop equations eq.(2-85) which has a topological expansion for which the $\mathcal{W}_n^{(g)}$'s have the poles given by lemma 2.2, can be obtained by the following recursion relation:

Theorem 2.2

$$\begin{aligned} \mathcal{W}_{n+1}^{(g)}(z_1, \dots, z_n, z_{n+1}) &= \frac{-1}{4} \mathop{\text{Res}}_{z \to 0} \frac{dz}{(z_{n+1}^2 - z^2)\; y(z)} \Big[\mathcal{W}_{n+2}^{(g-1)}(z, -z, J) \\ &\quad + \sum_{h=0}^{g} \sum_{I \subset J}{}' \mathcal{W}_{1+|I|}^{(h)}(z, I) \mathcal{W}_{1+n-|I|}^{(g-h)}(-z, J/I) \Big] \end{aligned} \tag{2-88}$$

where $J = \{z_1, \dots, z_n\}$, and $\sum_h \sum_I'$, means that we exclude the terms $(h, I) = (0, \emptyset)$ and $(h, I) = (g, J)$.

proof:

The proof proceeds exactly like in [35]. Write the Cauchy residue formula:

$$\mathcal{W}_{n+1}^{(g)}(z_1, \dots, z_n, z_{n+1}) = \mathop{\text{Res}}_{z \to z_{n+1}} \frac{dz}{z - z_{n+1}}\; \mathcal{W}_{n+1}^{(g)}(z_1, \dots, z_n, z) \tag{2-89}$$

and move the integration contour, to enclose all the other poles, i.e. only $z = 0$, and thus:

$$\mathcal{W}_{n+1}^{(g)}(z_1, \dots, z_n, z_{n+1}) = \mathop{\text{Res}}_{z \to 0} \frac{dz}{z_{n+1} - z}\; \mathcal{W}_{n+1}^{(g)}(z_1, \dots, z_n, z)$$

$$= \quad \operatorname*{Res}_{z\to 0} \frac{x'(z)\,dz}{z_{n+1}-z}\; W_{n+1}^{(g)}(x(z_1),\dots,x(z_n),x(z)) \qquad (2-90)$$

Then, insert in the right hand side eq.(2-85):

$$\begin{aligned}
-2W_1^{(0)}(x)W_{n+1}^{(g)}(x_1,\dots,x_n,x) \;=\; & \sum_{h=0}^{g}\sum_{I\subset J}' W_{1+|I|}^{(h)}(x,I)W_{1+n-|I|}^{(g-h)}(x,J/I) \\
& +\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\frac{W_n^{(g)}(x,J/\{x_j\})-W_n^{(g)}(x_j,J/\{x_j\})}{x-x_j} \\
& -P_n^{(g)}(x;x_1,\dots,x_n) \qquad (2\text{-}91)
\end{aligned}$$

and notice that the polynomial $P_n^{(g)}$ has no pole and doesn't contribute to the residue. All what remains is eq.(2-88). □

The recursion relation eq.(2-88) is precisely the definition of the symplectic invariant's correlators defined in [36]. In [36], it is explained how to associate an infinite family of $\mathcal{W}_n^{(g)}$'s, to any spectral curve defined by a pair of functions $(x(z),y(z))$.

Examples:

- eq.(2-88) gives:

$$\mathcal{W}_3^{(0)}(z_1,z_2,z_3)=\frac{1}{2\,y'(0)}\;\frac{1}{z_1^2\,z_2^2\,z_3^2} \qquad (2\text{-}92)$$

- eq.(2-88) gives:

$$\begin{aligned}
\mathcal{W}_1^{(1)}(z_1) &= \frac{-1}{4}\operatorname*{Res}_{z\to 0}\frac{dz}{(z_1^2-z^2)\;y(z)}\,\frac{1}{4z^2} \\
&= \frac{-1}{32}\,\frac{d^2}{dz^2}\left(\frac{z}{(z_1^2-z^2)\;y(z)}\right)_{z=0} \\
&= \frac{y'''(0)}{48\,y'(0)^2}\,\frac{1}{z_1^2}-\frac{1}{16\;y'(0)}\,\frac{1}{z_1^4} \qquad (2\text{-}93)
\end{aligned}$$

2.12 Double scaling limit and $(p,2)$ kernel

By comparison with theorem 1.2, we conclude that:

Theorem 2.3 *the $s\to s_c$ double scaling limit of (possibly formal) matrix integrals correlation functions $W_n^{(g)}$ are the determinantal formula correlation functions of the $(p,2)$ kernel:*

$$\boxed{\omega_n^{(g)}(x_1,\dots,x_n)=W_n^{(g)}(x_1,\dots,x_n).} \qquad (2\text{-}94)$$

where

$$\omega_n^{(g)}(x_1,\dots,x_n)=\lim_{s\to s_c}(s-s_c)^{(2g+n-2)\frac{2m+3}{2m+2}+\frac{n}{m+1}}\hat{\omega}_n^{(g)}((s-s_c)^{\frac{1}{m+1}}x_1,\dots,(s-s_c)^{\frac{1}{m+1}}x_n) \qquad (2\text{-}95)$$

is the double scaling limit of matrix integrals correlators, and $W_n^{(g)}(x_1,\dots,x_n)$ *is the* $g^{\rm th}$ *term in the BKW expansion of the determinantal correlator of the* $(2m+1,2)$ *minimal model (def 2.1).*

And similarly if $m>0$*:*

$$\lim_{s\to s_c}(s-s_c)^{(2g-2)\frac{2m+3}{2m+2}}\hat{f}_g = f_g = F_g \tag{2-96}$$

where $\hat{f}_g$ *are the free energies of the matrix model and* F_g *are the free energies of the* $(2m+1,2)$ *minimal model:*

$$\ln\tau = \sum_g N^{2-2g}F_g. \tag{2-97}$$

The double scaling limit is:

$$N\to\infty \quad , \quad s\to s_c \quad , \quad (s-s_c)N^{\frac{2m+2}{2m+3}} = O(1). \tag{2-98}$$

Therefore, we have proved that, as announced, the double scaling limit of matrix models, is given by the Liouville minimal models $(2m+1,2)$ coupled to gravity.

2.13 Parametrics of orthogonal polynomials and Baker-Akhiezer functions

Many approaches of matrix models use some orthogonal polynomials (see Mehta [57]):

$$p_n(x) = x^n+\dots \quad , \quad \int p_n(x)\,p_m(x)\,{\rm e}^{-\frac{N}{s}V(x)}\,dx = h_n\,\delta_{n,m}, \tag{2-99}$$

which we prefer to make orthonormal:

$$\psi_n(x) = \frac{{\rm e}^{-\frac{N}{2s}V(x)}}{\sqrt{h_n}}\,p_n(x), \tag{2-100}$$

as well as their Hilbert transforms:

$$\phi_n(x) = {\rm e}^{\frac{N}{2s}V(x)}\int\frac{\psi_n(x')\,{\rm e}^{-\frac{N}{2s}V(x')}}{x-x'}\,dx'. \tag{2-101}$$

The matrix

$$\Psi_n(x) = \begin{pmatrix}\psi_n(x) & \phi_n(x)\\ \psi_{n-1}(x) & \phi_{n-1}(x)\end{pmatrix} \tag{2-102}$$

satisfies a 2×2 differential system $\mathcal{D}_n(x)$ with polynomial coefficients of degree at most $\deg V'$:

$$\frac{1}{N}\frac{d}{dx}\Psi_n(x) = \mathcal{D}_n(x)\,\Psi_n(x). \tag{2-103}$$

It is easy to see that $\operatorname{tr}\mathcal{D}_n(x)=0$ and $\det\Psi_n(x)=\sqrt{h_{n-1}/h_n}$ is constant.

Since $\phi_n(x)$ is discontinuous across the integration contour of dx', the matrix $\Psi_n(x)$ is also discontinuous, and has jumps of the form:

$$\Psi_{n+}(x) = \Psi_{n-}(x) \begin{pmatrix} 1 & 2i\pi \\ 0 & 1 \end{pmatrix}. \tag{2-104}$$

Therefore $\Psi_n(x)$ satisfies an isomonodromic Riemann-Hilbert problem (the jump matrix, called the monodromy, is independent of n and of $V(x)$).

A general method was invented [24] to find large N asymptotic solution of isomonodromic Riemann-Hilbert problems. In the case where we approach a $(2m+1,2)$ singularity, the method of Deift& co [24] requires to have an ansatz for a parametrix asymptotics for $\Psi_n(x)$ in the vicinity of the singularity.

We claim that the correct parametrix for the $(2m+1,2)$ singularity, is the matrix of Baker-Akhiezer functions of eq.(2-26) for the $(2m+1,2)$ minimal model:

$$\Psi_n((s-s_c)^{\frac{1}{m+1}}x) \sim \Psi(x)\ (1+O((s-s_c)^{\frac{1}{m+1}})). \tag{2-105}$$

This should be checked by the steepest descent Riemann-Hilbert method of [24].

3 Kontsevich's integral

In this section, we also propose a combinatorical interpretation of the coefficients of the f_g's and $\omega_n^{(g)}$'s, based on the comparison with Kontsevich integral.

The spectral curve eq.(2-69) of the $(2m+1,2)$ minimal model

$$\mathcal{E}_{(2m+1,2)} = \left\{ \begin{array}{l} x(z) = z^2 - 2u_0 \\ y(z) = \sum_j \sum_l t_j z^{2j+1-2l}\,(-u_0/2)^l\ \frac{(2j+1)!}{j!}\ \frac{(j-l)!}{l!\,(2j+1-2l)!} \end{array} \right. \tag{3-1}$$

is of the same form as the Kontsevich integral's spectral curve (see in [37], or see below), and thus it has the same correlators and spectral invariants F_g as those of the Kontsevich integral. Therefore, the correlators and F_g's of the minimal model $(2m+1,2)$ can be written as integrals of tautological classes on the moduli spaces of Riemann surfaces. This can be viewed as a proof that the double scaling limits of matrix models, i.e. the limit of large maps generating function, indeed coincides with topological gravity, as claimed by Witten [71, 28] and then proved by Kontsevich [52].

The Kontsevich integral [52]:

$$Z_K(\Lambda) = \int dM \mathrm{e}^{-N\,\mathrm{Tr}\,\frac{M^3}{3} - M\Lambda^2} = \mathrm{e}^{\sum_g N^{2-2g} F_g} \quad , \quad \tau_k = \frac{1}{N}\,\mathrm{Tr}\,\Lambda^{-k} \tag{3-2}$$

(to simplify we assume $\tau_1 = 0$ here) is the generating function for intersection numbers of cotangent line bundles at marked points of Riemann surfaces of genus g:

$$F_g = W_0^{(g)} = \sum_{\sum_i d_i = 3g-3} \prod_i \frac{\hat{\tau}_i^{d_i}}{d_i!} \left\langle \prod_i \psi_i^{d_i} \right\rangle_{\overline{\mathcal{M}}_g} \qquad , \qquad \psi_i = c_1(\mathcal{L}_i) \tag{3-3}$$

where ψ_i is the Chern class of the cotangent line bundle at point i, and where

$$\hat{\tau}_i = (2i-1)!!\ \tau_{2i+1} \tag{3-4}$$

and more generally, correlation functions of the Kontsevich integral give access [37] to integrals of Mumford κ characteristic classes [60]:

$$\begin{aligned} W_n^{(g)}(z_1,\dots,z_n) \quad = \quad & 2^{-d_{g,n}} (\tau_3 - 2)^{2-2g-n} \sum_{d_0+d_1+\dots+d_n = d_{g,n}} \sum_{k=1}^{d_0} \frac{1}{k!} \sum_{b_1+\dots+b_k = d_0, b_i>0} \\ & \prod_{i=1}^n \frac{2d_i+1!}{d_i!} \frac{dz_i}{z_i^{2d_i+2}} \prod_{l=1}^k \tilde{\tau}_{b_l} < \prod_{l=1}^k \kappa_{b_l} \prod_{i=1}^n \psi_i^{d_i} >_{g,n} \end{aligned}$$

$(3-5)$

The class κ_0 is the Euler class, and $2\pi\kappa_1$ is the curvature form of the Weil-Petersson symplectic metrics. The dual times $\tilde{\tau}_k$ are closely related to the τ_k's, see the relation in [37] or eq.(3-9) below.

It was shown in [36, 37], that Kontsevich's integral's $W_0^{(g)} = F_g$'s and correlators $W_n^{(g)}$ can be computed as the symplectic invariants $F_g = F_g(\mathcal{E}_K)$ of a spectral curve:

$$\boxed{\mathcal{E}_{\mathrm{K}} = \left\{ \begin{array}{l} x(z) = z^2 \\ y(z) = z - \frac{1}{2}\sum_j \tau_{j+2} z^j \end{array} \right.} \tag{3-6}$$

We see that the minimal model $\mathcal{E}_{(p,2)}$ spectral curve eq.(2-69) can be identified with Kontsevich integral's spectral curve $\mathcal{E}_{\mathrm{K}}$, under the identification of times:

$$\delta_{k,0} - \frac{1}{2}\tau_{2k+3} = \frac{k!}{(2k+1)!} \sum_l t_{l+k} \, (-u_0/2)^l \ \frac{(2l+2k+1)!}{l!\,(l+k)!} \tag{3-7}$$

In particular

$$1 - \frac{1}{2}\tau_3 = y'(0) = \frac{1}{-2\,\dot{u}_0} \tag{3-8}$$

The dual times $\tilde{\tau}_k$ are given by their generating function $\tilde{g}(r) = \sum_k \tilde{\tau}_k r^k$ and $\tilde{g}(r) = -\ln(1 - g(r))$ with:

$$1 - g(r) = -2\dot{u}_0 \sum_{k\geq 0} r^k \sum_l t_{l+k} \, (-u_0/2)^l \ \frac{(2l+2k+1)!}{l!\,(l+k)!} = \mathrm{e}^{-\tilde{g}(r)} \tag{3-9}$$

$$\tilde{g}(r) = \sum_k \tilde{\tau}_k r^k \tag{3-10}$$

I.e.

$$\begin{aligned}
1 - g(r) &= -2\,\dot{u}_0 \sum_j \sum_{l=0}^{j} t_j\, r^{j-l}\, (-u_0/2)^l \;\frac{(2j+1)!}{l!\,j!} \\
&= -2\,\dot{u}_0 \sum_j \frac{(2j+1)!}{j!}\, t_j\, r^j \sum_{l=0}^{j} \frac{1}{l!}\, (-u_0/2r)^l \\
&= -2\,\dot{u}_0 \sum_j \frac{(2j+1)!}{j!}\, t_j\, \left(r^j\, \mathrm{e}^{-u_0/2r}\right)_+
\end{aligned}$$

$(3-11)$

4 Derivatives

The general method to compute derivatives of F_g and $W_n^{(g)}$'s with respect to any parameter entering the spectral curve is explained in [36].

Here, our spectral curve $\mathcal{E}_{(p,2)}$ depends on the parameters t_j's and $t = -2t_{-1}$. [36] says that we first have to study the variation of $y(z)x'(z)$ under variation of any such parameter, and we write it:

$$\frac{\partial y(z)}{\partial t_j}\, x'(z) - \frac{\partial x(z)}{\partial t_j}\, y'(z) = \Lambda_j'(z) \tag{4-1}$$

Here, we find for $j \geq 0$:

$$\Lambda_j(z) = -2\frac{(2j+1)!}{j!}\, u_0 \;\sum_l z^{2j+1-2l}\, (-u_0/2)^l \;\frac{(j-l)!}{(l+1)!\,(2j+1-2l)!} \tag{4-2}$$

and for $j = -1$:

$$\Lambda_{-1}(z) = -2\dot{u}_0\, y(z) = -\frac{\dot{u}_0}{z} \sum_{j=0}^{m} t_j \Lambda_j'(z) \tag{4-3}$$

The theorem 5.1 of [36], then shows that those functions are such that:

$$\frac{\partial W_n^{(g)}(z_1,\dots,z_n)}{\partial t_j} = \operatorname*{Res}_{\infty} W_{n+1}^{(g)}(z_1,\dots,z_n,z_{z+1})\, \Lambda_j(z_{n+1})\, x'(z_{n+1}) \tag{4-4}$$

and in particular for $n = 0$:

$$\frac{\partial F_g}{\partial t_j} = \operatorname*{Res}_{\infty} W_1^{(g)}(z)\, \Lambda_j(z)\, x'(z) = \operatorname*{Res}_{\infty} \mathcal{W}_1^{(g)}(z)\, \Lambda_j(z) \tag{4-5}$$

and:

$$\frac{\partial^k F_g}{\partial t_{j_1} \dots \partial t_{j_k}} = \operatorname*{Res}_{\infty} \dots \operatorname*{Res}_{\infty} \mathcal{W}_k^{(g)}(z_1,\dots,z_k)\, \Lambda_{j_1}(z_1) \dots \Lambda_{j_k}(z_k) \tag{4-6}$$

4.0.1 Example

The recursion relation eq.(2-88) gives:

$$\mathcal{W}_3^{(0)}(z_1,z_2,z_3) = W_3^{(0)}(z_1,z_2,z_3)\, x'(z_1)x'(z_2)x'(z_3) = \frac{1}{2y'(0)}\,\frac{1}{z_1^2\,z_2^2\,z_3^2} \qquad \text{(4-7)}$$

This implies:

$$\begin{aligned}
\frac{\partial^3 F_0}{\partial t_{j_1}\partial t_{j_2}\partial t_{j_3}} &= \mathop{\rm Res}_{\infty}\mathop{\rm Res}_{\infty}\mathop{\rm Res}_{\infty} \mathcal{W}_3^{(0)}(z_1,z_2,z_3)\,\Lambda_{j_1}(z_1)\,\Lambda_{j_2}(z_2)\,\Lambda_{j_3}(z_3) \\
&= \frac{-1}{2y'(0)}\,\Lambda'_{j_1}(0)\,\Lambda'_{j_2}(0)\,\Lambda'_{j_3}(0)
\end{aligned} \qquad (4-8)$$

Notice that

$$\Lambda'_j(0) = \frac{(2j+1)!}{j!\,(j+1)!}\,(-2u_0)^{j+1}2^{-2j} \qquad , \qquad \Lambda'_{-1}(0)=1 \qquad \text{(4-9)}$$

In particular this implies that:

$$\frac{\partial^3 F_0}{\partial t^3} = \frac{-1}{2y'(0)} = \dot{u}_0(t) \qquad \text{(4-10)}$$

and thus, as expected we recover:

$$\boxed{\frac{\partial^2 F_0}{\partial t^2} = u_0(t)} \qquad \text{(4-11)}$$

4.0.2 Example $W_1^{(1)}$

The recursion relation eq.(2-88) gives:

$$\begin{aligned}
\mathcal{W}_1^{(1)}(z) = W_1^{(1)}(z)x'(z) &= \frac{1}{8(\tau_3-2)}\left(\frac{1}{z^4} - \frac{\tau_5}{(\tau_3-2)\,z^2}\right) \\
&= \frac{-\dot{u}_0}{8}\left(\frac{1}{z^4} - \frac{\dot{u}_0}{3z^2}\sum_l t_{l+1}(-u_0/2)^l\,\frac{(2l+3)!}{l!(l+1)!}\right)
\end{aligned} \qquad (4-12)$$

and thus:

$$\begin{aligned}
\frac{\partial F_1}{\partial t_j} &= \mathop{\rm Res}_{\infty} \mathcal{W}_1^{(1)}(z)\,\Lambda_j(z) \\
&= \frac{2\dot{u}_0}{3!\;16}\left(\Lambda'''_j(0) - 2\dot{u}_0\,\Lambda'_j(0)\sum_l t_{l+1}(-u_0/2)^l\,\frac{(2l+3)!}{l!(l+1)!}\right)
\end{aligned} \qquad (4-13)$$

In particular:

$$\frac{\partial F_1}{\partial t} = -\frac{\dot{u}_0^2}{24}\left(y'''(0)+\sum_l t_{l+1}(-u_0/2)^l\,\frac{(2l+3)!}{l!(l+1)!}\right) \qquad (4-14)$$

4.0.3 Other examples

$$\begin{aligned}
\mathcal{W}_2^{(1)}(z_1,z_2) &= \frac{1}{4(\tau_3-2)^2\,z_1^6\,z_2^6}\Big[\frac{5!}{2!}(z_1^4 < \psi_2^2 > + z_2^4 < \psi_1^2 >) + 3!^2 z_1^2 z_2^2 < \psi_1\psi_2 > \\
&\quad + \tilde{\tau}_1 z_1^2 z_2^4 < \kappa_1\psi_1 > + \tilde{\tau}_1 z_1^4 z_2^2 < \kappa_1\psi_2 > + \frac{1}{2}\tilde{\tau}_1^2 z_1^4 z_2^4 < \kappa_1^2 > \\
&\quad + \tilde{\tau}_2 z_1^4 z_2^4 < \kappa_2 > \Big] \\
&= \frac{1}{8(\tau_3-2)^4 z_1^6 z_2^6}\Big[(\tau_3-2)^2(5z_1^4+5z_2^4+3z_1^2z_2^2)+6\tau_5^2 z_1^4 z_2^4 \\
&\quad -(\tau_3-2)(6\tau_5 z_1^4 z_2^2+6\tau_5 z_1^2 z_2^4+5\tau_7 z_1^4 z_2^4)\Big]
\end{aligned} \qquad (4\text{-}15)$$

$$\begin{aligned}
\mathcal{W}_1^{(2)}(z) &= -\frac{1}{128(2-\tau_3)^7 z^{10}}\Big[252\,\tau_5^4 z^8 + 12\,\tau_5^2 z^6(2-\tau_3)(50\,\tau_7 z^2+21\,\tau_5) \\
&\quad + z^4(2-\tau_3)^2(252\,\tau_5^2+348\,\tau_5\tau_7 z^2+145\,\tau_7^2 z^4+308\,\tau_5\tau_9 z^4) \\
&\quad + z^2(2-\tau_3)(203\,\tau_5+145\,z^2\tau_7+105\,z^4\tau_9+105\,z^6\tau_{11}) \\
&\quad + 105\,(2-\tau_3)^4\Big].
\end{aligned} \qquad (4-16)$$

$$\mathcal{W}_4^{(0)}(z_1,z_2,z_3,z_4) = 12\,\frac{1}{(\tau_3-2)^3\,z_1^2 z_2^2 z_3^2 z_4^2}\Big((\tau_3-2)(z_1^{-2}+z_2^{-2}+z_3^{-2}+z_4^{-2})-\tau_5\Big) \qquad (4\text{-}17)$$

and so on ...

4.0.4 Homogeneity relation

Theorem 4.7 of [36] gives another relation which we can apply here: the homogeneity equation. Let $\Phi(z)$ such that

$$\Phi'(z) = y(z)x'(z) = \frac{\Lambda_{-1}}{-2\dot{u}_0}\,2z = \sum_{j=0}^{m} t_j \Lambda'_j \qquad (4\text{-}18)$$

and thus:

$$\Phi = \sum_{j=0}^{m} t_j \Lambda_j \qquad (4\text{-}19)$$

We have:

$$(2-2g-n)\,\mathcal{W}_n^{(g)}(z_1,\dots,z_n) = \operatorname*{Res}_0 \mathcal{W}_{n+1}^{(g)}(z_1,\dots,z_n,z_{z+1})\,\Phi(z_{n+1})$$
$$= -\sum_{j=0}^{m} t_j \frac{\partial}{\partial t_j} \mathcal{W}_n^{(g)}(z_1,\dots,z_n)$$
$$(4-20)$$

In other words, $\mathcal{W}_n^{(g)}$ is homogeneous of degree $2-2g-n$.

5 The $(2m,1)$ minimal model

Another kind of universal limit of matrix models may arise when two connected components of the eigenvalues support merge, typically the equilibrium density of eigenvalues behaves as:

$$\hat{y} \sim x^{2m}. \tag{5-1}$$

The case $m=1$ was treated in [10, 11, 17] The results concerning general m, were described without proof in [12]. The universal limit is given by the $(2m,1)$ reduction of the mKdV hierarchy. All the results can be proven in a way very similar to the $(2m+1,2)$ case, and here we just summarize the results the results stated in [12].

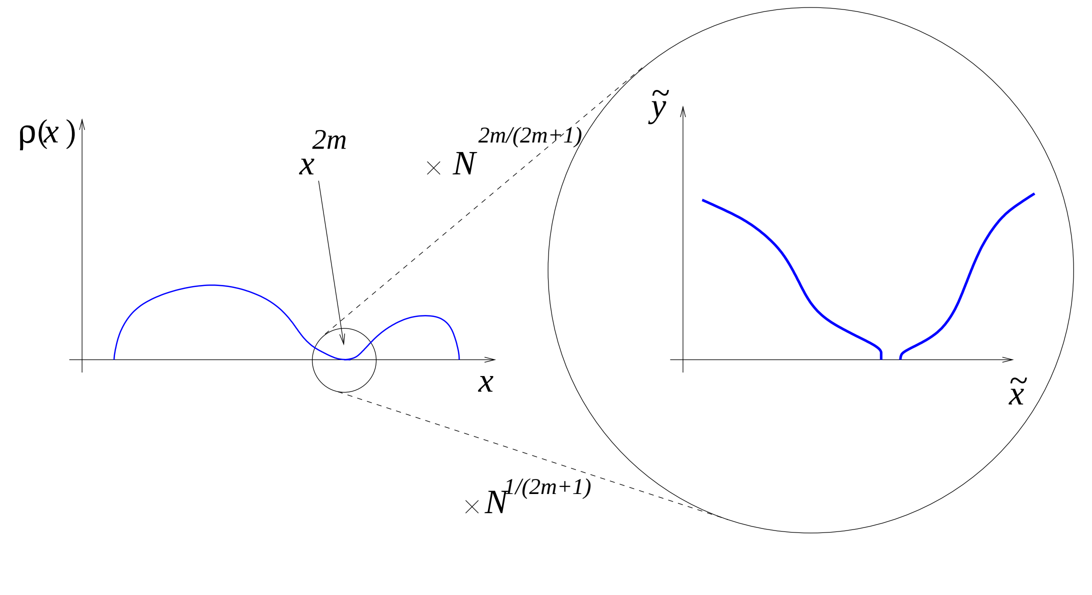

Figure 2: When two cuts merge, the equilibrium density of eigenvalues behaves like x^{2m} near the merging endpoints. The universal eigenvalues statistics is given by determinants of the $(2m,1)$ kernel, associated to the integrable mKdV hierarchy.

5.1 The $(2m,1)$ minimal model

Again, we shall write a Lax pair $\mathcal{D}(x,t)$ and $\mathcal{R}(x,t)$ where $\mathcal{D}(x,t)$ is polynomial in x of degree $2m$, and $\mathcal{R}(x,t)$ is polynomial of degree 1, and such that:

$$\left[\frac{1}{N}\frac{\partial}{\partial x} - \mathcal{D}(x,t), \mathcal{R}(x,t) - \frac{1}{N}\frac{\partial}{\partial t}\right] = 0 \tag{5-2}$$

We choose:

$$\mathcal{R}(x,t) = \begin{pmatrix} 0 & x+u(t) \\ -x+u(t) & 0 \end{pmatrix} \tag{5-3}$$

and the matrix $\mathcal{D}(x,t)$ is of the form:

$$\mathcal{D}(x,t) = \sum_k t_k \mathcal{D}_k(x,t) \tag{5-4}$$

with:

$$\mathcal{D}_k(x,t) = \begin{pmatrix} -A_k(x,t) & xB_k(x,t)+C_k(x,t) \\ xB_k(x,t)-C_k(x,t) & A_k(x,t) \end{pmatrix} \tag{5-5}$$

where A_k, B_k, C_k are even polynomials of x, of degree $\deg A_k = 2k-2$, $\deg B_k = 2k-2$, $\deg C_k = 2k$. They can be found by recursion:

$$\begin{aligned} A_0 &= 0 \quad , \quad B_0 = 0 \quad , \quad C_0 = 1, \\ C_{k+1} &= x^2 C_k + \check{R}_k(u) \\ B_{k+1} &= x^2 B_k + \hat{R}_k(u) \\ A_{k+1} &= x^2 A_k + \frac{1}{2}\dot{\hat{R}}_k(u) \end{aligned} \tag{5-6}$$

where $\hat{R}_k(u)$ and $\check{R}_k(u)$ are the modified Gelfand-Dikii differential polynomials:

$$\begin{aligned} \hat{R}_0(u) &= u \quad , \quad \check{R}_0(u) = \frac{u^2}{2} \\ \hat{R}_{k+1}(u) &= u\check{R}_k(u) - \frac{1}{4}\ddot{\hat{R}}_k(u) \\ \dot{\check{R}}_k(u) &= u\dot{\hat{R}}_k(u) \end{aligned} \tag{5-7}$$

For example:

$$\begin{aligned} \hat{R}_1(u) &= \frac{u^3}{2} - \frac{\ddot{u}}{4} \quad , \quad \check{R}_1(u) = \frac{3u^4}{8} - \frac{u\ddot{u}}{4} + \frac{\dot{u}^2}{8} \\ \hat{R}_2(u) &= \frac{3u^5}{8} - \frac{5u^2\ddot{u}}{8} - \frac{5u\dot{u}^2}{8} \quad , \quad \check{R}_2(u) = \frac{5u^6}{16} - \frac{5u^3\ddot{u}}{8} - \frac{5u^2\dot{u}^2}{16} - \frac{uu^{(4)}}{16} - \frac{1}{16}\dot{u}\dddot{u} \\ &\qquad + \frac{\ddot{u}^2}{32} \end{aligned} \tag{5-8}$$

The matrix $\mathcal{D}(x,t)$ satisfies

$$[\partial_x - \mathcal{D}(x,t), \partial_t - \mathcal{R}(x,t)] = 0 \tag{5-9}$$

if and only if $u(t)$ satisfies the string equation:

$$\boxed{\sum_{k=0}^{m} t_k \hat{R}_k(u) = -t\,u.} \tag{5-10}$$

The Baker-Akhiezer functions

$$\Psi(x,t) = \begin{pmatrix} \psi(x,t) & \phi(x,t) \\ \tilde{\psi}(x,t) & \tilde{\phi}(x,t) \end{pmatrix}, \tag{5-11}$$

are given by the common solutions of the two compatible systems:

$$\frac{1}{N}\frac{\partial}{\partial x}\Psi(x,t) = \mathcal{D}(x,t)\,\Psi(x,t) \qquad , \qquad \frac{1}{N}\frac{\partial}{\partial t}\Psi(x,t) = \mathcal{R}(x,t)\,\Psi(x,t). \tag{5-12}$$

It was claimed in [12] that the parametrics asymptotics of orthogonal polynomials near the singularity are given by:

$$\begin{aligned} \psi_n((s-s_c)^{\frac{1}{2m}}x) \qquad \sim \qquad & \cos((n+\frac{1}{2})\pi\epsilon)\,\psi(x) - \sin((n+\frac{1}{2})\pi\epsilon)\,\tilde{\psi}(x) \\ & +(s-s_c)^{\frac{1}{2m}}\frac{u(t)\,\cos\pi\epsilon}{4(\sin\pi\epsilon)^2}\Big(\cos(3(n+\frac{1}{2})\pi\epsilon)\,\psi(x) \\ & \qquad -\sin(3(n+\frac{1}{2})\pi\epsilon)\,\tilde{\psi}(x)\Big) + \dots \end{aligned} \tag{5 - 13}$$

and where $\epsilon/(1-\epsilon)$ is the ratio of the number of eigenvales in the 2 cuts which merge at the singularity ($\epsilon = 1/2$ is the symmetric case).

The Christoffel-Darboux kernel $K(x_1,x_2)$ is the same as eq.(2-28):

$$K(x_1,x_2) = \frac{\psi(x_1)\tilde{\phi}(x_2) - \tilde{\psi}(x_1)\phi(x_2)}{x_1 - x_2} \tag{5-14}$$

and the correlators are obtained by the same determinantal formulae eq.(2-30).

And again the claim is that the determinantal correlators of the $(2m,1)$ minimal model, are the limits of matrix models correlators.

6 Conclusion

In this article, we have summarized some properties of scaling limits of matrix models (formal or not), known for a long time. We have provided a mathematical proof that the asymptotics of the $\hat{\omega}_n^{(g)}$'s and $\hat{f}_g$'s are indeed those obtained from conformal field theory. Our proof is based on the fact that the limits $\omega_n^{(g)}$'s of matrix models correlators, are the spectral invariants of the limit spectral curve, and the fact that the determinantal correlators of the $(2m+1,2)$ minimal model kernel are also the spectral invariants of the same spectral curve.

We recall that those correlators can be interpreted in the Kontsevich integral's framework, and have a combinatorial interpretation as intersection numbers of some tautological classes on the moduli spaces of Riemann surfaces.

We claim that the same method can be applied to other sorts of universal limits, in particular the merging of two cuts like in [12], and hopefully, we can work out the same kind of proof for multi-matrix models, whose universal limits should be the (p,q) minimal models with arbitrary p and q. Unfortunately, one of the key points should be the equivalent of [8] (i.e. the fact that determinantal correlators obey loop equations), but this is not proved yet for differential systems of order $q>2$.

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References

[1] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, pp. 446-452, 1972.

[2] O. Babelon, D. Bernard, M. Talon, *Introduction to Classical Integrable Systems* (Cambridge University Press).

[3] A.A.Belavin, A.B. Zamolodchikov, On Correlation Numbers in 2D Minimal Gravity and Matrix Models, J.Phys.A42:304004,2009, arXiv:0811.0450.

[4] A.Belavin, V.Belavin, Four-point function in Super Liouville Gravity, J.Phys.A42:304003,2009, arXiv:0810.1023.

[5] A.A.Belavin, V.A.Belavin, A.V.Litvinov, Y.P.Pugai, Al.B.Zamolodchikov, On correlation functions in the perturbed minimal models M(2,2n+1), Nucl.Phys. B676 (2004) 587-614, arXiv:hep-th/0309137.

[6] M. Bertola, Boutroux curves with external field: equilibrium measures without a minimization problem, arXiv:0705.3062.

[7] Bergère M.C., Correlation functions of complex matrix models, J. Phys. A 39 8749-8773 (2006).

[8] M. Bergère, B. Eynard, Determinantal formulae and loop equations, math-ph: arxiv.0901.3273.

[9] P.M. Bleher and A.R. Its, eds., "Random Matrix Models and Their Applications", *MSRI Research Publications* **40**, Cambridge Univ. Press, (Cambridge, 2001).

[10] P. Bleher, A. Its, "Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model" *Ann. of Math.* (2) **150**, no. 1, 185–266 (1999).

[11] P. Bleher and A. Its, Double scaling limit in the matrix model: the Riemann-Hilbert approach, Preprint, 2002, (arXiv:math-ph/0201003).

[12] P. Bleher, B. Eynard, "Double scaling limit in random matrix models and a non-linear hierarchy of differential equations", J. Phys. A36 (2003) 3085-3106, xxx, hep-th/0209087.

[13] A. Boutet de Monvel, L. Pastur, and M. Shcherbina, On the statistical mechanics approach in the random matrix theory: Integrated density of states, J. Statist. Phys. **79** (1995), 585–611.

[14] E. Brezin, C. Itzykson, G. Parisi, and J. Zuber, Comm. Math. Phys. **59**, 35 (1978).

[15] E. Brézin and V. A. Kazakov. Exactly solvable field theories of closed strings. Phys. Lett. B **236** (1990), 144–150.

[16] E. Brézin and A. Zee. Universality of the correlations between eigenvalues of large random matrices, Nucl. Phys. B **402** (1993), 613–627.

[17] R. C. Brower, N. Deo, S. Jain, and C.-I Tan, Symmetry breaking in the double-well hermitian matrix models, Nucl. Phys. **B405** (1993) 166-187.

[18] L.Chekhov, B.Eynard, “Hermitian matrix model free energy: Feynman graph technique for all genera”, *J. High Energy Phys.* **JHEP03** (2006) 014, hep-th/0504116.

[19] L.Chekhov, B.Eynard and N.Orantin, “Free energy topological expansion for the 2-matrix model”, *J. High Energy Phys.* **JHEP12** (2006) 053, math-ph/0603003.

[20] G.M.Cicuta “Phase transitions and random matrices”, cond-mat/0012078.

[21] F. David, “Loop equations and nonperturbative effects in two-dimensional quantum gravity”. *Mod.Phys.Lett.* **A5** (1990) 1019.

[22] F. David, ”Planar diagrams, two-dimensional lattice gravity and surface models”, Nuclear Physics B, Volume 257, p. 45-58.

[23] J.M.Daul, V.Kazakov, I.Kostov, “Rational Theories of 2D Gravity from the Two-Matrix Model”, *Nucl.Phys.* **B409** (1993) 311-338, hep-th/9303093.

[24] P. Deift, T. Kriecherbauer, K. T-R. McLaughlin, S. Venakides, and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, Commun. Pure Appl. Math. **52** (1999) 1335-1425.

[25] P. Di Francesco, ”2D gravity, matrix models and graph combinatorics”, Lecture given in Les Houches, *applications of random matrices in physics*, june 2004, math-ph/0406013.

[26] P. Di Francesco, P. Mathieu, D. Sénéchal, ”Conformal field theory”, Springer, 1999.

[27] P. Di Francesco, P. Ginsparg, J. Zinn-Justin, “2D Gravity and Random Matrices”, *Phys. Rep.* **254**, 1 (1995).

[28] R.Dijkgraaf, “Intersection theory, integrable hierarchies and topological field theory”, in J.Frohlich and al. (eds), *New symmetry principles in quantum field theory*, Plenum Press, p.95; hep-th/9201003.

[29] M. Douglas, S. Shenker, Strings in less than one dimension. Nuclear Phys. B **335** (1990), no. 3, 635–654.

[30] M. Douglas, N. Seiberg, and S. Shenker, Flow and instability in quantum gravity, Phys. Lett. B **244** (1990), 381–386.

[31] Duplantier B., Sheffield S., Liouville Quantum Gravity and KPZ, arXiv:0808.1560v1.

[32] Duplantier B., Sheffield S., Duality and KPZ in Liouville Quantum Gravity, Phys. Rev. Lett. 102 150603 (2009).

[33] F.J. Dyson, "correlations between the eigenvales of a random matrix", Comm. Math. Phys. **19** (1970) 235-50.

[34] "Weil-Petersson volume of moduli spaces, Mirzakhani's recursion and matrix models", B. Eynard, N. Orantin, math-ph: arXiv:0705.3600v1.

[35] B. Eynard, "Topological expansion for the 1-hermitian matrix model correlation functions", JHEP/024A/0904, hep-th/0407261.

[36] B.Eynard, N.Orantin, "Invariants of algebraic curves and topological expansion", math-ph/0702045.

[37] B. Eynard, Recursion between Mumford volumes of moduli spaces , B. Eynard, math-ph: arXiv:0706.4403.

[38] B. Eynard, N. Orantin, Algebraic methods in random matrices and enumerative geometry, math-ph: arxiv.0811.3531, to appear in J.Phys.A.

[39] A. S. Fokas, A. R. Its, and A. V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, Commun. Math. Phys. **147** (1992), 395–430.

[40] A. Guionnet, E. Segala Maurel, "Combinatorial aspects of matrix models".

[41] *Two dimensional quantum gravity and random surfaces*, edited by D. Gross, T. Piran, and S. Weinberg (Jerusalem winter school, World Scientific, ., 1991).

[42] D. Gross and A. Migdal, Nonperturbative two-dimensional quantum gravity, Phys. Rev. Lett. **64** (1990) 127-130; A nonperturbative treatment of two-dimensional quantum gravity, Nucl. Phys. B **340** (1990) 333–365.

[43] K. Hosomichi, S. Ribault, Solution of the H3+ model on a disc, JHEP0701:057,2007, arXiv:hep-th/0610117.

[44] A. Its and V. Novokshenov, The isomonodromic deformation method in the theory of Painlevé equations, Lecture Notes in Math. **1191** (1986), Springer-Verlag, New York.

[45] K. Johansson, On fluctuations of eigenvalues of random hermitian matrices, Duke Math. J. **91** (1988), 151–204.

[46] VA Kazakov, IK Kostov, AA Migdal, "Critical properties of randomly triangulated planar random surfaces", Physics Letters B, 1985.

[47] V.A. Kazakov, "Bilocal regularization of models of random surfaces" Physics Letters B, Volume 150, Issue 4, p. 282-284.

[48] V.A. Kazakov, "The appearance of matter fields from quantum fluctuations of 2D-garvity", Modern Physics Letters A, Vol. 4, No. 22 (1989) 2125-2139.

[49] S.Kharchev, A.Marshakov, "On $p-q$ duality and explicit solutions in $c<1$ 2d gravity models", hep-th/9303100.

[50] A. Kitaev, Turning points of linear systems and double asymptotics of Painlevé transcendents, Diff. Geom. Lie Groups Mech. **12**, LOMI **187** (1991) 53-74.

[51] V.G. Knizhnik, A.M. Polyakov, A.B. Zamolodchikov, Mod. Phys. Lett **A3** (1988) 819.

[52] M. Kontsevich, "Intersection theory on the moduli space of curves and the matrix Airy function", Comm. Math. Phys. **147** (1992), no 1. 1-23.

[53] I.K.Kostov, " Conformal field theory techniques in random matrix models", hep-th/9907060.

[54] I.K. Kostov, "Bilinear functional equations in 2d quantum gravity", hep-th/9602117.

[55] I.Krichever "The τ-function of the universal Whitham hierarchy, matrix models and topological field theories", *Commun.Pure Appl.Math.* **47** (1992) 437; hep-th/9205110

[56] I.M. Krichever, "The τ-function of the universal Whitham hierarchy, matrix models and topological field theories", hep-th/9205110.

[57] M. L. Mehta, Random Matrices, 3rd ed., Elsevier Academic Press, 2004.

[58] P. Van Moerbeke, Random Matrices and their applications, MSRI-publications **40**, 4986 (2000).

[59] G. Moore, Geometry of the string equations, Commun. Math. Phys. **133** (1990) 261-304.

[60] D. Mumford, “Towards an enumerative geometry of the moduli space of curves”, in Arithmetic and Geometry (M. Artin, J. Tate, editors), vol. 2, Birkhauser, Boston, 1983, pp. 271328.

[61] S. Ribault, J. Teschner, H(3)+ correlators from Liouville theory, JHEP 0506 (2005) 014, arXiv:hep-th/0502048.

[62] G. Szegö. Orthogonal Polynomials, 3rd ed. AMS, Providence, 1967.

[63] J. Teschner, From Liouville Theory to the Quantum Geometry of Riemann Surfaces Proceedings of the ICMP 2003, Lisbon, arXiv:hep-th/0308031.

[64] G. ’t Hooft, *Nuc. Phys.* **B72**, 461 (1974).

[65] C. Tracy and H. Widom, Level-spacing distributions and the Airy kernel, Physics Letters B, Volume 305, Issue 1-2, p. 115-118.

[66] C. Tracy and H. Widom, Introduction to random matrices. In: Geometric and quantum aspects of integrable systems (Scheveningen, 1992), Lecture Notes in Phys. **424**, Springer-Verlag, Berlin, 1993, 103-130.

[67] W.T. Tutte, ”A census of planar triangulations”, *Can. J. Math.* **14** (1962) 21-38.

[68] W.T. Tutte, ”A census of planar triangulations”, *Can. J. Math.* **14** (1962) 21-38.

[69] W.T. Tutte, ”A census of planar maps”, *Can. J. Math.* **15** (1963) 249-271.

[70] E.P. Wigner, *Proc. Cambridge Philos. Soc.* **47**, 790 (1951).

[71] E. Witten, “Two-dimensional gravity and intersection theory on moduli space”, Surveys in Differential Geometry **1** (1991), 243-310.

[72] A.B. Zamolodchikov, Al.B.Zamolodchikov, ”Structure Constants and Conformal Bootstrap in Liouville Field Theory”, Journal-ref: Nucl.Phys. B477 (1996) 577-605, hep-th/9506136.

[73] Al. Zamolodchikov, ”On the Three-point Function in Minimal Liouville Gravity”, hep-th/0505063.