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Abstract: We prove resolvent estimates for semiclassical operators such as $-h^2\Delta + V(x)$ in scattering situations. Provided the set of trapped classical trajectories supports a chaotic flow and is sufficiently filamentary, the analytic continuation of the resolvent is bounded by h^{-M} in a strip whose width is determined by a certain topological pressure associated with the classical flow. This polynomial estimate has applications to local smoothing in Schrödinger propagation and to energy decay of solutions to wave equations.

KEY WORDS Quantum scattering, chaotic trapped set, semiclassical resolvent estimates

Received

1 Statement of Results

In this short note we prove a resolvent estimate in the pole free strip for operators whose classical Hamiltonian flows are hyperbolic on the sets of trapped trajectories (trapped sets), and the latter are assumed to be sufficiently filamentary – see (1.4) for the precise condition. The proof is based on the arguments of (21) and we refer to §3 of that paper for the preliminary material and assumptions on the operator.

The polynomial estimate on the resolvent in the pole free strip below the real axis (1.5) provides a direct proof of the estimate on the real axis (1.6), and that estimate is only logarithmically weaker than the similar bound in the non-trapping case (that is, the case where all classical trajectories escape to infinity). Through an argument going back to Kato, and more recently to Burq, that estimate is crucial for obtaining local smoothing and Strichartz estimates for the Schrödinger equation. These in turn are important in the investigation of nonlinear waves in non-homogeneous trapping media. Also, as has been known since the work of Lax-Phillips, the estimate in the complex domain is useful for obtaining exponential decay of solutions to wave equations (see the paragraph following (1.6) for some references to recent literature).

An example of an operator to which our methods apply is given by the semiclassical Schrödinger operator

$$Pu(x) = P(h)u(x) = -h^2 \frac{1}{\sqrt{\bar{g}}} \sum_{i,j=1}^{n} \partial_{x_j} \left(\sqrt{\bar{g}} g^{ij} \partial_{x_i} u(x) \right) + V(x) \,, \quad x \in \mathbb{R}^n \,, \tag{1.1}$$

$G(x) \stackrel{\text{def}}{=} (g^{ij}(x))_{i,j}$ is a symmetric positive definite matrix representing a (possibly nontrivial) metric on $\mathbb{R}^n$, $\bar{g} \stackrel{\text{def}}{=} 1/\det G(x)$, and $V(x)$ is a potential function. We assume that the geometry and the potential are “trivial” outside a bounded region:

$$g^{ij}(x) = \delta_{ij} \,, \quad V(x) = -1 \,, \quad \text{when } |x| > R \,.$$

This operator is hence associated with a short-range scattering situation. We refer to (21, §3.2) for the complete set of assumptions which allow long range perturbations, at the expense of some analyticity assumptionsWe note that for $V \equiv -1$, $P(h)u - 0$ is the Helmholtz equation for a Laplace-Beltrami operator, with $h = 1/\lambda$, playing the rôle of wavelength.

Such operators have a purely continuous spectrum near the origin, and their truncated resolvent $\chi(P(h) - z)^{-1}\chi$ $(\chi \in C_c^\infty(\mathbb{R}^n))$ can be meromorphically continued from $\operatorname{Im} z > 0$ to $\operatorname{Im} z < 0$, with poles of

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finite multiplicity called *resonances*. In the semiclassical limit $h \ll 1$, the distribution of resonances depends on the properties of the classical flow generated by the Hamiltonian

$$p(x,\xi) = \sum_{i,j=1}^{n} g^{ij}(x)\xi_i\xi_j + V(x),$$

that is the flow $(x,\xi) \mapsto \exp tH_p(x,\xi)$ associated with the Hamiltonian vector field

$$H_p(x,\xi) \stackrel{\rm def}{=} \sum_{k=1}^{n} \partial_{\xi_k} p\, \partial_{x_k} - \partial_{x_k} p\, \partial_{\xi_k}\,.$$

(when $V \equiv -1$ the Hamiltonian flow corresponds to the geodesic flow on $S^*\mathbb{R}^n$.) More precisely, the properties of the resolvent $\chi(P(h)-z)^{-1}\chi$ near $z=0$ are influenced by the nature of flow on the energy shell $\{p(x,\xi)=0\}$. A lot of attention has been given to *nontrapping flows*, that is flows for which the *trapped set*

$$K \stackrel{\rm def}{=} \{(x,\xi)\ :\ p(x,\xi)=0\,,\ \ \exp tH_p(x,\xi) \not\to \infty\,,\ t \to \pm\infty\} \tag{1.2}$$

is empty. In that case, for $\delta>0$ small enough and any $C>0$, the resolvent is pole free in a strip $[-\delta,\delta]-i[0,Ch]$, and satisfies the bound (17; 18)

$$\|\chi(P(h)-z)^{-1}\chi\|_{L^2\to L^2} = \mathcal{O}(h^{-1}), \quad z\in[-\delta,\delta]-i[0,Ch]\,.$$

On the opposite, there exist cases of "strong trapping" for which the trapped set has a positive volume; resonances can then be exponentially close to the real axis, and the norm of the resolvent be of order $e^{C/h}$ for $z\in[-\delta,\delta]$ (28; 3; 6).

In this note we are considering an intermediate situation, namely the case where the trapped set (1.2) is a (locally maximal) *hyperbolic set*. This means that K is a compact, flow-invariant set with no fixed point, such that at any point $\rho\in K$ the tangent space splits into the neutral ($\mathbb{R}H_p(\rho)$), stable (E^-_ρ), and unstable (E^+_ρ) directions:

$$T_\rho p^{-1}(0) = \mathbb{R}H_p(\rho)\oplus E^-_\rho \oplus E^+_\rho\,.$$

This decomposition is preserved through the flow. The (un)stable directions are characterized by the following properties:

$$\exists\,\lambda>0, \quad \|d\exp tH_p(\rho)v\| \le C\,e^{-\lambda|t|}\|v\|, \quad \forall\, v\in E^{\mp}_\rho,\ \pm t>0.$$

Such trapped sets are easy to construct. The simplest case consists in a single unstable periodic orbit, but we will rather consider the more general case where K is a *fractal* set supporting a chaotic flow; such a set contains countably many periodic orbits, which are dense on the set of nonwandering points $NW(K)\subset K$ (15).

Our results will depend on the "thickness" of the trapped set, defined formulated in terms of a certain dynamical object, the *topological pressure*. We refer to (21, §3.3) and texts on dynamical systems (15; 29) for the general definition of the pressure, recalling only a definition valid in the present case. Let $f\in C^0(K)$. Then the *pressure* of f with respect to the Hamiltonian flow on K is given by

$$\mathcal{P}(f) \stackrel{\rm def}{=} \lim_{T\to\infty}\frac{1}{T}\log\sum_{T_\gamma<T}\exp\int_0^{T_\gamma}(\exp tH_p)^*f(\rho_\gamma)\,dt\,, \tag{1.3}$$

where the sum runs over all periodic orbits γ of periods $T_\gamma\le T$, and ρ_γ is a point on the orbit γ. The function f we will be using is a multiple of the (infinitesimal) unstable Jacobian of the flow on K:

$$\varphi_+(\rho) \stackrel{\rm def}{=} \frac{d}{dt}\det(d\exp tH_p \restriction E^+_\rho)|_{t=0}\,, \qquad \rho\in K\,.$$

We can now formulate our main result:

Theorem. *Suppose that $P(h)$ satisfies* (1.1) *or the more general assumptions of (21, §3.2). Suppose also that the Hamiltonian flow is hyperbolic on the trapped set K, and that the topological pressure*

$$\mathcal{P}(-\varphi_+/2)<0\,, \qquad \varphi_+ \text{ the unstable Jacobian.} \tag{1.4}$$

Then for any $\chi \in \mathcal{C}^\infty_{\rm c}(\mathbb{R}^n)$ *and* $\epsilon > 0$*, there exist* $\delta(\epsilon) > 0$ *and* $h(\epsilon) > 0$ *such that the cut-off resolvent* $\chi(P(h) - z)^{-1}\chi$, $\operatorname{Im} z > 0$*, continues analytically to the strip*

$$\Omega_\epsilon(h) \stackrel{\rm def}{=} \{ z \; : \; \operatorname{Im} z > h(\mathcal{P}(-\varphi_+/2) + \epsilon) \,, \quad |\operatorname{Re} z| < \delta(\epsilon) \} \,, \quad 0 < h < h(\epsilon) \,.$$

For $z \in \Omega_\epsilon(h) \cap \{\operatorname{Im} z \leq 0\}$*, this resolvent is polynomially bounded in* h*:*

$$\begin{gathered} \| \chi(P(h) - z)^{-1}\chi \|_{L^2 \to L^2} \leq C(\epsilon, \chi) \, h^{-1 + c_E \operatorname{Im} z / h} \, \log(1/h) \,, \\ c_E \stackrel{\rm def}{=} \frac{n}{2|P(-\varphi_+/2) + \epsilon/2|} \,. \end{gathered} \tag{1.5}$$

For any $s \in [0, 1]$, the pressure $\mathcal{P}(-s\varphi_+)$ measures relative strengths of the complexity of the flow on K (i.e. the number of periodic orbits), and the instability of the trajectories (through the Jacobian). For $s = 0$, $\mathcal{P}(0)$ only measures the complexity, it is the topological entropy of the flow, which is generally positive. On the opposite, $\mathcal{P}(-\varphi_+)$ is negative, it represents the "classical decay rate" of the flow. The intermediate value $\mathcal{P}(-\varphi_+/2)$ can take either sign, depending on the "thickness" of K. In dimension $n = 2$ the condition (1.4) is equivalent to the statement that the Hausdorff dimension of $K \subset p^{-1}(0)$ is less than 2. Since the energy surface $p^{-1}(0)$ has dimension 3 and the minimal dimension of a non-empty K is 1, the condition means that we are less than "half-way" and K is *filamentary.* Trapped sets with dimensions greater than 2 are referred to as *bulky.*

The first part of the theorem is the main result of (21), see Theorem 3 there. Here we use the techniques developed in that paper to prove (1.5). For the Laplacian outside several convex obstacles on $\mathbb{R}^n$ (satisfying a condition guaranteeing strict hyperbolicity of the flow) with Dirichlet or Neumann boundary condition, the theorem was proved by Ikawa (14), with the pressure being only implicit in the statement.For more recent developments in that setting see (2),(22), and (19).

In particular, for z on the real axis the bound (1.5) gives

$$\| \chi(P(h) - z)^{-1}\chi \|_{L^2 \to L^2} \leq C \, \frac{\log(1/h)}{h} \,, \quad z \in [-\delta(\epsilon), \delta(\epsilon)], \quad 0 < h < h(\epsilon) \,. \tag{1.6}$$

This result was already given in (21, Theorem 5) with a less direct proof. It has been generalized to a larger class of manifolds in (9) and (1.5) provides no new insight in that setting.

One of the applications of (1.6) in the case of the Laplacian is a *local smoothing* with a minimal loss (7) in the Schrödinger evolution (see (4) for the original application in the setting of obstacle scattering):

$$\forall T > 0, \forall \epsilon > 0, \exists C = C(T, \epsilon), \qquad \int_0^T \| \chi e(-it\Delta_g) u \|^2_{H^{1/2 - \epsilon}} \, dt \leq C \, \|u\|^2_{L^2} \,.$$

One can also deduce from (1.6) a Strichartz estimate (7; 5) useful to prove existence of solutions for some related semilinear Schrödinger equations.

In the case of the Laplacian ($V \equiv -1$), the estimate in a strip (1.5) has important consequences regarding the energy decay for the wave equation – see (4; 8; 12) and references given there. In odd dimension $n \geq 3$, it implies that the local energy of the waves decays exponentially in time. The same type of energy decay (also involving a pressure condition) has been recently obtained by Schenck in the setting of the damped wave equation on a compact manifold of negative curvature (24).

To prove (1.5) we use several methods and intermediate results from (21). Using estimates from (21, §7), we show in §3 how to obtain a good parametrix for the complex-scaled operator, which leads to an estimate for the resolvent. As was pointed out to us by Burq the construction of the parametrix for the outgoing resolvent was the, somewhat implicit, key step in the work of Ikawa (14) on the resonance gap for several convex obstacle. That insight lead us to re-examine the consequences of (21).

We follow the notation of (21) with precise references given as we go along. For the needed aspects of semiclassical microlocal analysis (21, §3) and the references to (10) and (11) should be consulted.

2 Review of the hyperbolic dispersion estimate

The central "dynamical ingredient" of the proof is a certain dispersion estimate relative to a modification of $P(h)$, which we will now describe.

The first modification of $P(h)$ comes from the method of complex scaling reviewed in (21, §3.4). For any fixed sufficiently large $R_0 > 0$, it results in the operator $P_\theta(h)$, with the following properties. To formulate them put

$$\Omega_\theta \stackrel{\text{def}}{=} [-\delta, \delta] + i[-\theta/C, C] \,, \quad \theta = M_1 h \log(1/h) \,. \tag{2.1}$$

Then

$$P_\theta(h) - z \;:\; H^2_h(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \;\text{ is a Fredholm operator for } z \in \Omega_\theta, \tag{2.2}$$

$$\forall\, \chi \in \mathcal{C}^\infty_{\rm c}(B(0,R_0)) \,, \quad \chi R(z,h)\chi = \chi R_\theta(z,h)\chi \,. \tag{2.3}$$

Here and below we set the following notation for the resolvents:

$$R_\bullet(z,h) \stackrel{\text{def}}{=} (P_\bullet(h) - z)^{-1} \,, \quad \operatorname{Im} z > 0 \,,$$

and (2.3) shows the meromorphic continuation of $\chi R(z,h)\chi$ to Ω_θ, guaranteed by the Fredholm property of $P_\theta(h) - z$.

The operator $P_\theta(h)$ is further modified by an exponential weight, $G^w = G^w(x, hD)$,

$$G \in \mathcal{C}^\infty_{\rm c}(T^*\mathbb{R}^n) \,, \quad \operatorname{supp} G \subset p^{-1}((-2\delta, 2\delta)) \,, \quad \partial^\alpha G = \mathcal{O}(h \log(1/h)) \,,$$

where $\delta > 0$ is a fixed small number. The modified operator is obtained by conjugation:

$$P_{\theta,\epsilon}(h) \stackrel{\text{def}}{=} e^{-\epsilon G^w/h} P_\theta(h) e^{\epsilon G^w/h} \,, \quad \epsilon = M_2\theta \,, \quad \theta = M_1 h \log(1/h) \,. \tag{2.4}$$

This operator has the same spectrum as $P_\theta(h)$ and the following properties:

$$\text{if } \;\psi_0 \in S(T^*\mathbb{R}^n), \quad \operatorname{supp}\psi_0 \subset p^{-1}((-3\delta/2, 3\delta/2)) \,, \,,$$

$$\text{if} \quad \operatorname{Im} \psi_0^w(x,hD)\, P_{\theta,\epsilon}(h)\, \psi_0^w(x,hD) \le C\, h \tag{2.5}$$

$$\tag{2.6}$$

The main reason for introducing the weight G is to ensure the bound (2.5). The specific choice of G is explained in (21, §6.1). In particular G vanishes in some neibhbourhood of the trapped set K, and the operator $\exp(\epsilon G^w(x,hD))$ is an h-pseudodifferential operator $B^w(x,hD)$, with symbol satisfying

$$B \in h^{-N} S_\delta(T^*\mathbb{R}^n) \,, \quad B\restriction_{\complement \operatorname{supp} G} = 1 + \mathcal{O}_{S_\delta}(h^\infty) \,.$$

As a result, if the spatial cutoff χ is supported away from $\pi \operatorname{supp} G$, calculus of semiclassical pseudodifferential operators ensures that

$$\chi R(z,h)\chi = \chi R_{\theta,\epsilon}(z,h)\chi + \mathcal{O}_{L^2\to L^2}(h^\infty)\|R_{\theta,\epsilon}(z,h)\| \,. \tag{2.7}$$

From now on our objective will then be to estimate the norm $\|R_{\theta,\epsilon}(z,h)\|_{L^2\to L^2}$.

We consider a final modification of $P_{\theta,\epsilon}(h)$ near the zero energy surface. Let $\psi_0 \in S(T^*\mathbb{R}^n)$ be supported in $p^{-1}((-3\delta/2, 3\delta/2)$ and equal to 1 in $p^{-1}(-\delta,\delta)$. Define

$$\widetilde{P}_{\theta,\epsilon}(h) \stackrel{\text{def}}{=} \psi_0^w(x,hD)\, P_{\theta,\epsilon}\, \psi_0^w(x,hD) \,, \tag{2.8}$$

and the associated propagator

$$U(t) \stackrel{\text{def}}{=} \exp\{-it\widetilde{P}_{\theta,\epsilon}(h)/h\} \,. \tag{2.9}$$

The crucial ingredients in proving (1.5) are good upper bounds for the norms

$$\|U(t)\psi^w(x,hD)\|_{L^2\to L^2} \,, \quad \text{on time scales} \quad 0 \le t \le M\log(1/h) \,,$$

where $M > 0$ is fixed but large, and

$$\psi \in S(T^*\mathbb{R}^n), \quad \operatorname{supp} \psi \subset p^{-1}((-\delta/2, \delta/2)), \quad \psi = 1 \ \text{ on } \ p^{-1}((-\delta/4, \delta/4)) . \tag{2.10}$$

From the bound (2.5) on the imaginary part of $\widetilde{P}_{\theta,\epsilon}(h)$, we obviously get an exponential control on the propagator:

$$\|U(t)\|_{L^2 \to L^2} \leq \exp(C\, t), \qquad t \geq 0 . \tag{2.11}$$

The reason to conjugate P_θ with the weight G^w was indeed to ensure this exponential bound. Together with the hyperbolic dispersion bound (2.13), this exponential bound would suffice to get a polynomial bound $\mathcal{O}(h^{-L})$ in (1.5), for some (unknown) $L > 0$. To obtain the explicit value,

$$-1 + \frac{c_E \operatorname{Im} z}{h} ,$$

for the exponent, we need to improve (2.11) into the following uniform bound:

Lemma 2.1. *Let ψ satisfy the conditions* (2.10). *Then, there exist h_0, $C_0 > 0$ such that,*

$$\|U(t)\psi^w(x, hD)\|_{L^2 \to L^2} \leq C_0 , \quad 0 \leq t \leq M \log(1/h), \quad h < h_0 . \tag{2.12}$$

Before proving this Lemma, we state the major consequence of our dynamical assumptions for the classical flow on K, namely its hyperbolicity and the "filamentary" nature of K (expressed through (1.4)). It is a *hyperbolic dispersion estimate* which was explicitly written only in a model case (21, Proposition 9.1) (21), but can be easily drawn from (21, Proposition 6.3), in the spirit of (21, §6.4) As above, we take ψ as in (2.10). For any $\epsilon > 0$ we set $\lambda \stackrel{\text{def}}{=} -\mathcal{P}(\varphi_+/2) + \epsilon/2$. For any $0 < h < h(\epsilon)$, we then have

$$\begin{gathered} \|U(t)\psi^w(x, hD)\|_{L^2 \to L^2} \leq C\, h^{-n/2} \exp(-\lambda t) + \mathcal{O}(h^{M_3}) , \\ \text{uniformly in the time range} \quad 0 < t < M \log(1/h) . \end{gathered} \tag{2.13}$$

The constant M is arbitrarily large, and M_3 can be taken as large as we wish, provided we choose M_1 in (2.1) large enough depending on M. If the pressure $\mathcal{P}(\varphi_+/2)$ is negative, one can take ϵ small enough to ensure $\lambda > \epsilon/2 > 0$. The above estimate is then sharper than (2.12) for times beyond the *Ehrenfest time*

$$t_E \stackrel{\text{def}}{=} c_E \log(1/h), \qquad c_E \stackrel{\text{def}}{=} \frac{n}{2\lambda} . \tag{2.14}$$

The large constant M will always be chosen (much) larger than c_E.

Proof of Lemma 2.12. To motivate the proof we start with a heuristic argument for the bound (2.12). As mentioned above, the exponential bound (2.11) is due to the fact that the imaginary part of $\widetilde{P}_{\theta,\epsilon}(h)$ can take positive values of order $\mathcal{O}(h)$ (see (2.5)). However, the construction of the weight G shows that outside a bounded region of phase space of the form

$$V_{\text{pos}} = p^{-1}((-2\delta, 2\delta)) \cap T^*_{\{R_1 < |x| < R_2\}}\mathbb{R}^n ,$$

the imaginary part of $\widetilde{P}_{\theta,\epsilon}(h)$ is negative up to $\mathcal{O}(h^\infty)$ errors.

The radius R_1 above is large enough, so that V_{pos} lies at *finite distance* from the trapped set. As a result, any trajectory crossing the region V_{pos} will only spend a bounded time in that region. For this reason, the propagator $U(t)$ on a large time $t \gg 1$ will "accumulate" exponential growth during a uniformly bounded time only.

We now provide a rigorous proof, using ideas and results from (21, §6.3). The phase space $T^*\mathbb{R}^n$ is split using a smooth partition of unity:

$$1 = \sum_{b=0,1,2,\infty} \pi_b , \qquad \pi_b \in C^\infty(T^*\mathbb{R}^n, [0,1]) .$$

These four functions have specific localization properties:

- $\operatorname{supp}\pi_b \subset p^{-1}((-\delta,\delta))$ for $b = 0, 1, 2$
- π_∞ is localized outside $p^{-1}((-3\delta/4, 3\delta/4))$
- π_1 is supported near K, in particular, its support does not intersect $V_{\rm pos}$
- π_2 is supported away from K but inside $\{|x| < R_2 + 1\}$
- π_0 is supported near spatial infinity, that is on $\{|x| > R_2 - 1\}$ where the operator $\widetilde{P}_{\theta,\epsilon}(h)$ is *absorbing* (the imaginary part of its symbol is negative).

Employing a positive (Wick) quantization scheme (see for instance (16), and for the semiclassical setting (23, §3.3)), $\Pi_b = \mathrm{Op}_h^+(\pi_b)$, we produce a quantum partition of unity

$$\mathrm{Id} = \sum_{b=0,1,2,\infty} \Pi_b\,, \qquad \|\Pi_b\| \leq 1\,.$$

The evolution $U(t)$ is then split between time intervals of length t_0, where $t_0 > 0$ is large but independent of h. Using the partition of unity, we decompose the propagator at time $t = Nt_0$ into

$$U(Nt_0)\,\psi^w(x,hD) = \big(\sum_{b=0,1,2,\infty} U_b\big)^N \psi^w(x,hD), \quad \text{where} \quad U_b \stackrel{\rm def}{=} U(t_0)\,\Pi_b\,.$$

Expanding the power, we obtain a sum of terms $U_{b_N}\cdots U_{b_1}\psi^w$; to understand each such term semiclassically, we investigate whether there exist true classical trajectories following that "symbolic history", namely sitting in $\operatorname{supp}\pi_{b_1}$ at time 0, in $\operatorname{supp}\pi_{b_2}$ at time t_0, etc. up to time Nt_0.

Since the energy cutoffs ψ and π_∞ have disjoint support, no classical trajectory can spend time in both supports. As a result, any sequence containing at least one index $b_i = \infty$ is *irrelevant* (meaning that the corresponding term is $\mathcal{O}_{L^2\to L^2}(h^\infty)$) (21, Lemma 6.5).

Since any classical trajectory can travel in $\operatorname{supp}\pi_2$ at most for a finite time $\leq N_0 t_0$ before escaping, (21, Lemma 6.6) shows that the *relevant* sequences $b_1\cdots b_N$ are of the form

$$b_i = 1 \quad \text{for} \quad N_0 < i < N - N_0\,.$$

They correspond to trajectories spending most of the time near K. One then has

$$U(Nt_0)\,\psi^w(x,hD) = U(N_0t_0)\,(U_1)^{N-2N_0}\,U(N_0t_0)\,\psi^w(x,hD) + \mathcal{O}_{L^2\to L^2}(h^{M_5})\,,$$

uniformly for any $2N_0 \leq N < M\log(1/h)$, where $M_5 > 0$ is large if the previous M, M_i are.

Finally, using the fact that the weight G vanishes on $\operatorname{supp}\pi_1$, (21, Lemma 6.3) shows that

$$U_1 = U(t_0)\Pi_1 = U_0(t_0)\Pi_1 + \mathcal{O}_{L^2\to L^2}(h^\infty)\,,$$

where $U_0(t_0) = \exp(-it_0P(h)/h)$ is *unitary.* Hence, $\|U_1\| \leq 1 + \mathcal{O}(h^\infty)$, while $\|U(N_0t_0)\|$ is estimated using (2.11). ☐

3 Resolvent estimates

We can now prove the resolvent estimate (1.5) by constructing a parametrix for $P_{\theta,\epsilon}(h) - z$, $z \in \Omega_\epsilon(h)$ defined in the statement of the theorem. We will use the notation

$$\zeta \stackrel{\rm def}{=} z/h$$

to shorten some of the formulæ. We want to find an approximate solution to

$$(P_{\theta,\epsilon}(h) - z)u = f\,, \quad f \in L^2(\mathbb{R}^n)\,, \quad z \in \Omega_\epsilon(h)\,.$$

First, the ellipticity away from the energy surface $p^{-1}(0)$ shows that, for ψ as in (2.10), there exists an operator, $T_0 = \mathcal{O}(1) : L^2(\mathbb{R}^n) \to H_h^2(\mathbb{R}^n)$, such that

$$(P_{\theta,\epsilon}(h) - z)T_0 f = (1 - \psi^w(x,hD))f + R_0 f\,, \qquad R_0 = \mathcal{O}_{L^2\to L^2}(h^\infty)\,.$$

To treat the vicinity of $p^{-1}(0)$ we put

$$T_1 f = (i/h) \int_0^{t_M} dt\, e^{i\zeta t}\, U(t)\, \psi^w(x, hD) f\,, \quad t_M = M \log(1/h)\,,$$

which satisfies

$$(\widetilde{P}_{\theta,\epsilon}(h) - z)\, T_1 f = \psi^w(x, hD) f + R_1 f\,, \qquad R_1 \stackrel{\text{def}}{=} -e^{i\zeta t_M}\, U(t_M)\, \psi^w(x, hD)\,. \tag{3.1}$$

The estimate (2.13) shows that, if $\lambda + \operatorname{Im}\zeta > \epsilon/2$, and for arbitrary $M_4 > 0$, one can choose M and M_3 large enough such that $R_1 = \mathcal{O}_{L^2\to L^2}(h^{M_4})$. We can estimate the norm of T_1 by the triangle inequality,

$$\|T_1\|_{L^2\to L^2} \leq h^{-1} \int_0^{t_M} e^{-\operatorname{Im}\zeta t}\, \|U(t)\, \psi^w(x, hD)\|_{L^2\to L^2}\, dt\,, \tag{3.2}$$

and then use the bounds (2.12) for times $0 \leq t \leq t_E$ and (2.13) for times $t_E < t \leq t_M$.

When $\operatorname{Im}\zeta = 0$, the above integral can be estimated by the integral over the interval $t \in [0, t_E]$:

$$\operatorname{Im}\zeta = 0 \Longrightarrow \|T_1\|_{L^2\to L^2} \leq h^{-1}\Big(C_0\, t_E + \frac{1}{\lambda}\Big) \leq C\, h^{-1}\, \log h^{-1}\,.$$

In the case $0 > \operatorname{Im}\zeta > -\lambda + \epsilon/2$, the dominant part of the integral comes from $t = t_E$:

$$0 > \operatorname{Im}\zeta > -\lambda + \epsilon/2 \Longrightarrow \|T_1\|_{L^2\to L^2} \leq C_\epsilon h^{-1}\, e^{-\operatorname{Im}\zeta t_E} = C_\epsilon\, h^{-1+c_E \operatorname{Im}\zeta}\,.$$

We rewrite (3.1) as

$$\psi_0^w(x, hD)(P_{\theta,\epsilon}(h) - z)\psi_0^w(x, hD) T_1 f = \psi^w(x, hD) f + R_1 f\,.$$

From the inclusion $\psi_0|_{\operatorname{supp}\psi} \equiv 1$, one can show (as in (21, Lemma 6.5)) that

$$\psi_0^w(x, hD)(P_{\theta,\epsilon}(h) - z)\psi_0^w(x, hD) T_1 = (P_{\theta,\epsilon}(h) - z)\, T_1 + R_2\,, \qquad R_2 = \mathcal{O}_{L^2\to L^2}(h^\infty)\,,$$

and also that

$$\|T_1\|_{H_h^2} \leq C\, \|T_1\|_{L^2}\,.$$

Putting $T = T_0 + T_1$ and $R = R_0 + R_1 + R_2$, we obtain

$$(P_{\theta,\epsilon}(h) - z) T = \operatorname{Id} + R\,, \qquad R = \mathcal{O}_{L^2\to L^2}(h^{M_4})\,.$$

This means that $(P_{\theta,\epsilon}(h) - z)$ can be inverted, with

$$\|(P_{\theta,\epsilon}(h) - z)^{-1}\|_{L^2\to H_h^2} = (1 + \mathcal{O}(h^{M_4}))\|T\|_{L^2\to H_h^2}\,.$$

The above estimates on the norms of T_0 and T_1 can be summarized by

$$0 \geq \operatorname{Im}\zeta \geq \epsilon + \mathcal{P}(-\varphi_+/2) \Longrightarrow \|T\|_{L^2\to H_h^2} \leq C_\epsilon\, h^{-1+c_E \operatorname{Im}\zeta}\, \log h^{-1}\,. \tag{3.3}$$

Using (2.7), this proves the bound (1.5). □

Remark. By using a sharper energy cutoff ψ_h belonging to an exotic symbol class (see (27, §4)) and supported in the energy layer $p^{-1}((-h^{1-\delta}, h^{1-\delta}))$ (as in (1)), the bound (2.13) is likely to be improved to

$$\|U(t)\psi_h^w(x, hD)\|_{L^2\to L^2} \leq C h^{-(n-1+\delta)/2} \exp(-\lambda t) + \mathcal{O}(h^{M_3})\,. \tag{3.4}$$

This bound becomes sharper than (2.12) around the time $t'_E = c'_E \log(1/h)$, where

$$c'_E \stackrel{\text{def}}{=} \frac{n - 1 + \delta}{2\lambda} < c_E.$$

As a result, the bounds on the norm of the corresponding operator T_1' are modified accordingly. At the same time, as shown in (1, Prop. 5.4), the ellipticity away from the energy surface provides an operator T_0' satisfying

$$(P_{\theta,\epsilon}(h) - z)T_0' = (1 - \psi_h^w(x, hD)) + \mathcal{O}_{L^2\to L^2}(h^\infty)\,,$$

and of norm $\|T_0'\|_{L^2\to L^2} = \mathcal{O}(h^{-1+\delta})$. The norm of $T' = T_0' + T_1'$ is still dominated by that of T_1', so that we eventually get

$$\|\chi(P(h) - z)^{-1}\chi\|_{L^2\to H_h^2} \leq C_\epsilon\, h^{-1+c_E' \operatorname{Im} z/h} \log(1/h)\,, \quad z \in \Omega_\epsilon(h) \cap \{\operatorname{Im} z \leq 0\}.$$

Since it is not clear that even this bound is optimal, and that proving (3.4) would require some effort, we have limited ourselves to using the established bound (2.13).

One advantage of the approach presented in this note (compared with the method of (21, §9)) is that, to obtain the bound (1.6) we did not have to use the complex interpolation arguments of (4) and (28).

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References

[1] N. Anantharaman and S. Nonnenmacher, *Half-delocalization of eigenfunctions of the laplacian*, Ann. Inst. Fourier. **57**, 2465–2523 (2007)

[2] N. Burq, *Contrôle de l'équation des plaques en présence d'obstacle stictement convexes.* Mémoires de la Société Mathématique de France, Sér. 2, **55** (1993), 3-126

[3] N. Burq, *Lower bounds for shape resonances widths of long range Schrödinger operators*, Amer. J. Math. **124** (2002), 677–735

[4] N. Burq, *Smoothing effect for Schrödinger boundary value problems,* Duke Math. J. **123** (2004), 403–427

[5] N. Burq, C. Guillarmou, and A. Hassel, *Strichartz estimates without loss on manifolds with hyperbolic trapped geodesics*, preprint 2009, arXiv:0907.3545

[6] , F. Cardoso and G. Vodev, *High frequency resolvent estimates and energy decay of solutions to the wave equation*, Canadian Math. Bull. **47** (2004), 504-514

[7] H. Christianson, *Cutoff resolvent estimates and the semilinear Schrodinger equation*, Proc. Amer. Math. Soc. **136**(2008), 3513–3520.

[8] H. Christianson, *Applications of cutoff Resolvent estimates to the wave equation,* Math. Res. Lett. **16** (2009), 577–590

[9] K. Datchev, *Local smoothing for scattering manifolds with hyperbolic trapped sets,* Comm. Math. Phys. **286**(2009), 837-850

[10] M. Dimassi and J. Sjöstrand, *Spectral Asymptotics in the semi-classical limit,* Cambridge University Press, 1999.

[11] L.C. Evans and M. Zworski, *Lectures on Semiclassical Analysis,* `http://math.berkeley.edu/~zworski/semiclassical.pdf`

[12] C. Guillarmou and F. Naud, *Wave decay on convex co-compact hyperbolic manifolds*, Commun. Math. Phys. **287** (2009), 489–511

[13] L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Vol. I, II, Springer-Verlag, Berlin, 1983.

[14] M. Ikawa, *Decay of solutions of the wave equation in the exterior of several convex bodies,* Ann. Inst. Fourier, **38**(1988), 113-146

[15] A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems,* Cambridge University Press, 1997.

[16] N. Lerner, *The Wick calculus of pseudodifferential operators and some of its applications,* Cubo Mat. Edu. **5**(2003), 213–236

[17] A. Martinez, *Resonance Free Domains for Non Globally Analytic Potentials*, Ann. Henri Poincaré **4** (2002) 739–756

[18] S. Nakamura, P. Stefanov and M. Zworski, *Resonance expansions of propagators in the presence of potential barriers*, J. Funct. Anal. **205** (2003), 180–205

[19] S. Nonnenmacher, J. Sjöstrand, and M. Zworski, *Hyperbolic quantum monodromy operators,* in preparation.

[20] S. Nonnenmacher and M. Zworski, *Distribution of resonances for open quantum maps,* Comm. Math. Phys. **269** (2007), 311–365; ibid, *Fractal Weyl laws in discrete models of chaotic scattering,* Journal of Physics A, **38** (2005), 10683-10702

[21] S. Nonnenmacher and M. Zworski, *Quantum decay rates in chaotic scattering,* Acta Math., to appear.

[22] V. Petkov and L. Stoyanov, *Analytic continuation of the resolvent of the Laplacian and the dynamical zeta function,* preprint 2007,
`http://www.math.u-bordeaux.fr/~petkov/publications/publi2.html`

[23] K. Pravda-Starov, *Pseudo-spectrum for a class of semi-classical operators,* Bulletin de la Société Mathématique de France, **136** (2008), 329–372.

[24] E. Schenck, *Energy decay for the damped wave equation under a pressure condition*, preprint 2009.

[25] J. Sjöstrand, *A trace formula and review of some estimates for resonances*, in *Microlocal analysis and spectral theory* (Lucca, 1996), 377–437, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 490, Kluwer Acad. Publ., Dordrecht, 1997.

[26] J. Sjöstrand and M. Zworski, *Complex scaling and the distribution of scattering poles*, Journal of AMS **4**(1991), 729–769

[27] J. Sjöstrand and M. Zworski, *Asymptotic distribution of resonances for convex obstacles,* Acta Math. **183**(1999), 191-253.

[28] S.H. Tang and M. Zworski, *From quasimodes to resonances,* Math. Res. Lett. **5**(1998), 261-272.

[29] P. Walters, *An Introduction to Ergodic Theory*, Springer, Heidelberg, 1982.