Computing Critical Pairs in 2-Dimensional Rewriting Systems

Samuel Mimram

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April 17, 2010

Abstract

Rewriting systems on words are very useful in the study of monoids. In good cases, they give finite presentations of the monoids, allowing their manipulation by a computer. Even better, when the presentation is confluent and terminating, they provide one with a notion of canonical representative for the elements of the presented monoid. Polygraphs are a higher-dimensional generalization of this notion of presentation, from the setting of monoids to the much more general setting of n-categories. Here, we are interested in proving confluence for polygraphs presenting 2-categories, which can be seen as a generalization of term rewriting systems. For this purpose, we propose an adaptation of the usual algorithm for computing critical pairs. Interestingly, this framework is much richer than term rewriting systems and requires the elaboration of a new theoretical framework for representing critical pairs, based on contexts in compact 2-categories.[1]

Term rewriting systems have proven very useful to reason about terms modulo equations. In some cases, the equations can be oriented and completed in a way giving rise to a converging (i.e. confluent and terminating) rewriting system, thus providing a notion of canonical representative of equivalence classes of terms. Usually, terms are freely generated by a *signature* $(\Sigma_n)_{n\in\mathbb{N}}$, which consists of a family of sets Σ_n of generators of arity n, and one considers *equational theories* on such a signature, which are formalized by sets of pairs of terms called *equations*. For example, the equational theory of monoids contains two generators m and e, whose arities are respectively 2 and 0, and three equations

$$m(m(x,y),z) = m(x,m(y,z)) \qquad m(e,x) = x \quad \text{and} \quad m(x,e) = x$$

These equations, when oriented from left to right, form a rewriting system which is converging. The termination of this system can be shown by giving an interpretation of the terms in a well-founded poset, such that the rewriting rules are strictly decreasing. Since the system is terminating, the confluence can be deduced from the local confluence, which can itself be shown by verifying that the five critical pairs

$$m(m(m(x,y),z),t) \qquad m(m(e,x),y) \qquad m(m(x,e),y) \qquad m(m(x,y),e) \qquad m(e,e)$$

are joinable and these critical pairs can be computed using a unification algorithm. A more detailed presentation of term rewriting systems along with the classic techniques to prove their convergence can be found in [1].

As a particular case, when the generators of an equational theory are of arity one, the category of terms modulo the congruence generated by the equations is a monoid, with addition given by composition and neutral element being the identity. A *presentation* of a monoid $(M,\times,1)$ is such an equational theory, which is generating a monoid isomorphic to M. For example the

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[1]This work was started while I was in the PPS team (CNRS – Univ. Paris Diderot) and has been supported by the CHOCO ("Curry Howard pour la Concurrence", ANR-07-BLAN-0324) French ANR project.

monoid $\mathbb{N}/2\mathbb{N}$ is presented by the equational theory with only one generator a, of arity one, and the equation $a(a(x)) = x$. Presentations of monoids are particularly useful since they can provide finite description of monoids which may be infinite, thus allowing their manipulation with a computer. More generally, with generators of any arity, equational theories give rise to presentations of Lawvere theories [9], which are cartesian categories whose objects are the natural integers and such that product is given on objects by addition: a signature namely generates such a category, whose morphisms $f : m \to n$ are n-uples of terms with m free variables, composition being given by substitution.

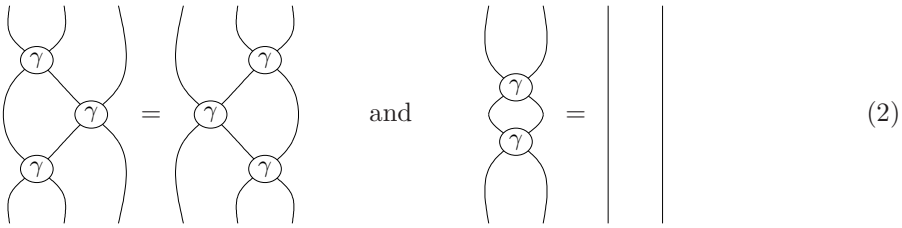
Term rewriting systems have been generalized by *polygraphs*, in order to provide a formal framework in which one can give presentations of any (strict) n-category. We are interested here in adapting the classical technique to study confluence of 3-polygraphs, which give rise to presentations of 2-categories, by computing their critical pairs. These polygraphs can be seen as term rewriting systems improved on the following points:

- the variables of terms are simply typed (this can be thought as generalizing from a Lawvere theory of terms to any cartesian category of terms),
- variables in terms cannot necessarily be duplicated, erased or swapped (the categories of terms are not necessarily cartesian but only monoidal),
- and the terms can have multiple outputs as well as multiple inputs.

Many examples of presentations of monoidal categories where studied by Lafont [8], Guiraud [4, 3] and the author [12, 14]. A fundamental example is the 3-polygraph S, presenting the monoidal category **Bij** (the category of finite ordinals and bijections). This polygraph has one generator for objects 1, one generator for morphisms $\gamma : 2 \to 2$ (where 2 is a notation for $1 \otimes 1$) and two equations

$$(\gamma \otimes 1) \circ (1 \otimes \gamma) \circ (\gamma \otimes 1) = (1 \otimes \gamma) \circ (\gamma \otimes 1) \circ (1 \otimes \gamma) \qquad \text{and} \qquad \gamma \circ \gamma = 1 \otimes 1 \tag{1}$$

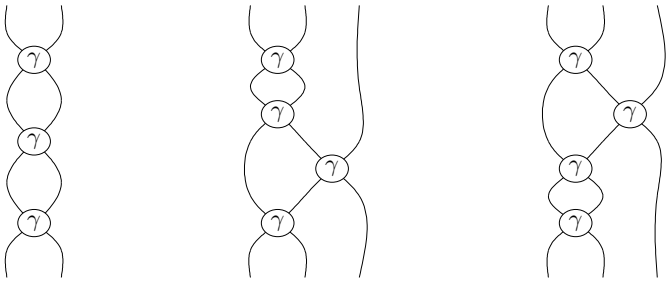
where the morphism 1 is a short notation for id_1. That this polygraph is a presentation of the category **Bij** means that this category is isomorphic to the free monoidal category containing an object 1 and a generator γ, quotiented by the smallest congruence generated by the equations (1). This result can be seen as a generalization of the presentation of the symmetric groups by transpositions. These equations can be better understood with the graphical notation provided by *string diagrams*, which is a diagrammatic notation for morphisms in monoidal categories, introduced formally in [6]. The morphism γ should be thought as a device with two inputs and two outputs of type 1, and the two equations (1) can thus be represented graphically by



(2)

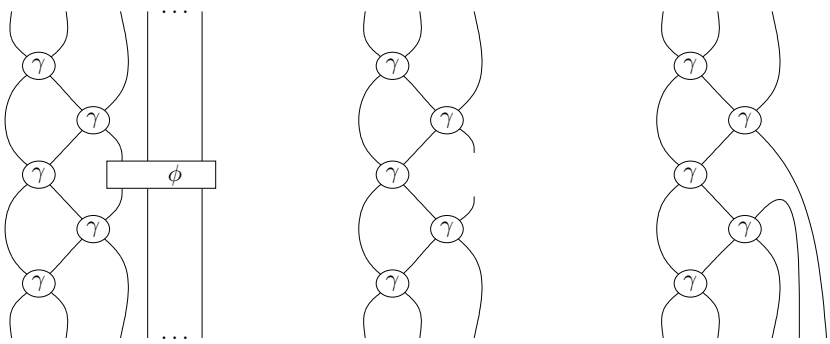
In this notation, wires represent identities (on the object 1), horizontal juxtaposition of diagrams corresponds to tensoring, and vertical linking of diagrams corresponds to composition of morphisms. Moreover, these diagrams should be considered modulo planar continuous deformations, so that the axioms of monoidal categories are verified. These diagrams are conceptually important because they allow us to see morphisms in monoidal categories either as algebraic objects or as geometric objects (some sort of planar graphs). If we orient both equations from left to right, we get a rewriting system which can be shown to be convergent. It has the three following critical

pairs [8]:



(3)

Moreover, for every morphism $\phi : 1 \otimes m \to 1 \otimes n$, the morphism on the left of (4)



(4)

can be rewritten in two different ways, thus giving rise to an infinite number of critical pairs for the rewriting system. This phenomenon was first observed by Lafont [8] and later on studied by Guiraud and Malbos [5]. Interestingly, we can nevertheless consider that there is a finite number of critical pairs if we allow ourselves to consider the "diagram" on the center of (4) as a critical pair. Of course, this diagram does not make sense at first. However, we can give a precise meaning to it if we embed our terms in a larger category, which is compact: in such a category every object has a dual, which corresponds graphically to having the ability to bend wires (see the figure on the right). This observation was the starting point of this paper which is devoted to formalizing these intuitions in order to propose an algorithm for computing critical pairs in polygraphs.

We believe that this is a major area of higher-dimensional algebra where computer scientists should step in: typical presentations of categories can give rise to a very large number of critical pairs and having automated tools to compute them seems to be necessary in order to push further the study of those systems. The present paper constitutes a first step in this direction, by defining the structures necessary to manipulate algorithmically the morphisms in categories generated by polygraphs and by proposing an algorithm to compute the critical pairs in polygraphic rewriting systems. Conversely, algebra provides strong indications about technical choices that should be made in order to generalize rewriting theory in higher dimensions. We have done our possible to provide an overview of the theoretical tools used here, as well as intuitions about them. A preliminary detailed version of this work is available in [13].

We begin by recalling the definition of polygraphs, describe the categories they generate, and formulate the unification problem in this framework using the notion of context in a 2-category. Then, we show that 2-categories can be fully and faithfully embedded into the free compact 2-category they generate, which allows us to describe a unification algorithm for polygraphic rewriting systems.

1 Presentations of 2-categories

Because of space limitations, we have to omit the basic definitions in category theory and refer the reader to MacLane's reference book [11]. We only recall that a *2-category* is a generalization in

dimension 2 of the concept of category. It consists essentially of a class of 0-*cells* A, a class of 1-*cells* $f : A \to B$ (with 0-cells A and B as source and target) and a class of 2-*cells* $\alpha : f \Rightarrow g : A \to B$ (with parallel 1-cells $f : A \to B$ and $g : A \to B$ as source and target), together with a *vertical composition*, which to every pair of 2-cells $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$ associates a 2-cell $\beta \circ \alpha : f \Rightarrow h$, and a *horizontal composition*, which to every pair of 2-cells $\alpha : f \Rightarrow g$ and $\beta : h \Rightarrow i$ associates a 2-cell $\alpha \otimes \beta : (f \otimes h) \Rightarrow (g \otimes i)$, such that vertical and horizontal composition are associative, admit neutral elements (the identities) and the *exchange law* is satisfied: for every four 2-cells

$$\alpha : f \Rightarrow f' : A \to B, \quad \alpha' : f' \Rightarrow f'' : A \to B, \quad \beta : g \Rightarrow g' : B \to C, \quad \beta' : g' \Rightarrow g'' : B \to C$$

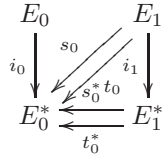
the following equality holds

$$(\alpha' \circ \alpha) \otimes (\beta' \circ \beta) \quad = \quad (\alpha' \otimes \beta') \circ (\alpha \otimes \beta) \tag{5}$$

as well as a nullary version of this law: $\mathrm{id}_{A \otimes B} = \mathrm{id}_A \otimes \mathrm{id}_B$ for every objects A and B. In a 2-category, two n-cells are *parallel* when they have the same source and the same target. We also recall that two 0-cells A and B of a 2-category $\mathcal{C}$, induce a category $\mathcal{C}(A, B)$, called *hom-category*, whose objects are the 1-cells $f : A \to B$ of $\mathcal{C}$ and whose morphisms $\alpha : f \Rightarrow g$ are 2-cells of $\mathcal{C}$, composition being given by vertical composition. A (strict) *monoidal category* is a 2-category with exactly one 0-cell.

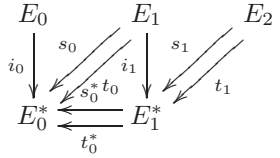
Polygraphs are algebraic structures which were introduced in their 2-dimensional version by Street [16] under the name *computads*, generalized to higher dimensions by Power [15], and independently rediscovered by Burroni [2]. We are specifically interested in 3-polygraphs, which give rise to presentations of 2-categories, and briefly recall their definition here. This definition is a bit technical but conceptually clear: it consists of sets of 0-, 1-, 2-generators for "terms", each 2-generator having a list of 1-generators as source and as target, each 1-generator having itself a 0-generator as source and as target, together with a set of equations which are pairs of terms (generated by the 2-generators).

Suppose that we are given a set E_0 of *0-generators*, such a set will be called a *0-polygraph*. We write $E_0^* = E_0$ and $i_0 : E_0 \to E_0^*$ the identity function. A 1-polygraph on these generators is a graph, that is a diagram $E_0^* \underset{t_0}{\overset{s_0}{\leftleftarrows}} E_1$ in **Set**, with E_0^* as vertices, the elements of E_1 being called *1-generators*. We can construct a free category on this graph: its set E_1^* of morphisms is the set of paths in the graph (identities are the empty paths), the source $s_0^*(f)$ (resp. target $t_0^*(f)$) of a morphism $f \in E_1^*$ being the source (resp. target) of the path. If we write $i_1 : E_1 \to E_1^*$ for the injection of the 1-generators into morphisms of this category, which to every 1-generator associates the corresponding path of length one, we thus get a diagram



(6)

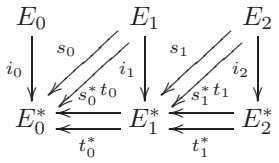
in **Set**, which is commutative in the sense that $s_0^* \circ i_1 = s_0$ and $t_0^* \circ i_1 = t_0$. A *2-polygraph* on this 1-polygraph consists of a diagram



(7)

in **Set**, such that $s_0^* \circ s_1 = s_0^* \circ t_1$ and $t_0^* \circ s_1 = t_0^* \circ t_1$. The elements of E_2 are called *2-generators*. Again we can generate a free 2-category on this data, whose underlying category is the category

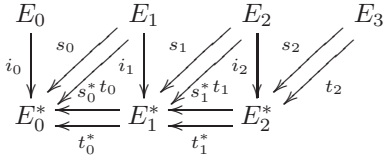
generated in (6) and which has the 2-generators as morphisms. If we write E_2^* for its set of morphisms and $i_2 : E_2 \to E_2^*$ for the injection of the 2-generators into morphisms, we thus get a diagram



(8)

We can now formulate the definition of 3-polygraphs as follows.

Definition 1 A *3-polygraph* consists of a diagram

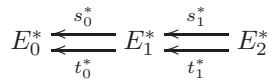


(9)

(where E_i^*, s_i^* and t_i^* are freely generated as previously explained), such that

$$s_i^* \circ s_{i+1} = s_i^* \circ t_{i+1} \quad \text{and} \quad t_i^* \circ s_{i+1} = t_i^* \circ t_{i+1}$$

for $i = 0$ and $i = 1$, together with a structure of 2-category on the 2-graph



Again, a 3-polygraph freely generates a 3-category $\mathcal{C}$ whose underlying 2-category is the underlying 2-category of the polygraph and whose 3-cells are generated by the 3-generators of the polygraph. A quotient 2-category $\tilde{\mathcal{C}}$ can be constructed from this 2-category: it is defined as the underlying 2-category of $\mathcal{C}$ quotiented by the congruence identifying two 2-cells whenever there exists a 3-cell between them in $\mathcal{C}$. A 3-polygraph P *presents* a 2-category $\mathcal{D}$ when $\mathcal{D}$ is isomorphic to the 2-category $\tilde{\mathcal{C}}$ induced by the polygraph P. In this sense, the underlying 2-polygraph of a 3-polygraph is a *signature* generating terms which are to be considered modulo the *equations* described by the 3-generators; these equations $r \in E_3$ being oriented, they will be called *rewriting rules*, the source $s_2(r)$ (resp. the target $t_2(r)$) being the *left member* (resp. *right member*) of the rule. A polygraph is *finite* when all the sets E_i are; in the following, we only consider such polygraphs.

A *morphism of polygraphs* $F = (F_0, F_1, F_2, F_3)$ between two 3-polygraphs P and Q consists of functions $F_i : E_i^P \to E_i^Q$, such that the obvious diagrams commute (for example, for every i, $s_i^Q \circ F_{i+1} = F_i^* \circ s_i^P$, where $F_i^* : E_i^{P^*} \to E_i^{Q^*}$ is the monoid morphism induced by F_i). We write n-**Pol** for the category of n-polygraphs (this construction can be carried on to any dimension $n \in \mathbb{N}$ but we will only consider cases with $n \leqslant 3$). These categories have many nice properties, amongst which being cocomplete. The free n-category generated by an n-polygraph P is denoted $\mathcal{C}_n(P)$. Given an integer $k \leqslant n$, we write $U_k : n\text{-}\mathbf{Pol} \to k\text{-}\mathbf{Pol}$ for the forgetful functor which simply forgets about the sets of generators of dimension higher than k. This functor admits a left adjoint $F_n : k\text{-}\mathbf{Pol} \to n\text{-}\mathbf{Pol}$ which adds empty sets of generators of dimension higher than k. We sometimes leave implicit the inclusion of k-**Pol** into n-**Pol** induced by F_n.

Example 1 The theory of *symmetries* mentioned in the introduction is the polygraph S whose generators are

$$E_0 = \{*\} \qquad E_1 = \{1 : * \to *\} \qquad E_2 = \{\gamma : 1 \otimes 1 \Rightarrow 1 \otimes 1\}$$
$$E_3 = \{y : (\gamma \otimes 1) \circ (1 \otimes \gamma) \circ (\gamma \otimes 1) \Rrightarrow (1 \otimes \gamma) \circ (\gamma \otimes 1) \circ (1 \otimes \gamma),\ s : \gamma \circ \gamma \Rrightarrow 1 \otimes 1\}$$

Example 2 The theory of *monoids* is the polygraph M defined by

$$E_0 = \{*\} \qquad E_1 = \{1 : * \to *\} \qquad E_2 = \{\mu : 1 \otimes 1 \Rightarrow 1,\ \eta : * \Rightarrow 1\}$$
$$E_3 = \{a : \mu \circ (\mu \otimes 1) \Rrightarrow \mu \circ (1 \otimes \mu),\ l : \mu \circ (\eta \otimes 1) \Rrightarrow 1,\ r : (1 \otimes \eta) \to 1\}$$

This polygraph presents the augmented simplicial category (the category of finite ordinals and non-decreasing functions).

2 Formal representation of free 2-categories

The definition of 3-polygraphs involves the construction of free categories and free 2-categories, which are abstractly defined in category theory by universal constructions. Here, we need a more concrete representation of these mathematical objects. As already mentioned, the free category (6) on a graph is easy to describe: its objects are the vertices of the graph and morphisms are paths of the graph with composition given by concatenation. However, describing the free 2-category on a 2-polygraph in an effective way (which can be implemented) is much less straightforward. Of course, following the definition given in Section 1, one could describe the 2-cells of this 2-category as formal vertical and horizontal compositions of 2-generators up to a congruence imposing associativity and absorption of units for both compositions and the exchange law (5). However, given an object A in a 2-category $\mathcal{C}$ and two 2-cells $\alpha, \beta : \mathrm{id}_A \Rightarrow \mathrm{id}_A : A \to A$ of this category, the equality $\alpha \otimes \beta = \beta \otimes \alpha$ can be deduced from the following sequence of equalities:

$$\alpha \otimes \beta = (\mathrm{id}_A \circ \alpha) \otimes (\beta \circ \mathrm{id}_A) = (\mathrm{id}_A \otimes \beta) \circ (\alpha \otimes \mathrm{id}_A) = (\beta \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \alpha) = (\beta \circ \mathrm{id}_A) \otimes (\mathrm{id}_A \circ \alpha) = \beta \otimes \alpha$$

It requires inserting and removing identities, and using the exchange law in both directions. So, it seems to be very hard to find a generic way to handle formal composites of generators modulo the congruence described above. We will therefore define an alternative construction of these morphisms which doesn't require such a quotienting.

Consider the morphism $\gamma \circ \gamma : (1 \otimes 1) \Rightarrow (1 \otimes 1) : * \to *$ in the theory S of symmetries (Example 1), depicted on the left of (10):

(10)

Graphically, in this morphism, the two 2-cells are γ, wires are typed by the 1-cell 1 and regions of the plane are typed by the 0-cell $*$. Now, if we give a different name to each *instance* of a generator used in this morphism, for example by numbering them as in the right of (10), the morphism itself can be described as the 2-polygraph P defined by

$$E_0 = \{*_0, \ldots, *_4\} \qquad E_1 = \{1_0 : *_1 \to *_0,\ 1_1 : *_0 \to *_2, \ldots,\ 1_5 : *_4 \to *_2\}$$

and

$$E_2 = \{\gamma_0 : 1_0 \otimes 1_1 \Rightarrow 1_2 \otimes 1_3,\ \gamma_1 : 1_2 \otimes 1_3 \Rightarrow 1_4 \otimes 1_5\}$$

together with a function ℓ which to every i-generator of this polygraph associates a label, which is an i-generator of S, so that $\ell : P \to S$ is a morphism of polygraphs (ℓ is defined by $\ell(*_i) = *$, $\ell(1_i) = 1$ and $\ell(\gamma_i) = \gamma$). Formulated in categorical terms, (P, ℓ) is an object in the slice category 2-$\mathbf{Pol}{\downarrow}U_2(S)$. Of course, the naming of the instances of the generators occurring in nets is arbitrary, so we have to consider these labeled polygraphs up to bijections, which correspond to injective renaming of instances. Notice that not every such labeled polygraph is the representation of a morphism: we need an inductive construction of those (it seems to be difficult to give a direct characterization of the suitable polygraphs).

Based on these ideas, we describe the category generated by a polygraph S as a category whose cells are polygraphs labeled by S. We suppose fixed a signature 2-polygraph S and write S_i for $U_i(S)$. This is a generalization of the constructions of labeled transition systems, and is reminiscent of pasting schemes [15] and of proof-nets, which is why we call them *polygraphic nets* (or *nets* for short).

The category of 0-*nets* 0-$\mathbf{Net}_{S_0}$ on the 0-polygraph S_0 is the full subcategory of 0-$\mathbf{Pol}{\downarrow}S_0$ whose objects are 0-polygraphs with exactly one 0-cell, labeled by S_0. Concretely, its objects are pairs (n, A), often written A_n, where n is the *name* of the instance (an integer for example) and A an element of $E_0^{S_0}$, called its *label*, and there is a morphism between two objects whenever they have the same label (all those morphisms are invertible). The category of 1-*nets* 1-$\mathbf{Net}_{S_1}$ is the smallest category whose objects are the 0-nets A_i, whose morphisms $(s^f, f, t^f) : A_i \to B_j$ are triples consisting of a 1-polygraph f labeled by S_1 (i.e. an object in 1-$\mathbf{Pol} \downarrow S_1$) and two morphisms of labeled polygraphs $s^f : A_i \to f$ and $t^f : B_j \to f$, called *source* and *target*, which are either a 1-polygraph f such that $E_0^f = \{A_i, B_j\}$ and E_1^f contains only one 1-cell $n \in \mathbb{N}$ with A_i as source and B_j as target (and the obvious injections for s^f and t^f), or $A_i = B_j$, $f = A_i$ and $s^f = t^f = \mathrm{id}_{A_i}$ (this is the identity on A_i), or a composite $f \otimes g : A_i \to B_j$ of two morphisms $f : A_i \to C_k$ and $g : C_k \to B_j$. Here, the composite of two such morphisms is defined as the pushout of the diagram $f \xleftarrow{t^f} C_k \xrightarrow{s^g} g$, that is the disjoint union of the polygraphs f and g quotiented by a relation identifying the 0-cell in C_k in the two components of the union.

Example 3 If S is the polygraph of symmetries, the composite of the two morphisms $f : *_0 \to *_1$ and $g : *_1 \to *_2$ defined by

$$E_0^f = \{*_0,\ *_1\}\quad E_1^f = \{1_0 : *_0 \to *_1\}\quad E_0^g = \{*_0,\ *_1,\ *_2\}\quad E_1^g = \{1_1 : *_1 \to *_0,\ 1_0 : *_0 \to *_2\}$$

is the morphism $h = f \otimes g$ such that

$$E_0^h = \{*_0, \ldots, *_3\} \quad \text{and} \quad E_1^h = \{1_0 : *_0 \to *_1,\ 1_1 : *_1 \to *_3,\ 1_2 : *_3 \to *_2\}$$

Graphically,

$$*_0 \xrightarrow{1_0} *_1 \quad \otimes \quad *_1 \xrightarrow{1_1} *_0 \xrightarrow{1_0} *_2 \quad = \quad *_0 \xrightarrow{1_0} *_1 \xrightarrow{1_1} *_3 \xrightarrow{1_2} *_2$$

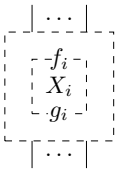
Since composition is defined by a pushout construction, it involves a renaming of some instances (it is the case in the example above) and this renaming is arbitrary. So, composition is not strictly associative but only associative up to isomorphism of polygraphs. Therefore, what we have built is not precisely a category but only a bicategory: this is a well-known fact, this construction being a particular instance of the general construction of cospan bicategories. We can iterate this construction one step further and define the tricategory (that is a 2-category whose compositions are associative up to isomorphism) of 2-*nets* 2-$\mathbf{Net}_S$ as the smallest tricategory whose 0-cells are 0-nets A_i, whose 1-cells $f : A_i \to B_j$ contain 1-nets, and whose 2-cells $\alpha : f \Rightarrow g$ are triples $(s^\alpha, \alpha, t^\alpha)$, consisting of a 2-polygraph α labeled by S and two morphisms of labeled polygraphs $s^\alpha : f \to \alpha$ and $t^\alpha : g \to \alpha$, containing all the 2-polygraphs with one 2-generator $n \in \mathbb{N}$ whose source $f = s_1^\alpha(n)$ and target $g = t_1^\alpha(n)$ are 1-nets which are "disjoint" in the sense they only have their own source and target as common generators, with the obvious injections for s^α and t^α. Moreover, we requires this tricategory to contain identities and to be closed under both vertical and horizontal compositions, which are defined by pushout constructions in a way similar to 1-nets. If we quotient this tricategory and identify cells which are isomorphic labeled polygraphs, we get a proper 2-category, that we still write 2-$\mathbf{Net}_S$.

Proposition 2 *The* 2*-category* 2-$\mathbf{Net}_S$ *described above is equivalent to the free category generated by the* 2*-polygraph* S*.*

This construction has the advantage to be simple to implement and manipulate: we have for example given the data needed to describe the morphism (10).

3 Critical pairs in polygraphs

In order to formalize the notion of critical pair for a polygraph, we need to formalize first the notion of context of a morphism in the 2-category $\mathcal{C}_2(S)$ generated by a 2-polygraph S, which may be thought as a 2-cell with multiple typed "holes". These contexts have multiples "inputs" (one for each hole) and will therefore organize into a multicategory, which is a notion generalizing categories in the sense that morphisms $f : (A_1, \ldots, A_n) \to A$ have one output of type A, and *a list* of inputs of type A_i instead of only one input. Composition is also generalized in the sense that we compose such a morphism f with n morphisms f_i with A_i as target, what we write $f \circ (f_1, \ldots, f_n)$. Multicategories should moreover have identities $\mathrm{Id}_A : (A) \to A$ and satisfy coherence axioms [10].



Suppose that we are given a signature 2-polygraph S. Suppose moreover that we are given a list of n pairs of parallel 1-cells (f_i, g_i) in the category generated by the 1-polygraph $U_1(S)$. We write $S[X_1 : f_1 \Rightarrow g_1, \ldots, X_n : f_n \Rightarrow g_n]$, for the polygraph obtained from S by adding $X_1, \ldots, X_n$ as 2-generators, with f_i as the source and g_i as the target of X_i (we suppose that the X_i were not already present in the 2-generators of S). The X_i should be thought as typed variables for 2-cells and we can easily define a notion of *substitution* of a variable $X_i : f_i \Rightarrow g_i$ by a 2-cell $\alpha : f_i \Rightarrow g_i$ in a 2-cell of the 2-category generated by $S[X_1 : f_1 \Rightarrow g_1, \ldots, X_n : f_n \Rightarrow g_n]$.

Given a signature S, we build a multicategory $\mathcal{K}(S)$ whose objects are pairs (f, g) of parallel 1-cells in the 2-category generated by S and whose morphisms $K : ((f_1, g_1), \ldots, (f_n, g_n)) \to (f, g)$, called *contexts*, are the 2-cells $\alpha : f \Rightarrow g$ in the 2-category which is generated by the polygraph $S[X_1 : f_1 \Rightarrow g_1, \ldots, X_n : f_n \Rightarrow g_n]$, which are *linear* in the sense that each of the variables X_i appears exactly once in the morphism α. Composition in this multicategory is induced by the substitution operation. This multicategory can be canonically equipped with a structure of symmetric multicategory, which essentially means that, for every permutation σ on n elements, the sets of morphisms of type $((f_1, g_1), \ldots, (f_n, g_n)) \to (f, g)$ is isomorphic to the set of morphisms of type $((f_{\sigma(1)}, g_{\sigma(1)}), \ldots, (f_{\sigma(n)}, g_{\sigma(n)})) \to (f, g)$ in a coherent way. Any 2-cell $\alpha : f \Rightarrow g$ in the 2-category generated by S, can be seen as a nullary context of type $() \to (f, g)$ that we still write α. A concrete and implementable definition of the multicategory $\mathcal{K}(S)$ of contexts of S can be given by adapting the construction of polygraphic nets given in the previous section.

This construction enables us to reformulate usual notions of rewriting theory in our framework as follows. We suppose fixed a rewriting system given by a 3-polygraph R. We write $S = U_2(R)$ for the underlying signature of R and $\mathcal{C}$ for the 2-category it generates.

Definition 3 A *unifier* of two 2-cells

$$\alpha_1 : f_1 \Rightarrow g_1 \quad \text{and} \quad \alpha_2 : f_2 \Rightarrow g_2$$

in $\mathcal{C}$ is a pair of cofinal unary contexts

$$K_1 : ((f_1, g_1)) \to (f, g) \quad \text{and} \quad K_2 : ((f_2, g_2)) \to (f, g)$$

such that $K_1 \circ (\alpha_1) = K_2 \circ (\alpha_2)$. A unifier is a *most general unifier* when it is

- *non-trivial*: there exists no binary context $K : ((f_1, g_1), (f_2, g_2)) \to (f, g)$ which satisfies $K_1 = K \circ (\mathrm{Id}_{(f_1,g_1)}, \alpha_2)$ and $K_2 = K \circ (\alpha_1, \mathrm{Id}_{(f_2,g_2)})$. Informally, the morphisms α_1 and α_2 should not appear in disjoint positions in the morphism $K_1 \circ (\alpha_1) = K_2 \circ (\alpha_2)$.
- *minimal*: for every unifier K'_1, K'_2 of α_1 and α_2, such that $K_1 = K''_1 \circ K'_1$ and $K_2 = K''_2 \circ K'_2$, for some contexts K''_1 and K''_2, the contexts K''_1 and K''_2 should be invertible.

Remark 1 If we write $\alpha = K_1 \circ (\alpha_1) = K_2 \circ (\alpha_2)$ and represent the 2-cells α_1, α_2 and α by 2-nets, the fact that α is a unifier of the morphisms means that there exist two injective morphisms of labeled polygraphs $i_1 : \alpha_1 \to \alpha$ and $i_2 : \alpha_2 \to \alpha$, and the non-triviality condition means that there exists at least one 2-generator which is both in the image of i_1 and i_2.

For example, the last two morphisms of (3) are both unifiers of the left members of the rules (2). By extension, a unifier of two 3-generators $r_1 : \alpha_1 \Rrightarrow \beta_1$ and $r_2 : \alpha_2 \Rrightarrow \beta_2$ of R is a unifier of their sources α_1 and α_2. A *critical pair* (K_1, r_1, K_2, r_2) consists of a pair of 3-generators r_1, r_2 and a most general unifier K_1, K_2 of those.

Remark 2 In Definition 3, the 2-cell α_1, can be seen as a context $\alpha_1 : () \to (f_1, g_1)$ in $\mathcal{K}(\mathcal{C})$, and similarly for α_2. In fact, the notion of *unifier* can be generalized to any pair of morphisms in the multicategory $\mathcal{K}(\mathcal{C})$.

A 2-cell $\alpha : f \Rightarrow g$ *rewrites* to a 2-cell $\beta : f \Rightarrow g$, by a 3-generator $r : \alpha' \Rrightarrow \beta' : f' \Rightarrow g'$, when there exists a context $K : ((f', g')) \to (f, g)$ such that $\alpha = K \circ \alpha'$ and $\beta = K \circ \beta'$. In this case, we write $\alpha \Rrightarrow^{K,r} \beta$. The rewriting system R is *terminating* when there is no infinite sequence $\alpha_1 \Rrightarrow^{K_1,r_1} \alpha_2 \Rrightarrow^{K_2,r_2} \dots$. A *peak* is a triple $(\alpha_1, r_1, \alpha, r_2, \alpha_2)$, where α, α_1 and α_2 are 2-cells and r_1 and r_2 are 3-generators, such that $\alpha \Rrightarrow^{K_1,r_1}$ and $\alpha \Rrightarrow^{K_2,r_2} \alpha_2$. In particular, with the notations of Definition 3, every critical pair induces a peak $(K_1 \circ (\beta_1), r_1, K_1 \circ (\alpha_1), r_2, K_2 \circ (\beta_2))$. A peak is *joinable* when there exist a 2-cell β and 3-cells $\rho_1 : \alpha_2 \Rrightarrow \beta$ and $\rho_2 : \alpha_2 \Rrightarrow \beta$. A rewriting system is *locally* confluent if every peak is joinable. Newman's Lemma is valid for 3-polygraphs [5]:

Proposition 4 *A terminating rewriting system is confluent if it is locally confluent.*

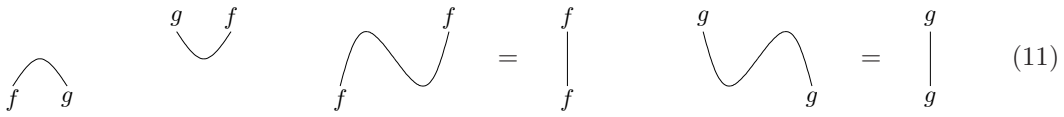
Moreover, local confluence can be tested using critical pairs:

Proposition 5 *A rewriting system is locally confluent if all its critical pairs are joinable.*

So, in order to test whether a terminating polygraphic rewriting system is confluent, it would be tempting to compute all its critical pairs and test whether they are joinable, as in term rewriting systems. However, as explained in the introduction, even a finite polygraphic rewriting system might admit an infinite number of critical pairs. In the next section, we introduce a theoretical setting which allows us to compute a finite number of generating families of critical pairs.

4 An embedding in compact 2-categories

The notion of adjunction in the 2-category **Cat** of categories, functors and natural transformations can be generalized to any 2-category as follows. Suppose that we are given a 2-category $\mathcal{C}$. A 1-cell $f : A \to B$ is *left adjoint* to a 1-cell $g : B \to A$ (or g is *right adjoint* to f) when there exist two 2-cells $\eta : \mathrm{id}_A \Rightarrow f \otimes g$ and $\varepsilon : g \otimes f \Rightarrow \mathrm{id}_B$, called respectively the *unit* and the *counit* of the adjunction and depicted respectively on the left of (11), such that $(f \otimes \varepsilon) \circ (\eta \otimes f) = \mathrm{id}_f$ and $(\varepsilon \otimes g) \circ (g \otimes \eta) = \mathrm{id}_g$. These equations are called the *zig-zag laws* because of their graphical representation, given on the right of (11):



(11)

A 2-category is *compact* (sometimes also called *autonomous* or *rigid*) when every 1-cell admits both a left and a right adjoint. Given a 2-category $\mathcal{C}$, we write $\overline{\mathcal{C}}$ for the free compact 2-category on $\mathcal{C}$. An explicit description of this 2-category can be given [7]:

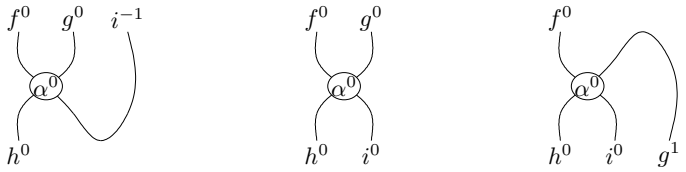
– its 0-cells are the 0-cells of $\mathcal{C}$,

– its 1-cells are pairs $f^n : A \to B$ consisting of an integer $n \in \mathbb{Z}$, called *winding number*, and a 1-cell $f : A \to B$ (resp. $f : B \to A$) of $\mathcal{C}$ if n is even (resp. odd),

– a 2-cell is either $\alpha^0 : f^0 \Rightarrow g^0$, where $\alpha : f \Rightarrow g$ is a 2-cell of $\mathcal{C}$, or $\eta_f^n : \mathrm{id}_B \Rightarrow f^n \otimes f^{n+1}$ or $\varepsilon_f^n : f^{n+1} \otimes f^n \Rightarrow \mathrm{id}_A$, where $f^n : A \to B$ is a 1-cell, or a formal vertical or horizontal composite of those,

– 1- and 2-cells are quotiented by a suitable congruence imposing the axioms of 2-categories, compatibility of vertical and horizontal compositions in $\overline{\mathcal{C}}$ with those of $\mathcal{C}$ (for example $(\beta \circ \alpha)^0 = \beta^0 \circ \alpha^0$ and $(\mathrm{id}_f)^0 = \mathrm{id}_{f^0}$) and the zig-zag laws (11).

Given a 1-cell f in this category, we often write f^m for the 1-cell which is defined inductively by $(f \otimes g)^m = f^m \otimes g^m$ and $(f^n)^m = f^{n+m}$ (notice that f^{-1} does not denote the inverse of f in this context). This algebraic construction is important in order to formally define the 2-category $\overline{\mathcal{C}}$ but this construction might be better grasped graphically, with the help of string diagrams: the compact structure adds to $\mathcal{C}$ the possibility to bend wires, without creating loops. For example, consider a 2-cell $\alpha : f \otimes g \Rightarrow h \otimes i$ in a 2-category $\mathcal{C}$. This 2-cell can be seen as a 2-cell $\alpha^0 : f^0 \otimes g^0 \Rightarrow h^0 \otimes i^0$ of $\overline{\mathcal{C}}$, as pictured in the center of (12).



(12)

From this morphism, we can deduce a 2-cell $\rho_{f^0,g^0,h^0\otimes i^0}(\alpha) : f^0 \Rightarrow h^0 \otimes i^0 \otimes g^1$, pictured on the right of (12), defined by $\rho_{f^0,g^0,h^0\otimes i^0}(\alpha) = (\alpha \otimes \mathrm{id}_{g^1}) \circ (\mathrm{id}_{f^0} \otimes \eta_g^0)$: the wire corresponding to g^0 can be bent on the right and the winding number is increased by one (the output is of type g^1) to "remember" that we have bent the wire once on the right. Similarly, one can define from α the morphism $\rho'_{f^0\otimes g^0,i^0,h^0}(\alpha) : f^0 \otimes g^0 \otimes i^{-1} \Rightarrow h^0$, which corresponds to bending the wire of type i^0 on the left, so its winding number is decreased by 1 (similar transformations can be defined for bending the wires of type f^0 and h^0 in α). Interestingly, by the definition of adjunctions, these two transformations provide mutual inverses: $\rho^{-1}_{f,g,h} = \rho'_{f,g,h}$. We call *rotations* these bijections between the hom-categories of $\overline{\mathcal{C}}$.

Remark 3 The notions of source and target of a 2-cell in a compact 2-category is really artificial since, given a pair of parallel 1-cells $f, g : A \to B$, the rotations induce a bijection between the hom-categories $\mathcal{C}(f, g)$ and $\mathcal{C}(\mathrm{id}_B, f^{-1} \otimes g)$.

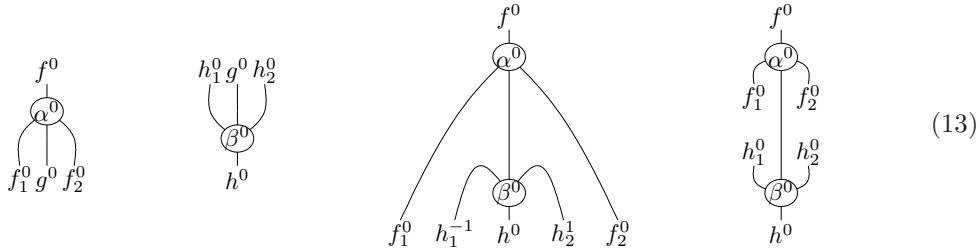
It can be shown that the winding numbers on the 1-cells provide enough information about the bending of wires, so that

Proposition 6 *Given a 2-category $\mathcal{C}$, the embedding functor $E : \mathcal{C} \to \overline{\mathcal{C}}$ defined as the identity on 0-cells, as $f \mapsto f^0$ on 1-cells and as $\alpha \mapsto \alpha^0$ on 2-cells is full and faithful.*

This means that given two 0-cells A and B of $\mathcal{C}$, the hom-categories $\mathcal{C}(A, B)$ and $\overline{\mathcal{C}}(A, B)$ are isomorphic in a coherent way. The 2-category $\overline{\mathcal{C}}$ thus provides a "larger world" in which we can embed the 2-category $\mathcal{C}$ without losing information.

The interest of this embedding is that there are "extra morphisms" in $\overline{\mathcal{C}}$ that can be used to represent "partial compositions" in $\mathcal{C}$. For example, consider two 2-cells $\alpha : f \Rightarrow f_1 \otimes g \otimes f_2$ and $\beta : h_1 \otimes g \otimes h_2 \Rightarrow h$ in $\mathcal{C}$. These can be seen as the morphisms of $\overline{\mathcal{C}}$ depicted on the left of (13) by

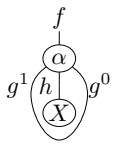
the previous embedding.



(13)

From these two morphisms, the morphism $\alpha \otimes_g \beta : f^0 \Rightarrow f_1^0 \otimes h_1^{-1} \otimes h^0 \otimes h_2^1 \otimes f_2^0$, depicted in the center right of (13), can be constructed. This morphism represents the *partial composition* of the 2-cells α and β on the 1-cell g: up to rotations, this 2-cell is fundamentally a way to give a precise meaning to the diagram depicted on the right of (13).

The notion of 2-polygraph can easily be adapted to generate compact 2-categories instead of 2-categories. Instead of generating a free category from the underlying 1-polygraph, we generate a free category with winding numbers: with the notations of Section 1, its objects are the elements of E_0 and its morphisms $f_1^{n_1} \cdot f_2^{n_2} \cdots f_k^{n_k} : A \to B$ are the paths $e(f_1^{n_1}) \cdot e(f_2^{n_2}) \cdots e(f_k^{n_k}) : A \to B$ in the graph described by the 1-polygraph, the edge $e(f^n)$ being f is $n \in \mathbb{Z}$ is even or f taken backwards if f is odd. Similarly, instead of generating a 2-category from the polygraph, we generate a free compact 2-category on the previously generated category with winding numbers with the 2-generators given by the 2-polygraph. Such "polygraphs" are called *compact polygraphs* and we write 2-**CPol** for the category of compact 2-polygraphs. The embedding given in Proposition 6 can be extended into an embedding of 2-**Pol** into 2-**CPol**: every 2-polygraph can be seen as a compact 2-polygraph. Given a compact 2-polygraph S, the definition given in Section 3 can be adapted in order to define the multicategory of *compact contexts* $\mathcal{K}(S)$ of S. Finally, the construction of nets given in Section 2 can also be adapted in order to give a concrete and implementable description of the multicategory $\mathcal{K}(S)$ – this essentially amounts to suitably adding winding numbers to 1-cells in the polygraphs involved.



Interestingly, the setting of compact contexts provides a generalization of partial composition by allowing a "partial composition of a morphism with itself". Namely, from a context $\alpha : (\dots, (f_i, g_i), \dots) \to (f, g^1 \otimes h \otimes g^0)$ with $f : A \to A$ and $h : B \to B$ one can build the context depicted on the left $\varepsilon_g^0 \circ (g^1 \otimes X \otimes g^0) \circ \alpha : (\dots, (f_i, g_i), \dots, (h, \mathrm{id}_B)) \to (f, \mathrm{id}_A)$, where $X : h \to \mathrm{id}_B$ is a fresh variable. This operation amounts to *merging* the outputs of type g^1 and g^0 of α.

5 The unification algorithm

Now that the theoretical setting has been established, we can describe our unification algorithm. Suppose that we are given a polygraphic rewriting system $R \in$ 3-**Pol** whose underlying signature is $S = U_2(R)$. By the previous remarks, S can be seen as a compact 2-polygraph $\overline{S}$. Now, suppose that r_1 and r_2 are two rewriting rules (i.e. 3-generators) in R whose left member are respectively 2-cells $\alpha : f \Rightarrow g$ and $\beta : h \Rightarrow i$. The 2-cell $\alpha : f \Rightarrow g$ in the 2-category generated by S can be seen as a 2-cell $\alpha^0 : f^0 \Rightarrow g^0$ in the compact 2-category $\mathcal{C}$ generated by $\overline{S}$, and therefore as a nullary context $\alpha : () \to (f^0, g^0)$ in the multicategory of contexts $\mathcal{K}(\mathcal{C})$. Similarly, β can be seen as a context $\beta : () \to (h^0, i^0)$. In the multicategory $\mathcal{K}(\mathcal{C})$, we can compute a most general unifier of α and β (see Remark 2) from which we will be able to generate critical pairs of the rules r_1 and r_2. Because of space limitations, we don't provide here a fully detailed and formal presentation of the algorithm: the purpose of this paper was to introduce the formal framework necessary to define the algorithm, whose in-depth description will be given in subsequent works.

We first introduce some terminology and notations on nets. Given a 2-net α, an instance of a 2-generator y is the *father* (resp. *son*) of an instance of a 1-generator x if x occurs in the target (resp. source) of y. For example, in (10), γ_0 is a son of 1_0 and 1_1 and a father of 1_2 and 1_3. It is

easy to show that a given instance of a 1-generator admits at most one father and one son. An instance of 1-generator is *dangling* when it has no father or no son. An instance of a generator is in the *border* of a net if it is in its source or its target.

The algorithm proceeds as follows. We suppose that we have represented the 2-cells α and β as polygraphic 2-nets. Our goal is to construct a 2-net ω together with two injective morphisms of labeled polygraphs $i_1 : \alpha \to \omega$ and $i_2 : \beta \to \omega$ satisfying the properties required for unifiers as reformulated in Remark 2. The algorithm is quite similar to the rule-based formulation of the unification algorithm for terms [1]. It begins by setting $\omega = \alpha$ and $i_1 = \mathrm{id}_\alpha$, and then iterates a procedure that will progressively propagate the unification and make ω grow, by adding cells to it, until it is big enough so that there exists an injection $i_2 : \beta \to \omega$. The procedure which is iterated is non-deterministic and the critical pairs will be obtained as the collection of the results of the non-failed branches of computation. During the iteration two sets are maintained, T and U, which both contains pairs (x, x') consisting of an n-cell x of β and an n-cell x' of ω for some integer $n \in \{0, 1, 2\}$. The set U (for Unified) contains the injection i_2 which is being constructed: if $(x, x') \in U$ and the branch succeeds then the resulting map $i_2 : \beta \to \omega$ will be such that $i_2(x) = x'$. The set T (as in Todo) contains the pairs (x, x') such that x is a cell of β which is to be unified with the cell x' of ω.

Initially, $\omega = \alpha$, $U = \emptyset$ and $T = \{(x, x')\}$, where x and x' are instances of 2-generators in β and in ω respectively, both chosen non-deterministically. Then the algorithm iterates over the following rules, updating the values of ω, U and T by executing the first rule which applies (updating a value is denoted with the symbol $:=$).

- *Duplicate.* If $T = \{(x, x')\} \uplus T'$ with $(x, x') \in U$ then $T := T'$.
- *Clash.* If $(x, x') \in T$ and $(x, x'') \in U$ and $x' \neq x''$ then fail.
- *Typecheck.* If $(x, x') \in T$ with $\ell(x) \neq \ell(x')$ then fail.
- *Propagate-0.* If $T = \{(x, x')\} \uplus T'$, where x and x' are 0-cells then

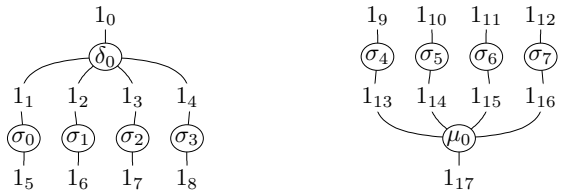
  $T := T'$ and $U := \{(x, x')\} \cup U$.
- *Propagate-1.* If $T = \{(x, x')\} \uplus T'$, where x and x' are 1-cells, then

  $T := T'$ and
  if x has a father y then
  if x' has a father y' then
  $T := \{(y, y')\} \cup T$ and $U := \{(x, x')\} \cup U$
  else either
  add a fresh generator y' of type $\ell(y)$ in ω,
  $T := \{(y, y')\} \cup T$ and $U := \{(x, x')\} \cup U$
  or
  merge x' with some other 1-cell x'' in the border of ω in ω,
  $T := \{(x, x')\} \cup T$
  if x has a son y then
  similar to the previous case.
- *Propagate-2.* If $T = \{(x, x')\} \uplus T'$, where x and x' are 2-cells, then

  $T := T'$, $U := \{(x, x')\} \cup U$, we add in T that the 0- and 1-cells in the source of x should be matched with the corresponding cells in the source of x', and the 0- and 1-cells of the target of x should be matched with those in the target of x'.

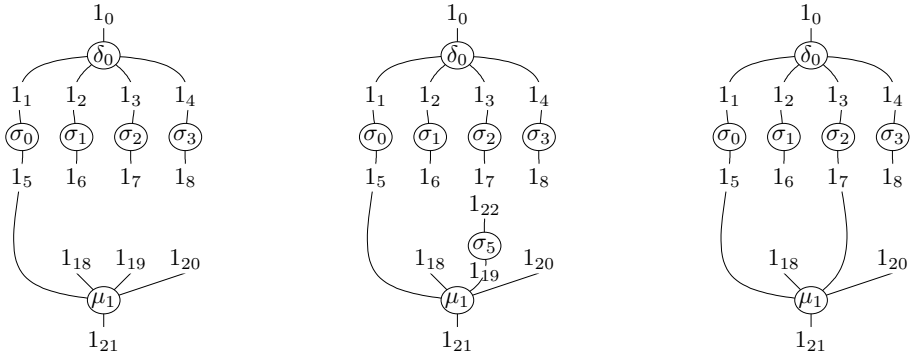
The "either... or" construction above denotes a non-deterministic choice and the "merge" refers to the merging operation introduced in Section 4 (this operation might fail if the labels or the winding numbers of x' and x'' are not suitable).

The way this algorithm works is maybe best understood with an example. Consider the signature S with one 0-cell $*$, one 1-cell $1 : * \to *$ and three 2-cells $\delta : 1 \to 4$, $\mu : 4 \to 1$ and $\sigma : 1 \to 1$ (where 4 denotes $1 \otimes 1 \otimes 1 \otimes 1$). We write $\varsigma = \sigma \otimes \sigma \otimes \sigma \otimes \sigma$. Now, consider a rewriting system on this signature containing two rules r_1 and r_2 whose left members are respectively $\alpha = \varsigma \circ \delta$ and $\beta = \mu \circ \varsigma$, that we represent respectively as the compact nets



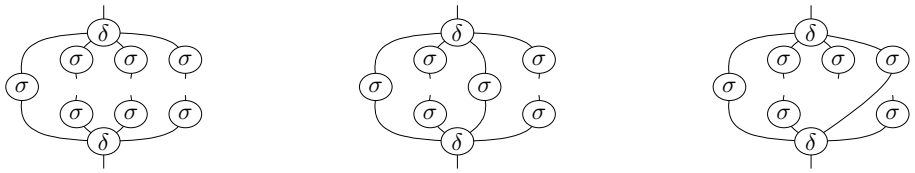
(14)

(for simplicity, we omitted the instances of 0-cells). We describe here a few possible non-deterministic branches of the execution of the algorithm. For example, if we begin with $T = \{(\sigma_4, \delta_0)\}$, the algorithm will immediately fail by Typecheck because the label σ of σ_4 differs from the label δ of δ_0. Consider another execution beginning with $T = \{(\sigma_4, \sigma_0)\}$, this time the label matches so Propagate-2 will propagate the unification by setting $T = \{(1_9, 1_1), (1_{13}, 1_5)\}$ and $U = \{(\sigma_4, \sigma_0)\}$. Since 1_9 is dangling, Propagate-1 will move the pair $(1_9, 1_1)$ from T to U. Then the pair $(1_{13}, 1_5)$ will be handled by Propagate-1. Since 1_5 is dangling but 1_{13} is not, a new generator μ_1 will be added to ω (now pictured on the left of (15)) and after a few propagations $(1_{13}, 1_5)$ will be moved from T to U, (μ_0, μ_1) will be added to U and T will contain $(1_{11}, 1_{19})$. By Propagate-1, this unification pair can lead to multiple non-deterministic executions: a new generator σ_5 can be added (in the middle of (15)), or the 1-generator 1_{19} can be merged with another 1-generator (1_7 for example as pictured in the right of (15)). Notice that in this last case, the morphism contains a "hole" of type $1_6 \Rightarrow 1_{18}$, which is handled by a context variable.



(15)

By executing fully the algorithm, the three morphisms of (16) will be obtained as unifiers (as well as many others).



(16)

It can be shown that the algorithm terminates and generates all the critical pairs in compact contexts, and these are in finite number. It is important to notice that the algorithm generates the

critical pairs of a rewriting system R in the "bigger world" of compact contexts, from which we can generate the critical pairs in the 2-category generated by R (which are not necessarily in finite number as explained in the introduction). If joinability of the critical pairs in compact contexts implies that the rewriting system is confluent, the converse is unfortunately not true: a similar situation is well known in the study of λ-calculus with explicit substitution, where a rewriting system might be confluent without being confluent on terms with metavariables.

We have realized a toy implementation of the algorithm in less than 2000 lines of OCaml, with which we have been able to successfully recover the critical pairs of rewriting systems in [8]. Even though we did not particularly focus on efficiency, the execution times are good, typically less than a second, because the morphisms involved in polygraphic rewriting systems are usually small (but they can generate a large number of critical pairs)

Future works. This paper lays the theoretical foundations for unification in polygraphic 2-dimensional rewriting systems and leaves many research tracks open for future works. We plan to study the precise links between our algorithm and the usual unification for terms (every term rewriting system can be seen as a polygraphic rewriting system [2]) as well as algorithms for (planar) graph rewriting. Concerning concrete applications, since these rewriting systems essentially transform circuits made of operators (the 2-generators) linked by a bunch of wires (the 1-generators), it would be interesting to see if these methods can be used to optimize electronic circuits. Finally, we plan investigating the generalization of these methods in dimension higher than 2, which seems to be very challenging.

Acknowledgements. The author is much indebted to John Baez, Albert Burroni, Jonas Frey, Emmanuel Haucourt, Martin Hyland, Yves Lafont, Paul-André Melliès and François Métayer.

References

[1] F. Baader and T. Nipkow. *Term Rewriting and All That.* Cambridge University Press, 1999.

[2] A. Burroni. Higher-dimensional word problems with applications to equational logic. *Theor. Comput. Sci.*, 115(1):43–62, 1993.

[3] Y. Guiraud. The three dimensions of proofs. *Ann. pure appl. logic*, 141(1-2):266–295, 2006.

[4] Y. Guiraud. Two polygraphic presentations of Petri nets. *TCS*, 360(1-3):124–146, 2006.

[5] Y. Guiraud and P. Malbos. Higher-dimensional categories with finite derivation type. *Theor. Appl. Cat.*, 22(18):420–478, 2009.

[6] A. Joyal and R. Street. The Geometry of Tensor Calculus, I. *Adv. Math.*, 88:55–113, 1991.

[7] G.M. Kelly and M.L. Laplaza. Coherence for compact closed categories. *Journal of Pure and Applied Algebra*, 19:193–213, 1980.

[8] Y. Lafont. Towards an algebraic theory of Boolean circuits. *J. Pure Appl. Alg.*, 184:257–310, 2003.

[9] F. W. Lawvere. *Functorial Semantics of Algebraic Theories and Some Algebraic Problems in the context of Functorial Semantics of Algebraic Theories.* PhD thesis, Columbia University, 1963.

[10] T. Leinster. *Higher Operads, Higher Categories.* Cambridge University Press, 2004.

[11] S. MacLane. *Categories for the Working Mathematician.* Springer Verlag, 1971.

[12] S. Mimram. *Sémantique des jeux asynchrones et réécriture 2-dimensionnelle.* PhD thesis, 2008.

[13] S. Mimram. Computing Critical Pairs in Polygraphs. Preprint, 2009.

[14] S. Mimram. The Structure of First-Order Causality. In *LICS'09*, pages 212–221, 2009.

[15] J. Power. An n-categorical pasting theorem. *Proc. Int. Conf. Como*, pages 326–358, 1990.

[16] R. Street. Limits indexed by category-valued 2-functors. *J. Pure Appl. Alg.*, 8(2):149–181, 1976.