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INVISCID INSTABILITY OF A STABLY STRATIFIED BOUNDARY LAYER

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Summary The three-dimensional stability of an inflection-free boundary layer flow of length scale L and maximum velocity U_0 in a stably stratified fluid of constant Brunt-Väisälä frequency N is examined in an inviscid Boussinesq framework. The plane of the boundary layer is assumed to be inclined with an angle θ with respect to the vertical direction of stratification. The stability analysis is performed using both numerical and theoretical methods for all the values of θ and Froude number $F = U_0/(LN)$. The boundary layer flow is found to be unstable whatever F as soon as $\theta \neq 0$. The growth rate of the most unstable mode is shown to increase with the inclination angle to reach its maximum for a vertical boundary layer $\theta = \pi/2$. The mechanism of instability is shown to be associated with internal gravity wave emission. In the strongly stratified limit, frequency and growth rate become independent of the Froude number and proportional to the sine of the inclination angle (as long as $F/\sin\theta \ll 1$).

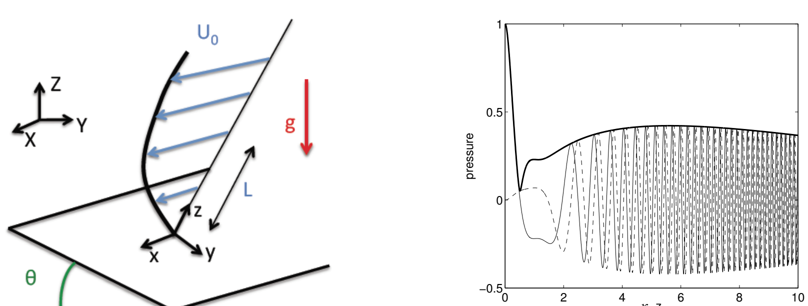
INTRODUCTION

Tollmien-Schlichting waves are viscous perturbations which are usually responsible for the destabilisation of inflection-free boundary layers such as the Blasius boundary layer. These unstable waves do not exist in an inviscid framework. However, in the presence of stratification, inflection-free boundary layer flows can become unstable with respect to new inviscid instability modes. The goal of the present study is to analyse the characteristics of such instability modes. We also want to understand the effect of the inclination angle between the shear plane and the direction of stratification on the stability properties.

In stratified fluids, most studies have considered configurations where shear and stratification are aligned in the same direction. Moreover, they have mainly been concerned with the effects of stratification on the shear instability in relation with the Richardson criterion for instability [8]. Yet, the presence of a ground has also been shown to create other less unstable modes [5]. Although the presence of an inflection point is not a necessary condition for instability when a solid ground is present [3], inflection-free profiles generally remain stable in a inviscid framework when shear and stratification are aligned in the same direction [4]. Here we demonstrate that when shear and stratification are not aligned, the flow exhibits unstable radiative modes owing to the stratification. These radiative modes, whose unstable character is associated with the wave emission are similar to the acoustic waves of compressible flows [7, 12]. They have also been obtained in shallow water shear flows (e.g. [10]). In a continuously stratified fluid, they have been found in columnar vortices [11, 6] and rotating flows [9]. The instability mechanism is directly related to the mechanism of over-reflection [1, 6].

FRAMEWORK

We consider a 2D boundary layer flow of length scale L and maximum velocity U_0 with a hyperbolic tangent profile in a stably stratified fluid of constant Brunt-Väisälä frequency N. The plane of the boundary layer is assumed to be inclined with an angle θ with respect to the vertical direction of stratification as indicated in figure 1(a). The stability



(a) (b)

Figure 1. (a) Sketch of the base flow and definition of the angle θ. (b) Eigenfunction (pressure amplitude) of the most unstable mode (obtained for $k_x/(k_y F) \approx 4.9$ and large k_x and k_y) when $F/\sin\theta \ll 1$. Absolute value (thick solid line); real part (solid line); imaginary part (dashed line).

characteristics of this base flow, which is defined by its Froude number $F = U_0/(LN)$ and the inclination angle θ, are determined in a inviscid, incompressible, Boussinesq framework using a temporal normal mode approach. The eigenvalue problem for the complex frequency ω is solved using a shooting method. We denote by k_x and k_y, the wavenumbers in

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the x and y directions, respectively (see figure 1(a)). $\Re e(\omega)$ and $\Im m(\omega)$ are the (real) frequency and the growth rate of the perturbation.

RESULTS

We demonstrate that as soon as $\theta > 0$, that is the boundary surface is not horizontal, the flow is unstable for any Froude number. The most unstable modes are obtained for large values of k_x and k_y with a fixed ratio k_y/k_x. Their characteristics as a function of F are displayed in figure 2 for various θ. For all F, we observe that the flow is the most unstable for a vertical boundary layer ($\theta = \pi/2$). The flow is also the most unstable when the stratification is strong (small F). In this

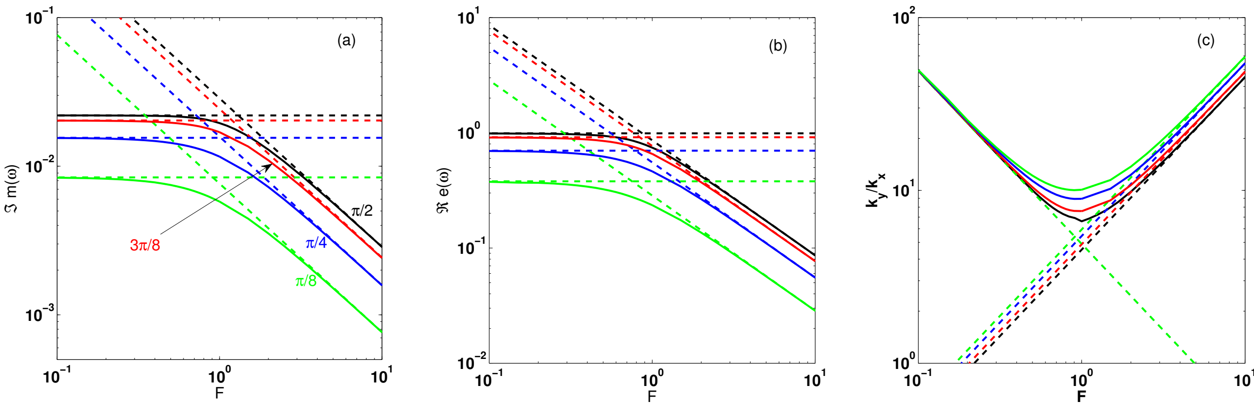

Figure 2. Characteristics of the most unstable mode over all k_y/k_x as a function of F for $\theta = \pi/8, \pi/4, 3\pi/8, \pi/2$. (a) Maximum growth rate $\Im m(\omega)$. (b) Frequency $\Re e(\omega)$. (c) Maximizing ratio k_y/k_x. The dotted lines are the predictions obtained from small and large F analyses.

limit, the stability problem can be simplified if $F/\sin\theta$ is also small: the dependence with respect to F and θ becomes trivial as the complex frequency of the most unstable modes can be written as

$$\omega(k_x, k_y, F, \theta) \sim \omega_{max}(k_y F/k_x) \sin\theta. \tag{1}$$

The most unstable mode is obtained for $k_y F/k_x \approx 4.9$, and its spatial structure is represented in figure 1(b). The radiative structure of the eigenmode is clearly visible on this plot. This internal wave emission is responsible for the growth of the mode by the mechanism of over-reflection explained in [1] and [6].

Because the most unstable modes are obtained for large k_x and k_y the instability characteristics only depend on the local shear rate of the flow near the wall. Therefore, they do not depend on the precise form of the boundary layer profile as long as it is not inflexional and with a non-vanishing shear rate at the boundary. In particular, the results apply to the Blasius boundary layer profile as well as other more realistic local boundary layer profiles. A more detailed account of this work can be found in [2].

References

[1] BASOVICH, A. Y. & TSIMRING, L. S. 1984 Internal waves in a horizontally inhomogeneous flow. *J. Fluid Mech.* **142**, 223–249.

[2] CANDELIER, J., LE DIZÈS, S. & MILLET, C. 2012 Inviscid instability of a stably stratified compressible boundary layer on an inclined surface. *J. Fluid Mech.* (in press).

[3] CHIMONAS, G. 1974 Consideration of the stability of certain heterogenrous shear flows including some inflexion-free profiles. *J. Fluid Mech.* **65**, 65–69.

[4] CHIMONAS, G. 2002 On internal gravity waves associated with the stable boundary layer. *Boundary-Layer Met.* **102**, 139–155.

[5] DAVIS, P. A. & PELTIER, W. R. 1976 Resonant parallel shear instability in the stratified planetary boundary layer. *J. Atmos. Sci.* **33**, 1287–1300.

[6] LE DIZÈS, S. & BILLANT, P. 2009 Radiative instability in stratified vortices. *Phys. Fluids* **21**, 096602.

[7] MACK, L. M. 1990 On the inviscid acoustic-mode instability of supersonic shear flows. Part 1: Two-dimensional waves. *Theo. Comp. Fluid Dyn.* **2**, 97–123.

[8] MILES, J. W. 1961 On the stability of heterogeneous shear flows. *J. Fluid Mech.* **10**, 496–508.

[9] RIEDINGER, X., LE DIZÈS, S. & MEUNIER, P. 2011 Radiative instability of the flow around a rotating cylinder in a stratified fluid. *J. Fluid Mech.* **672**, 130–146.

[10] SATOMURA, T. 1981 An investigation of shear instability in a shallow water. *J. Met. Soc. Japan* **59**, 148–167.

[11] SCHECTER, D. A. & MONTGOMERY, M. T. 2004 Damping and pumping of a vortex Rossby wave in a monotonic cyclone: Critical layer stirring versus inertia–buoyancy wave emission. *Phys. Fluids* **16**, 1334–48.

[12] TAM, C. K. W. & HU, F. Q. 1989*b* On the three families of instability waves of high-speed jets. *J. Fluid Mech.* **201**, 447–483.