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Abstract. *Consider a coil above a conducting plate. The interaction between the probe-coil and the plate is modeled by a quasi-static approximation of Maxwell's equations: the eddy current model. The associated electromagnetic transmission boundary-value problem can be solved by the integral equations method. However, the discretization of integral operators gives dense, complex and ill-conditioned linear systems. We present here a method to compute the reaction field and the coil impedance variation by solving only surface partial differential equations.*

Keywords: Eddy currents; integral equations; Padé approximation.

1 INTRODUCTION

We are interested in eddy current non-destructive testing of a conducting plate. Eddy current method is based on the principle of measuring changes in the impedance of an electromagnetic coil as it is scanned over a surface of a conductive material. This method is modeled as a low-frequency electromagnetic transmission problem between two media: the air containing the probe-coil and the conducting workpiece. Integral equations method is one of the principal tools to solve this problem (e.g. [3, 5, 6]). This approach applies the Green's function formalism to reduce equivalently the governing boundary value problem to an integral equation set on the interface between the media. The discretization of boundary integral formulations leads to fully-populated and in general, ill-conditioned complex linear systems. We propose an alternative to compute the impedance variation of the coil in a simplified configuration. This approach is based on the knowledge of the Steklov-Poincaré operator (or Dirichlet-to-Neumann map) which is expressed by non-local square-root operators. To localize this operator, we propose the so-called Padé approximation. This leads to solve only a finite number of sparse linear systems.

2 AN ELECTROMAGNETIC TRANSMISSION PROBLEM

2.1 Configuration

Consider the following configuration. The interface $\Gamma = \{\mathbf{x} \in \mathbb{R}^3 | x_3 = 0\}$ divides the whole space into two media: a non-conducting air domain $\Omega^{\mathrm{ext}} = \mathbb{R}^3_+$ containing the coil, and a conducting plate $\Omega^c = \mathbb{R}^3_-$ (without defect). We denote by $\mathbf{n}$ the outgoing unit normal to the surface Γ.

We make the following assumptions. The coil creates a current source $\mathbf{J_s}$ with sinusoidal angular frequency ω and fixed amplitude. The impressed current $\mathbf{J_s}$ has only support in Ω^{ext} and $\mathrm{div}\mathbf{J_s} = 0$ and $\mathbf{J_s} \cdot \mathbf{n} = 0$ on $\partial\Omega_s$ with $\Omega_s := supp(\mathbf{J_s})$. Assume that the coil does not affect the electromagnetic parameters of Ω^{ext}. The exterior domain Ω^{ext} is homogeneous with magnetic permeability $\mu = \mu_0$, electric permittivity $\epsilon = \epsilon_0$ and zero conductivity $\sigma = 0$. As a first step, assume further that magnetic permeability μ_c and electric permittivity ϵ_c of the conductor are constants. We are only interested in the variation of the electrical conductivity σ_c. For a workpiece Ω^c without defect, the conductivity may be assumed constant.

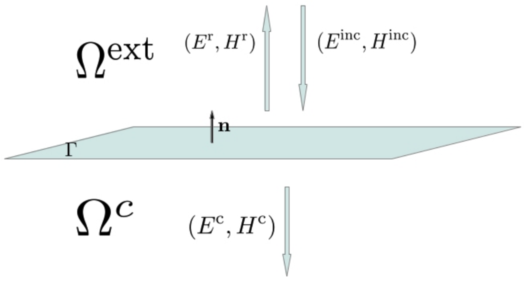

Figure 1: Configuration

2.2 Functional setting

The results of this section can be found in [5]. Ω is equally Ω^c and Ω^{ext}. The normal vector $\mathbf{n}$ is the outward normal to Ω^c. We define the trace operators: let $\mathbf{u} \in (C_0^\infty(\overline{\Omega}))^3$

$$\begin{aligned}
\gamma_\times(\mathbf{u}) &= \mathbf{u}_{|\Gamma} \times \mathbf{n} \text{ (Dirichlet vectorial trace)},\\
\gamma_T(\mathbf{u}) &= \mathbf{n} \times (\mathbf{u}_{|\Gamma} \times \mathbf{n}) \text{ (tangential component)},\\
\gamma_N(\mathbf{u}) &= \mathbf{curl}(\mathbf{u})_{|\Gamma} \times \mathbf{n} \text{ (Neumann vectorial trace)}.
\end{aligned}$$

We extend the trace operators to $\mathbf{H}(\mathbf{curl}, \Omega) = \{\mathbf{u} \in (L^2(\Omega))^3 | \, \mathbf{curl}(\mathbf{u}) \in (L^2(\Omega))^3\}$.

Definition 1 *We define Y the range of $\gamma_\times$ as follows*

$$Y = \{\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}(\Gamma); \exists \mathbf{E} \in \mathbf{H}(\mathbf{curl}, \Omega); \mathbf{E} \times \mathbf{n} = \boldsymbol{\lambda}\}.$$

The space Y with the norm $\|\boldsymbol{\lambda}\|_Y = \inf\limits_{\mathbf{E}\times\mathbf{n}=\boldsymbol{\lambda}} \|\mathbf{E}\|_{\mathbf{H}(\mathbf{curl},\Omega)}$ is a Hilbert space.

Theorem 1 *We have*

$$Y = \{\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}(\Gamma); \boldsymbol{\lambda} \cdot \mathbf{n} = 0 \textit{ et } \mathrm{div}_\Gamma \boldsymbol{\lambda} \in \mathrm{H}^{-1/2}(\Gamma)\}.$$

We define the Hilbert space Z by

$$\begin{array}{rccl}
\gamma_T: & \mathbf{H}(\mathbf{curl}, \Omega) & \to & Z = \{\boldsymbol{\lambda} \in \mathbf{H}^{-1/2}(\Gamma),\ \mathrm{curl}_\Gamma \boldsymbol{\lambda} \in \mathrm{H}^{-1/2}(\Gamma)\}\\
& \mathbf{E} & \mapsto & \mathbf{n} \times \mathbf{E} \times \mathbf{n}.
\end{array}$$

Spaces Y and Z are in duality by the following duality product

$$< \gamma_\times(\mathbf{E}), \gamma_T(\mathbf{v}) >= \int \mathbf{curl}\,\mathbf{E} \cdot \mathbf{v} - \int \mathbf{E} \cdot \mathbf{curl}\,\mathbf{v} \ \ \forall \mathbf{E}, \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega).$$

2.3 The electromagnetic transmission problem

We consider the $\mathbf{E}$-field formulation. We introduce the notations:

- the total exterior field $\mathbf{E}_{|\Omega^{\text{ext}}} = \mathbf{E}^r + \mathbf{E}^{\text{inc}}$, with $\mathbf{E}^r$ and $\mathbf{E}^{\text{inc}}$ the reaction and incident fields,
- the total interior field $\mathbf{E}_{|\Omega^c} = \mathbf{E}^c$.

Consider the following time-harmonic model (with a time dependence in $e^{i\omega t}$) with the condition $\omega\dfrac{\epsilon}{\sigma_c} \ll 1$. Under this condition, we can consider the quasi-static approximation of Maxwell's equations: the eddy current model (see [1]). The dielectric displacement currents are neglected in the Maxwell's equations in the exterior domain. Furthermore, the assumptions on the current source and $\epsilon \simeq 0$ yield $\mathrm{div}\mathbf{E}^r = 0$ in Ω^{ext}.

- Input : an incident electric field $\mathbf{E}^{\text{inc}}$ generated by the coil and such that

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^{\text{inc}} = -i\omega\mu_0 \mathbf{J}_s \delta_{coil}, \text{div}\mathbf{E}^{\text{inc}} = 0, \text{ in } \Omega^{\text{ext}}.$$

- **E**-field formulation : find $(\mathbf{E}^r, \mathbf{E}^c)$ such that

$$(\mathscr{P}) \begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^r = 0, \text{div}\mathbf{E}^r = 0, & \text{in } \Omega^{\text{ext}}, \\ \mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^c + k^2\mathbf{E}^c = 0, & \text{in } \Omega^c, \\ \mathbf{E}^c_{|\Gamma} \times \mathbf{n} = \mathbf{E}^r_{|\Gamma} \times \mathbf{n} + \mathbf{E}^{\text{inc}}_{|\Gamma} \times \mathbf{n}, & \text{on } \Gamma, \\ \mathbf{curl}\,\mathbf{E}^c_{|\Gamma} \times \mathbf{n} = \mathbf{curl}\,\mathbf{E}^r_{|\Gamma} \times \mathbf{n} + \mathbf{curl}\,\mathbf{E}^{\text{inc}}_{|\Gamma} \times \mathbf{n}, & \text{on } \Gamma, \\ \mathbf{E}(x) = O(\frac{1}{|x|}), \mathbf{curl}\,\mathbf{E}(x) = O(\frac{1}{|x|^2}), |x| \to +\infty, & \end{cases}$$

with the complex wavenumber $k = \frac{\sqrt{2}}{2}(1+i)\sqrt{\omega\mu_0\sigma_c}$. To overcome the unboundedness of the computational domain, the method of integral equations (e.g. [5, 6]) can be applied. The Stratton-Chu integral representation formula allows to express the field $\mathbf{E}^r$ (resp. $\mathbf{E}^c$) in terms of the surfacic unknowns $\mathbf{E}^r_{|\Gamma} \times \mathbf{n}$ and $\mathbf{curl}\,\mathbf{E}^r_{|\Gamma} \times \mathbf{n}$ (resp. $\mathbf{E}^c_{|\Gamma} \times \mathbf{n}$ and $\mathbf{curl}\,\mathbf{E}^c_{|\Gamma} \times \mathbf{n}$). The surfacic fields become the new unknows of the problem. Applying the trace operators to the Stratton-Chu representation formula, and translating the boundary transmission conditions yield boundary integral equations on Γ. We have to solve these integral equations to obtain:

- Output :

$$\mathbf{E}^r_{|\Gamma} \times \mathbf{n} \text{ and } \mathbf{curl}\,\mathbf{E}^r_{|\Gamma} \times \mathbf{n}.$$

These surface data are not experimentally observable. The only quantity which is accessible is the coil impedance variation. This quantity of interest can be expressed with these data. We have:

- Impedance variation [2] :

$$\Delta Z = -\frac{i}{I^2\omega\mu_0}\left(\int_\Gamma \mathbf{curl}\,\mathbf{E}^{\text{inc}}_{|\Gamma} \times \mathbf{n} \cdot \mathbf{E}^r_{|\Gamma} \times \mathbf{n} ds - \int_\Gamma \mathbf{curl}\,\mathbf{E}^{\text{r}}_{|\Gamma} \times \mathbf{n} \cdot \mathbf{E}^{\text{inc}}_{|\Gamma} \times \mathbf{n} ds\right), \quad (1)$$

with $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^3 u_i v_i,\ \mathbf{uv} \in \mathbb{C}^3$ and I the current energizing the coil.

3 SOLVING THE PROBLEM

In order to solve the electromagnetic transmission problem $(\mathscr{P})$, we define the exterior and interior Steklov-Poincaré operators

$$\begin{aligned} \mathcal{SP}^{\text{int}}_k : \quad & Z \to Y \\ & \gamma_T^-(\mathbf{E}^c) \mapsto \gamma_N^-(\mathbf{E}^c), \end{aligned}$$

$$\begin{aligned} \mathcal{SP}^{\text{ext}}_0 : \quad & Z \to Y \\ & \gamma_T^+(\mathbf{E}^r) \mapsto \gamma_N^+(\mathbf{E}^r). \end{aligned}$$

In the configuration described in Fig. 1, Fourier analysis allows us to obtain expressions in terms of surface differential operators:

$$\mathcal{SP}^{\text{int}}_k = \left(k^2\text{Id}_\Gamma - \mathbf{\Delta}_\Gamma\right)^{-1/2}\left[k^2\text{Id}_\Gamma + \mathbf{curl}_\Gamma\,\text{curl}_\Gamma\right],$$

and

$$\mathcal{SP}^{\text{ext}}_0 = -\left(-\mathbf{\Delta}_\Gamma\right)^{-1/2}\mathbf{curl}_\Gamma\,\text{curl}_\Gamma,$$

with $\mathbf{\Delta}_\Gamma \mathbf{v} = \begin{pmatrix} \Delta_\Gamma v_1 \\ -\Delta_\Gamma v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_1^2 v_1 + \partial_2^2 v_1 \\ \partial_1^2 v_2 + \partial_2^2 v_2 \\ 0 \end{pmatrix}$, $\mathbf{curl}_\Gamma\, v = \begin{pmatrix} \partial_2 v \\ -\partial_1 v \\ 0 \end{pmatrix}$ et $\text{curl}_\Gamma\,\mathbf{v} = \partial_1 v_2 - \partial_2 v_1$. Then, the transmission problem $(\mathscr{P})$ is equivalently reduced to solving the surface problem: find $\gamma_T^+(\mathbf{E}^{\text{r}}) \in \mathbf{H}^{-1/2}(\text{curl}_\Gamma, \Gamma)$ solution to

$$\boxed{(\mathcal{SP}^{\text{int}}_k - \mathcal{SP}^{\text{ext}}_0)\gamma_T^+(\mathbf{E}^r) = -(\mathcal{SP}^{\text{int}}_k + \mathcal{SP}^{\text{ext}}_0)\gamma_T^+(\mathbf{E}^{\text{inc}}), \text{ on } \Gamma.} \quad (2)$$

To this end, we use the Helmholtz decomposition of the fields. In the case of the interface $\Gamma = \{x_3 = 0\}$, the Helmholtz decomposition of $\gamma_T^+(\mathbf{E}^r)$ and $\gamma_T^+(\mathbf{E}^{\text{inc}}) \in \mathbf{H}^{-1/2}(\text{curl}_\Gamma, \Gamma)$ are given respectively by

$$\gamma_T^+(\mathbf{E}^{\text{inc}}) = \nabla_\Gamma e_1 + \mathbf{curl}_\Gamma\, e_2, \ \ (e_1, e_2) \in \mathrm{H}^{1/2}(\Gamma) \times \mathrm{H}^{3/2}(\Gamma),$$
$$\gamma_T^+(\mathbf{E}^r) = \nabla_\Gamma \varphi_1 + \mathbf{curl}_\Gamma\, \varphi_2, \ \ (\varphi_1, \varphi_2) \in \mathrm{H}^{1/2}(\Gamma) \times \mathrm{H}^{3/2}(\Gamma).$$

Solving equation (2) means to seek $(\varphi_1, \varphi_2) \in \mathrm{H}^{1/2}(\Gamma) \times \mathrm{H}^{3/2}(\Gamma)$ that is solution of the differential system

$$\begin{cases} \varphi_1 = -e_1 \text{ on } \Gamma. & (3)\\ \varphi_2 = \left(-1 + 2\dfrac{\Delta_\Gamma}{k^2} + 2\left(\dfrac{-\Delta_\Gamma}{k^2}\right)^{1/2}\left(1 - \dfrac{\Delta_\Gamma}{k^2}\right)^{1/2}\right) e_2, \text{ on } \Gamma, & (4)\end{cases}$$

where (e_1, e_2) are known. The operators involved in equation (4) are non-local operators. We propose to localize the square-root operators thanks to a Padé paraxial approximation of order N_p with a rotating branch-cut technique of angle θ [7, 4]

$$\left(1 - \frac{\Delta_\Gamma}{k^2}\right)^{1/2} \approx C_0 - \sum_{k=1}^{N_p} A_j \Delta_\Gamma \left(k^2 - B_j \Delta_\Gamma\right)^{-1}, \tag{5}$$

with $C_0,\ A_j, B_j,\ j = 1, \ldots, N_p$ complex coefficients depending on θ. Now, we can compute the impedance variation using the following formula

$$\Delta Z = \frac{2}{I^2} \frac{\sigma_c}{k^2} \int_\Gamma \mathbf{curl}_\Gamma (-\Delta_\Gamma)^{1/2} e_2 \cdot \mathbf{curl}_\Gamma\, \varphi_2 ds. \tag{6}$$

4 CONCLUSIONS

In this paper, we propose an approach to solve a low-frequency electromagnetic transmission problem by solving only a finite number of sparse linear systems. On the contrary, the usual method of integral equations involves the solution of dense complex and ill-conditioned linear systems. We have to implement the method in order to study its accuracy. The solution of this direct problem is the first step of the research program PINCEL which is devoted to the solution of an inverse problem related to the location of defects in a conducting plate.

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