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Linking partial and quasi dynamical symmetries in rotational nuclei

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We establish a link between two distinct symmetry concepts, partial dynamical symmetry (PDS) and quasi dynamical symmetry (QDS). The connection is illustrated in the framework of the interacting boson model of nuclei. Quantum-number fluctuations reveal a previously unrecognized region of Hamiltonians that have both $O(6)$ PDS (purity) and $SU(3)$ QDS (coherence) in the ground band. Many rare-earth nuclei can be identified approximately satisfying both symmetry requirements.

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Understanding the structure and dynamics of complex many-body systems can often be obtained from the observation and analysis of symmetries. Symmetry considerations are particularly significant for addressing a key question in such systems, namely, how do simple features emerge within a complicated environment. A notable example is the collective behavior of nuclei which stems from the complex interactions among the constituent nucleons. Despite the complex nature of the low-energy effective forces at work and the large number of participating particles, collective nuclei give rise to strikingly regular excitation spectra, signaling the presence of underlying symmetries [1]. The theme of "simplicity out of complexity" and the understanding of simple emergent behavior are major challenges facing the study of almost any many-body system, from atomic nuclei to nanoscale and macroscopic systems [2].

Although, usually, a many-body Hamiltonian does not conform to a dynamical symmetry (DS) limit [3], the possibility exists that certain symmetries are obeyed by only a subset of its eigenstates. This situation, referred to as partial dynamical symmetry (PDS) [4], was shown to be relevant to specific nuclei and molecules [4–13]. In parallel, the notion of quasi dynamical symmetry (QDS) was introduced and discussed in the context of nuclear models [14–21]. While QDS can be defined mathematically in terms of embedded representations [22, 23], its physical meaning is that several observables associated with a particular subset of eigenstates, may be consistent with a certain symmetry which in fact is broken in the Hamiltonian. This typically occurs for a Hamiltonian transitional between two DS limits which retains, for a certain range of its parameters, the characteristics of one of those limits. This "apparent" symmetry is due to a coherent mixing of representations in selected states, imprinting an adiabatic motion and increased regularity [19–21].

PDS and QDS are applicable to any many-body problem (bosonic and fermionic) endowed with an algebraic structure. They play a role in diverse phenomena including nuclear and molecular spectroscopy, quantum phase transitions and mixed regular and chaotic dynamics. In this Letter, a hitherto unnoticed link is established between these two different symmetry concepts and it is shown that coherent mixing of one symmetry (QDS) can result in the partial conservation of a different, incompatible symmetry (PDS). An empirical manifestation of such a linkage is presented.

Algebraic models provide a convenient framework for exploring the role of symmetries [24]. One such framework is the interacting boson model (IBM) [25], which has been widely used to describe quadrupole collective states in nuclei in terms of N monopole ($s^\dagger$) and quadrupole ($d^\dagger$) bosons, representing valence nucleon pairs. The model has $U(6)$ as a spectrum generating algebra and exhibits three DS limits, associated with chains of nested subalgebras, starting with $U(5)$, $O(6)$, and $SU(3)$, respectively. These solvable limits correspond to known benchmarks of the geometric description of nuclei [26], involving vibrational [$U(5)$], γ-soft [$O(6)$], and rotational [$SU(3)$] types of dynamics. In what follows we employ the IBM as test ground for connecting the PDS and QDS notions. The particular example considered, namely, $SU(3)$ QDS as an emanation of $O(6)$ PDS, is shown to have approximate validity in many deformed rare-earth nuclei.

One particularly successful approach within the IBM is the extended consistent-Q formalism (ECQF) [27, 28], which is frequently used for the interpretation and classification of nuclear data. It uses the same quadrupole operator, $\hat{Q}^\chi = d^\dagger s + s^\dagger \tilde{d} + \chi\,(d^\dagger \tilde{d})^{(2)}$, in the $E2$ transition operator and in the Hamiltonian, the latter being written as

$$\hat{H}_{\rm ECQF} = \omega\left[(1-\xi)\,\hat{n}_d - \frac{\xi}{4N}\,\hat{Q}^\chi\cdot\hat{Q}^\chi\right]\,, \qquad (1)$$

where $\hat{n}_d$ is the d-boson number operator, $\hat{Q}^\chi\cdot\hat{Q}^\chi$ is the quadrupole interaction, and the dot implies a scalar product. The parameters ω, ξ, and χ are fitted to empirical data or calculated microscopically if possible; ξ and χ are the sole structural parameters of the model since ω

is a scaling factor. The parameter ranges $0 \leq \xi \leq 1$ and $-\frac{\sqrt{7}}{2} \leq \chi \leq 0$ interpolate between the $U(5)$, $O(6)$, and $SU(3)$ DS limits, which are reached for $(\xi, \chi) = (0, \chi)$, $(1, 0)$, and $(1, -\frac{\sqrt{7}}{2})$, respectively. It is customary to represent the parameter space by a symmetry triangle [29], whose vertices correspond to these limits. The ECQF has been used extensively for the description of nuclear properties (see, *e.g.*, Ref. [30]) and it was found that rotational nuclei are best described by ECQF parameters in the interior of the triangle, away from the naively expected $SU(3)$ DS limit. The SU(3) mixing was found to be strong and coherent, *i.e.*, the same for all rotational states in a band, exemplifying a $SU(3)$-QDS [19–21]. In what follows we examine the $O(6)$ symmetry properties of ground-band states in such nuclei, in the rare-earth region, using the ECQF of the IBM.

The $O(6)$ DS basis states are specified by quantum numbers N, σ, τ, and L, related to the algebras in the chain $U(6) \supset O(6) \supset O(5) \supset O(3)$ [31]. Given an eigenstate Ψ of the ECQF Hamiltonian (1), its expansion in the O(6) basis reads

$$|\Psi(\xi, \chi)\rangle = \sum_i \alpha_i(\xi, \chi) \, |N, \sigma_i, \tau_i, L\rangle \ , \qquad (2)$$

where the sum is over all basis states and, for simplicity, the dependence of Ψ and α_i on the boson number N and the angular momentum L is suppressed. The degree of $O(6)$ symmetry of the state Ψ is inferred from the fluctuations in σ which can be calculated as

$$\Delta\sigma_\Psi = \sqrt{\sum_i \alpha_i^2 \, \sigma_i^2 - \left(\sum_i \alpha_i^2 \, \sigma_i\right)^2} \ . \qquad (3)$$

If Ψ carries an exact $O(6)$ quantum number, σ fluctuations are zero, $\Delta\sigma_\Psi = 0$. If Ψ contains basis states with different $O(6)$ quantum numbers, then $\Delta\sigma_\Psi > 0$, indicating that the $O(6)$ symmetry is broken. Note that $\Delta\sigma_\Psi$ also vanishes for a state with a mixture of components with the same σ but different $O(5)$ quantum numbers τ, corresponding to a Ψ with good $O(6)$ but mixed $O(5)$ character. This method of quantifying the $O(6)$ purity of states has already been applied to ^{124}Xe [32]. Also, $\Delta\sigma_\Psi$ has the same physical content as wave-function entropy which, upon averaging over all eigenstates, discloses the global DS content of a given Hamiltonian [33]. We examine here the fluctuations $\Delta\sigma_\Psi$ for the entire parameter space of the ECQF Hamiltonian (1) for values of N up to 60, using the ArbModel code [34].

Results of this calculation for the ground state, $\Psi = 0^+_{\rm gs}$, with $N = 14$ and parameters $\xi \in [0, 1]$, $\chi \in [-\frac{\sqrt{7}}{2}, 0]$, are shown in Fig. 1. At the $O(6)$ DS limit ($\xi = 1$, $\chi = 0$) $\Delta\sigma_{\rm gs}$ vanishes per construction whereas it is greater than zero for all other parameter pairs. Towards the $U(5)$ DS limit ($\xi = 0$), the fluctuations reach a saturation value of $\Delta\sigma_{\rm gs} \approx 2.47$. At the $SU(3)$ DS limit ($\xi = 1$, $\chi = -\frac{\sqrt{7}}{2}$) the fluctuations are $\Delta\sigma_{\rm gs} \approx 1.25$. In both cases the $O(6)$ symmetry is completely dissolved as measured by $\sigma_{\rm crit} = 0.849$ [32]. Surprisingly, there is a previously unrecognized valley of almost vanishing $\Delta\sigma_{\rm gs}$ values, two orders of magnitude lower than at saturation. This region represents a parameter range of the IBM, outside the $O(6)$ DS limit, where the ground-state wave function exhibits an exceptionally high degree of purity with respect to the $O(6)$ quantum number σ.

The ground-state wave functions in the valley of low $\Delta\sigma_{\rm gs}$ can be analyzed with the help of the $O(6)$ decomposition (2). At the $O(6)$ DS limit only one $O(6)$ basis state, with $\sigma = N$ and $\tau = 0$ contributes, while outside this limit the wave function consists of multiple $O(6)$ basis states. Investigation of the wave function for parameter combinations inside the valley reveals an overwhelming dominance of the $O(6)$ basis states with $\sigma = N$. This is seen in Fig. 2 for the ground-state wave function of

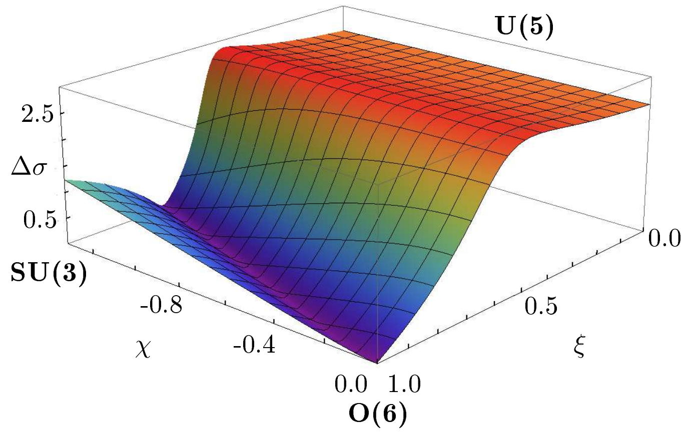

FIG. 1. (Color online) Ground-state fluctuations $\Delta\sigma_{\rm gs}$ (3) for the ECQF Hamiltonian (1) with $N = 14$ bosons. The fluctuations vanish at the $O(6)$ DS limit, saturate towards the $U(5)$ DS limit, and are of the order 10^{-2} in the valley.

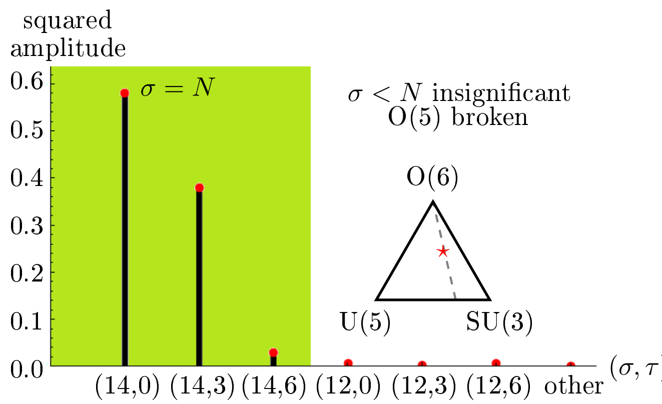

FIG. 2. (Color online) Squared amplitudes α_i^2 in the expansion (2) of the $0^+_{\rm gs}$ ground state of the ECQF Hamiltonian (1) for $\xi = 0.84$ and $\chi = -0.53$ (indicated by the red star in the symmetry triangle and appropriate for ^{160}Gd).

the ECQF Hamiltonian (1) at $\xi = 0.84$ and $\chi = -0.53$ with $N = 14$, parameter values that apply to the nucleus ^{160}Gd discussed below. The $\sigma = N$ states comprise more than 99% of the ground-state wave function at the bottom of the valley and their dominance causes $\Delta\sigma_{\rm gs}$ to be small. Furthermore, it is evident that at the same time the $O(5)$ symmetry is broken, as basis states with different quantum number τ contribute significantly to the wave function. Consequently, the valley can be identified as an entire region in the symmetry triangle with an approximate PDS of type III [4], which means that some of the eigenstates exhibit some of the symmetries. Outside this valley the ground state is a mixture of several σ values and $\Delta\sigma_{\rm gs}$ increases. In the $SU(3)$ DS limit the $\sigma = N$ components constitute 67% of the wave function and in the $U(5)$ DS limit and throughout the plateau of saturated $\Delta\sigma_{\rm gs}$ this contribution drops below 1%. This region of approximate ground-state $O(6)$ symmetry is similar to the previously established "arc of regularity" [35] which is a region of reduced mixing inside the IBM parameter space attributed to an approximate $SU(3)$ symmetry [36].

An argument for the existence of the valley of ground-state $O(6)$ symmetry can be given in terms of the following Hamiltonian [7]:

$$\hat{H}_{\rm M} = -\hat{C}_{O(6)} + \hat{N}(\hat{N}+4) + 2\alpha\hat{C}_{O(5)} - \alpha\hat{C}_{O(3)} + 2\alpha\hat{n}_d(\hat{N}-2) + \sqrt{14}\alpha(d^\dagger s + s^\dagger\tilde{d})\cdot(d^\dagger\tilde{d})^{(2)}\,, \quad (4)$$

where $\hat{C}_G$ denotes the quadratic Casimir operator of the group G [25], $\hat{N}$ is the total boson number operator, and α is a parameter. The Hamiltonian (4) generates a PDS of type III [4]. For $\alpha = 0$, $\hat{H}_{\rm M}$ has exact $O(6)$ symmetry whereas for $\alpha > 0$ the last two terms introduce $O(6)$-symmetry breaking. However, the yrast states of this Hamiltonian, projected from the IBM intrinsic state with intrinsic variables [37] $\beta = 1$ and $\gamma = 0$, keep exact $O(6)$ symmetry ($\sigma = N$) but break the $O(5)$ symmetry (mixed τ) for all values of $\alpha > 0$ [7]. Interestingly, although $\hat{H}_{\rm M}$ differs from $\hat{H}_{\rm ECQF}$, the overlap between their $0^+_{\rm gs}$ ground states maximizes (more than 99%) in extended regions of (ξ,χ) inside the valley of low $\Delta\sigma_{\rm gs}$. This suggests that the $(\beta = 1, \gamma = 0)$ intrinsic state provides a good approximation, in a variational sense, to the ground band of $\hat{H}_{\rm ECQF}$ along the valley. The equilibrium deformations for a given IBM Hamiltonian are found by minimizing an energy surface, $E(\beta,\gamma)$, obtained by its expectation value in an intrinsic state which is a condensate of N bosons, $b^\dagger_c \propto \beta\cos\gamma d^\dagger_0 + \beta\sin\gamma(d^\dagger_2 + d^\dagger_{-2})/\sqrt{2} + s^\dagger$, that depends parametrically on (β,γ) [38, 39]. Apart from a constant, $E(\beta,\gamma) \propto (1+\beta^2)^{-2}\beta^2\left[a - b\beta\cos3\gamma + c\beta^2\right]$, where a, b, and c are coefficients depending on the Hamiltonian. The two extremum equations, $\partial E/\partial\beta = \partial E/\partial\gamma = 0$, have $\beta = 1$ and $\gamma = 0$ as a solution, provided $b = 2c$. For large N, the coefficients of $\hat{H}_{\rm ECQF}$ are $b = -\omega\xi\sqrt{\frac{2}{7}}\chi/N$ and $c = \omega\left[1 - \xi - \xi\chi^2/14\right]/N$. Thus, in the valley of low $\Delta\sigma_{\rm gs}$ the desired condition, $b = 2c$, fixes ξ to be

$$\xi = \frac{1}{1 - \sqrt{\frac{1}{14}}\chi + \frac{1}{14}\chi^2}\,. \quad (5)$$

As seen in Fig. 3, this relation predicts the location of the region of approximate ground-state $O(6)$ symmetry for large N very precisely. For small N its precision decreases somewhat due to finite-N effects, causing a more pronounced curvature of the region close to the $O(6)$ DS limit.

Detailed ECQF fits for energies and electromagnetic transitions of rare-earth nuclei, performed by McCutchan *et al.* [30], allow one to relate the structure of collec-

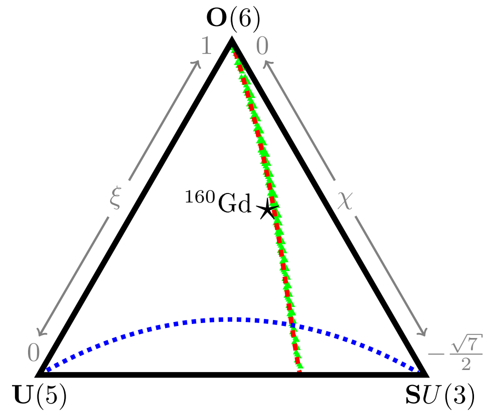

FIG. 3. (Color online) The ECQF symmetry triangle with the position of the nucleus ^{160}Gd indicated by a star. The green area shows the region of low $\Delta\sigma_{\rm gs}$, calculated from Eq. (3) for $N = 60$. The red dashed line shows the same region of approximate ground-state $O(6)$ symmetry, as predicted by Eq. (5) for large N. The blue dotted line shows the "arc of regularity" [35].

TABLE I. Calculated σ fluctuations $\Delta\sigma_L$, Eq. (3), for rare earth nuclei in the vicinity of the identified region of approximate ground-state-$O(6)$ symmetry. Also shown are the fraction $f^{(L)}_{\sigma={\rm N}}$ of $O(6)$ basis states with $\sigma = N$ contained in the $L=0,2,4$ states, members of the ground band. The structure parameters ξ and χ are taken from [30].

Nucleus	N	ξ	χ	$\Delta\sigma_0$	$f^{(0)}_{\sigma={\rm N}}$	$\Delta\sigma_2$	$f^{(2)}_{\sigma={\rm N}}$	$\Delta\sigma_4$	$f^{(4)}_{\sigma={\rm N}}$
^{156}Gd	12	0.72	-0.86	0.46	95.3%	0.43	95.8%	0.38	96.6%
^{158}Gd	13	0.75	-0.80	0.35	97.2%	0.33	97.5%	0.30	97.9%
^{160}Gd	14	0.84	-0.53	0.19	99.1%	0.19	99.2%	0.17	99.3%
^{162}Gd	15	0.98	-0.53	0.41	96.0%	0.40	96.0%	0.40	96.1%
^{160}Dy	14	0.81	-0.49	0.44	96.2%	0.39	96.4%	0.36	96.8%
^{162}Dy	15	0.92	-0.31	0.07	99.9%	0.07	99.9%	0.06	99.9%
^{164}Dy	16	0.98	-0.26	0.13	99.6%	0.13	99.6%	0.13	99.6%
^{164}Er	14	0.84	-0.37	0.39	96.5%	0.37	96.7%	0.35	97.1%
^{166}Er	15	0.91	-0.31	0.12	99.7%	0.11	99.7%	0.10	99.7%

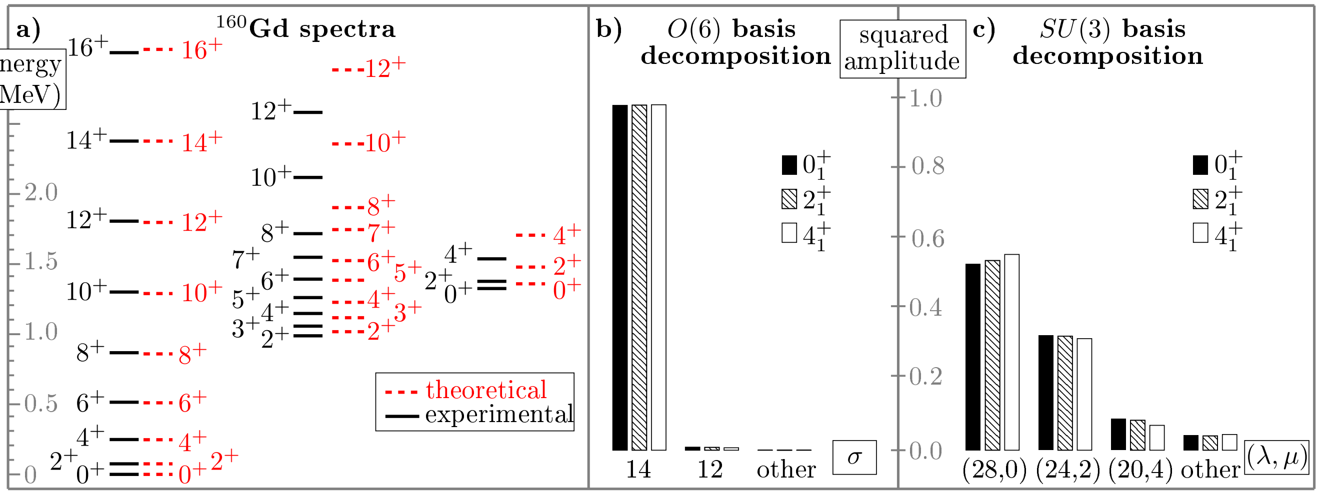

FIG. 4. (Color online) a) The experimental spectrum of ^{160}Gd compared with the IBM calculation using the ECQF Hamiltonian (1) with parameters $\xi = 0.84$ and $\chi = -0.53$ taken from Ref. [30]. b) The $O(6)$ decomposition in σ components of yrast states with $L = 0, 2, 4$. c) The $SU(3)$ decomposition in (λ, μ) components of the same yrast states.

tive nuclei to the parameter space of the ECQF Hamiltonian (1). Examining the extracted (ξ, χ) parameters, one finds that several rotational nuclei in this region, such as ^{160}Gd, commonly interpreted as $SU(3)$-like nuclei, are actually located in the valley of small σ fluctuations. They can be identified as candidate nuclei with approximate ground-state $O(6)$ symmetry. The experimental spectrum of ^{160}Gd, along with its ECQF description with $\xi = 0.84$ and $\chi = -0.53$ taken from Ref. [30], is shown in the left panel of Fig. 4. The middle and right panels show the decomposition into $O(6)$ and $SU(3)$ basis states, respectively, for yrast states with $L = 0, 2, 4$. It is evident that the $SU(3)$ symmetry is broken, as significant contributions of basis states with different $SU(3)$ quantum numbers (λ, μ) occur. It is also clear from Fig. 4c that this mixing occurs in a coherent manner with similar patterns for the different members of the ground-state band. This is the hallmark of a QDS [18] and it results from the existence of a single intrinsic wave function for the members of this band. On the other hand, as seen in Fig. 4b, the yrast states with $L = 0, 2, 4$ are almost entirely composed out of $O(6)$ basis states with $\sigma = N = 14$ which implies small fluctuations $\Delta\sigma_\Psi$ and the preservation of $O(6)$ symmetry in the ground-state band.

Other rare-earth nuclei with ground-state bands with approximate $O(6)$ symmetry can be identified by the same arguments. Their structure parameters ξ and χ can be taken from Ref. [30], from where the fluctuations $\Delta\sigma_\Psi$ and the fractions $f_{\sigma=N}$ of squared $\sigma = N$ amplitude can be calculated. Nuclei with $\Delta\sigma_{\rm gs} < 0.5$ and $f_{\sigma=N} > 95\%$ are listed in Table I. These quantities are also calculated for yrast states with $L > 0$ and exhibit similar values in each nucleus. It is evident that the IBM predicts a high degree of $O(6)$ purity in the ground-state-band, for a large set of rotational rare-earth nuclei.

These results show that the approximate $O(6)$ PDS does hold not only for the ground state but also for the members of the band built on top of it. Since the entire band corresponds to a single intrinsic state, the $SU(3)$ wave-function decomposition is similar for the different members of the band and therefore the notion of $SU(3)$ QDS applies. In addition, provided the indicated intrinsic state has $\beta \approx 1$ and $\gamma = 0$, the notion of $O(6)$ PDS applies. Thus a link is established between $SU(3)$ QDS and $O(6)$ PDS.

To summarize, the method of quantum-number fluctuations reveals the existence of a region of almost exact ground-state-band $O(6)$ symmetry outside the $O(6)$ DS limit of the IBM. The existence of a valley of small σ fluctuations can be understood in terms of an approximate $O(6)$ PDS of type III. The same wave functions display coherent (L-independent) mixing of $SU(3)$ representations and hence comply with the conditions of an $SU(3)$ QDS. Coherent mixing of one symmetry may therefore result in the purity of a quantum number associated with partial conservation of a different, incompatible symmetry. Previously established ECQF systematics show that many rare-earth nuclei do exhibit these approximate partial $O(6)$ and quasi $SU(3)$ dynamical symmetries. We conclude that partial dynamical symmetries are more abundant than previously recognized, may lead to coherent mixing and quasi dynamical symmetries, and hence play a role in understanding the regular behavior of complex nuclei. This example serves to illustrate a fundamental linkage between two distinct types of intermediate symmeteries, PDS and QDS, with potential implications to algebraic modeling of diverse dynamical systems.

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