PARAMETERS OF THE EIGENELLIPSOID FOR SEPARATED SECTOR CYCLOTRONS

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ABSTRACT

The analytical expressions of the elements of the beam matrix corresponding to the eigenellipsoid for a beam injected on an equilibrium orbit of a cyclotron are presented. The four dimensional phase space of the horizontal plane is only considered. Some restrictive hypotheses are made: there is no acceleration, and space charge effects are not taken into account. The beam matrix has been computed for the general case of spiraled sectors cyclotrons, and is valid for any given azimuth.

1. INTRODUCTION

The problem of beam matching at the injection into a separated sector cyclotron has been solved with a good approximation by many authors. We will mainly make reference to the well documented publication of G. Hinderer.[1)]

There exist presently computer programs which give with a great accuracy the numerical value of the elements of the transfer matrix corresponding to an equilibrium orbit for a given magnetic field map. Incidentally, these programs work also if acceleration is present. This opens the possibility of computing accurately the elements of the beam matrix of the eigenellipsoid corresponding to a given equilibrium orbit.

We have undertaken such a study for the horizontal plane[2)] . The problem in the vertical plane, supposed uncoupled, is straight-forward and will not be discussed.

By definition, the eigenellipsoid must recover its original characteristics after one complete revolution along an equilibrium orbit. This definition can be written, using the formalism of program TRANSPORT as follows:

$$R \cdot \sigma_E \cdot R^T = \sigma_E \qquad (1)$$

where R represents the transfer matrix for one revolution along a given equilibrium orbit, R^T the transpose of R, and σ_E the eigenellipsoid beam matrix.

The same formula applies to any individual section of a cyclotron composed of N identical sections, R being in this case the transfer matrix corresponding to one section.

In addition to the Rij elements of the transfer matrix R, the following parameters of the ellipsoid are supposed to be given:

- ε, which determines the transverse emittance $\pi.\varepsilon$
- z_o, the value of the longitudinal extension of the ellipsoid, should the longitudinal emittance be represented by an upright ellipse
- δ_o, the momentum dispersion

2. SOLVING THE PROBLEM

Solving equation (1) in the four dimensional phase space x, θ, z, δ is not straightforward and laborious developments can be avoided using the following guide line. Firstly the approach of the four dimensional problem must be progressive. The two dimensional problem x, θ being solved, the third dimension z can then be introduced, observing that the elements of the beam matrix for the two dimensional phase space do not include any R3j element of the transfer matrix. Secondly, in this progressive approach, the following formula, to be commented in the next paragraph, should be used in conjunction with formula (1) to help solve easily the four dimensional problem:

$$\sigma_E = C \cdot \sigma_o \cdot C^T \qquad (2)$$

From a general point of view, some remarks can be made. Equation (1) leads to a system of 10 elementary equations to be solved to get the expressions of the 10 elements of the beam matrix.

In fact, these equations depend on a total of 8 independent parameters: the 3 above mentioned parameters ε, z_o, δ_o, plus 5 parameters for the transfer matrix R. Normally, matrix R includes 9 different parameters, but there are between these parameters 3 symplectic relations and one relation for isochronism. Consequently, one must be aware that some degree of redundancy is inherent in the set of the 10 equations.

3. THE FOUR DIMENSIONAL BEAM MATRIX

The solution for the four dimensional eigenellipsoid beam matrix is presented as follows, according to formula (2) and is valid for a spiraled sectors cyclotron, at any given azimuth on an equilibrium orbit:

$$\sigma_E = \overbrace{\begin{vmatrix} 1 & 0 & 0 & C_{14} \\ 0 & 1 & 0 & C_{24} \\ C_{31} & C_{32} & 1 & C_{34} \\ 0 & 0 & 0 & 1 \end{vmatrix}}^{C} * \overbrace{\begin{vmatrix} S_{11} & S_{21} & 0 & 0 \\ S_{21} & S_{22} & 0 & 0 \\ 0 & 0 & z_0^2 & 0 \\ 0 & 0 & 0 & \delta_0^2 \end{vmatrix}}^{\sigma_0} * \overbrace{\begin{vmatrix} 1 & 0 & C_{31} & 0 \\ 0 & 1 & C_{32} & 0 \\ 0 & 0 & 1 & 0 \\ C_{14} & C_{24} & C_{34} & 1 \end{vmatrix}}^{C^T}$$

σ_0 is a beam matrix in which Sij's are the elements of the matrix of the eigenellipsoid in the x, θ phase space if $\delta_0 = 0$

C is a symplectic transfer matrix. Consequently:

$$C_{14} = - C_{31} \qquad C_{24} = C_{32}$$

C^T is the transpose of C

The Rij elements of the transfer matrix R being known, the elements of matrices σ_0 and C are expressed as follows:

$$S_{11} = S_{21} \cdot 2 \cdot R_{12}/(R_{22} - R_{11})$$

$$S_{22} = S_{21} \cdot 2 \cdot - R_{21}/(R_{22} - R_{11})$$

$$S_{21}{}^2 = \varepsilon^2 \cdot (R_{22} - R_{11})^2/(4 - (R_{22} + R_{11})^2)$$

$S_{21}{}^2$ must remain positive, this establishes the criterion of stability in the x, θ phase space: $(R_{22} + R_{11})^2 < 4$. The sign to be taken for S_{21} is the one which makes S_{11} and S_{22} positive, because all S_{ii} elements of the beam matrix are positive by definition.

$$C_{24} = - C_{31} = (R_{31} \cdot (1 - R_{22}) + R_{21} \cdot R_{32}) / (2 - (R_{22} + R_{11}))$$

$$C_{14} = C_{32} = (- R_{32} \cdot (1 - R_{11}) - R_{12} \cdot R_{31}) / (2 - (R_{22} + R_{11}))$$

C_{34} is found to be a free parameter. Its position in the C matrix shows that this element is related to the tilt of the eigenellipsoid in the longitudinal phase space. The value zero for C_{34} corresponds to an upright ellipsoid. This condition gives the smallest beam envelope longitudinally and must be selected in order to minimize the aberrations caused by the accelerating system. But, to the first order, there are theoretically no constraints on the value of C_{34} which remains a free parameter.

It seems obvious that the longitudinal confinement cannot be achieved if the magnetic field is not isochronous, the definition of an isochronous magnetic field being expressed by the statement that all equilibrium orbits have the same revolution frequency. But, from a formal point of view, the necessity of an isochronous field cannot be introduced as an hypothesis.

Actually, when solving the system of equations (1), the condition to be fulfilled for ensuring the existence of a longitudinal beam envelope is found to be:

$$- R_{32} \cdot C_{31} + R_{31} \cdot C_{32} + R_{34} = 0$$

and this condition, which involves the element R_{34} of the transfer matrix, appears to be identical to the condition of isochronism for the magnetic field:

$$- R_{14} \cdot C_{31} - R_{24} \cdot C_{32} + R_{34} = 0$$

It can also be noted that the Cij elements of transfer matrix C represent intrinsically the various couplings involving the longitudinal coordinates of individual particles in a matched beam.

4. SINGULAR AZIMUTH FOR SPIRALED SECTORS CYCLOTRONS

If the starting (and also final) point for the computation of the transfer matrix R is moved along the equilibrium orbit, the Rij elements become variable. They behave as periodic functions of the azimuth of the starting point, the period being $2\pi/N$ if the cyclotron is composed of N identical elementary sections. By convention, the first section begins at the starting point. As far as elements R_{11} and R_{22} are concerned, their sum remains constant because the betatron phase shift μ over one revolution does not depend on the choice of the starting point:

$$R_{11} + R_{22} = 2 \ . \ \cos \mu$$

From the above statements, and with the help of the first order optical model applied to each section, it can be shown that functions R_{11} and R_{22} do intersect, and they do this 2 N times during one complete revolution. The intersection points, where $R_{11} = R_{22}$, have no other peculiar geometric location, except that they form two series of equidistant points along the equilibrium orbit. Their position should be found by numerical computation, using an iterative process.

The eigenellipsoid matrix computed at any intersection point ($R_{11} = R_{22}$) has the same expression as in the general case except for the Sij elements which become:

$$S_{11} = \varepsilon \cdot \mathrm{SQRT}\ (- R_{12}/R_{21})$$

$$S_{22} = \varepsilon \cdot \mathrm{SQRT}\ (- R_{21}/R_{12})$$

$$S_{21} = 0$$

5. THE CASE OF CYCLOTRONS WITH SYMMETRIC SECTORS

The above mentioned intersection points for functions R_{11} and R_{22} are located on the 2 N symmetry axes of a cyclotron having N identical "straight" sectors.

For any of these points, the Sij elements have the same expressions as mentioned in the preceding paragraph.

In addition, the solution of the set of equations (1) gives:

$$C_{24} = - C_{31} = 0$$

$$C_{14} = C_{32} = R_{31}/R_{21}$$

It can be observed on practical cases that the expression r_0/γ^2 proposed by G. Hinderer is quite close (a few per cent) to the expression C_{32}.

The isochronism relation is thus:

$$R^2_{31}/R_{21} + R_{34} = 0$$

and due to the symmetry of the magnetic field matrix:

$$R_{31} = - R_{24} \quad \text{and} \quad R_{32} = - R_{14}.$$

6. CONCLUSION

We have established the analytical expression of the parameters of the eigenellipsoid corresponding to a given equilibrium orbit of a cyclotron, in the four dimensional horizontal phase space. The method does not apply to the case of an accelerated beam. However, as far as the radial gain per turn remains small in comparison with the mean radius of rotation, the parameters of the eigenellipsoid computed for the equilibrium orbit can be used conveniently. This has been confirmed by numerical computation of beam envelopes in presence of acceleration. To extend the principle of the method to an accelerated beam will require first, to formulate the definition of a matched beam in presence of acceleration. Moreover, it should be noted that in most practical cases there exists a lack of symmetry between the magnetic field and the accelerating system (for instance: 4 magnetic sectors and 2 RF resonators), resulting in a slight imbalance of the turn pattern. Under these circumstances the concept of a well matched beam becomes even more complex to formulate rigorously.

1) Hinderer, G. "Phase space matching between preaccelerators and cyclotrons," in Proceedings of the Ninth Conference on Cyclotrons and their Applications, 1981, pp. 327-335.
2) Fermé, J. GANIL internal report, R.89-03, February 7, 1989