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ScienceDirect

Nuclear Physics B 885 (2014) 266–279

NUCLEAR PHYSICS B

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Current reflection and transmission at conformal defects: Applying BCFT to transport process

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Received 6 March 2014; received in revised form 20 May 2014; accepted 21 May 2014

Available online 27 May 2014

Editor: Hubert Saleur

Abstract

We study reflection/transmission process at conformal defects by introducing new transport coefficients for conserved currents. These coefficients are defined by using BCFT techniques thanks to the folding trick, which turns the conformal defect into the boundary. With this definition, exact computations are demonstrated to describe reflection/transmission process for a class of conformal defects. We also compute the boundary entropy based on the boundary state.

1. Introduction

A wide range of physicists – cosmologists, condensed matter physicists, and particle physicists – have been attracted by anomalous scaling behavior of matter caused by critical phenomena. Studying critical phenomena with conformal defects is of great interest, because most of realistic situations inevitably contain impurities. A powerful method for studying critical phenomena with conformal defects is boundary conformal field theory (BCFT). There are many applications of BCFT especially to one-dimensional quantum systems with impurities, e.g., the Heisenberg spin chain, the Kondo model, and so on. See [1] for a review along this direction.

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http://dx.doi.org/10.1016/j.nuclphysb.2014.05.026
0550-3213/

However, it has not been completely understood how BCFT describes the reflection/transmission at conformal defects. For this purpose, Quella, Runkel, and Watts proposed the reflection/transmission coefficient to characterize the transport phenomena at the conformal defect [2]. Their proposal is quite natural and generic in the sense of CFT because their coefficients are based on the gluing condition for the energy–momentum tensor. However, it is not obvious how the proposed coefficient is related to transport coefficients used in other contexts, such as quantum wire junctions and experiments. Our goal is to further investigate the meaning of the proposed reflection/transmission coefficient and to obtain a more detailed description of the reflection/transmission process.

In this paper, we define the reflection/transmission coefficient for conserved currents, as a natural generalization of that proposed in [2] and also in [3]. Our definition involves current algebras and boundary states, which characterize boundary conditions of fields at conformal defects. We demonstrate exact computations for two systems: the system having permutation boundary conditions and the system partially breaking the $SU(2)_{k_1} \times SU(2)_{k_2}$ symmetry into the $SU(2)_{k_1+k_2}$. Our definition also reveals which current penetrates the conformal defects as well as how much it does. In addition, we compute the boundary entropies to identify the amount of information carried by the boundary. In general, it is difficult to distinguish the contributions of the boundary and the bulk CFTs to the entropy. In our analysis, since the boundary state is explicitly constructed, we can separate them more efficiently, and obtain results consistent with previous works.

2. Reflection and transmission coefficients

We shall briefly review the reflection/transmission coefficient proposed in [2] and give a more detailed meaning to that. That is to say, we claim that the proposed reflection/transmission coefficient corresponds to the energy transport. Besides, by generalizing their proposal, we define the reflection/transmission coefficient for a conserved current with conformal weight $h = 1$.

2.1. Conformal defect and the junction

We consider two one-dimensional quantum systems connected by a junction, which can be considered as an impurity interacting with the bulk. Let us assume that the first system is in the positive domain $x > 0$, the second is in the negative $x < 0$, and they are connected at the origin as depicted in Fig. 1(a). If these systems obey symmetry algebras $\mathcal{A}_i$, the Hamiltonian density for each domain is obtained by Sugawara construction at the conformal fixed point[1]

$$\mathcal{H}^1(x) = \frac{1}{2\pi(k_1 + h_1^\vee)} d_{AB}^1 J^{1,A}(x) J^{1,B}(x) \quad (x > 0) \tag{2.1}$$

$$\mathcal{H}^2(x) = \frac{1}{2\pi(k_2 + h_2^\vee)} d_{AB}^2 J^{2,A}(x) J^{2,B}(x) \quad (x < 0) \tag{2.2}$$

where d_{AB}^i is the inverse of the Cartan–Killing form and $h_i^\vee$ is the dual Coxeter number of the algebra $\mathcal{A}_i$. The current $J^{i,A}$ takes value in the Lie algebra $\mathcal{A}_i$ and the index A runs over $A = 1, \cdots, \dim \mathcal{A}_i$. In general, $\mathcal{A}_1$ and $\mathcal{A}_2$ can be different algebras. The Fourier modes of $J^{i,A}$ satisfy the Kac–Moody algebra $\widehat{\mathcal{A}}_i$:

1 Here we omit the anti-holomorphic part.

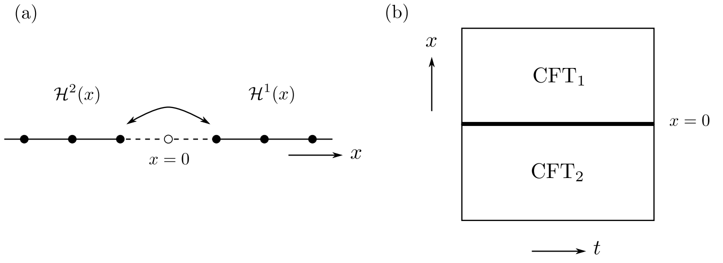

Fig. 1. From the *impurity* to the *defect*. (a) Two one-dimensional systems are connected through the impurity at $x=0$. (b) Adding the time direction and taking the continuum limit, that system is mapped into the two-dimensional system with the defect along the line $x=0$.

$$\left[j_m^{i,A}, j_n^{i,B}\right] = \left(f^i\right)^{AB}{}_C\, j_{n+m}^{i,C} + k_i m d^{i,AB} \delta_{m+n,0}, \tag{2.3}$$

where f^i is the structure constant of $\mathcal{A}_i$ and k_i is the level of $\widehat{\mathcal{A}}_i$. Especially for the $SU(2)$ theory, this level corresponds to the electron spin as $k=2s$ for the multi-critical spin chain [4,5] and to the number of channels for the Kondo model [6–9]. Note that the anti-holomorphic parts satisfy the same Kac–Moody algebras.

In general, the impurity breaks the symmetry of the bulk theory, and couples to a common subalgebra $\mathcal{C}$ of $\mathcal{A}_1$ and $\mathcal{A}_2$. One possibility for the interaction term is

$$\mathcal{H}^{\text{int}}(x) = \delta(x) d_{ab}\left(\lambda_1 J^{1,a} + \lambda_2 J^{2,a}\right) S^b, \tag{2.4}$$

where λ_i are the coupling constants and $J^{i,a}$ takes value in the subalgebra $\mathcal{C}$. Here, S^a stands for the impurity *spin* and d^{ab} is the Cartan–Killing form for $\mathcal{C}$. For this kind of interaction, as well discussed in the Kondo problem, we can complete the square by shifting the current $\mathcal{J}^{i,a} = J^{i,a} + 2\pi S^a \delta(x)$ when the coupling constant takes the critical value. Then we obtain a quadratic Hamiltonian again. This observation indicates the existence of a non-trivial conformal fixed point at low energy with the impurity spin absorbed. We remark that although (2.4) is written in terms of the currents, there are models whose interaction terms should be written in terms of fundamental fields rather than currents, e.g., the spin chain with a single impurity model [10]. Even in such a case, it is expected that the conformal fixed point obtained by the RG flow is described by the Hamiltonian (2.1) and (2.2) with boundary conditions, which are specified in the following sections.

Now we shall describe the above system in terms of BCFT. Corresponding to the two quantum systems, the BCFT picture involves two CFTs: CFT$_1$ and CFT$_2$. These CFTs are defined in the upper and lower half planes respectively as depicted in Fig. 1(b). The real axis, which divides the two CFTs, stands for the world line of the impurity, or the defect. We can reformulate this system to obtain CFT$_1$ $\times$ $\overline{\text{CFT}}_2$ in the upper half plane thanks to the folding trick [11–13], as shown in Fig. 2. In this way, the junction of the one-dimensional quantum systems can be mapped into a CFT boundary condition.

2.2. *Energy reflection and transmission*

Let us then introduce the reflection/transmission coefficient to characterize the transport phenomena at the conformal defect. For this purpose, there are two key ingredients. The first is a boundary state. The boundary state is a state of BCFT which characterizes a boundary condi-

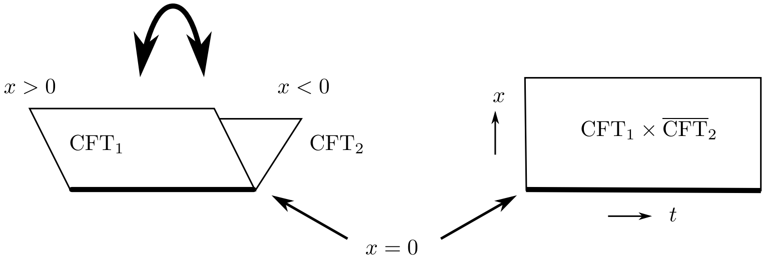

Fig. 2. From the *defect* to the *boundary*. By using the folding trick, a system with the defect is mapped into another system defined on the upper half plane with the boundary.

tion at the defect. For example, the boundary condition for the energy–momentum tensor, which implies the energy conservation at the defect, gives the so-called Virasoro gluing condition:

$$\left(L_n^{\mathrm{tot}} - \overline{L}_{-n}^{\mathrm{tot}}\right)|B\rangle = 0. \tag{2.5}$$

Here L_n^{tot} is the sum of Virasoro generators of $\mathrm{CFT}_{1,2}$:

$$L_n^{\mathrm{tot}} = L_n^1 + L_n^2. \tag{2.6}$$

The Virasoro gluing condition also ensures that the junction preserves the conformal symmetry.[2]

The second is the R-matrix, the 2 by 2 matrix defined as [2]

$$R^{ij} = \frac{\langle 0|L_2^i \overline{L}_2^j|B\rangle}{\langle 0|B\rangle}, \quad i, j = 1, 2, \tag{2.7}$$

where $|0\rangle$ is the conformal vacuum. Although the R-matrix has four components, it has only one degree of freedom due to the following three constraints. The first constraint is given by the Virasoro gluing condition:

$$\langle 0|L_2^{\mathrm{tot}} \overline{L}_2^{\mathrm{tot}}|B\rangle = \langle 0|L_2^{\mathrm{tot}} L_{-2}^{\mathrm{tot}}|B\rangle = \frac{c_1 + c_2}{2}\langle 0|B\rangle, \tag{2.8}$$

where $c_{1,2}$ are the central charges for $\mathrm{CFT}_{1,2}$, respectively. The second and the third constraints originate from the existence of two primary fields with respect to the total energy–momentum tensor $T^{\mathrm{tot}} = T^1 + T^2$ and its Hermitian conjugate: $W = c_2 T_1 - c_1 T_2$ and $\overline{W} = c_2 \overline{T}_1 - c_1 \overline{T}_2$. Thus we have

$$\langle 0|L_2^{\mathrm{tot}} \overline{W}_2|B\rangle = \langle 0|W_2 \overline{L}_2^{\mathrm{tot}}|B\rangle = 0. \tag{2.9}$$

We remark that these constraints show that R^{ij} is symmetric:

$$0 = \langle 0|\left(L_2^{\mathrm{tot}} \overline{W}_2 - W_2 \overline{L}_2^{\mathrm{tot}}\right)|B\rangle = -(c_1 + c_2)\langle 0|\left(L_2^1 \overline{L}_2^2 - L_2^2 \overline{L}_2^1\right)|B\rangle. \tag{2.10}$$

As a result, the R-matrix is parametrized by a single real parameter

$$\omega_B = \frac{2}{c_1 c_2 (c_1 + c_2)} \frac{\langle 0|W_2 \overline{W}_2|B\rangle}{\langle 0|B\rangle}, \tag{2.11}$$

[2] Beside, in string theory context, this condition ensures that the energy flow vanishes at the open string endpoints.

as

$$R = \frac{c_1 c_2}{2(c_1+c_2)} \left[\begin{pmatrix} \frac{c_1}{c_2} & 1 \\ 1 & \frac{c_2}{c_1} \end{pmatrix} + \omega_B \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right]. \tag{2.12}$$

Now we give the definition of the reflection/transmission coefficient $\mathcal{R}/\mathcal{T}$. The proposal for $\mathcal{R}$ and $\mathcal{T}$ is [2]

$$\mathcal{R} = \frac{2}{c_1+c_2}\left(R^{11}+R^{22}\right), \tag{2.13}$$

$$\mathcal{T} = \frac{2}{c_1+c_2}\left(R^{12}+R^{21}\right). \tag{2.14}$$

It is easy to show that the sum is given by $\mathcal{R}+\mathcal{T}=1$ for any ω_B, which means the energy conservation. Because the R-matrix is written in terms of Virasoro generators, we suggest that $\mathcal{R}$ and $\mathcal{T}$ are the reflection and transmission coefficients for the energy transport at the defect. We shall see that this interpretation is consistent with our definition of the current reflection/transmission coefficient.

2.3. *Current reflection and transmission*

We generalize the above construction of $\mathcal{R}$ and $\mathcal{T}$ to the reflection/transmission coefficient for a conserved current with $h=1$. When we define the energy reflection/transmission coefficient, there are two key ingredients: the boundary state and the R-matrix. In addition, there are three constraints, which originated from the total energy–momentum tensor T^{tot} and the primary fields W and $\overline{W}$, play an important role in counting the effective degrees of freedom of the R-matrix. Here we shall take the similar process.

We assume that $\text{CFT}_{1,2}$ have the same symmetry subalgebra $\mathcal{C}$, which is preserved at the conformal defect. For such a defect, we choose the following current gluing condition

$$\left(j_n^{\text{tot},a} + \bar{j}_{-n}^{\text{tot},a}\right)|B\rangle = 0, \tag{2.15}$$

where $j_n^{\text{tot},a} = j_n^{1,a} + j_n^{2,a}$ takes values in the Kac–Moody algebra $\hat{\mathcal{C}}$. (Here j_n^a is the Fourier mode of J^a.) Notice that the signs in front of the anti-holomorphic sectors are opposite between energy and current gluing conditions due to the different parity of their conformal weights [14].

The straightforward generalization of the R-matrix is

$$R[\mathcal{C}]^{ij,ab} = -\frac{\langle 0|j_1^{i,a}\bar{j}_1^{j,b}|B\rangle}{\langle 0|B\rangle}. \tag{2.16}$$

The extra minus sign is due to the sign difference in the gluing conditions. Note that since the gluing condition becomes rather complicated in terms of $j_n^{1,a}$ and $j_n^{2,a}$, this R-matrix gives a nontrivial value, as we will show later. Here we take the $n=1$ component of $j_n^{i,a}$ in contrast to L_2^i. In fact, any positive choice of n gives the same R-matrix. To see this fact, let us consider the following equation derived from the Virasoro gluing condition:

$$0 = \langle 0|j_n^{i,a}\bar{j}_{n+1}^{j,b}\left(L_1^{\text{tot}} - \bar{L}_{-1}^{\text{tot}}\right)|B\rangle. \tag{2.17}$$

Together with the commutator $[L_m^i, j_n^{i,a}] = -n j_{m+n}^{i,a}$, this leads to the recursion relation

$$0 = n\langle 0|j_{n+1}^{i,a}\bar{j}_{n+1}^{j,b}|B\rangle - (n+1)\langle 0|j_n^{i,a}\bar{j}_n^{j,b}|B\rangle. \tag{2.18}$$

This relation implies that if we defined the R-matrix with mode n, the matrix element $\langle 0| j_n^{i,a} \bar{j}_n^{j,d} |B\rangle$ could be written in terms of (2.16). In addition, due to the symmetry, we have

$$R[\mathcal{C}]^{ij,ab} = -\frac{\langle 0|G j_1^{i,a} \bar{j}_1^{j,b} G^{-1}|B\rangle}{\langle 0|B\rangle}, \tag{2.19}$$

where $G = \exp\{\alpha_a (j_0^{\mathrm{tot},a} + \bar{j}_0^{\mathrm{tot},a})\}$. If $\mathcal{C}$ is a simple Lie algebra, this symmetry factorizes the R-matrix:

$$R[\mathcal{C}]^{ij,ab} = d^{ab} R[\mathcal{C}]^{ij}. \tag{2.20}$$

We remark that although the R-matrix can be defined with a generic algebra rather than the common subalgebra $\mathcal{C}$, the contribution only from $\mathcal{C}$ gives a non-trivial value.[3]

Now let us see that there are three constraints which reduce the degrees of freedom of the R-matrix. The first constraint is associated with the current gluing condition. Eq. (2.15) leads to

$$\langle 0| j_1^{\mathrm{tot},a} \bar{j}_1^{\mathrm{tot},b} |B\rangle = -(k_1 + k_2) d^{ab} \langle 0|B\rangle. \tag{2.21}$$

This constraint is similar to the constraint (2.8), which is given by the Virasoro gluing condition. To find the other two constraints, we introduce primary fields with respect to the total current $J^{\mathrm{tot}} = J^1 + J^2$ (and its conjugate $\bar{J}^{\mathrm{tot}}$):

$$\begin{aligned} K^a(z) &= k_2 J^{1,a}(z) - k_1 J^{2,a}(z), \\ \bar{K}^a(z) &= k_2 \bar{J}^{1,a}(z) - k_1 \bar{J}^{2,a}(z). \end{aligned} \tag{2.22}$$

It is easy to show that these satisfy

$$\langle 0| K_1^a \bar{j}_1^{\mathrm{tot},b} |B\rangle = \langle 0| j_1^{\mathrm{tot},a} \bar{K}_1^b |B\rangle = 0. \tag{2.23}$$

Interestingly, these constraints ensure that $R[\mathcal{C}]^{ij}$ is symmetric. In fact, we have

$$0 = \langle 0| K_1^a \bar{j}_1^{\mathrm{tot},b} |B\rangle - \langle 0| j_1^{\mathrm{tot},a} \bar{K}_1^b |B\rangle = (k_1 + k_2) \langle 0| \big(j_1^{1,a} \bar{j}_1^{2,b} - j_1^{2,a} \bar{j}_1^{1,b} \big) |B\rangle. \tag{2.24}$$

Because of the above three constraints, $R[\mathcal{C}]^{ij}$ has only one degree of freedom. Now let us define $\omega_B[\mathcal{C}]$ as

$$d^{ab} \omega_B[\mathcal{C}] = -\frac{1}{k_1 k_2 (k_1 + k_2)} \frac{\langle 0| K_1^a \bar{K}_1^b |B\rangle}{\langle 0|B\rangle}. \tag{2.25}$$

With this $\omega_B[\mathcal{C}]$, the R-matrix $R[\mathcal{C}]^{ij}$ is given by

$$R[\mathcal{C}] = \frac{k_1 k_2}{k_1 + k_2} \left(\begin{pmatrix} \frac{k_1}{k_2} & 1 \\ 1 & \frac{k_2}{k_1} \end{pmatrix} + \omega_B[\mathcal{C}] \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right). \tag{2.26}$$

Obviously, this expression is similar to (2.12). The level k_i plays essentially the same role to the central charge.

[3] We can show that the reflection coefficient (2.28) gives $\mathcal{R} = 1$ for $\mathcal{A}_{1,2}/\mathcal{C}$, and thus it implies the full reflection process.

Now we can define the reflection and transmission coefficients: $\mathcal{R}[\mathcal{C}]$ and $\mathcal{T}[\mathcal{C}]$.

$$\mathcal{R}[\mathcal{C}] = \frac{1}{k_1+k_2}\left(R^{11}+R^{22}\right) = \frac{1}{(k_1+k_2)^2}\left(\left(k_1^2+k_2^2\right)+2k_1k_2\omega_B[\mathcal{C}]\right), \tag{2.27}$$

$$\mathcal{T}[\mathcal{C}] = \frac{1}{k_1+k_2}\left(R^{12}+R^{21}\right) = \frac{2k_1k_2}{(k_1+k_2)^2}\left(1-\omega_B[\mathcal{C}]\right). \tag{2.28}$$

From (2.26), it is easy to show that $\mathcal{R}+\mathcal{T}=1$, which ensures the current conservation. We remark that the identification of $\mathcal{R}/\mathcal{T}$ as the reflection/transmission coefficient is available provided that both of $\mathcal{R}$ and $\mathcal{T}$ are nonnegative. Although it is unclear that the nonnegative condition holds in general, we shall see that it holds for all examples considered in the present paper.

Before ending this section, let us comment on the case with $\mathcal{C}=su(2)$ for later use. In this case, the Cartan–Killing form is $d^{ab}=-\delta^{ab}/2$. By defining $J^{\pm}=J^1\pm iJ^2$, the R-matrix (2.20) can be rewritten as

$$R[\mathcal{C}]^{ij} = -R[\mathcal{C}]^{ij,-+} = \frac{\langle 0|j_1^{i,-}\bar{j}_1^{j,+}|B\rangle}{\langle 0|B\rangle}, \tag{2.29}$$

as well as $\omega_B[\mathcal{C}]$:

$$\omega_B[\mathcal{C}] = \frac{1}{k_1k_2(k_1+k_2)}\frac{\langle 0|K_1^-\overline{K}_1^+|B\rangle}{\langle 0|B\rangle}, \tag{2.30}$$

where we have used $d^{-+}=-1$.

3. Application to some models

We evaluate reflection and transmission coefficients for conserved currents by using the above definition. In Section 3.1, we consider the simpler case with the permutation boundary condition, where we know the explicit form of the boundary conditions for currents. On the other hand, in Section 3.2, we study the case where $SU(2)_{k_1}\times SU(2)_{k_2}$ is broken into $SU(2)_{k_1+k_2}$ thanks to the non-trivial interaction at the boundary.

3.1. Permutation boundary condition for a sub-symmetry

Let us first consider the simpler example where we impose the following boundary condition:

$$\begin{aligned} &J^{1,\alpha_1}(z)=\bar{J}^{1,\alpha_1}(z), \qquad J^{2,\alpha_2}(z)=\bar{J}^{2,\alpha_2}(z), \\ &J^{1,a}(z)=\bar{J}^{2,a}(z), \qquad J^{2,a}(z)=\bar{J}^{1,a}(z), \end{aligned} \tag{3.1}$$

where $\alpha_{1,2}$ and a stand for the labels for $\mathcal{A}_{1,2}/\mathcal{C}$ and $\mathcal{C}$ respectively. To be consistent with the boundary condition, we have to impose $k_1=k_2\equiv k_c$. In this example, degrees of freedom associated with $\mathcal{C}$ completely penetrate the defect, while the others are completely reflected. This observation suggests $\mathcal{T}[\mathcal{C}]=1$. Let us show this as follows.

Using the boundary condition, the off-diagonal elements of the R-matrix are

$$\langle 0|j_1^{1,a}\bar{j}_1^{2,b}|B\rangle = \langle 0|j_1^{2,a}\bar{j}_1^{1,b}|B\rangle = -d^{ab}k_c\langle 0|B\rangle. \tag{3.2}$$

This and (2.26) immediately show that $\omega_B=-1$, $\mathcal{R}^{11}=\mathcal{R}^{22}=0$, and $\mathcal{R}^{12}=\mathcal{R}^{21}=k_c$. This proves the full transmission: $\mathcal{T}[\mathcal{C}]=1$. This result is in contrast to the energy transmission coefficient $\mathcal{T}=2c/(c_1+c_2)$ [2] where $c_{1,2}$ and c are the central charges associated with $\mathcal{A}_{1,2}$ and $\mathcal{C}$.

This is because only the fields associated with $\mathcal{C}$ contribute to the transmission. In other words, we found that among the total degrees of freedom $c_1 + c_2$, $2c$ degrees of freedom completely penetrate and the others are completely reflected. (The factor 2 of $2c$ stems from the fact that both $j^{1,a}$ and $j^{2,a}$ contribute to the energy transport.) This argument supports our identification of $\mathcal{T}$ (2.14) as the total energy transmission coefficient. The benefit of our current transmission is that we can see more microscopic information about the transmission process.

3.2. $SU(2)_{k_1} \times SU(2)_{k_2} \to SU(2)_{k_1+k_2}$

Let us consider the more general case where $\widehat{\mathcal{A}}_{1,2} = su(2)_{k_{1,2}}$ and $\widehat{\mathcal{C}} = su(2)_{k_1+k_2}$. The symmetry can be rewritten as

$$SU(2)_{k_1} \times SU(2)_{k_2} = \frac{SU(2)_{k_1} \times SU(2)_{k_2}}{SU(2)_{k_1+k_2}} \times SU(2)_{k_1+k_2}. \tag{3.3}$$

Hereafter, we use $G = SU(2)_{k_1} \times SU(2)_{k_2}$ and $H = SU(2)_{k_1+k_2}$. The $SU(2)_{k_1+k_2}$-preserving boundary states are characterized by three parameters (ρ_1, ρ_2, ρ) which run over $2\rho_i = 0, 1, \cdots, k_i$ and $2\rho = 0, 1, \cdots, k_1 + k_2$ with the identification $(\rho_1, \rho_2, \rho) \sim (\frac{k_1}{2} - \rho_1, \frac{k_2}{2} - \rho_2, \frac{k_1+k_2}{2} - \rho)$ [15]:

$$\big|B(\rho_1, \rho_2, \rho)\big\rangle = \sum_{\mu_1+\mu_2+\mu\in\mathbb{Z}} \frac{S^{(k_1+k_2)}_{\rho\mu} S^{(k_1)}_{\rho_1\mu_1} S^{(k_2)}_{\rho_2\mu_2}}{S^{(k_1+k_2)}_{0\mu} \sqrt{S^{(k_1)}_{0\mu_1} S^{(k_2)}_{0\mu_2}}} \big|(\mu_1, \mu_2, \mu)\big\rangle\!\big\rangle \otimes |\mu\rangle\!\rangle, \tag{3.4}$$

with $2\mu_i \in \{0, 1, \cdots, k_i\}$ and $2\mu \in \{0, 1, \cdots, k_1 + k_2\}$. Here $|(\mu_1, \mu_2, \mu)\rangle\!\rangle$ is an Ishibashi state for G/H and $|\mu\rangle\!\rangle$ is a current Ishibashi state for H. $S^{(k)}_{\rho\mu}$ is the modular S-matrix of $SU(2)_k$ [16,17]

$$S^{(k)}_{\rho\mu} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}(2\rho + 1)(2\mu + 1)\right). \tag{3.5}$$

Because $J^{\text{tot},a} = J^{H,a}$, the above boundary state satisfies the current gluing condition (2.15).

In order to compute the R-matrix, we have to deal with $\langle 0| j^{i,a}_1 \bar{j}^{j,b}_1 |B\rangle$, whose non-trivial part is reduced to $\omega_B[\mathcal{C}]$ as shown in (2.26). Since $|B\rangle = |B(\rho_1, \rho_2, \rho)\rangle$ is spanned by the Hilbert space basis for $G/H \otimes H$, we first need to expand $j^{i,a}_{-1} \bar{j}^{j,b}_{-1} |0\rangle$ or $K^+_{-1} \overline{K}^-_{-1} |0\rangle$ with them as (3.4), and then identify the highest weight vectors with respect to $G/H \otimes H$.

To begin with, the ground state $|0\rangle$ is mapped to the tensor product of the ground states:

$$|0_G\rangle = \big|(0, 0, 0)\big\rangle \otimes |0_H\rangle. \tag{3.6}$$

Notice that $|0_G\rangle$ in the left hand side is the ground state for G, while $|0_H\rangle$ in the right hand side is that for H. To be more specific, let us focus on the holomorphic sector and consider $j^{i,+}_{-1}|0\rangle$. There should be two independent states corresponding to $i = 1, 2$. The first one can be easily found,

$$\frac{1}{\sqrt{k_1 + k_2}} j^{\text{tot},+}_{-1} |0_G\rangle = \frac{1}{\sqrt{k_1 + k_2}} \big|(0, 0, 0)\big\rangle \otimes j^{H,+}_{-1} |0_H\rangle \equiv |w_1\rangle. \tag{3.7}$$

Here we normalized the state: $\||w_1\rangle\|^2 = 1$. Another state that is orthogonal to $|w_1\rangle$ is

$$|w_2\rangle \equiv \frac{1}{\sqrt{k_1 k_2 (k_1 + k_2)}} K^+_{-1} |0_G\rangle. \tag{3.8}$$

Due to the commutation relation $[K^a_m, j^{\mathrm{tot},b}_n] = f^{ab}{}_c K^c_{m+n}$, we find that $|w_2\rangle$ is the current primary state with respect to H. Because $|w_2\rangle$ is killed when $j^{\mathrm{tot},-}_0$ acts three times, it belongs to a spin-1 representation. Therefore, the H-part of $|w_2\rangle$ is determined:

$$|w_2\rangle = |w_{G/H}\rangle \otimes |1_H\rangle. \tag{3.9}$$

In order to find $|w_{G/H}\rangle$, we shall investigate $L^{G/H}_1$. Because $|1_H\rangle$ is a primary state of H,

$$L^{G/H}_1|w_2\rangle = \left(L^G_1 - L^H_1\right)|w_2\rangle = L^G_1|w_2\rangle \propto \left(L^1_1 + L^2_1\right)K^+_{-1}|0_G\rangle = 0. \tag{3.10}$$

In the last equality, we have used $[L^i_m, j^{i,a}_n] = -nj^{i,a}_{m+n}$. Thus $|w_{G/H}\rangle$ is a primary state of G/H. Since the conformal weight of $|w_2\rangle$ is 1, the primary state of G/H is uniquely determined:

$$|w_2\rangle = \big|(0,0,1)\big\rangle \otimes |1_H\rangle. \tag{3.11}$$

Here, we normalized the states: $\||(0,0,1)\rangle\|^2 = \||1_H\rangle\|^2 = 1$.

According to [18], Ishibashi states are expressed as

$$\begin{aligned}
&\big|(0,0,0)\big\rangle\!\big\rangle \otimes |0_H\rangle\!\rangle = |0\rangle \otimes |\widetilde{0}\rangle + |w_1\rangle \otimes |\widetilde{Uw_1}\rangle + \cdots,\\
&\big|(0,0,1)\big\rangle\!\big\rangle \otimes |1_H\rangle\!\rangle = |w_2\rangle \otimes |\widetilde{Uw_2}\rangle + \cdots,
\end{aligned} \tag{3.12}$$

where tildes stand for the anti-holomorphic parts. Dots involve states with higher weights and the current descendant states such as $j^{H,-}_0|1_H\rangle$. U is an antiunitary operator that acts on H:

$$Uj^{H,+}_n U^{-1} = j^{H,-}_n, \qquad Uj^{H,3}_n U^{-1} = j^{H,3}_n. \tag{3.13}$$

By substituting (3.12) into (3.4), we obtain

$$\begin{aligned}
&\big|B(\rho_1,\rho_2,\rho)\big\rangle\\
&= \frac{S^{(k_1)}_{\rho_1 0} S^{(k_2)}_{\rho_2 0}}{\sqrt{S^{(k_1)}_{00} S^{(k_2)}_{00}}}\left(\frac{S^{(k_1+k_2)}_{\rho 0}}{S^{(k_1+k_2)}_{00}}\big|(0,0,0)\big\rangle\!\big\rangle \otimes |0\rangle\!\rangle + \frac{S^{(k_1+k_2)}_{\rho 1}}{S^{(k_1+k_2)}_{01}}\big|(0,0,1)\big\rangle\!\big\rangle \otimes |1\rangle\!\rangle + \cdots\right)\\
&= \frac{S^{(k_1)}_{\rho_1 0} S^{(k_2)}_{\rho_2 0}}{\sqrt{S^{(k_1)}_{00} S^{(k_2)}_{00}}}\frac{S^{(k_1+k_2)}_{\rho 0}}{S^{(k_1+k_2)}_{00}}\left(|0\rangle + |w_1\rangle \otimes |\widetilde{Uw_1}\rangle + \frac{S^{(k_1+k_2)}_{00} S^{(k_1+k_2)}_{\rho 1}}{S^{(k_1+k_2)}_{\rho 0} S^{(k_1+k_2)}_{01}}|w_2\rangle \otimes |\widetilde{Uw_2}\rangle + \cdots\right),
\end{aligned} \tag{3.14}$$

where $|0\rangle$ in the second line stands for $|0\rangle \otimes |\widetilde{0}\rangle$. The dots in the second line represent the states with higher weights as well as the descendants.

Now we proceed to the computation of the R-matrix. Using (2.30), ω_B is

$$\omega_B\big[su(2)\big] = \frac{1}{k_1 k_2(k_1+k_2)}\frac{\langle 0|K^-_1 \overline{K}^+_1|B\rangle}{\langle 0|B\rangle} = \frac{\langle w_2| \otimes \langle\widetilde{Uw_2}|B\rangle}{\langle 0|B\rangle} = \frac{S^{(k_1+k_2)}_{00} S^{(k_1+k_2)}_{\rho 1}}{S^{(k_1+k_2)}_{\rho 0} S^{(k_1+k_2)}_{01}}. \tag{3.15}$$

By substituting this into (2.28), the transmission coefficient is obtained as

$$\mathcal{T}\big[su(2)\big] = \frac{2k_1 k_2}{(k_1+k_2)^2}\left(1 - \frac{S^{(k_1+k_2)}_{00} S^{(k_1+k_2)}_{\rho 1}}{S^{(k_1+k_2)}_{\rho 0} S^{(k_1+k_2)}_{01}}\right). \tag{3.16}$$

Notice that $\mathcal{T}$ is independent of $\rho_{1,2}$ as in the case of the energy transmission [2]. In the case with $k_1 = k_2 = 1$, corresponding to the junction of $s = \frac{1}{2}$ Heisenberg spin chains, this transmission coefficient only gives 0 or 1. This is consistent with the fact that there are only full reflection and full transmission fixed points [10]. Another property of $\mathcal{T}$ is that $\mathcal{T} = 0$ when $\rho = 0$. We can explain this property as follows. It was found [15] that for $\rho = 0$ the original symmetry $G = SU(2)_{k_1} \times SU(2)_{k_2}$ is restored and the boundary state can be written as

$$\big|B(\rho_1, \rho_2, 0)\big\rangle = |\rho_1\rangle \otimes |\rho_2\rangle, \tag{3.17}$$

where $|\rho_i\rangle$ is Cardy's boundary state [14] for $\mathrm{CFT_i}$:

$$|\rho_i\rangle = \sum_{\mu_i} \frac{S^{(k_i)}_{\rho_i \mu_i}}{\sqrt{S^{(k_i)}_{\rho_i 0}}} |\mu_i\rangle\rangle, \tag{3.18}$$

with $2\mu_i \in \{0, 1, \cdots, k_i\}$. The right hand side of (3.17) immediately leads to the current gluing conditions for both $j^{1,a}$ and $j^{2,a}$. Thus we obtain $R^{12} = R^{21} = 0$, and hence $\mathcal{T} = 0$. The full reflection, or $\mathcal{T} = 0$, implies that the conformal defect is decoupled from the bulk system at the critical point.

4. Boundary entropy

In this section, we focus on the conformal defect as the impurity in the one-dimensional quantum system. In general, the current in the bulk theory interacts with the impurity as (2.4), and thus this impurity contributes to the total free energy of the system. This means that there is also the impurity contribution to the thermodynamic entropy. This impurity entropy, also called boundary entropy [19], can be detected, for example, by estimating the entanglement entropy [20]. See also [21].

In order to define the boundary entropy, let us set the total length of the system $2L$ and the temperature T by compactifying the time direction. Under this condition, the boundary entropy is defined as

$$S_{\mathrm{bdry}} = \lim_{L\to\infty} \big[S(L, T) - S_0(L, T)\big], \tag{4.1}$$

where $S_0(L, T)$ is the bulk entropy which is obtained in the absence of the impurity.

We are especially interested in the zero temperature limit $T \to 0$. In this case it is enough to consider the ground state contribution. If the boundary has no interaction with the bulk, the boundary entropy at $T = 0$ must be given by the degeneracy of the boundary ground state. For example, if the non-interacting impurity belongs to the spin s representation of $SU(2)$, we have $S_{\mathrm{bdry}} = \ln(2s + 1)$. On the other hand, when there exists an interaction between the impurity and the bulk, that interaction leads to non-trivial entropy in general.

Let us compute the boundary entropy for the models considered in Section 3.2. It was shown that the boundary entropy is given by the overlap between the boundary state and the conformal vacuum [19]:

$$S_{\mathrm{bdry}} = \ln\langle 0|B\rangle - \ln\langle 0|B_0\rangle. \tag{4.2}$$

Here $|B_0\rangle$ represents the situation in the absence of the interaction between the impurity and the bulk [19,1,9]:

$$|B_0\rangle = |0\rangle \otimes |0\rangle, \tag{4.3}$$

where $|0\rangle$'s are Cardy's boundary states (3.18) with $\rho_i = 0$. Therefore, we demand that the contribution from $|B_0\rangle$ corresponds to the bulk contribution S_0. Through (3.17), we can rewrite $|B_0\rangle$ as

$$|B_0\rangle = \big|B(0,0,0)\big\rangle. \tag{4.4}$$

From the expression (3.14) the overlap between the vacuum and the boundary state is given by

$$\big\langle 0\big|B(\rho_1,\rho_2,\rho)\big\rangle = \frac{S^{(k_1)}_{\rho_1 0} S^{(k_2)}_{\rho_2 0}}{\sqrt{S^{(k_1)}_{00} S^{(k_2)}_{00}}} \frac{S^{(k_1+k_2)}_{\rho 0}}{S^{(k_1+k_2)}_{00}}. \tag{4.5}$$

With the above identification of $|B_0\rangle$, we have

$$W_{\mathrm{bdry}} \equiv \exp(S_{\mathrm{bdry}}) = \frac{S^{(k_1)}_{\rho_1 0} S^{(k_2)}_{\rho_2 0} S^{(k_1+k_2)}_{\rho 0}}{S^{(k_1)}_{00} S^{(k_2)}_{00} S^{(k_1+k_2)}_{00}}. \tag{4.6}$$

Here W_{bdry} stands for the degeneracy of the ground state. Interestingly, this depends on $\rho_{1,2}$ in contrast to the reflection/transmission coefficients. As with the Kondo problem, we encounter non-integer degeneracies for generic (ρ_1, ρ_2, ρ), which are indications of non-Fermi liquid behavior, and some of them may be related to Majorana-like excitation [22–26]. Since the boundary entropy can be detected in the entanglement entropy [20], it can be a convenient criterion for such a behavior, e.g., in numerical analysis.

In addition to the non-integer degeneracies as discussed above, we also encounter integer ones for some (ρ_1, ρ_2, ρ). A remarkable example is $W_{\mathrm{bdry}} = 2$ for $\rho_1 = \rho_2 = \rho = 1$ with $k_1 = k_2 = 2$ that has the same symmetry as the two-channel Kondo model. It is known that the two-channel Kondo model usually gives a non-integer degeneracy. However this example indicates that the ground state of the Kondo impurity can have an integer degeneracy when the interaction involves the channel current in addition to the electron spin current. To well understand the origins of these integer degeneracies as well as the physical meaning of ρ's, further investigation is necessary.

The result obtained here provides an interesting implication also for the spin chain models. The situation we have discussed corresponds to the junction of $SU(2)$ chains with arbitrary spins $s_{1,2} = k_{1,2}/2$. Thus the expression (4.6) gives a quite general formula for the impurity entropy of the spin chain junction. It is interesting to check that the formula (4.6) can be obtained from the spin chain models by using another analytical method, e.g., Bethe ansatz.

5. Summary and discussion

We have defined the reflection/transmission coefficient for the conserved current at conformal defects. The BCFT approach offers an analytic and exact method to describe the reflection/transmission process. In addition, our definition provides a microscopic description of the reflection/transmission process. Namely, it reveals which and how much the current penetrates the defect. We have also computed the boundary entropy and observed a non-integer degeneracy.

We add some comments on the Kondo problem, to which our analysis is directly applicable. In particular, for $k_1 = k_2 = 2$, the model considered in Section 3.2 has the same symmetry as the two-channel Kondo model. In this case the two $SU(2)_2$'s in $SU(2)_2 \times SU(2)_2$ have different meanings: the first one is for the spin and the second is for the channel. Hence the transmission process means exchanging of spin and channel currents at the defect. As in the case of Kondo impurities, it is interesting to compute the specific heat and the resistivity. That computation could give further information in order to understand the physical meaning of $(\rho_{1,2}, \rho)$.

Let us comment on some possibilities beyond this work. It is interesting to extend our analysis of $SU(2)_{k_1} \times SU(2)_{k_2}$ into $SU(N)_{k_1} \times SU(N)_{k_2}$. This generalization attracts attention from not only theoretical, but also experimental point of view. It is because such a situation could be realized experimentally with, e.g., a quantum dot [27–29], or an ultracold atomic system [30–34]. Although the Kac–Moody algebra is more complicated for $N > 2$, one can use the formal expression of boundary states given in [15], and we can compute the R-matrix defined in (2.16) as in the case of the $SU(2)$ theory in principle. The $SU(N)$ theory may give richer results corresponding to the non-trivial fixed points since its representation theory is rather complicated, although some of the fixed points can be unstable. In addition, if we could take the large N limit, it is interesting to compare with the holographic methods for BCFT [35,36] and for the Kondo problem [37]. Furthermore, by applying the folding trick a number of times, we can straightforwardly generalize our analysis to the multiple junction of CFTs. In this case, the R-matrix becomes $M \times M$ matrix with the M multiplicity of the junction. On top of that, it turns out that the level-rank duality allows us to regard this system as the multi-channel Kondo model. We are preparing a paper in this direction.

Although we have focused on the impurity preserving the $SU(2)$ symmetry, we can also consider the situation where $SU(2)$ is partly broken to $U(1)$. Such a situation could be applicable to spin transport, which is driven by the spin–orbit interaction. Since the spin–orbit interaction breaks $SU(2)$ spin symmetry, the non-$SU(2)$ symmetric, or non-magnetic impurity plays an important role in the spin transport at the junction, especially with the Rashba effect induced at the surface. In this way we expect that our transport coefficients can be experimentally observed.

Another challenging issue is to connect critical phenomena including conformal defects to string field theory. String field theory derives non-trivial boundary states from its solutions through the proposed formulas [38,39]. Therefore, a new boundary state could be presented by string field theory to describe a non-trivial reflection/transmission process. For this purpose, the level truncation technique demonstrated in [40,41] may be helpful. In addition, it is interesting to find the interpretation of reflection/transmission coefficient from string theory point of view.

Acknowledgements

We would like to thank S.C. Furuya and M. Schnabl for reading the manuscript and giving useful comments. We are also grateful to T. Procházka for valuable discussions. The work of TK is supported in part by Grant-in-Aid for JSPS Fellows (#25-4302). The research of MM was supported by grant GACR P201/12/G028.

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