# Difference equations for graded characters from quantum cluster algebra

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# DIFFERENCE EQUATIONS FOR GRADED CHARACTERS FROM QUANTUM CLUSTER ALGEBRA

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Abstract. In this paper, we construct graded tensor product characters in terms of generalized difference Macdonald raising operators which form a representation of the quantum Q-system. Characters for the graded tensor product of Kirillov-Reshetikhin modules were expressed as the constant term of a non-commutative generating function [DFK14]. This function is written in terms of the generators of a quantum cluster algebra, subject to recursion relations known as the quantum Q-system. The latter form a discrete non-commutative integrable system, with a set of commuting conserved quantities. In type A such conserved quantities can be treated as analogues of the Casimir elements of $U_q(\mathfrak{s}l_{r+1})$. We show that the graded tensor product character is the analogue of a class I Whittaker function (to which it reduces in the Demazure case), and that the difference equations which follow from the action of the conserved quantities on characters generalize the difference quantum Toda equations of Etingof [Eti99]. Finally, we construct the graded characters as solutions of these equations, by introducing a representation of the quantum Q-system via difference operators which generalize the Macdonald raising difference operators of Kirillov–Noumi [KN99] in the dual Whittaker limit.

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## 1. Introduction

Let $\mathfrak{g} = \mathfrak{sl}_{r+1}$ and consider the recursion relation known as the $Q$-system [KNS94] associated with $\mathfrak{g}$:

$$Q_{\alpha,k+1}Q_{\alpha,k-1} = Q_{\alpha,k}^2 - Q_{\alpha-1,k}Q_{\alpha+1,k}, \quad Q_{0,k} = Q_{r+1,k} = 1, \tag{1.1}$$

with $\alpha \in [1, r]$ and $k \in \mathbb{Z}$. Here, $Q_{\alpha,k}$ are commuting, invertible variables.

Considered as a discrete evolution in the variable $k$, any $Q_{\alpha,k}$ (1.1) is determined once an appropriate set of $2r$ initial data is specified. For example, given the initial data set

$$S_0 = \{Q_{\alpha,0}, Q_{\alpha,1} : \alpha \in [1, r]\},$$

any $Q_{\alpha,k}$ can be expressed as a function of the elements of $S_0$: In fact, it is Laurent polynomial in this data [Ked08].

In the context of the representation theory of $\mathfrak{g}$, the most commonly used initial data set is

$$Q_{\alpha,0} = 1, \quad Q_{\alpha,1} = \mathrm{ch}V(\omega_\alpha), \quad \alpha \in [1, r], \tag{1.2}$$

where $\omega_\alpha$ are the fundamental weights of $\mathfrak{g}$, and $V(\omega_\alpha)$ are the fundamental irreducible modules. In this case, for any $k > 0$, $Q_{\alpha,k} = \mathrm{ch}V(k\omega_\alpha)$ for $k > 0$. Here, $V(\lambda)$ is the irreducible highest weight module of $\mathfrak{g}$ with highest weight $\lambda$ with $\lambda \in P^+$ a dominant highest weight of $\mathfrak{g}$.

By a slight abuse of terminology, we refer to irreducible $\mathfrak{g}$-modules with highest weights of the form $k\omega_\alpha$ as Kirillov-Reshetikhin (KR)-modules in this section. These are the restriction of the usual KR-modules of $\mathfrak{g}[t]$ to $\mathfrak{g}$, in the special case of $\mathfrak{g} = \mathfrak{sl}_{r+1}$. For any $\lambda$ in the weight lattice of $\mathfrak{g}$ we define $\ell_\alpha$ by $\lambda = \sum_\alpha \ell_\alpha \omega_\alpha$.

The $Q$ system is integrable, in the sense that there exist $r$ linearly independent quantities $C_1, ..., C_r$, which are Laurent polynomials in $\{Q_{\alpha,k}, Q_{\alpha,k+1} : \alpha \in [1, r]\}$, independent of $k$. That is, they are preserved under the discrete evolution (1.1), and are also called conserved quantities.

Since the $Q$-system (1.1) is a mutation in a cluster algebra [FZ02] of geometric type [Ked08], it has a natural Poisson structure [GSV10], with respect to which the conserved quantities are in involution. The associated quantization [BZ05] is called the *quantum $Q$-system* [DFK11], see Equation (1.5). The quantum $Q$-system is also an integrable discrete evolution of *noncommuting* variables. It has $r$ conserved quantities in involution with each other, which are $q$-deformations of those of the $Q$-system.

The first purpose of this paper is to derive quantum difference equations satisfied by characters of graded tensor products of KR-modules. The integrals of motion of the non-commutative $Q$-system play the key role in the construction of these difference equations. In the simplest case, one can show that characters of Demazure or Weyl modules [CL06, FL07], i.e. the graded tensor products of fundamental KR-modules, satisfy difference equations known as the (quantum) difference Toda equations of [Eti99], corresponding to $U_q(\mathfrak{sl}_{r+1})$.

Our next goal is to solve the difference equations satisfied by the graded characters by introducing a representation of the quantum $Q$-system via difference operators that act on (symmetric) polynomials of a variable $\mathbf{z} = (z_1, z_2, ..., z_{r+1})$. In terms of the associated quantum cluster algebra, the initial seed is made of the degenerate Macdonald difference operators for $\mathfrak{s}l_{r+1}$ corresponding to the limit $t \to \infty$ (dual to the Whittaker limit $t \to 0$) [Mac95], as well as the corresponding degenerate Kirillov-Noumi raising difference operators [KN99].

## 1.1. **Background.** Let us recall some relevant definitions.

1.1.1. *Graded tensor products of KR-modules.* Let $\{V_i : i \in [1, N]\}$ be a finite set of $\mathfrak{g}$-modules, which are restrictions to $\mathfrak{g}$ of KR-modules of $\mathfrak{g}[t]$. Thus, each $V_i$ is parameterized by two integers, $\alpha \in [1, r]$ and $j \in \mathbb{N}$, such that the highest weight of $V_i$ is $j\omega_\alpha$. The set $\{V_i : i \in [1, N]\}$ is thus specified by a multi-index

$$\mathbf{n} = (n_j^{(\alpha)} : \alpha \in [1, r], j \in \mathbb{N})$$

where $n_j^{(\alpha)}$ is the number of modules with highest weight $j\omega_\alpha$ in the set. As we only consider tensor products of finitely many such modules, we define the *level* of any given tensor product to be the integer

$$k = \mathrm{Max}\{j \,|\, \exists \alpha \,\text{such that}\, n_j^{(\alpha)} > 0\} \tag{1.3}$$

Define the tensor product multiplicity

$$M_{\mathbf{n},\lambda} = \dim\left(\mathrm{Hom}_{\mathfrak{g}}(V_1 \otimes \cdots \otimes V_N, V(\lambda))\right),$$

where $V(\lambda)$ is a finite-dimensional, irreducible $\mathfrak{g}$-module with highest weight $\lambda$.

The combinatorial Kirillov-Reshetikhin formula [KR87, HKO$^+$99] is an explicit, fermionic formula for this integer. It has its origin in the Bethe anstaz solution of the generalized, inhomogeneous Heisenberg spin chain, where it counts the number of Bethe vectors, conjecturally the dimension of the Hilbert space, in a fixed sector of the Hamiltonian.

The Bethe ansatz solution of the Heisenberg spin chain is also the origin of the graded version of the multiplicities, $M_{\mathbf{n},\lambda}(q)$, as well as the original physical interpretation of the grading [KKMM93]. The Bethe ansatz quite generally leads to such fermionic formulas for the linearized spectrum for any model which can be solved using this method. In condensed matter physics, this is called Haldane statistics.

In its ungraded version, the KR-conjecture was proven in a series of steps, originating in the work of [HKO$^+$99], then using a function space realization in the spirit of [SF94] in [AK07], and finally proving [DFK08] the identity of the different fermionic sums found in [HKO$^+$99] and [AK07]. See [Ked11] for a full explanation of this sequence of steps.

1.1.2. *Constant term identities.* A key feature of the proof in [DFK08] is the re-expression of $M_{\mathbf{n},\lambda}$ as an evaluation of the constant term in a monomial expression in solutions of the $Q$-system (1.1). Define for any level $k$ tensor product:

$$M^{(m)}_{\mathbf{n},\lambda} = \left\langle \prod_{\alpha=1}^{r} Q_{\alpha,1} \left( \prod_{j=1}^{k} Q_{\alpha,j}^{n_j^{(\alpha)}} \right) \left( Q_{\alpha,m} Q_{\alpha,m+1}^{-1} \right)^{\ell_\alpha+1} \right\rangle, \quad m \geq k, \tag{1.4}$$

where $\langle\cdot\rangle$ means the following:

**Definition 1.1.** *Given a Laurent series $f(S_0)$ in the initial data of the $Q$-system, the expression $\langle f(S_0)\rangle$ is obtained by the following two step procedure:*

1. *Evaluate $f$ at $Q_{\alpha,0} = 1$ for all $\alpha$;*
2. *Extract the constant term in $Q_{\alpha,1}$ for all $\alpha$.*

The expression (1.4) is thus obtained by expressing all $Q_{\alpha,j}$ as Laurent polynomials in the variables of $S_0$, and then following these two steps.

For any fixed finite $\mathbf{n}$, there exists $p \geq k$ such that $M^{(p+\ell)}_{\mathbf{n},\lambda} = M^{(p)}_{\mathbf{n},\lambda}$ for any $\ell \geq 0$. In that sense we have the following:

**Theorem 1.2.** [DFK08] *With $M^{(m)}_{\mathbf{n},\lambda}$ defined by equation* (1.4)*, the tensor product multiplicities are given by:*

$$\lim_{m\to\infty} M^{(m)}_{\mathbf{n},\lambda} = M_{\mathbf{n},\lambda}.$$

In [DFK14] we generalized this identity to the case of the quantum $Q$-system, thus obtaining a constant term identity for the graded multiplicities of [KR87, FL99], see Equation (1.11). This provides yet another interpretation for the grading. We note that in especially simple cases, the graded tensor products are the characters of Demazure modules [FL07] or Weyl modules [CL06].

1.1.3. *The quantum $Q$-system.* A constant term identity for $M_{\mathbf{n},\lambda}(q^{-1})$ was derived in [DFK14] in terms of solutions of the quantum $Q$-system, defined as the natural $q$-deformation of the $Q$-system (1.1) in terms of its cluster algebra structure.

Thus, let $\mathcal{A}$ be the algebra generated by the non-commuting variables $\{\mathcal{Q}_{\alpha,k}^{\pm 1} : \alpha \in [1,r], k \in \mathbb{Z}\}$ over $\mathbb{Z}[t,t^{-1}]$ modulo the ideal generated by the relations (1.5) and (1.6) below. The first is called the quantum $Q$-system:

$$t^{\Lambda_{\alpha,\alpha}}\mathcal{Q}_{\alpha,k+1}\mathcal{Q}_{\alpha,k-1} = \mathcal{Q}_{\alpha,k}^2 - \mathcal{Q}_{\alpha-1,k}\mathcal{Q}_{\alpha+1,k}, \qquad \mathcal{Q}_{0,k} = \mathcal{Q}_{r+1,k} = 1. \tag{1.5}$$

The variables $\mathcal{Q}_{\alpha,k}$ are subject to the following commutation relations:

$$\mathcal{Q}_{\alpha,k}\mathcal{Q}_{\beta,k'} = t^{\Lambda_{\alpha,\beta}(k'-k)}\mathcal{Q}_{\beta,k'}\mathcal{Q}_{\alpha,k} \qquad (|k-k'| \leq |\alpha-\beta|+1). \tag{1.6}$$

The matrix $\Lambda$ is proportional to the inverse of the Cartan matrix of $\mathfrak{g}$:

$$\Lambda_{\alpha,\beta} = \mathrm{Min}(\alpha,\beta)\,(r+1-\mathrm{Max}(\alpha,\beta))\,, \quad (\alpha,\beta \in [1,r]). \tag{1.7}$$

Here, $t$ is an invertible central element of the algebra, which is related to $q$ via:

$$q = t^{-r-1}. \tag{1.8}$$

Note that all the variables $\mathcal{Q}_{\alpha,k}$ for different $\alpha$ and fixed $k$ commute among each other. In Section 2.1, we provide a brief review of the discrete integrable structure of the quantum $Q$-system. Explicit solutions were worked out in detail in Ref. [DF11].

As a consequence of the Laurent property of quantum cluster algebras, we have

**Lemma 1.3.** *Given a set of initial data $\mathcal{S}_0 = \{\mathcal{Q}_{\alpha,0}, \mathcal{Q}_{\alpha,1} : \alpha \in [1,r]\}$, any element $\mathcal{Q}_{\alpha,k}$ can be expressed as a Laurent polynomial of the elements of $\mathcal{S}_0$, with coefficients in $\mathbb{Z}[t,t^{-1}]$.*

**Definition 1.4.** *The* normal ordered expression *of a Laurent series in the variables of $\mathcal{S}_0$ is the expression obtained, using the commutation relations* (1.6)*, when all the $\mathcal{Q}_{\alpha,0}$ are written to the left of all the $\mathcal{Q}_{\beta,1}$, for all $\alpha,\beta$.*

To define the non-commutative analogue of step (1) in Definition 1.1 of the "evaluation of $Q_{\alpha,0}$ at 1" of an expression in the quantum cluster variables, we need the following definitions.

**Definition 1.5.** *The map $ev : \mathbb{Z}[t,t^{-1}][\mathcal{Q}_{\alpha,0}^{\pm 1}, \mathcal{Q}_{\alpha,1}^{\pm 1}] \to \mathbb{Z}[t,t^{-1}][\mathcal{Q}_{\alpha,1}^{\pm 1}]$ is given by (1) normal ordering a Laurent polynomial in the variables in $\mathcal{S}_0$, and then (2) setting $\mathcal{Q}_{\alpha,0} = 1$ for all $\alpha$ in the normal-ordered expression.*

**Definition 1.6.** *The map* $ev_0 : \mathbb{Z}[t,t^{-1}][\mathcal{Q}_{\alpha,0}^{\pm 1}, \mathcal{Q}_{\alpha,1}^{\pm 1}] \to \mathbb{Z}[t,t^{-1}][\mathcal{Q}_{\alpha,1}^{\pm 1}]$ *is given by (1) normal ordering a Laurent polynomial in the variables in* $\mathcal{S}_0$, *and then (2) setting* $\mathcal{Q}_{\alpha,0} = t^{-\sum_\beta \Lambda_{\alpha,\beta}}$ *for all* $\alpha$ *in the normal-ordered expression.*

These two definitions are related in the following manner.

**Lemma 1.7.** *For any Laurent polynomial* $f \in \mathbb{Z}[t,t^{-1}][\mathcal{Q}_{\alpha,0}^{\pm 1}, \mathcal{Q}_{\alpha,1}^{\pm 1}]$, *we have:*

$$ev\left(\prod_{\beta=1}^{r} \mathcal{Q}_{\beta,1}\, f\right) = \left(\prod_{\beta=1}^{r} \mathcal{Q}_{\beta,1}\right) ev_0(f)$$

*Proof.* We simply note that the commutation relations (1.6) imply: $\left(\prod_{\beta=1}^{r} \mathcal{Q}_{\beta,1}\right) \mathcal{Q}_{\alpha,0} = t^{-\sum_\beta \Lambda_{\alpha,\beta}} \mathcal{Q}_{\alpha,0} \prod_{\beta=1}^{r} \mathcal{Q}_{\beta,1}$. □

The above definitions extend to Laurent series as well, and will allow us to define the quantum analogue of step (2) in Definition 1.1.

Note that there is a stronger version of Lemma 1.3, which is a *polynomiality* property due to the specific form of the quantum $Q$-system (see Corollary 5.13 of [DFK14]):

**Lemma 1.8.** *Let* $f$ *be a polynomial of the variables* $\{\mathcal{Q}_{\alpha,k}, \alpha \in [1,r], k \geq 1\}$, *obeying the quantum* $Q$*-system. Then* $ev_0(f) \in \mathbb{Z}[t,t^{-1}][\{\mathcal{Q}_{\beta,1}, \beta \in [1,r]\}]$, *namely it is a* polynomial *of the variables* $\{\mathcal{Q}_{\beta,1}\}_{\beta \in [1,r]}$, *with coefficients which are Laurent polynomials in* $t$. *Equivalently,* $ev(\prod_{\beta=1}^{r} \mathcal{Q}_{\beta,1}\, f) \in \left(\prod_{\beta=1}^{r} \mathcal{Q}_{\beta,1}\right) \mathbb{Z}[t,t^{-1}][\{\mathcal{Q}_{\beta,1}, \beta \in [1,r]\}]$ *is also a polynomial of the variables* $\{\mathcal{Q}_{\beta,1}\}_{\beta \in [1,r]}$, *which is a multiple of* $\prod_{\beta=1}^{r} \mathcal{Q}_{\beta,1}$.

Note that the second statement is an immediate consequence of Lemma 1.7.

**Definition 1.9.** *Given a Laurent series* $f$ *in* $\{\mathcal{Q}_{\alpha,1}^{-1}, \alpha \in [1,r]\}$ *with coefficients in* $\mathbb{Z}[t,t^{-1}]$, *the map CT(f) is the constant term in* $\mathcal{Q}_{\alpha,1}$ *for all* $\alpha$.

Finally we define the non-commutative analogue of the map $\langle \cdot \rangle$:

**Definition 1.10.** *Given a Laurent series* $f$ *in* $\{\mathcal{Q}_{\alpha,1}^{-1}, \alpha \in [1,r]\}$ *with coefficients in* $\mathbb{Z}[t,t^{-1}][\mathcal{Q}_{\alpha,0}^{\pm 1}]$, *we define the map* $\phi = CT \circ ev$ *that sends such a series to a polynomial in* $\mathbb{Z}[t,t^{-1}]$ *by first evaluating the normal ordered expression of* $f$ *at all* $\mathcal{Q}_{\alpha,0} = 1$, *and then extracting the constant term in all* $\mathcal{Q}_{\alpha,1}$.

1.1.4. *Constant term identity.* Let $\lambda = \sum_\alpha \ell_\alpha \omega_\alpha \in P^+$. In [DFK14] we proved the following theorem:

**Theorem 1.11.** [DFK14] *The Graded multiplicities of the irreducible components in the level $k$ $M$-sum formula of* [KR87] *can be expressed as*

$$\begin{aligned} M_{\mathbf{n},\lambda}(q^{-1}) &= t^{\sum_{\alpha,\beta,i} n_i^{(\alpha)}\Lambda_{\alpha,\beta}+\frac{1}{2}(\sum_\alpha \ell_\alpha \Lambda_{\alpha,\alpha}+\sum_{i,j,\alpha,\beta} n_i^{(\alpha)}\mathrm{Min}(i,j)\Lambda_{\alpha,\beta}n_j^{(\beta)})} \\ &\quad \phi\left(\left(\prod_{\alpha=1}^r \mathcal{Q}_{\alpha,1}\mathcal{Q}_{\alpha,0}^{-1}\right)\left(\prod_{i=1}^k\prod_{\alpha=1}^r \mathcal{Q}_{\alpha,i}^{n_i^{(\alpha)}}\right)\prod_{\alpha=1}^r \lim_{k\to\infty}(\mathcal{Q}_{\alpha,k}\mathcal{Q}_{\alpha,k+1}^{-1})^{\ell_\alpha+1}\right), \end{aligned} \tag{1.9}$$

*where $\mathcal{Q}_{\alpha,n}$ are the solution of the quantum $Q$-system* (1.5).

Note that in Equation (1.9), the monomial involving the $\mathcal{Q}_{\alpha,i}$ becomes a *polynomial* of the initial data variables $\mathcal{Q}_{\alpha,1}$ after the evaluation step of Definition 1.5, as a consequence of the polynomiality Lemma 1.8, and the obvious property that $ev(fg) = ev(ev(f)g)$. However the "tails" $\lim_{k\to\infty}\mathcal{Q}_{\alpha,k}\mathcal{Q}_{\alpha,k+1}^{-1}$ are Laurent series in $Q_{\alpha,1}^{-1}$, for $\alpha \in [1, r]$. Therefore, the constant terms pick only finitely many contributions.

1.2. **Main results.** In this paper, we use the integrable structure of the quantum $Q$-system to derive difference equations for the graded characters $\chi_{\mathbf{n}}(q; z) = \sum_\lambda M_{\mathbf{n},\lambda}(q)s_\lambda(\mathbf{z})$, where $s_\lambda$ are the Schur functions. The derivation uses the explicit expressions for the conserved quantities of the quantum $Q$-system and their action on the constant term type matrix elements.

For the case of level 1, when $V_1, ..., V_N$ are chosen to be fundamental $\mathfrak{g}$-modules and the fusion product is a Weyl module, these difference equations reduce to the q-deformed (relativistic) Toda equations. This allows in particular to identify the graded characters as Macdonald polynomials in the so-called Whittaker limit $t \to 0$.

The higher level equations are new. Their general structure suggests to view the graded characters as multi-partition generalization of the Macdonald polynomials in the Whittaker limit.

Subsequently, we present the general solution of these equations, by expressing them as the iterated action of a new set of raising difference operators in the variable $\mathbf{z} = (z_1, ..., z_{r+1})$ on the constant function 1. These form a generalization of the Macdonald difference operators in the limit $t \to \infty$, dual to the Whittaker limit, and include in particular the limit of the Macdonald polynomial raising operators introduced by Kirillov and Noumi [KN99].

Our first result is that these difference operators provide a representation of the dual quantum $Q$-system. In particular, they obey the quantum commutation relations inherited from the quantum cluster algebra structure.

1.2.1. *Difference equations.* We consider the graded characters $\chi_{\mathbf{n}}(q^{-1}, \mathbf{z})$, defined as

$$\chi_{\mathbf{n}}(q^{-1}, \mathbf{z}) = \sum_{\lambda\in P^+} M_{\mathbf{n},\lambda}(q^{-1})s_\lambda(z_1, ..., z_{r+1}) \tag{1.10}$$

where $\mathbf{z} := (z_1, ..., z_{r+1})$ and $z_1 z_2 ... z_{r+1} = 1$. The Schur function $s_\lambda$ is the $\mathfrak{sl}_{r+1}$ character corresponding to the Young diagram corresponding to the weight $\lambda = \sum_i \ell_i \omega_i$, i.e. the Young diagram $\overline{\lambda}_i = \ell_i - \ell_{i+1}$.[1]

In Section 2.4, we derive a set of difference equations for the characters (1.10), by using the conserved quantities of the quantum $Q$-system. The main result is the following:

**Theorem 2.15.** *The graded characters $\chi_{\mathbf{n}} \equiv \chi_{\mathbf{n}}(q^{-1}, \mathbf{z})$, $\mathbf{n} = (n_i^{(\alpha)})_{\alpha \in [1,r]; i \in [1,k]}$ at level $k$ satisfy the following difference equation for $k \geq 1$:*

$$\sum_{\alpha=1}^{r+1} \chi_{\mathbf{n}+\epsilon_{\alpha-1,k-1}-\epsilon_{\alpha,k-1}+\epsilon_{\alpha,k}-\epsilon_{\alpha-1,k}}$$
$$- \sum_{\alpha=1}^{r} q^{k-1-\sum_{i=1}^k i n_i^{(\alpha)}} \chi_{\mathbf{n}+\epsilon_{\alpha-1,k-1}-\epsilon_{\alpha,k-1}+\epsilon_{\alpha+1,k}-\epsilon_{\alpha,k}} = e_1(\mathbf{z})\, \chi_{\mathbf{n}}$$

*where the vector $\epsilon_{\alpha,i}$ defined so that $(\epsilon_{\alpha,i})_j^{(\beta)} = \delta_{\beta,\alpha}\delta_{j,i}$, and $e_1(\mathbf{z}) = z_1 + z_2 + \cdots + z_{r+1}$.*

We study the particular case of level $k = 1$ in Section 3, where characters are identified as special Whittaker functions, which are eigenfunctions for the $U_q(\mathfrak{sl}_{r+1})$ difference Toda equation [Eti99].

1.2.2. *Raising operators.* Section 4 introduces a solution of the difference equations of Theorem 2.15 in the form of iterated action of raising operators on the constant function 1. We define difference operators which generalize the difference Macdonald raising operators [KN99], in the limit $t \to \infty$.

**Remark 1.12.** *Note that our $t$ in eq.*(1.8) *is not to be confused with the $t$ variable in the theory of Macdonald polynomials and difference operators, for which the limit $t \to \infty$ may be thought of as a "dual Whittaker limit". Indeed, as pointed out below, the duality of Macdonald polynomials $P_\lambda^{q^{-1},t^{-1}} = P_\lambda^{q,t}$, allows to relate our limit $t \to \infty$ to the so-called Whittaker limit $t \to 0$.*

Define the maps $T_i(z_1, ..., z_{r+1}) = (z_1, ..., qz_i, ..., z_{r+1})$ (recall that $q = t^{-r-1}$), $\Delta(z_1, ..., z_{r+1}) = (tz_1, ..., tz_{r+1})$ and $D_i = T_i \circ \Delta$, extended by linearity to act on the space of functions in the variables $\mathbf{z}$ with coefficients in $\mathbb{Z}[t, t^{-1}]$. Let $I$ be any subset of $[1, r+1]$, and $\overline{I} = [1, r+1] \setminus I$ its complement. Let

$$z_I = \prod_{i \in I} z_i, \quad D_I = \prod_{i \in I} D_i, \quad \text{and} \quad a_I(\mathbf{z}) = \prod_{\substack{i \in I \\ j \in \overline{I}}} \frac{z_i}{z_i - z_j}.$$

[1]Note that the restriction that the product of $z_i$'s is equal to 1 means that we can use $\lambda$ to index the Schur function rather than $\overline{\lambda}$.

Then we define $\mathcal{D}_{\alpha,n}$ to be operators acting on the space of functions in $\mathbf{z}$ with coefficients in $\mathbb{Z}[t,t^{-1}]$ as follows:

$$\mathcal{D}_{\alpha,n} = t^{-\frac{\Lambda_{\alpha,\alpha}}{2}n-\sum_{\beta=1}^{r}\Lambda_{\alpha,\beta}} \sum_{\substack{I\subset[1,r+1]\\|I|=\alpha}} (z_I)^n a_I(\mathbf{z})\, D_I \qquad \alpha\in[0,r+1],\ n\in\mathbb{Z}. \tag{1.11}$$

Thus, $\mathcal{D}_{0,n}=1$ and $\mathcal{D}_{r+1,n}=1$ since $z_1z_2...z_{r+1}=1$.

The crucial property satisfied by the operators $\mathcal{D}_{\alpha,k}$ is the following:

**Theorem 4.2.** *The operators $\mathcal{D}_{\alpha,n}$ satisfy the dual quantum $Q$-system for $A_r$:*

$$\begin{aligned} \mathcal{D}_{\alpha,n}\,\mathcal{D}_{\beta,p} &= t^{-\Lambda_{\alpha,\beta}(p-n)}\,\mathcal{D}_{\beta,p}\,\mathcal{D}_{\alpha,n} \qquad (|p-n|\le|\beta-\alpha|+1)\\ t^{-\Lambda_{\alpha,\alpha}}\,\mathcal{D}_{\alpha,n+1}\,\mathcal{D}_{\alpha,n-1} &= (\mathcal{D}_{\alpha,n})^2 - t^{-r-1}\mathcal{D}_{\alpha+1,n}\,\mathcal{D}_{\alpha-1,n}\end{aligned}$$

The dual quantum $Q$-system is obtained from the equations (1.5) and (1.6) by application of an involutive antihomorphism that simply reverses the order of products and preserves $t$. The main result of Section 4 is the following expression for the graded characters:

**Theorem 4.6.** *The graded characters for $\mathfrak{sl}_{r+1}$ at level $k$ are given by:*

$$\begin{aligned}\chi_{\mathbf{n}}(q^{-1},\mathbf{z}) &= t^{\frac{1}{2}\sum_{i,j,\alpha,\beta} n_i^{(\alpha)}\mathrm{Min}(i,j)\Lambda_{\alpha,\beta}n_j^{(\beta)}+\sum_{i,\alpha,\beta}n_i^{(\alpha)}\Lambda_{\alpha,\beta}+\frac{1}{2}\sum_\alpha\Lambda_{\alpha,\alpha}+\sum_{\alpha<\beta}\Lambda_{\alpha,\beta}}\\ &\quad\times\prod_{\alpha=1}^{r}(\mathcal{D}_{\alpha,k})^{n_k^{(\alpha)}}\prod_{\alpha=1}^{r}(\mathcal{D}_{\alpha,k-1})^{n_{k-1}^{(\alpha)}}\cdots\prod_{\alpha=1}^{r}(\mathcal{D}_{\alpha,1})^{n_1^{(\alpha)}}\ 1\end{aligned}$$

This may be reformulated as follows. Expressing the graded characters as $\chi_{\mathbf{n}}(q^{-1},\mathbf{z}) = \Psi(\prod_{\alpha,i} Q_{\alpha,i}^{n_i^{(\alpha)}})$ in terms of some linear form $\Psi$ defined on polynomials $p\in\mathbb{Z}[t,t^{-1}][\{\mathcal{Q}_{\alpha,i}\}]$ (see Definition 4.14), we have a natural action of $\mathcal{Q}_{\alpha,k}$ by right multiplication: $\mathcal{Q}_{\alpha,k}\circ\Psi(p) = \Psi(p\mathcal{Q}_{\alpha,k})$. We will show that this action is represented by the left action of the difference operator $\mathcal{D}_{\alpha,k}$, namely

$$\mathcal{Q}_{\alpha,k}\circ\Psi(p) = \mathcal{D}_{\alpha,k}\,\Psi(p)$$

Theorem 4.6 will follow by noting the normalization condition that $\Psi(1)=1$.

Finally we focus on the case of $\mathfrak{sl}_2$ in Section 5, where we also give alternative proofs of the theorems above, and find some explicit compact formulas for the graded characters.

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## 2. From conserved quantities to difference equations

In this section, we derive difference equations for the graded characters, by using the explicit conserved quantities of the quantum $Q$-system (1.5). The quantum $Q$-system may indeed be viewed as an evolution equation in a discrete time for non-commuting variables, which is integrable, namely possesses sufficiently many conservation laws.

The section is organized as follows. We first describe in detail the conserved quantities of the quantum $Q$-system (Sect. 2.1), as described in [DFK11]. After introducing the suitable generating functions (Sect. 2.2), we first show how conserved quantities act at infinite time by multiplication by a scalar (Sect. 2.3). Finally the action at finite time is shown to lead to difference equations (Sect. 2.4).

2.1. **Conserved quantities of the quantum $Q$-system.** The quantum $Q$-system (1.5) is a discrete, non-commutative integrable system [DFK11]. In particular, we have $r$ discrete independent conserved quantities, $C_m$, which are the coefficients of a linear recursion relation satisfied by $\{\mathcal{Q}_{1,k}\}$ with $r+2$ terms, and which commute with each other.

We will need explicit expressions for these conserved quantities. This is can be done combinatorially by first defining the hard particle model with non-commutative weights. Let $\mathcal{G}$ be an unoriented graph with $N$ vertices $\mathcal{G}_0$ labelled by $i \in [1, N]$, and unoriented edges $\mathcal{G}_1$ connecting the vertices. To each vertex $i$ we associate a weight $y_i$, where the set $\{y_1, y_2, ..., y_N\}$ are non-commuting variables.

**Definition 2.1.** *A hard particle configuration $\mathcal{H}$ on $\mathcal{G}$ is a subset of $\mathcal{G}_0$ such that the pair $i, j \in \mathcal{H}$ only if there is no edge in $\mathcal{G}_1$ connecting $i$ and $j$.*

We may think of such a subset of $\mathcal{G}_0$ as being "occupied" by particles, with the hard core condition that edges of $\mathcal{G}$ can be adjacent to *at most* one such particle. Denote the set of all hard particle configurations on $\mathcal{G}$ by $\mathcal{C}_{\mathcal{G}}^{\mathrm{HP}}$.

**Definition 2.2.** *The weight of a hard particle configuration $\mathcal{H}$ on $\mathcal{G}$ is the product of the weights of the vertices in $\mathcal{H}$, written in descending order. That is, the weight of the set $\{i_1, ..., i_k\}$ with $i_1 < i_2 < \cdots < i_k$ is $y_{i_k} y_{i_{k-1}} \cdots y_{i_1}$.*

**Definition 2.3.** *The hard-particle partition function $Z_{\mathcal{G}}(y_1, ..., y_N)$ on $\mathcal{G}$ is the sum over all hard particle configurations on $\mathcal{G}$ of the weights of the configuration.*

$$Z_{\mathcal{G}}(y_1, ..., y_N) = 1 + \sum_{k \geq 1} \sum_{\substack{\mathcal{H} \in \mathcal{C}_{\mathcal{G}}^{\mathrm{HP}} : |\mathcal{H}| = k \\ 1 \leq i_1 < i_2 \cdots < i_k}} y_{i_k} y_{i_{k-1}} \cdots y_{i_1} \tag{2.1}$$

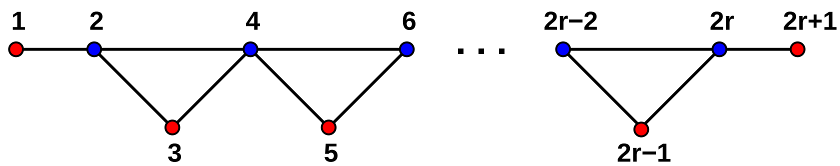
*The $k$-particle partition function is given by the term with fixed $k$ in* (2.1)*, and is denoted by $Z_{\mathcal{G}}^{(k)}(y_1, ..., y_N)$.*

Now let $\{\mathcal{Q}_{\alpha,n} : \alpha \in [1,r], n \in \mathbb{Z}\}$ be a set of solutions of the quantum $Q$-system. Define the weights $y_i(n)$ as the following ordered monomials:

$$y_{2\alpha-1}(n) = \mathcal{Q}_{\alpha,n+1}\mathcal{Q}_{\alpha-1,n+1}^{-1}\mathcal{Q}_{\alpha,n}^{-1}\mathcal{Q}_{\alpha-1,n} \tag{2.2}$$

$$y_{2\alpha}(n) = -\mathcal{Q}_{\alpha+1,n+1}\mathcal{Q}_{\alpha,n+1}^{-1}\mathcal{Q}_{\alpha,n}^{-1}\mathcal{Q}_{\alpha-1,n} \tag{2.3}$$

Moreover, define the graph $\mathcal{G}_r$ with $2r+1$ vertices as follows:

$\mathcal{G}_r =$ 


**Theorem 2.4.** [DFK11] *The following Laurent polynomials are independent of $n$:*

$$C_m = Z_{\mathcal{G}_r}^{(m)}(y_1(n), y_2(n), ..., y_{2r+1}(n)), \qquad (m \in [0, r+1]), \tag{2.4}$$

*The $m^{\text{th}}$ quantity is the hard-particle partition function on $\mathcal{G}_r$ with $m$ particles. Then $C_1, ..., C_r$ are algebraically independent, commuting elements of $\mathcal{A}$, which are called the conserved quantities of the quantum $Q$-system. In particular, we have $C_0 = 1$, and $C_{r+1} = t^{\frac{r(r+1)}{2}}$ is a central element.*

**Example 2.5.** *For $r = 1$ (case of $\mathfrak{sl}_2$), where $\mathcal{Q}_{0,n} = \mathcal{Q}_{2,n} = 1$, we have $C_0 = 1, C_2 = t$ and*

$$C_1 = y_1(n) + y_2(n) + y_3(n),$$

*where $y_1(n) = \mathcal{Q}_{1,n+1}\mathcal{Q}_{1,n}^{-1}$, $y_2(n) = -\mathcal{Q}_{1,n+1}^{-1}\mathcal{Q}_{1,n}^{-1}$ and $y_3(n) = \mathcal{Q}_{1,n+1}^{-1}\mathcal{Q}_{1,n}$.*

**Example 2.6.** *For $r = 2$ (case of $\mathfrak{sl}_3$), where $\mathcal{Q}_{0,n} = \mathcal{Q}_{3,n} = 1$, we have $C_0 = 1, C_3 = t^3$ and*

$$\begin{aligned} C_1 &= y_1(n) + y_2(n) + y_3(n) + y_4(n) + y_5(n), \\ C_2 &= (y_3(n) + y_4(n) + y_5(n))\, y_1(n) + y_5(n)\, (y_2(n) + y_3(n)), \end{aligned}$$

*where*

$$y_1(n) = \mathcal{Q}_{1,n+1}\mathcal{Q}_{1,n}^{-1}, \quad y_2(n) = -\mathcal{Q}_{2,n+1}\mathcal{Q}_{1,n+1}^{-1}\mathcal{Q}_{1,n}^{-1}, \quad y_3(n) = \mathcal{Q}_{2,n+1}\mathcal{Q}_{1,n+1}^{-1}\mathcal{Q}_{2,n}^{-1}\mathcal{Q}_{1,n},$$
$$y_4(n) = -\mathcal{Q}_{2,n+1}^{-1}\mathcal{Q}_{2,n}^{-1}\mathcal{Q}_{1,n}, \quad y_5(n) = \mathcal{Q}_{2,n+1}^{-1}\mathcal{Q}_{2,n}.$$

Let us denote by $G$ the $2r+1 \times 2r+1$ adjacency matrix of the graph $\mathcal{G}_r$. The variables $y_\alpha(n)$ have particularly simple commutation relations, as a consequence of (1.6). We have for any $1 \le \alpha < \beta \le 2r+1$ [DF11]:

$$y_\alpha(n)\, y_\beta(n) = q^{-G_{\alpha,\beta}}\, y_\beta(n)\, y_\alpha(n).$$

For later use, let us introduce the quantities

$$\theta_{\alpha,k} = \mathcal{Q}_{\alpha,k}\mathcal{Q}_{\alpha,k+1}^{-1}, \qquad \xi_{\alpha,k} = t^{\frac{\Lambda_{\alpha,\alpha}}{2}}\theta_{\alpha,k}. \tag{2.5}$$

As shown in reference [DFK14], $\xi_{\alpha,k}$ is expressible as a formal power series of the variables $\mathcal{Q}_{\alpha,1}^{-1}$ with coefficients which are Laurent polynomials of the $\mathcal{Q}_{\beta,0}$'s, and the limit $k \to \infty$ exists. We denote it by

$$\xi_\alpha = \lim_{k\to\infty} \xi_{\alpha,k}. \tag{2.6}$$

The conserved quantities $C_i$ are independent of $n$, and they are easy to evaluate in the limit $n \to \infty$:

$$y_{2\alpha} := \lim_{n\to\infty} y_{2\alpha}(n) = 0 \quad \text{and} \quad y_{2\alpha-1} := \lim_{n\to\infty} y_{2\alpha-1}(n) = t^{\frac{r}{2}}\, \xi_{\alpha-1}\xi_\alpha^{-1} \tag{2.7}$$

Thus, all conserved quantities may be written in terms of the functions $\xi_\alpha$. Since the odd vertices $1, 3, 5, ..., 2r+1$ of $\mathcal{G}_r$ are not connected by edges, the odd-numbered weights $\{y_{2i+1}(n) : i \in [0, r]\}$ commute among themselves. The hard particle partition functions are simply the elementary symmetric functions in the odd variables of Equation (2.7):

$$\begin{aligned} C_m &= \sum_{1\le i_1<i_2<\cdots<i_m\le r+1} y_{2i_m-1}y_{2i_{m-1}-1}\cdots y_{2i_1-1} \\ &= t^{\frac{mr}{2}} \sum_{1\le i_1<i_2<\cdots<i_m\le r+1} \prod_{\ell=1}^{m} \xi_{\alpha_{i_\ell}-1}\xi_{\alpha_{i_\ell}}^{-1} \end{aligned} \tag{2.8}$$

The formula is also valid for $C_0 = 1$ and $C_{r+1} = y_{2r+1}y_{2r-1}\cdots y_1 = t^{\frac{r(r+1)}{2}}$. Moreover, it is immediate from (2.8) that all the conserved quantities $C_m$ commute among themselves, as the $y_{2\alpha-1}$ do. Thus, the quantum $A_r$ $Q$-system is a discrete integrable equation.

On the other hand, using Equation (2.4) for the conserved quantities, each $C_m$ may be expressed as a Laurent polynomial in $\{\mathcal{Q}_{\alpha,n}, \mathcal{Q}_{\alpha,n+1} : \alpha \in [1, r]\}$ for any fixed $n \in \mathbb{Z}$. In particular,

**Lemma 2.7.** *The conserved quantity $C_1$ for fixed $r$ is*

$$C_1 = \sum_{\alpha=1}^{r+1} \mathcal{Q}_{\alpha,n}^{-1}\mathcal{Q}_{\alpha-1,n}\left(t^r\mathcal{Q}_{\alpha,n+1}\mathcal{Q}_{\alpha-1,n+1}^{-1} - t^{-1}\mathcal{Q}_{\alpha+1,n+1}\mathcal{Q}_{\alpha,n+1}^{-1}\right), \tag{2.9}$$

*where $\mathcal{Q}_{r+2,n} = 0$.*

*Proof.* Fix $n$ and use the expression for the weights (2.2) as well as the commutation relations (1.6):

$$
\begin{aligned}
C_1 &= \sum_{\alpha=1}^{r+1} y_{2\alpha-1}(n) + \sum_{\alpha=1}^{r} y_{2\alpha}(n) \\
&= \sum_{\alpha=1}^{r+1} t^{\Lambda_{\alpha,\alpha}-2\Lambda_{\alpha,\alpha-1}+\Lambda_{\alpha-1,\alpha-1}} \mathcal{Q}_{\alpha,n}^{-1}\mathcal{Q}_{\alpha-1,n}\mathcal{Q}_{\alpha,n+1}\mathcal{Q}_{\alpha-1,n+1}^{-1} \\
&\quad -\sum_{\alpha=1}^{r} t^{\Lambda_{\alpha,\alpha+1}-\Lambda_{\alpha-1,\alpha+1}-\Lambda_{\alpha,\alpha}+\Lambda_{\alpha-1,\alpha}} \mathcal{Q}_{\alpha,n}^{-1}\mathcal{Q}_{\alpha-1,n}\mathcal{Q}_{\alpha+1,n+1}\mathcal{Q}_{\alpha,n+1}^{-1}.
\end{aligned}
\tag{2.10}
$$

The Lemma follows from the identities:

$$
\Lambda_{\alpha,\alpha} - 2\Lambda_{\alpha,\alpha-1} + \Lambda_{\alpha-1,\alpha-1} = r,
$$
$$
\Lambda_{\alpha,\alpha+1} - \Lambda_{\alpha-1,\alpha+1} - \Lambda_{\alpha,\alpha} + \Lambda_{\alpha-1,\alpha} = -1.
$$

□

2.2. **Generating functions.** We now turn to generating functions of graded characters. Our aim is to rexpress eq. (1.10) by use of the quantum constant term identity (1.9). Define the function $\tau(\mathbf{z})$, dependent on the variables $\mathbf{z} = \{z_1, ..., z_{r+1}\}$ with $z_1 \cdots z_{r+1} = 1$ as

$$
\begin{aligned}
\tau(\mathbf{z}) &:= t^{\frac{1}{2}\sum_\alpha \Lambda_{\alpha,\alpha}+\sum_{\alpha<\beta}\Lambda_{\alpha,\beta}} \sum_{\lambda\in P^+}\prod_{\alpha=1}^{r}(\xi_\alpha)^{\ell_\alpha+1} s_\lambda(\mathbf{z}) \\
&= t^{\frac{r+1}{4}\binom{r+2}{3}} \sum_{\lambda\in P^+}\prod_{\alpha=1}^{r}(\xi_\alpha)^{\ell_\alpha+1} s_\lambda(\mathbf{z})
\end{aligned}
\tag{2.11}
$$

with $\xi_\alpha$ as in (2.6). We use the usual convention for $\mathfrak{sl}_{r+1}$, where $P^+$ is the set of dominant integral weights or partitions of length $r$ or less, and $\ell_i = \lambda_i - \lambda_{i+1}$.

Fix $k \geq 1$ and define the generating series of characters as a series in the indeterminates $\mathbf{y} = \{y_{\alpha,i} : \alpha \in [1, r], i \in [1, k]\}$:

$$
G^{(k)}(\mathbf{y}) = \phi\left(\left(\prod_{\alpha=1}^{r}\mathcal{Q}_{\alpha,1}\right)\left(\prod_{i=1}^{k}\left(\prod_{\alpha=1}^{r}\frac{1}{1-y_{\alpha,i}\mathcal{Q}_{\alpha,i}}\right)\right)\tau(\mathbf{z})\right).
\tag{2.12}
$$

Here, each rational function is defined to be a series in the variables $y_{\alpha,i}$, and the product over $i$ is ordered from left to right.

The coefficient of $\prod_{\alpha,i} y_{\alpha,i}^{n_i^{(\alpha)}}$ in the formal series expansion of $G^{(k)}(\mathbf{y})$ is defined to be $G^{(k)}_{\mathbf{n}}$, with $\mathbf{n}=\{n_{\alpha,i}\}_{\alpha\in[1,n];i\in[1,k]}$:

$$G^{(k)}_{\mathbf{n}} = \phi\left(\left(\prod_{\alpha=1}^{r} \mathcal{Q}_{\alpha,1}\right)\left(\prod_{i=1}^{k}\left(\prod_{\alpha=1}^{r} \mathcal{Q}_{\alpha,i}^{n_i^{(\alpha)}}\right)\right)\tau(\mathbf{z})\right) \tag{2.13}$$

The normalization of $\tau(\mathbf{z})$ in (2.11) is chosen so that $G^{(1)}_0 = G^{(1)}(0) = 1$.

Comparing with Equation (1.9) and using the commutation relations between $\mathcal{Q}_{\alpha,0}$ and $\mathcal{Q}_{\beta,1}$, we see that these coefficients are the renormalized characters of Equation (1.10):

$$\chi_{\mathbf{n}}(q^{-1},\mathbf{z}) = t^{\sum_{\alpha,\beta,i} n_i^{(\alpha)}\Lambda_{\alpha,\beta}+\frac{1}{2}\mathbf{n}\cdot(\Lambda\otimes A)\mathbf{n}}\, G^{(k)}_{\mathbf{n}}. \tag{2.14}$$

2.3. **The action of the conserved quantities on $\tau$.** The following theorem will be instrumental in the derivation of the graded characters. Let

$$e_m(\mathbf{z}) = s_{1^m}(\mathbf{z}), \quad m\in[1,r] \tag{2.15}$$

denote the characters of the fundamental representations of $SL_{r+1}$. They are the elementary symmetric functions in $\{z_1,...,z_{r+1}\}$.

**Theorem 2.8.** *The conserved quantities of the $A_r$ quantum Q-system act on the function $\tau(\mathbf{z})$ as:*

$$C_m\,\tau(\mathbf{z}) = t^{\frac{mr}{2}}\, e_m(\mathbf{z})\,\tau(\mathbf{z}) + R_m(\mathbf{z}) \tag{2.16}$$

*where $R_m(\mathbf{z})$ is a sum of power series of the $\xi_\alpha$, with each of the summands independent of at least one of the $\{\xi_\alpha\}_{\alpha\in[1,r]}$.*

*Proof.* Recall the expression (2.11) for $\tau(\mathbf{z})$. Using (2.8) and (2.8), we write explicitly:

$$\begin{aligned}
t^{-\frac{mr}{2}}C_m\,t^{-\frac{r+1}{4}\binom{r+2}{3}}\tau(\mathbf{z}) &= \sum_{\ell_1,..,\ell_r\geq 0}\prod_{\alpha=1}^{r}\xi_\alpha^{\ell_\alpha+1}\sum_{1\leq i_1<i_2<\cdots<i_m} s_{\ell_1,..\ell_{i_1-2},\ell_{i_1-1}-1,\ell_{i_1}+1,\ldots,\ell_{i_m-1}-1,\ell_{i_m}+1,\ldots\ell_r}(\mathbf{z})\\
&= \sum_{\ell_1,..,\ell_r}\prod_{\alpha=1}^{r}\xi_\alpha^{\ell_\alpha+1}\sum_{1\leq i_1<i_2<\cdots<i_m} s_{\lambda+\epsilon_{i_1}+\cdots+\epsilon_{i_m}}(\mathbf{z}) + u_m(\mathbf{z})\\
&= \sum_{\ell_1,..,\ell_r}\prod_{\alpha=1}^{r}\xi_\alpha^{\ell_\alpha+1}e_m(\mathbf{z})\,s_\lambda(\mathbf{z}) + u_m(\mathbf{z}) = e_m(\mathbf{z})\,t^{-\frac{r+1}{4}\binom{r+2}{3}}\tau(\mathbf{z}) + u_m(\mathbf{z})
\end{aligned}$$

where $u_m(\mathbf{z})$ has the same property as $R_m(\mathbf{z})$ (it adds counterterms for all cases where some of the $\ell_{i_j}$'s vanish in the first term of the first line, and each such term is independent of the corresponding $\xi_{i_j}$). We have used the fact that $\epsilon_i=\omega_i-\omega_{i-1}$ correspond to $\ell_\alpha=\delta_{\alpha,i}-\delta_{\alpha,i-1}$ and transcribed the result using the Pieri rule for $SL_{r+1}$. The theorem follows. $\square$

We deduce the following:

**Corollary 2.9.** *When evaluated inside the generating function* (2.12), *each conserved quantity $C_m$ acts on $\tau(\mathbf{z})$ as the scalar $t^{\frac{mr}{2}} e_m(\mathbf{z})$, namely:*

$$\phi\left(\left(\prod_{\alpha=1}^{r} \mathcal{Q}_{\alpha,1}\right)\left(\prod_{i=1}^{k}\prod_{\alpha=1}^{r} \frac{1}{1-y_{\alpha,i}\mathcal{Q}_{\alpha,i}}\right) C_m \tau(\mathbf{z})\right) = t^{\frac{mr}{2}} e_m(\mathbf{z})\, G^{(k)}(\mathbf{y}) \tag{2.17}$$

*Proof.* Each summand of $w_m(\mathbf{z})$ has at least one missing $\xi_\alpha$, hence the corresponding constant term in $\mathcal{Q}_{\alpha,1}$ must vanish, as the rest of the power series only generates positive powers of $\mathcal{Q}_{\alpha,1}$, once left evaluated at $\mathcal{Q}_{\alpha,0}=1$. □

2.4. **Difference equations from conserved quantities.** We now use a standard argument to reformulate the conserved quantities of the quantum Q-system into difference equations for the quantities $G_{\mathbf{n}}^{(k)}$ of (2.13).

**Theorem 2.10.** *For $k \geq 2$, the coefficients $G_{\mathbf{n}}^{(k)}$, $\mathbf{n} = \{n_i^{(\alpha)}\}_{i\in[1,k];\alpha\in[1,r]}$ obey the following difference equation:*

$$\begin{aligned} t^{\frac{r}{2}} e_1(\mathbf{z})\, G_{\mathbf{n}}^{(k)} &= \sum_{\alpha=1}^{r+1} t^{r+\sum_\beta (\Lambda_{\alpha,\beta}-\Lambda_{\alpha-1,\beta}) n_k^{(\beta)}}\, G^{(k)}_{\mathbf{n}+\epsilon_{\alpha-1,k-1}-\epsilon_{\alpha,k-1}+\epsilon_{\alpha,k}-\epsilon_{\alpha-1,k}} \\ &\quad - \sum_{\alpha=1}^{r} t^{-1+\sum_\beta (\Lambda_{\alpha,\beta}-\Lambda_{\alpha-1,\beta}) n_k^{(\beta)}}\, G^{(k)}_{\mathbf{n}+\epsilon_{\alpha-1,k-1}-\epsilon_{\alpha,k-1}+\epsilon_{\alpha+1,k}-\epsilon_{\alpha,k}} \end{aligned} \tag{2.18}$$

*where we use the notation $\epsilon_{\beta,m}$ for the vector with entries $(\epsilon_{\beta,m})_i^{(\alpha)} = \delta_{\alpha,\beta}\delta_{i,m}$. For $k=1$, the coefficients $G_{\mathbf{n}}^{(1)}$, $\mathbf{n} = \{n^{(\alpha)}\}_{\alpha\in[1,r]}$, obey the difference equation:*

$$\begin{aligned} t^{\frac{r}{2}} e_1(\mathbf{z})\, G_{\mathbf{n}}^{(1)} &= t^{r+\sum_\beta \Lambda_{1,\beta}(n^{(\beta)}+1)}\, G^{(1)}_{\mathbf{n}+\epsilon_1} \\ &\quad + \sum_{\alpha=2}^{r+1}\left(t^{r+\sum_\beta (\Lambda_{\alpha,\beta}-\Lambda_{\alpha-1,\beta})(n^{(\beta)}+1)} - t^{-1+\sum_\beta (\Lambda_{\alpha+1,\beta}-\Lambda_{\alpha,\beta})(n^{(\beta)}+1)}\right) G^{(1)}_{\mathbf{n}+\epsilon_\alpha-\epsilon_{\alpha-1}} \end{aligned} \tag{2.19}$$

*with the notation $\epsilon_\beta$ for the vector with entries $(\epsilon_\beta)^{(\alpha)} = \delta_{\alpha,\beta}$.*

*Proof.* We compute in two ways the quantity

$$B = \phi\left(\left(\prod_{\alpha=1}^{r} \mathcal{Q}_{\alpha,1}\right)\left(\prod_{i=1}^{k}\prod_{\alpha=1}^{r} \mathcal{Q}_{\alpha,i}^{n_i^{(\alpha)}}\right) C_1 \tau(\mathbf{z})\right)$$

First, we find $B = t^{\frac{r}{2}} e_1(\mathbf{z})\, G_{\mathbf{n}}^{(k)}$ by direct application of (2.17). Second, we use the expression (2.9) with $n = k-1$ for $C_1$. Using the notation:

$$\langle M \rangle_{\mathbf{n}} = \phi\left(\left(\prod_{\alpha=1}^{r} \mathcal{Q}_{\alpha,1}\right)\left(\prod_{i=1}^{k}\prod_{\alpha=1}^{r} (\mathcal{Q}_{\alpha,i})^{n_i^{(\alpha)}}\right) M\, \tau(\mathbf{z})\right)$$

for any Laurent monomial $M$ of the $Q$'s, we have:

$$\begin{aligned}\langle \mathcal{Q}_{\alpha,k-1}^{-1}\mathcal{Q}_{\alpha-1,k-1}\mathcal{Q}_{\alpha,k}\mathcal{Q}_{\alpha-1,k}^{-1}\rangle_{\mathbf{n}} &= t^{\sum_\beta(\Lambda_{\alpha,\beta}-\Lambda_{\alpha-1,\beta})n_k^{(\beta)}}\langle 1\rangle_{\mathbf{n}+\epsilon_{\alpha-1,k-1}-\epsilon_{\alpha,k-1}+\epsilon_{\alpha,k}-\epsilon_{\alpha-1,k}}\\ \langle \mathcal{Q}_{\alpha,k-1}^{-1}\mathcal{Q}_{\alpha-1,k-1}\mathcal{Q}_{\alpha+1,k}\mathcal{Q}_{\alpha,k}^{-1}\rangle_{\mathbf{n}} &= t^{\sum_\beta(\Lambda_{\alpha,\beta}-\Lambda_{\alpha-1,\beta})n_k^{(\beta)}}\langle 1\rangle_{\mathbf{n}+\epsilon_{\alpha-1,k-1}-\epsilon_{\alpha,k-1}+\epsilon_{\alpha+1,k}-\epsilon_{\alpha,k}}\end{aligned}$$

The case $k=1$ must be treated separately, as the insertion of $\mathcal{Q}_{\alpha,0}$ amounts to a factor $t^{-\sum_\beta \Lambda_{\alpha,\beta}}$, coming from commutation of $\mathcal{Q}_{\alpha,k-1}^{-1}\mathcal{Q}_{\alpha-1,k-1}$ through $\prod_\beta(\mathcal{Q}_{\beta,k})^{n_k^{(\beta)}}$. The Theorem follows. □

**Example 2.11.** *When $r=1$ (case of $\mathfrak{sl}_2$), we have for $n_i\equiv n_i^{(1)}$, and $\Lambda_{1,1}=1$:*

$$t^{n_k+1}G^{(k)}_{n_1,\ldots,n_{k-1}-1,n_k+1}+t^{1-n_k}G^{(k)}_{n_1,\ldots,n_{k-1}+1,n_k-1}-t^{n_k-1}G^{(k)}_{n_1,\ldots,n_{k-1}-1,n_k-1}=t^{\frac12}(z+z^{-1})\,G^{(k)}_{n_1,\ldots,n_{k-1},n_k}$$

*with $z=z_1=z_2^{-1}$, whereas for $k=1$, $n\equiv n_1$:*

$$t^{n+2}G^{(1)}_{n+1}+(t^{-n}-t^n)G^{(1)}_{n-1}=t^{\frac12}(z+z^{-1})\,G^{(1)}_n \tag{2.20}$$

More generally, repeating this with the other conserved quantities $C_m$, $m\geq 2$ leads to higher difference equations of the form $\mathcal{D}^{(m)}G^{(k)}_{\mathbf{n}}=t^{\frac{mr}{2}}e_mG^{(k)}_{\mathbf{n}}$, where the difference operators $\mathcal{D}^{(m)}$ form a commuting family for $m=1,2,...,r$, and $\mathcal{D}^{(1)}$ acts on the function $G^{(k)}_{\mathbf{n}}$ of $\mathbf{n}$ via the l.h.s. of eq.(2.18).

**Example 2.12.** *When $r=2$ and $k=2$ (case of $\mathfrak{sl}_3$, level 2), we have the following recursion relations in the variables $n_1^{(\alpha)}=n_\alpha$ and $n_2^{(\alpha)}=p_\alpha$, $\alpha=1,2$, obtained respectively by inserting the conserved quantities $C_1$ and $C_2$ of Example 2.6:*

$$\begin{aligned}&G^{(2)}_{n_1-1,p_1;n_2+1,p_2}+t^{-3n_2}G^{(2)}_{n_1+1,p_1-1;n_2-1,p_2+1}+t^{-3n_2-3p_2}G^{(2)}_{n_1,p_1+1;n_2,p_2-1}\\&\qquad -t^{-3}G^{(2)}_{n_1-1,p_1;n_2-1,p_2+1}-t^{-3-3n_2}G^{(2)}_{n_1+1,p_1-1;n_2,p_2-1}=t^{-1-2n_2-p_2}e_1(\mathbf{z})\,G^{(2)}_{n_1,p_1;n_2,p_2}\\&G^{(2)}_{n_1,p_1-1;n_2,p_2+1}+t^{-3p_2}G^{(2)}_{n_1-1,p_1+1;n_2+1,p_2-1}+t^{-3n_2-3p_2}G^{(2)}_{n_1+1,p_1;n_2-1,p_2}\\&\qquad -t^{-3}G^{(2)}_{n_1,p_1-1;n_2+1,p_2-1}-t^{-3-3p_2}G^{(2)}_{n_1-1,p_1+1;n_2-1,p_2}=t^{-1-n_2-2p_2}e_2(\mathbf{z})\,G^{(2)}_{n_1,p_1;n_2,p_2}\end{aligned}$$

*with $e_1(\mathbf{z})=z_1+z_2+z_3$ and $e_2(\mathbf{z})=z_1z_2+z_2z_3+z_1z_3$, $z_1z_2z_3=1$.*

For later use, let us focus on the level 1 higher difference equations obtained by inserting $C_m$, $m\in[1,r]$ into the bracket $\langle\cdots\rangle_{\mathbf{n}}$ defined above. As apparent from eq.(2.19) of Theorem 2.10, the difference equation for $G^{(1)}$ allows to express $G^{(1)}_{\mathbf{n}+\epsilon_1}$ as a linear combination of the shifted functions $G^{(1)}_{\mathbf{n}+\epsilon_{\alpha+1}-\epsilon_\alpha}$, $\alpha=1,2,...,r$, as well as $G^{(1)}_{\mathbf{n}}$. Similarly, due to the form of the conserved quantities as functions of the $\mathcal{Q}_{\alpha,n}$'s, the level 1 $C_m$ difference equation allows to express $G^{(1)}_{\mathbf{n}+\epsilon_m}$ as a linear combination of shifted functions of the form: $G^{(1)}_{\mathbf{n}+\sum_{1\leq i\leq m}\epsilon_{\alpha_i+1}-\epsilon_{\alpha_i}}$ with $1\leq\alpha_1<\cdots<\alpha_m\leq r$, as well as $G^{(1)}_{\mathbf{n}}$. Combining all the equations for $m=1,2,...,r$ provides therefore a recursive method for computing all $G^{(1)}_{\mathbf{n}}$. Indeed, defining $\sigma(\mathbf{n})=$

$\sum_\alpha n^{(\alpha)}$, we see that each equation is a three term recursion in the variable $\sigma(\mathbf{n})$, as the term $\mathbf{n}+\epsilon_m$ has a value of $\sigma$ 1 or 2 larger than all other terms. If we know all the values of $G^{(1)}_{\mathbf{n}}$ for $\sigma(\mathbf{n})\leq N$, we therefore deduce $G^{(1)}_{\mathbf{n}}$ for all values $\sigma(\mathbf{n})=N+1$. We have the following:

**Theorem 2.13.** *The difference equations obtained by inserting $C_m$, $m=1,2,...,r$ at level 1 determine the functions $G^{(1)}_{\mathbf{n}}$ uniquely.*

*Proof.* We must examine the initial conditions for $G^{(1)}_{\mathbf{n}}$. We note that for any $\mathbf{n}$ with some $n^{(\alpha)}=-1$, the function $G^{(1)}_{\mathbf{n}}$ must vanish. Indeed, by definition it is the constant term in $\mathcal{Q}_{\alpha,1}$ of an expression with no non-negative power of $\mathcal{Q}_{\alpha,1}$ (as the insertion of $\mathcal{Q}_{\alpha,1}^{-1}$ cancels the prefactor $\mathcal{Q}_{\alpha,1}$, and the contributions from $\tau(\mathbf{z})$ only provide strictly negative powers of $\mathcal{Q}_{\alpha,1}$). We conclude that all values of $G^{(1)}_{\mathbf{n}}=0$ for $\sigma(\mathbf{n})=-1,0$ except $G^{(1)}_{0,0,\ldots,0}=1$ by the normalization of $\phi$. With these initial data, the $r$ difference equations determine a unique solution $G^{(1)}_{\mathbf{n}}$ for all $\mathbf{n}=(n^{(\alpha)})_{\alpha\in[1,r]}$ and $n^{(\alpha)}\geq 0$ for all $\alpha$. □

**Example 2.14.** *When $r=2$ and $k=1$ (case of $\mathfrak{sl}_3$, level 1), we have $\Lambda_{1,1}=\Lambda_{2,2}=2$ and $\Lambda_{1,2}=\Lambda_{2,1}=1$. Denoting by $n=n_1^{(1)}$ and $p=n_1^{(2)}$, we have the following recursion relation for $G_{n,p}\equiv G^{(1)}_{n,p}$:*

$$t^3G_{n+1,p}+(t^{-3n}-1)G_{n-1,p+1}+t^{-3-3n}(t^{-3p}-1)G_{n,p-1}=t^{-2n-p-1}e_1(\mathbf{z})\,G_{n,p} \tag{2.21}$$

*This equation does not determine $G_{n,p}$ entirely. We also have to consider the "conjugate equation", obtained by insertion of the second conserved quantity $C_2$:*

$$t^3G_{n,p+1}+(t^{-3p}-1)G_{n+1,p-1}+t^{-3-3p}(t^{-3n}-1)G_{n-1,p}=t^{-n-2p-1}e_2(\mathbf{z})\,G_{n,p} \tag{2.22}$$

*These two equations are readily seen to be three-term linear recursion relations in the variable $j=\sigma(n,p)=n+p$, namely allow to express a single function with $\sigma=j+1$ in terms of functions with $\sigma=j,j-1$. Together with the initial data $G_{-1,p}=G_{n,-1}=0$ for all $n,p\geq 0$ and $G_{0,0}=1$ which determine all functions with $\sigma=-1,0$, the two above equations therefore determine $G_{n,p}$ completely. For instance, using the equations for all values of $\sigma=n+p$ indicated, we get:*

$$\begin{array}{lll}
\sigma=0: & G_{1,0}=t^{-4}e_1 \quad G_{0,1}=t^{-4}e_2 & \\
\sigma=1: & G_{2,0}=t^{-7}(t^{-3}e_1^2+(1-t^{-3})e_2) & G_{1,1}=t^{-6}(t^{-3}e_1e_2+1-t^{-3}) \\
 & G_{0,2}=t^{-7}(t^{-3}e_2^2+(1-t^{-3})e_1) &
\end{array}$$

*with the shorthand $e_1=z_1+z_2+z_3$ and $e_2=z_1z_2+z_2z_3+z_1z_3$. Note that the two equations determining $G_{1,1}$ are compatible, as a consequence of the commutation of $C_1$ and $C_2$ which implies $e_2G_{1,0}=e_1G_{0,1}$.*

Theorem 2.10 may be immediately translated in terms of graded characters $\chi_{\mathbf{n}}(q^{-1},\mathbf{z})$ by use of the formula (2.14), which results straightforwardly into the following:

**Theorem 2.15.** *The graded characters* $\chi_{\mathbf{n}} \equiv \chi_{\mathbf{n}}(q^{-1}, \mathbf{z})$, $\mathbf{n} = (n_i^{(\alpha)})_{\alpha\in[1,r];i\in[1,k]}$ *at "level" $k$ satisfy the following difference equation for $k \geq 1$:*

$$\begin{aligned} &\sum_{\alpha=1}^{r+1} \chi_{\mathbf{n}+\epsilon_{\alpha-1,k-1}-\epsilon_{\alpha,k-1}+\epsilon_{\alpha,k}-\epsilon_{\alpha-1,k}} \\ &\qquad -\sum_{\alpha=1}^{r} q^{k-1-\sum_{i=1}^k i n_i^{(\alpha)}} \chi_{\mathbf{n}+\epsilon_{\alpha-1,k-1}-\epsilon_{\alpha,k-1}+\epsilon_{\alpha+1,k}-\epsilon_{\alpha,k}} = e_1(\mathbf{z})\,\chi_{\mathbf{n}} \end{aligned} \tag{2.23}$$

*with the convention that* $(\epsilon_{\alpha,i})_j^{(\beta)} = \delta_{\beta,\alpha}\delta_{j,i}$, *for* $\beta \in [1, r]$ *and* $j \in [1, k]$, $\epsilon_{0,i} = \epsilon_{r+1,i} = 0$ *for all* $i$, *and* $e_1(\mathbf{z}) = z_1 + z_2 + ... + z_{r+1}$.

The higher conserved quantities give rise to higher difference equations for $\chi_{\mathbf{n}}$.

**Example 2.16.** *In the case $r = 1$ ($\mathfrak{s}l_2$), we have:*

$$\chi_{n_1,...,n_{k-1}-1,n_k+1} + \chi_{n_1,...,n_{k-1}+1,n_k-1} - q^{k-1-\sum_{i=1}^k i n_i}\chi_{n_1,...,n_{k-1}-1,n_k-1} = (z + z^{-1})\,\chi_{n_1,...,n_{k-1},n_k}$$

*For $k = 1$, this reduces to:*

$$\chi_{n+1} + (1 - q^{-n})\chi_{n-1} = (z + z^{-1})\,\chi_n \tag{2.24}$$

## 3. Difference equations for characters of level-1 (Demazure) modules

### 3.1. Level-1 difference equations and the $q$-deformed open Toda chain.

When specialized to level $k = 1$, the difference equation of Theorem 2.15 takes a particularly simple form. In this section, we show that this difference equation is the eigenvalue equation for the q-deformed Toda operator for $U_q(\mathfrak{sl}_{r+1})$ of [Eti99], after applying a suitable automorphism, and performing a number of specializations.

We start with a few definitions.

**Definition 3.1.** *We introduce the following difference operators acting on functions of the variable $\mathbf{x} = (x_1, ..., x_r)$:*

$$S_\alpha(f)(\mathbf{x}) = f(x_1, ..., x_{\alpha-1}, x_\alpha - \frac{1}{2}, x_{\alpha+1}, ..., x_r) \qquad (\alpha = 1, 2, ..., r) \tag{3.1}$$

$$S_0(f)(\mathbf{x}) = f(x_1 + \frac{1}{2}, x_2, ..., x_r) \tag{3.2}$$

$$S_{r+1}(f)(\mathbf{x}) = f(x_1, ..., x_{r-1}, x_r - \frac{1}{2}) \tag{3.3}$$

$$T_\alpha = S_{\alpha+1}S_\alpha^{-1} \qquad (\alpha = 0, 1, ..., r) \tag{3.4}$$

**Definition 3.2.** *The $\tilde{q}$-deformed difference (open) Toda Hamiltonian* [Eti99] *for $U_{\tilde{q}}(\mathfrak{sl}_{r+1})$ is the following operator acting on functions of $\mathbf{x}$, for fixed parameters $\tilde{q}, \nu_\alpha \in \mathbb{C}^*$:*

$$H_{\tilde{q}} = \sum_{\alpha=0}^{r} T_\alpha^2 + (\tilde{q} - \tilde{q}^{-1})^2 \sum_{\alpha=1}^{r} \nu_\alpha \, \tilde{q}^{-2x_\alpha} T_{\alpha-1} T_\alpha \tag{3.5}$$

In [Eti99], this Hamiltonian is related to the so-called relativistic Toda operator by use of an automorphism.

**Definition 3.3.** *We introduce the following automorphism $\tau$ of the algebra $\mathfrak{T}$ generated by $T_\alpha$, $\alpha = 0, 1, ..., r$ and $U_\alpha = \tilde{q}^{-2x_\alpha}$, $\alpha = 1, 2, ..., r$:*

$$\tau(T_\alpha) = T_\alpha \qquad \tau(U_\alpha) = U_\alpha \, T_\alpha \, T_{\alpha-1}^{-1} \tag{3.6}$$

*In particular, the automorphism $\tau$ respects the commutation relations $T_\alpha \, U_\beta = q^{\delta_{\beta,\alpha+1} - \delta_{\beta,\alpha}} U_\beta \, T_\alpha$.*

The image of $H_q$ under $\tau$ is the following Hamiltonian:

$$H'_{\tilde{q}} = \tau(H_{\tilde{q}}) = T_0^2 + \sum_{i=1}^{r} \left(1 + (\tilde{q} - \tilde{q}^{-1})^2 \nu_\alpha \, \tilde{q}^{-2x_\alpha}\right) T_\alpha^2 \tag{3.7}$$

On the other hand, it is easy to rewrite the level-1 difference equation (2.23) of Theorem 2.15 for $\chi_{\mathbf{n}} \equiv \chi_{\mathbf{n}}(q^{-1}, \mathbf{z})$, $n = \{n^{(\alpha)}\}_{\alpha \in [1,r]}$ as:

$$\left( \sum_{\alpha=0}^{r} T_\alpha^{-2} - \sum_{\alpha=1}^{r} q^{-n^{(\alpha)}} T_\alpha^{-2} \right) \chi_{\mathbf{n}} = e_1(\mathbf{z}) \chi_{\mathbf{n}} \tag{3.8}$$

where the operator $T_\alpha^2$ acts on functions of $\mathbf{n}$ by $(T_\alpha^2 f)(\mathbf{n}) = f(\mathbf{n} + \epsilon_{\alpha-1} - \epsilon_\alpha)$.

We see that if we pick

$$\mathbf{x} = -\mathbf{n}, \quad \tilde{q}^2 = q^{-1}, \quad (\tilde{q} - \tilde{q}^{-1})^2 \nu_\alpha = -1 \tag{3.9}$$

then the difference equation turns into an eigenvector equation for the $\tilde{q}$-Toda Hamiltonian (3.7), with eigenvalue $e_1(\mathbf{z})$. The level-1 graded character is therefore a $\tilde{q}$-Whittaker function.

3.2. **Whittaker functions and fusion products.** In [DFKT14], we have obtained the so-called fundamental q-Whittaker functions $W_\lambda(\mathbf{x})$ for $U_q(\mathfrak{sl}_{r+1})$ by explicitly constructing Whittaker vectors in a Verma module $V_\lambda$ with generic highest weight $\lambda$, using a path model. These form a basis of the eigenspace of the q-Toda Hamiltonian (3.5) for eigenvalue $E_\lambda = \sum_{i=0}^{r} q^{2(\lambda+\rho|\omega_{i+1}-\omega_i)}$ where $\omega_i$ are the fundamental weights of $A_r$. The dimension of this eigenspace is the order of the Weyl group, here $(r+1)!$, as we may generate other independent solutions $W_{s(\lambda+\rho)-\rho}(\mathbf{x})$ by Weyl group reflections $s$, while preserving $E_{s(\lambda+\rho)-\rho} = E_\lambda$.

Identifying $z_i = \tilde{q}^{2(\lambda+\rho|\omega_i-\omega_{i-1})}$ for $i = 1, 2, ..., r+1$, we deduce that the graded level-1 character $\chi_{\mathbf{n}}$ is a linear combination of the image of the fundamental $\tilde{q}$-Whittaker functions under the automorphism $\tau$, with the additional specialization (3.9).

Let us illustrate this in the case of $U_q(\mathfrak{sl}_2)$. The $\tilde{q}$-Toda eigenvector equation is:

$$W_\lambda(x-1)+(1+(\tilde{q}-\tilde{q}^{-1})^2\nu\tilde{q}^{-2x})W_\lambda(x+1)=(p+p^{-1})W(x), \qquad p=\tilde{q}^{-\frac{\lambda+1}{2}}$$

Applying the automorphism $\tau$ and using the specialization (3.9), we obtain a transformed fundamental $\tilde{q}$-Whittaker function $W'_\lambda(n)$, with the following series expansion (valid for $|q|>1$):

$$W'_\lambda(n)=\tau(W_\lambda)(-n)=p^{n-\frac{1}{2}}\sum_{a\in\mathbb{Z}_+}\frac{q^{-a(n+1)}}{\prod_{i=1}^a(1-q^{-i})(1-p^2q^{-i})}$$

Analogously, we have the Weyl-reflected fundamental q-Whittaker function:

$$W'_{-\lambda-2}(n)=p^{\frac{1}{2}-n}\sum_{a\in\mathbb{Z}_+}\frac{q^{-a(n+1)}}{\prod_{i=1}^a(1-q^{-i})(1-p^{-2}q^{-i})}$$

The functions $W'_\lambda(n), W'_{-\lambda-2}(n)$ form a basis of the eigenspace of the transformed $\tilde{q}$-Toda Hamiltonian with same eigenvalue, namely

$$\tau(H_{\tilde{q}})W'(n)=W'(n+1)+(1-q^{-n})W'(n-1)=(p+p^{-1})W'(n) \tag{3.10}$$

This coincides with the level-1 difference equation (2.24) with $p=z$. Looking for a linear combination $\chi_n=c_\lambda(p,q)\,W'_\lambda(n)+c_{-\lambda-2}(p,q)\,W'_{-\lambda-2}(n)$ for say $n=0,1$, we find the coefficients:

$$c_\lambda(p,q)=\frac{p^{\frac{1}{2}}}{(1-p^{-2})\prod_{i=1}^\infty(1-p^{-2}q^{-i})},\quad c_{-\lambda-2}(p,q)=c_\lambda(p^{-1},q)=\frac{p^{-\frac{1}{2}}}{(1-p^{2})\prod_{i=1}^\infty(1-p^{2}q^{-i})}$$

Remarkably, we have realized the graded character, which is polynomial in $q^{-1}$, $p,p^{-1}$ as a linear combination of two infinite series of $q^{-1}$ (the fundamental q-Whittaker functions). The cancellations occurring are the q-deformed version of the so-called class 1 regularity condition on Whittaker functions. So we may view the graded character as a class one specialized q-Whittaker function. We expect this to generalize to $U_q(\mathfrak{sl}_{r+1})$ (see also [GLO10, GLO11] for analogous considerations).

## 4. The solution for $\mathfrak{sl}_{r+1}$

In this section, we introduce a generalization of the specialized Macdonald difference operators (corresponding to their "dual Whittaker limit" $t\to\infty$), and use them to construct a solution of the difference equations for the $\mathfrak{sl}_{r+1}$ graded characters, by iterated action on the constant function 1. We shall proceed in several steps. After introducing the new difference operators, we show that they satisfy the dual quantum $Q$-system. This allows to consider them as raising operators for graded characters, Theorems 4.5 and 4.6, which are proved in two separate steps, first only for level $k=1$ and then for general level $k\geq 2$.

### 4.1. **A realization of the dual quantum $Q$-system via generalized Macdonald operators.**

**Definition 4.1.** *We introduce the following generalizations of the specialized Macdonald difference operators, acting on functions $f(\mathbf{z})$, where $\mathbf{z} = (z_1, ..., z_{r+1})$. We first define the shift operators $\Delta, \Gamma_i, D_i$:*

$$\Delta(f)(\mathbf{z}) = f(t\mathbf{z}) \tag{4.1}$$

$$\Gamma_i(f)(z_1, ..., z_{r+1}) = f(z_1, ..., z_{i-1}, qz_i, z_{i+1}, ..., z_{r+1}) \tag{4.2}$$

$$D_i = \Gamma_i \circ \Delta \tag{4.3}$$

*For any set $I \subset [1, r+1]$, with complement $\bar{I} = [1, r+1] \setminus I$, we define*

$$z_I = \prod_{i \in I} z_i \tag{4.4}$$

$$D_I = \prod_{i \in I} D_i \tag{4.5}$$

$$a_I(\mathbf{z}) = \prod_{\substack{i \in I \\ j \in \bar{I}}} \frac{z_i}{z_i - z_j} \tag{4.6}$$

*We have the following sequence of operators $\mathcal{D}_{\alpha,n}$, $\alpha = 0, 1, ..., r+1$ and $n \in \mathbb{Z}$:*

$$\mathcal{D}_{\alpha,n} = t^{-\frac{\Lambda_{\alpha,\alpha}}{2} n - \sum_{\beta=1}^{r} \Lambda_{\alpha,\beta}} \sum_{\substack{I \subset [1,r+1] \\ |I| = \alpha}} (z_I)^n a_I(\mathbf{z})\, D_I \tag{4.7}$$

*In particular we have*

$$\mathcal{D}_{0,n} = 1 \quad \text{and} \quad \mathcal{D}_{r+1,n} = (z_1 z_2 \cdots z_{r+1})^n D_1 D_2 \cdots D_{r+1} = (z_1 z_2 \cdots z_{r+1})^n = 1$$

Recall the standard definition of the difference Macdonald operators for $\mathfrak{sl}_{r+1}$ [Mac95]:

$$M_\alpha^{q,t} = \sum_{\substack{I \subset [1,r+1] \\ |I| = \alpha}} \prod_{\substack{i \in I \\ j \notin I}} \frac{t z_i - z_j}{z_i - z_j} \Gamma_I \tag{4.8}$$

where $\Gamma_I = \prod_{i \in I} \Gamma_i$. This expression allows to identify our operators $\mathcal{D}_{\alpha,n}$ for $n = 0$ as:

$$\begin{aligned} \mathcal{D}_{\alpha,0} &= t^{-\sum_{\beta=1}^{r} \Lambda_{\alpha,\beta}} M_\alpha \, \Delta^\alpha \\ M_\alpha &:= \lim_{t \to \infty} t^{-\alpha(r+1-\alpha)} M_\alpha^{q,t} = \sum_{\substack{I \subset [1,r+1] \\ |I| = \alpha}} a_I(\mathbf{z})\, \Gamma_I \end{aligned} \tag{4.9}$$

The operators $M_\alpha^{q,t}$ as well as their limits $M_\alpha$, $\alpha = 0, 1, ..., r+1$ are known to form a commuting family.

Let $\mathcal{A}^*$ be the algebra generated by $\{Q^*_{\alpha,k} : \alpha \in [1,r], k \in \mathbb{Z}\}$ over $\mathbb{Z}[t,t^{-1}]$ modulo the ideal generated by the relations

$$\begin{aligned} \mathcal{Q}^*_{\alpha,n}\,\mathcal{Q}^*_{\beta,p} &= t^{-\Lambda_{\alpha,\beta}(p-n)}\,\mathcal{Q}^*_{\beta,p}\,\mathcal{Q}^*_{\alpha,n} \qquad (|p-n|\leq|\beta-\alpha|+1) \\ t^{\Lambda_{\alpha,\alpha}}\,\mathcal{Q}^*_{\alpha,n-1}\,\mathcal{Q}^*_{\alpha,n+1} &= (\mathcal{Q}^*_{\alpha,n})^2-\mathcal{Q}^*_{\alpha+1,n}\,\mathcal{Q}^*_{\alpha-1,n} \end{aligned} \tag{4.10}$$

Equivalently, the second relation may be rewritten, using (4.10) as:

$$t^{-\Lambda_{\alpha,\alpha}}\,\mathcal{Q}^*_{\alpha,n+1}\,\mathcal{Q}^*_{\alpha,n-1}=(\mathcal{Q}^*_{\alpha,n})^2-t^{-r-1}\mathcal{Q}^*_{\alpha+1,n}\,\mathcal{Q}^*_{\alpha-1,n} \tag{4.11}$$

We refer to this as the dual quantum $Q$-system. The algebra $\mathcal{A}^*$ is isomorphic t o the algebra $\mathcal{A}^{\rm op}$, with the opposite multiplication to $\mathcal{A}$.

We have the following main result.

**Theorem 4.2.** *We have a polynomial representation $\pi$ of $\mathcal{A}^*$, with $\pi(\mathcal{Q}^*_{\alpha,n})=\mathcal{D}_{\alpha,n}$ of (4.7). That is, acting by left multiplication on the space $\mathbb{C}[\mathbf{z}]$, the operators $\mathcal{D}_{\alpha,n}$ obey the dual quantum $Q$-system relations for $A_r$:*

$$\mathcal{D}_{\alpha,n}\,\mathcal{D}_{\beta,p} = t^{-\Lambda_{\alpha,\beta}(p-n)}\,\mathcal{D}_{\beta,p}\,\mathcal{D}_{\alpha,n} \qquad (|p-n|\leq|\beta-\alpha|+1) \tag{4.12}$$
$$t^{-\Lambda_{\alpha,\alpha}}\,\mathcal{D}_{\alpha,n+1}\,\mathcal{D}_{\alpha,n-1} = (\mathcal{D}_{\alpha,n})^2-t^{-r-1}\mathcal{D}_{\alpha+1,n}\,\mathcal{D}_{\alpha-1,n} \tag{4.13}$$

Note that when $n=p=0$ the relation (4.12) boils down to the commutation of the specialized Macdonald operators at $t\to\infty$, as $M_\alpha\,\Delta=\Delta\,M_\alpha$.

The remainder of this section is devoted to the proof of this theorem.

Let us define for any disjoint sets $I$, $J$ of indices the quantities:

$$a_{I,J}(\mathbf{z}) = \prod_{\substack{i\in I\\ j\in J}}\frac{z_i}{z_i-z_j} \tag{4.14}$$

$$b_{I,J}(\mathbf{z}) = \prod_{\substack{i\in I\\ j\in J}}\frac{z_i}{z_i-qz_j} \tag{4.15}$$

$$c_{I,J}(\mathbf{z}) = \prod_{\substack{i\in I\\ j\in J}}\frac{qz_i}{qz_i-z_j} \tag{4.16}$$

Note that in this notation $a_I(\mathbf{z})$ of eq.(4.6) is simply $a_{I,\bar{I}}(\mathbf{z})$.

We have the following two lemmas.

**Lemma 4.3.** *Fix integers $0\leq a\leq b$ and $\mathbf{z}=(z_1,z_2,...,z_{a+b})$. Then we have:*

$$\sum_{\substack{I\cup J=[1,a+b],\ I\cap J=\emptyset\\ |I|=a,\ |J|=b}}(z_J)^p\,(a_{I,J}(\mathbf{z})b_{J,I}(\mathbf{z})-q^{pa}a_{J,I}(\mathbf{z})b_{I,J}(\mathbf{z}))=0 \quad (|p|\leq b-a+1) \tag{4.17}$$

**Lemma 4.4.** *Fix an integer $a \geq 1$ and $\mathbf{z} = (z_1, z_2, ..., z_{2a})$. Then we have:*

$$\sum_{\substack{I\cup J=[1,2a],\ I\cap J=\emptyset \\ |I|=|J|=a}} a_{I,J}(\mathbf{z}) b_{J,I}(\mathbf{z}) \left(1 - q^a \frac{z_I}{z_J}\right) = \sum_{\substack{I\cup J=[1,2a],\ I\cap J=\emptyset \\ |I|=a+1,\ |J|=a-1}} a_{I,J}(\mathbf{z}) b_{J,I}(\mathbf{z}) \tag{4.18}$$

The above Lemmas 4.3 and 4.4 are proved in Appendix A below. Let us now turn to the proof of Theorem 4.2. Let us first compute the quantity $\mathcal{D}_{\alpha,n}\, \mathcal{D}_{\beta,p}$. Substituting the definition (4.7), we get:

$$\begin{aligned}
\mathcal{D}_{\alpha,n}\, \mathcal{D}_{\beta,p} &= \sum_{\substack{I,J\subset[1,r+1] \\ |I|=\alpha,\ |J|=\beta}} (z_I)^n\, a_I(\mathbf{z})\, D_I\, (z_J)^p\, a_J(\mathbf{z})\, D_J \\
&= \sum_{\substack{K\subset L\subset[1,r+1] \\ |L|\leq\alpha+\beta,\ |K|\leq\alpha,\beta}} \sum_{\substack{I_0\cup J_0=L\setminus K, I_0\cap J_0=\emptyset \\ (I=K\cup I_0,\ J=K\cup J_0)}} (z_{I_0} z_K)^n (z_{J_0} z_K)^p\, a_{K\cup I_0,\bar{L}\cup J_0}\, t^{\alpha\beta p} D_K D_{I_0} a_{K\cup J_0,\bar{L}\cup I_0} D_J \\
&= t^{\alpha\beta p} \sum_{\substack{K\subset L\subset[1,r+1] \\ |L|\leq\alpha+\beta,\ |K|\leq\alpha,\beta}} (z_L z_K)^n \sum_{\substack{I_0\cup J_0=L\setminus K, I_0\cap J_0=\emptyset \\ |I_0|=\alpha-|K|,\ |J_0|=\beta-|K|}} (z_{J_0})^{p-n} a_{K,\bar{L}} a_{I_0,\bar{L}} a_{K,J_0} a_{I_0,J_0} c_{K,\bar{L}} a_{K,I_0} a_{J_0,\bar{L}} b_{J_0,I_0} D_I D_J \\
&= t^{\alpha\beta p} \sum_{\substack{K\subset L\subset[1,r+1] \\ |L|\leq\alpha+\beta,\ |K|\leq\alpha,\beta}} (z_L z_K)^n a_{K,\bar{L}} c_{K,\bar{L}} a_{K,L\setminus K} a_{L\setminus K,\bar{L}} \left( \sum_{\substack{I_0\cup J_0=L\setminus K, I_0\cap J_0=\emptyset \\ |I_0|=\alpha-|K|,\ |J_0|=\beta-|K|}} (z_{J_0})^{p-n} a_{I_0,J_0} b_{J_0,I_0} \right) D_K D_L
\end{aligned}$$

where we have replaced the sum over $I, J$ by one over $K = I \cap J$ and $L = I \cup J$ first, and then written the disjoint unions $I = K \cup I_0$, $J = K \cup J_0$, $\bar{I} = \bar{L} \cup J_0$, and $\bar{J} = \bar{L} \cup I_0$. Note that we have isolated a factor $u_{K,L}(n) := (z_L z_K)^n a_{K,\bar{L}} c_{K,\bar{L}} a_{K,L\setminus K} a_{L\setminus K,\bar{L}}$ which does not depend on $I_0, J_0$. We may now write:

$$\begin{aligned}
& t^{\frac{n\Lambda_{\alpha,\alpha}+p\Lambda_{\beta,\beta}}{2}+\sum_\gamma \Lambda_{\alpha,\gamma}+\Lambda_{\beta,\gamma}} \left\{ \mathcal{D}_{\alpha,n}\, \mathcal{D}_{\beta,p} - t^{-\Lambda_{\alpha,\beta}(p-n)} \mathcal{D}_{\beta,p}\, \mathcal{D}_{\alpha,n} \right\} \\
&= \sum_{\substack{K\subset L\subset[1,r+1] \\ |L|\leq\alpha+\beta,\ |K|\leq\alpha,\beta}} u_{K,L}(n) \sum_{\substack{I_0\cup J_0=L\setminus K, I_0\cap J_0=\emptyset \\ |I_0|=\alpha-|K|,\ |J_0|=\beta-|K|}} (z_{J_0})^{p-n} \Big( t^{\alpha\beta p} a_{I_0,J_0} b_{J_0,I_0} - t^{\alpha\beta n-\Lambda_{\alpha,\beta}(p-n)} a_{J_0,I_0} b_{I_0,J_0} \Big) D_K D_L \\
&= t^{\alpha\beta p} \sum_{\substack{K\subset L\subset[1,r+1] \\ |L|\leq\alpha+\beta,\ |K|\leq\alpha,\beta}} u_{K,L}(n) \sum_{\substack{I_0\cup J_0=L\setminus K, I_0\cap J_0=\emptyset \\ |I_0|=\alpha-|K|,\ |J_0|=\beta-|K|}} (z_{J_0})^{p-n} \Big( a_{I_0,J_0} b_{J_0,I_0} - q^{\alpha(p-n)} a_{J_0,I_0} b_{I_0,J_0} \Big) D_K D_L = 0
\end{aligned}$$

where we have first used $\Lambda_{\alpha,\beta} + \alpha\beta = \alpha(r+1)$ for $\alpha \leq \beta$, $q = t^{-(r+1)}$, and then applied Lemma 4.3 for every fixed pair $K, L$ to the second summation, with $a = \alpha - |K| \leq b = \beta - |K|$ and $|p - n| \leq b - a + 1 = \beta - \alpha + 1$. The relation (4.12) follows.

Analogously, we compute:

$$t^{n\Lambda_{\alpha,\alpha}+2\sum_\beta \Lambda_{\alpha,\beta}}\left\{(\mathcal{D}_{\alpha,n})^2 - t^{-\Lambda_{\alpha,\alpha}}\,\mathcal{D}_{\alpha,n+1}\,\mathcal{D}_{\alpha,n-1}\right\}$$

$$= \sum_{\substack{K\subset L\subset[1,r+1]\\ |L|\leq 2\alpha,\ |K|\leq\alpha}} u_{K,L}(n) \sum_{\substack{I_0\cup J_0=L\setminus K, I_0\cap J_0=\emptyset\\ |I_0|=|J_0|=\alpha-|K|}} a_{I_0,J_0}b_{J_0,I_0}\left(t^{n\alpha^2} - t^{(n-1)\alpha^2-\Lambda_{\alpha,\alpha}}\frac{z_{I_0}}{z_{J_0}}\right) D_K D_L$$

$$= t^{n\alpha^2} \sum_{\substack{K\subset L\subset[1,r+1]\\ |L|\leq 2\alpha,\ |K|\leq\alpha}} u_{K,L}(n) \sum_{\substack{I_0\cup J_0=L\setminus K, I_0\cap J_0=\emptyset\\ |I_0|=|J_0|=\alpha-|K|}} a_{I_0,J_0}b_{J_0,I_0}\left(1 - q^{\alpha}\frac{z_{I_0}}{z_{J_0}}\right) D_K D_L \tag{4.19}$$

and finally

$$t^{n\Lambda_{\alpha,\alpha}+2\sum_\beta \Lambda_{\alpha,\beta}-r-1}\mathcal{D}_{\alpha+1,n}\,\mathcal{D}_{\alpha-1,n} = t^n t^{\frac{1}{2}(\Lambda_{\alpha+1,\alpha+1}+\Lambda_{\alpha-1,\alpha-1})n+\sum_\beta \Lambda_{\alpha+1,\beta}+\Lambda_{\alpha-1,\beta}}\mathcal{D}_{\alpha+1,n}\,\mathcal{D}_{\alpha-1,n}$$

$$= t^{n\alpha^2} \sum_{\substack{K\subset L\subset[1,r+1]\\ |L|\leq 2\alpha,\ |K|\leq\alpha}} u_{K,L}(n) \sum_{\substack{I_0\cup J_0=L\setminus K, I_0\cap J_0=\emptyset\\ |I_0|=\alpha+1-|K|, |J_0|=\alpha-1-|K|}} a_{I_0,J_0}b_{J_0,I_0} D_K D_L \tag{4.20}$$

where we have used the relations

$$\begin{aligned} 2+\Lambda_{\alpha+1,\alpha+1}+\Lambda_{\alpha-1,\alpha-1}-2\Lambda_{\alpha,\alpha} &= 0\\ \Lambda_{\alpha+1,\beta}+\Lambda_{\alpha-1,\beta}-2\Lambda_{\alpha,\beta} &= -(r+1)\delta_{\alpha,\beta}\end{aligned}$$

The relation (4.13) follows by identifying equations (4.19) and (4.20) by applying Lemma 4.4 for $a=\alpha-|K|$ to the second summation for $K,L$ fixed. This completes the proof of Theorem 4.2.

## 4.2. **Graded characters and difference raising operators.**

4.2.1. *The main results.* In this section, we show that, in a way analogous to how the Kirillov-Noumi difference operators are raising operators for Macdonald polynomials [KN99], our generalized degenerate Macdonald operators are raising operators for the graded characters.

**Theorem 4.5.** *For $\mathbf{n}=\{n_i^{(\alpha)}\}_{\alpha\in[1,r];i\in\mathbb{Z}_{>0}}$, the coefficients $G_{\mathbf{n}}^{(k)}$ (2.13) for $A_r$ at level $k$ are given by the iterated action of the generalized Macdonald operators (4.7) on the constant function 1:*

$$G_{\mathbf{n}}^{(k)} = \prod_{\alpha=1}^{r}(\mathcal{D}_{\alpha,k})^{n_k^{(\alpha)}}\prod_{\alpha=1}^{r}(\mathcal{D}_{\alpha,k-1})^{n_{k-1}^{(\alpha)}}\cdots\prod_{\alpha=1}^{r}(\mathcal{D}_{\alpha,1})^{n_1^{(\alpha)}}\,1 \tag{4.21}$$

Using the relation (2.14), we immediately deduce the following:

**Theorem 4.6.** *The graded characters for $\mathfrak{sl}_{r+1}$ at level $k$ are given by:*

$$\begin{aligned}\chi_{\mathbf{n}}(q^{-1},\mathbf{z}) &= t^{\frac{1}{2}\sum_{i,j,\alpha,\beta} n_i^{(\alpha)}\mathrm{Min}(i,j)\Lambda_{\alpha,\beta}n_j^{(\beta)}+\sum_{i,\alpha,\beta} n_i^{(\alpha)}\Lambda_{\alpha,\beta}+\frac{1}{2}\sum_\alpha \Lambda_{\alpha,\alpha}+\sum_{\alpha<\beta}\Lambda_{\alpha,\beta}} \\ &\quad\times \prod_{\alpha=1}^r (\mathcal{D}_{\alpha,k})^{n_k^{(\alpha)}} \prod_{\alpha=1}^r (\mathcal{D}_{\alpha,k-1})^{n_{k-1}^{(\alpha)}} \cdots \prod_{\alpha=1}^r (\mathcal{D}_{\alpha,1})^{n_1^{(\alpha)}}\, 1\end{aligned} \tag{4.22}$$

Relaxing the condition $z_1z_2\cdots z_{r+1}=1$, we may restate this result in terms of the family of difference operators $M_{\alpha,n}$ defined as:

$$M_{\alpha,n} = \sum_{\substack{I\subset[1,r+1]\\ |I|=\alpha}} (z_I)^n a_I(\mathbf{z})\,\Gamma_I = t^{\frac{\Lambda_{\alpha,\alpha}}{2}n+\sum_\beta \Lambda_{\alpha,\beta}}\,\mathcal{D}_{\alpha,n}\Delta^{-\alpha} \tag{4.23}$$

These satisfy a renormalized version of the dual quantum $Q$-system:

$$M_{\alpha,n}\,M_{\beta,p} = q^{\mathrm{Min}(\alpha,\beta)(p-n)}\,M_{\beta,p}\,M_{\alpha,n} \qquad (|p-n|\le|\beta-\alpha|+1) \tag{4.24}$$

$$q^\alpha\,M_{\alpha,n+1}\,M_{\alpha,n-1} = (M_{\alpha,n})^2 - M_{\alpha+1,n}\,M_{\alpha-1,n} \qquad (\alpha\in[1,r];n\in\mathbb{Z}) \tag{4.25}$$

with $M_{0,n}=1$ and $M_{r+1,n}=(z_1z_2\cdots z_{r+1})^n\Delta^{-r-1}$. Note also that $M_{\alpha,0}$ is equal to the degenerate Macdonald operator $M_\alpha$ of eq.(4.9). We have:

**Corollary 4.7.** *The graded characters for $\mathfrak{sl}_{r+1}$ at level $k$ are given by:*

$$\begin{aligned}\chi_{\mathbf{n}}(q^{-1},\mathbf{z}) &= q^{-\frac{1}{2}\sum_{i,j,\alpha,\beta} n_i^{(\alpha)}\mathrm{Min}(i,j)\mathrm{Min}(\alpha,\beta)n_j^{(\beta)}+\frac{1}{2}\sum_{i,\alpha} i\alpha n_i^{(\alpha)}} \\ &\quad\times \prod_{\alpha=1}^r (M_{\alpha,k})^{n_k^{(\alpha)}} \prod_{\alpha=1}^r (M_{\alpha,k-1})^{n_{k-1}^{(\alpha)}} \cdots \prod_{\alpha=1}^r (M_{\alpha,1})^{n_1^{(\alpha)}}\, 1\end{aligned} \tag{4.26}$$

*Proof.* We use the relation (4.23) to rewrite the result of Theorem 4.6. We make use of the commutation relation $\Delta M_{\alpha,n}=t^{n\alpha}M_{\alpha,n}\Delta$, and of $\Lambda_{\alpha,\beta}+\alpha\beta=\mathrm{Min}(\alpha,\beta)$. □

**Remark 4.8.** *The iterated action of the raising operators $M_{\alpha,n}$ on the function 1 results clearly in a symmetric polynomial of the $z$'s with coefficients that are polynomial in $q$. On the other hand, the prefactor is a negative integer power of $q$, as*

$$\begin{aligned}&\frac{1}{2}\sum_{i,j,\alpha,\beta} n_i^{(\alpha)}\mathrm{Min}(i,j)\mathrm{Min}(\alpha,\beta)n_j^{(\beta)} - \frac{1}{2}\sum_{i,\alpha} i\alpha n_i^{(\alpha)} \\ &\quad= \sum_{i,\alpha} i\alpha\frac{n_i^{(\alpha)}(n_i^{(\alpha)}-1)}{2} + \sum_{i<j \text{ or } \alpha<\beta} n_i^{(\alpha)}\mathrm{Min}(i,j)\mathrm{Min}(\alpha,\beta)n_j^{(\beta)} \in \mathbb{Z}_+\end{aligned}$$

*We deduce that $\chi_{\mathbf{n}}(q^{-1},\mathbf{z})$ is a polynomial of $q,q^{-1}$. Moreover the graded characters have the limit $\lim_{q\to\infty}\chi_{\mathbf{n}}(q^{-1},\mathbf{z})=s_\lambda(\mathbf{z})$ which sends the graded tensor product to its top component, with $\lambda_\alpha=\sum_{i=1}^k i n_i^{(\alpha)}$, hence $\chi_{\mathbf{n}}(q^{-1},\mathbf{z})$ is a polynomial of $q^{-1}$, as expected from its definition.*

4.2.2. *Proof in the case of level* 1. Let us now turn to the proof of Theorem 4.5. We will proceed in two steps. First, we will show the theorem in the case $k=1$ only. The idea is to show that the expression (4.21) satisfies all the difference equations that determine $G_{\mathbf{n}}^{(1)}$. To this end we use the conserved quantities of the dual quantum $Q$-system, easily obtained by applying the anti-homorphism

$$*: \mathcal{A} \to \mathcal{A}^*, \qquad \mathcal{Q}_{\alpha,k} \mapsto \mathcal{Q}^*_{\alpha,k}$$

such that $(AB)^* = B^* A^*$ for all $A, B \in \mathcal{A}$ and $t^* = t$, and then evaluating in the polynomial representation where $\pi(\mathcal{Q}^*_{\alpha,k}) = \mathcal{D}_{\alpha,k}$. The quantities $\mathcal{Y}_\alpha(n) = \pi(y_\alpha(n)^*)$ of (2.2-2.3) are expressed in terms of $\mathcal{D}_{\alpha,n}$, $\mathcal{D}_{\alpha,n+1}$ as:

$$\begin{aligned}
\mathcal{Y}_{2\alpha-1}(n) &= \mathcal{D}_{\alpha-1,n}\mathcal{D}_{\alpha,n}^{-1}\mathcal{D}_{\alpha-1,n+1}^{-1}\mathcal{D}_{\alpha,n+1} \\
\mathcal{Y}_{2\alpha}(n) &= -\mathcal{D}_{\alpha-1,n}\mathcal{D}_{\alpha,n}^{-1}\mathcal{D}_{\alpha,n+1}^{-1}\mathcal{D}_{\alpha+1,n+1}
\end{aligned}$$

Here, we use the formal (left and right) inverse $\mathcal{D}_{\alpha,k}^{-1}$ of the difference operator $\mathcal{D}_{\alpha,k}$ defined as follows. If $|t|>1$, setting $I_\alpha = \{1,2,\ldots,\alpha\}$, we write the convergent series:

$$\begin{aligned}
\mathcal{D}_{\alpha,k}^{-1} &= \left( z_{I_\alpha}^k a_{I_\alpha}(\mathbf{z}) D_{I_\alpha} \left( \sum_{I\subset[1,r+1],\, |I|=\alpha} D_{I_\alpha}^{-1} \frac{z_I^k a_I(\mathbf{z})}{z_{I_\alpha}^k a_{I_\alpha}(\mathbf{z})} D_I \right)\right)^{-1} \\
&= \sum_{n\geq 0} \left( \sum_{I\subset[1,r+1],\, |I|=\alpha} D_{I_\alpha}^{-1} \frac{z_I^k a_I(\mathbf{z})}{z_{I_\alpha}^k a_{I_\alpha}(\mathbf{z})} D_I \right)^n D_{I_\alpha}^{-1} \frac{1}{z_{I_\alpha}^k a_{I_\alpha}(\mathbf{z})}
\end{aligned}$$

If $|t|<1$, we must use $\bar{I}_{r-\alpha+1} = \{r+1, r, \ldots, r-\alpha+2\}$ instead of $I_\alpha$.

Noting that $*$ is an anti-homorphism which inverts the order of weights, we get the following:

**Lemma 4.9.** *The conserved quantities $\mathcal{C}_m = \pi(C_m^*)$, $m=0,1,\ldots,r+1$ of the dual quantum $Q$-system are expressed in terms of the operators $\mathcal{D}_{\alpha,n}$, $\mathcal{D}_{\alpha,n+1}$ as:*

$$\mathcal{C}_m = \sum_{\substack{\text{Hard Particle configurations} \\ i_1<i_2<\cdot<i_m \text{ on } \mathcal{G}_r}} \mathcal{Y}_{i_1}(n)\mathcal{Y}_{i_2}(n)\cdots\mathcal{Y}_{i_m}(n) \tag{4.27}$$

For instance, we have the first non-trivial conserved quantity, obtained from (2.9):

$$\mathcal{C}_1 = \sum_{\alpha=1}^{r+1} t^r \mathcal{D}_{\alpha-1,n+1}^{-1}\mathcal{D}_{\alpha,n+1}\mathcal{D}_{\alpha-1,n}\mathcal{D}_{\alpha,n}^{-1} - \sum_{\alpha=1}^{r} t^{-1}\mathcal{D}_{\alpha,n+1}^{-1}\mathcal{D}_{\alpha+1,n+1}\mathcal{D}_{\alpha-1,n}\mathcal{D}_{\alpha,n}^{-1} \tag{4.28}$$

All quantities $\mathcal{C}_m$ (4.27) are conserved i.e. they are independent of $n$, and we may in particular express them in the limit $n\to\infty$ as we did before.

**Theorem 4.10.** *For all $m = 0, 1, ..., r + 1$, the conserved quantity $\mathcal{C}_m$ (4.27) of the dual quantum $Q$-system acts on functions of $\mathbf{z}$ by multiplication by $t^{\frac{mr}{2}}$ times the $m$-th elementary symmetric function $e_m(\mathbf{z})$, namely*

$$\mathcal{C}_m = t^{\frac{mr}{2}} \sum_{1\leq i_1<i_2<\cdots<i_m\leq r+1} z_{i_1} z_{i_2} \cdots z_{i_m} = t^{\frac{mr}{2}} e_m(\mathbf{z})$$

*Proof.* We will compute the action of $\mathcal{C}_m$ expressed as (4.27) in the limit when $n \to \infty$. We must estimate the operator $\mathcal{D}_{\alpha,n}$ when $n$ becomes large. To this end, and without loss of generality, let us assume the modules of the $z_i$'s are strictly ordered, say $|z_1| > |z_2| > \cdots > |z_{r+1}| > 0$. Then for large $n$ the expression for $\mathcal{D}_{\alpha,n}$ is dominated by the contribution of the subset $I_\alpha = \{1, 2, ..., \alpha\}$, and we have

$$\mathcal{D}_{\alpha,n} \sim t^{-\frac{\Lambda_{\alpha,\alpha}}{2}n - \sum_\beta \Lambda_{\alpha,\beta}} (z_{I_\alpha})^n a_{I_\alpha}(\mathbf{z}) D_{I_\alpha} \quad \text{hence} \quad \lim_{n\to\infty} \mathcal{D}_{\alpha,n+1}^{-1} \mathcal{D}_{\alpha,n} = t^{-\frac{\Lambda_{\alpha,\alpha}}{2}} z_{I_\alpha}^{-1}$$

This gives

$$\begin{aligned} \lim_{n\to\infty} \mathcal{Y}_{2\alpha-1}(n) &= t^{\frac{r}{2}} z_\alpha \qquad (\alpha = 1, 2, ..., r + 1) \\ \lim_{n\to\infty} \mathcal{Y}_{2\alpha}(n) &= 0 \qquad (\alpha = 1, 2, ..., r) \end{aligned}$$

as the latter is proportional to $(z_{\alpha+1}/z_\alpha)^n \to 0$ when $n \to \infty$. As before, the hard particle model reduces to that on the odd vertices of $\mathcal{G}_r$ which are not connected by edges, hence the partition functions are simply the elementary symmetric functions of the variables $t^{\frac{r}{2}} z_\alpha$, $\alpha = 1, 2..., r + 1$ and the theorem follows. □

We are now ready to prove Theorem 4.5 in the case of level $k = 1$. We will show that the function (4.21) for $k = 1$ satisfies the *same* difference equation (2.19) as in Theorem 2.10 and its higher $m$ versions.

First, we may identify the action of the conserved quantity $C_m$ on the function $\tau(\mathbf{z})$ within the constant term evaluation of Corollary 2.9 with that of the conserved quantity $\mathcal{C}_m$ on functions of $\mathbf{z}$ of Theorem 4.10 above: in both cases, the action is by multiplication by $t^{\frac{mr}{2}} e_m(\mathbf{z})$. This involves writing the conserved quantity at $n \to \infty$ in both cases.

Second, if we use the expression of the conserved quantity $C_m$ (resp. $\mathcal{C}_m$) as a function of $\mathcal{Q}_{\alpha,0}, \mathcal{Q}_{\alpha,1}$ (resp. $\mathcal{D}_{\alpha,0}, \mathcal{D}_{\alpha,1}$), we obtain the exact *same* combinations of shift operators.

This shows that the difference equations obeyed by (2.13) and (4.21) at level $k = 1$ are identical. To complete the analysis, we should in principle examine the initial conditions. We have seen that $G^{(1)}_{\mathbf{n}} = 0$ as soon as any of the $n^{(\alpha)}$ are equal to $-1$. Let us now show that these conditions are not necessary to fix the solution, as each such term comes with a vanishing prefactor, and therefore drops out of the difference equation.

This fact relies on an important result of Ref. [DFK14], which was instrumental in proving the polynomiality property for the associated quantum cluster algebra. It relies on the Laurent polynomiality property which asserts that any cluster variable may be expressed as a Laurent polynomial of any seed variables. The following Lemma was derived by combining the Laurent property of the quantum cluster algebra for initial data $\mathcal{S}_0 = \{\mathcal{Q}_{\alpha,0}, \mathcal{Q}_{\alpha,1}\}$ as well as for initial data $\mathcal{S}_{-1} = \{\mathcal{Q}_{\alpha,-1}, \mathcal{Q}_{\alpha,0}\}$.

**Lemma 4.11.** *([DFK14], Lemma 5.9 and its proof.) For any polynomial $p$ of the $\{\mathcal{Q}_{\alpha,i}\}$ with coefficients in $\mathbb{Z}[t,t^{-1}]$, there exists a unique expression of the form:*

$$p = \sum_{A\cup B=[1,r];A\cap B=\emptyset;m_\alpha(A,B)\geq 0} \left(\prod_{\alpha\in A} \mathcal{Q}_{\alpha,-1}^{m_\alpha(A,B)}\right) c_{\mathbf{m}}^{A,B}(\{\mathcal{Q}_{\gamma,0}\}) \left(\prod_{\beta\in B} \mathcal{Q}_{\beta,1}^{m_\beta(A,B)}\right) \tag{4.29}$$

*where the coefficients $c_{\mathbf{m}}^{A,B}$ are Laurent polynomials of the variables $\{Q_{\gamma,0}\}$.*

In other words, any occurrence of $\mathcal{Q}_{\alpha,1}^{-1}$ in the Laurent polynomial expression of $p$ may be replaced by a term $\mathcal{Q}_{\alpha,-1}$, for which coefficients remain Laurent polynomials of the variables $\{\mathcal{Q}_{\gamma,0}\}$. This powerful property can be applied to the conserved quantities as well. Indeed, each quantity $C_m$ of (2.4) is a Laurent polynomial of the initial data $\mathcal{S}_0$ as well as of $\mathcal{S}_{-1}$ depending on whether it is expressed at $n = 0$ or $n = -1$. Repeating the argument leading to Lemma 4.11, we also find that each $C_m$ may be expressed in a unique way in the form (4.29). Let us examine the expression of $C_m$ as a Laurent polynomial of the initial data $\mathcal{S}_0$ more closely. From the hard particle condition and the explicit form of $y_i(0)$ (2.2-2.3), we see that the terms containing negative powers of $\mathcal{Q}_{\alpha,1}$ in $C_m$ must be of the form $c_{A,B}(\{\mathcal{Q}_{\gamma,0}\})\Big(\prod_{\alpha\in A}\mathcal{Q}_{\alpha,1}^{-1}\Big)\Big(\prod_{\beta\in B}\mathcal{Q}_{\beta,1}\Big)$, for some disjoint subsets $A, B \subset [1,r]$, as each particle is exclusive of its neighbors on the graph. Such terms may be rewritten as $\Big(\prod_{\alpha\in A}\mathcal{Q}_{\alpha,-1}\Big)c'_{A,B}(\{\mathcal{Q}_{\gamma,0}\})\Big(\prod_{\beta\in B}\mathcal{Q}_{\beta,1}\Big)$ according to the above. Now consider the level 1 quantity $G_{\mathbf{n}}^{(1)} = \phi\left(\Big(\prod_\beta \mathcal{Q}_{\beta,1}\Big)\Big(\prod_\alpha \mathcal{Q}_{\alpha,1}^{n^{(\alpha)}}\Big)\tau(z)\right)$ and insert $C_m$ as before. We get:

$$t^{\frac{mr}{2}} e_m(\mathbf{z}) G_{\mathbf{n}}^{(1)} = \phi\left(\left(\prod_{\beta=1}^r \mathcal{Q}_{\beta,1}\right)\left(\prod_{\alpha=1}^r \mathcal{Q}_{\alpha,1}^{n^{(\alpha)}}\right) C_m\, \tau(z)\right)$$

Suppose some $n^{(\alpha)} = 0$. The insertion of $C_m$, expressed in terms of $\mathcal{S}_0$ variables, will introduce terms of the form $G_{\mathbf{n}}^{(1)}$ with $n^{(\alpha)} = -1$, whenever $\mathcal{Q}_{\alpha,1}^{-1}$ occurs in $C_m$. These are precisely the unwanted terms, for which we showed that $G_{\mathbf{n}}^{(1)} = 0$. However, we need not impose this condition. Indeed, by the above argument we may replace the terms with $\mathcal{Q}_{\alpha,1}^{-1}$ in $C_m$ with $\mathcal{Q}_{\alpha,-1}$, up to a change of coefficient $c_{A,B} \to c'_{A,B}$. This gives a contribution of the form:

$$\phi\left(\Big(\prod_\beta \mathcal{Q}_{\beta,1}\Big)\Big(\prod_{\gamma\neq\alpha}\mathcal{Q}_{\gamma,1}^{n^{(\gamma)}}\Big)\mathcal{Q}_{\alpha,-1}p\,\tau(z)\right)$$
$$= t^{-2\sum_{\gamma\neq\alpha}\Lambda_{\alpha,\gamma}n^{(\gamma)}}\phi\left(\Big(\prod_\beta \mathcal{Q}_{\beta,1}\Big)\mathcal{Q}_{\alpha,-1}\Big(\prod_{\gamma\neq\alpha}\mathcal{Q}_{\gamma,1}^{n^{(\gamma)}}\Big)p\,\tau(z)\right)=0$$

by the evaluation. Hence the terms which would have created $G_{\mathbf{n}}^{(1)}$ with $n^{(\alpha)}=-1$ drop from the equation.

This phenomenon is examplified in the expressions of Examples 2.12 and 2.14 showing the difference equations for respectively $\mathfrak{sl}_2$ (2.20) (where the coefficient of the unwanted term vanishes for $n=0$), and for $\mathfrak{sl}_3$ (2.21-2.22) (where the coefficients of the unwanted terms vanish when $n=0$ or $p=0$).

The same holds for the difference equations satisfied by (4.21) at $k=1$. To prove it, we repeat the above argument, and note that unwanted terms from $\mathcal{C}_m$ take the form

$$\pi(p^*)\left(\prod_{\gamma\neq\alpha}\mathcal{D}_{\gamma,1}^{n^{(\gamma)}}\right)\mathcal{D}_{\alpha,-1}\,1=0$$

for some polynomials $p^*$ of the $\{\mathcal{Q}^*_{\alpha,i}\}$. This is due to the fact that $\mathcal{D}_{\alpha,-1}\,1$=0. This latter property is a consequence of the following lemma, proved in Appendix B below, and of its immediate corollary.

**Lemma 4.12.** *For any $\alpha\in[1,r]$, we have the following identity:*

$$\sum_{\substack{I\subset[1,r+1]\\|I|=\alpha}}(z_I)^p a_I(\mathbf{z})=\begin{cases}1 & \text{if } p=0\\ 0 & \text{for } p=-1,-2,\dots,\alpha-r-1\,.\end{cases}\tag{4.30}$$

This implies immediately the following:

**Corollary 4.13.** *We have $\mathcal{D}_{\alpha,-p}\,1=0$ for all $p=1,2,\dots,r+1-\alpha$, and $\mathcal{D}_{\alpha,0}\,1=t^{-\sum_\beta\Lambda_{\alpha,\beta}}$.*

The only initial data needed to feed the level 1 difference equations is therefore $G_0^{(1)}=1$, and the solution is uniquely determined by the equations. The corresponding function (4.21) for $\mathbf{n}=0$ is also trivially equal to 1, and Theorem 4.5 follows in the level 1 case.

4.2.3. *Proof for general level $k\geq 2$.* Let $V=\mathbb{Z}[t,t^{-1}][\mathbf{z}]^{S_{r+1}}$, the space of symmetric polynomials in $\mathbf{z}$ with coefficients in $\mathbb{Z}[t,t^{-1}]$. Using the map $\phi$ of Definition 1.10, we construct the map $\Psi$ from $A_+$, the space of polynomials in $\mathcal{Q}_{\alpha,k}$ with coefficients in $\mathbb{Z}[t,t^{-1}]$, to $V$ as follows.

**Definition 4.14.** *For all $p \in \mathbb{Z}[t,t^{-1}][\{Q_{\alpha,k}|\alpha\in[1,r],k\geq 1\}]$, we define:*

$$\Psi(p) := \phi\left(\prod_{\beta=1}^{r} \mathcal{Q}_{\beta,1}\, p\, \tau(z)\right) \tag{4.31}$$

In particular, this allows to rewrite (2.13) as:

$$G_{\mathbf{n}}(q^{-1},\mathbf{z}) = \Psi\left(\prod \mathcal{Q}_{\alpha,i}^{n_i^{(\alpha)}}\right) \tag{4.32}$$

and we have the normalization condition $\Psi(1)=1$.

Let $V_0$ denote the image of $A_+$ under $\Psi$. $V_0$ is a right module over $A_+^{\mathrm{op}}$ where the superscript op denotes the opposite multiplication, under the action:

$$\mathcal{Q}_{\alpha,k} \circ \Psi(p) = \Psi(p\mathcal{Q}_{\alpha,k})$$

**Theorem 4.15.** *The operators $\mathcal{D}_{\alpha,k}$ act on $V_0$ by left-multiplication, and form a representation of the action of $A_+^{\mathrm{op}}$ on $V_0$, such that: $\mathcal{Q}_{\alpha,k}\circ\Psi(p) = \mathcal{D}_{\alpha,k}\,\Psi(p)$.*

*Proof.* We use the anti-homomorphism $*$ that maps $\mathcal{Q}_{\alpha,k}\mapsto \mathcal{Q}^*_{\alpha,k}$ and reverses the order of multiplication, while preserving $t$, and compose it with the representation $\pi$. To any polynomial $p$ of the $\{\mathcal{Q}_{\alpha,i}\}$ with coefficients in $\mathbb{Z}[t,t^{-1}]$ we associate the polynomial $p^*$ of the $\mathcal{Q}^*_{\alpha,i}$ by $p^*(\{\mathcal{Q}^*_{\alpha,i}\}) = p(\{\mathcal{Q}_{\alpha,i}\})^*$, and finally $\pi(p^*)$ by the substitution $\mathcal{Q}^*_{\alpha,i}\to\mathcal{D}_{\alpha,i}$, namely $\pi(p^*) = p^*(\{\mathcal{D}_{\alpha,i}\})$. We wish to prove that $\Psi(p)=\pi(p^*)\,1$. By Lemma 4.11 we may write:

$$p = \sum_{A\cup B=[1,r];A\cap B=\emptyset;m_\alpha(A,B)\geq 0} \left(\prod_{\alpha\in A}\mathcal{Q}_{\alpha,-1}^{m_\alpha(A,B)}\right) c_{\mathbf{m}}^{A,B}(\{\mathcal{Q}_{\gamma,0}\}) \left(\prod_{\beta\in B}\mathcal{Q}_{\beta,1}^{m_\beta(A,B)}\right)$$

where the coefficients $c_{\mathbf{m}}^{A,B}$ are Laurent polynomials of the $\{\mathcal{Q}_{\gamma,0}\}$, for any polynomial $p$ of the $\{\mathcal{Q}_{\alpha,i}\}$ obeying the quantum $Q$-system relations. As moreover $\Psi(\mathcal{Q}_{\alpha,-1}f)=0$ for any polynomial $f$, we see that

$$\Psi(p) = \Psi(\varphi(p))$$

where

$$\varphi(p) = \sum_{m_\alpha(\emptyset,[1,r])\geq 0} c_{\mathbf{m}}^{\emptyset,[1,r]}(\{\mathcal{Q}_{\gamma,0}\}) \prod_{\alpha=1}^{r}\mathcal{Q}_{\alpha,1}^{m_\alpha(\emptyset,[1,r])}$$

The map $\varphi$ is simply the truncation to the polynomial part of $p$ in the variables $\mathcal{Q}_{\alpha,1}$. Let $\varphi^*$ the corresponding truncation of any Laurent polynomial of $\{\mathcal{Q}^*_{\alpha,0},\mathcal{Q}^*_{\alpha,1}\}$ to it polynomial part in $\{\mathcal{Q}^*_{\alpha,1}\}$. We have

$$\pi(p^*)\,1 = \varphi^*(\pi(p^*))\,1$$

where we have used $\varphi(p)^* = \varphi^*(p^*)$, and $\pi(f^*)\mathcal{D}_{\alpha,-1}\,1 = 0$ (by Corollary 4.13) for all polynomials $f^*$ of the $\mathcal{Q}^*_{\alpha,i}$. By definition of $\Psi$ and $\phi$ and the evaluation $ev_0$, we may now evaluate $\varphi(p)$ at $\mathcal{Q}_{\alpha,0} = t^{-\sum_\beta \Lambda_{\alpha,\beta}}$ without altering $\Psi(p) = \Psi(ev_0(\varphi(p)))$. Note that

$ev_0(f)^* = ev_0^*(f^*)$ where $ev_0^*$ is the right evaluation at $\mathcal{Q}^*_{\alpha,0} = t^{-\sum_\beta \Lambda_{\alpha,\beta}}$ (after the dual normal ordering that puts all $\mathcal{Q}^*_{\alpha,0}$ to the right). Finally, from Corollary 4.13 we have: $\pi(p^*)\,1 = ev_0^*(\varphi(\pi(p^*)))\,1$. The two polynomials $ev_0(\varphi(p))$ and $ev_0^*(\varphi(\pi(p^*)))$ are the *same* polynomial of respectively $\{Q_{\alpha,1}\}$ and $\{\mathcal{D}_{\alpha,1}\}$ with coefficients in $\mathbb{Z}[t,t^{-1}]$. Therefore, the statement $\Psi(p) = \pi(p^*)\,1$ needs only be proved for a polynomial $p \in \mathbb{Z}[t,t^{-1}][\{\mathcal{Q}_{\alpha,1}, \alpha \in [1,r]\}]$, and in fact for any monomial of the form $\prod_\alpha \mathcal{Q}_{\alpha,1}^{m_\alpha}$ with $m_\alpha \geq 0$. This is exactly the level 1 case of Theorem 4.5, which was proved in Sect. 4.2.2 above. The Theorem follows by using the anti-homomorphism property $\pi((p\mathcal{Q}_{\alpha,k})^*) = \mathcal{D}_{\alpha,k}\pi(p^*)$. □

Finally, noting that $\Psi(1) = 1$, and applying Theorem 4.15 iteratively, leads straightforwardly to Theorem 4.5 for arbitrary level $k$.

4.3. **Level one case and degenerate Macdonald polynomials.** When restricted to level 1, the formula of Corollary 4.7 for graded characters reduces to the following, for $\mathbf{n} = \{n^{(\alpha)}\}_{\alpha\in[1,r]}$:

$$\chi_{\mathbf{n}}(q^{-1},\mathbf{z}) = q^{-\frac{1}{2}\sum_{\alpha,\beta} n^{(\alpha)}\mathrm{Min}(\alpha,\beta)n^{(\beta)} + \frac{1}{2}\sum_\alpha \alpha n^{(\alpha)}} \prod_{\alpha=1}^{r} (M_{\alpha,1})^{n^{(\alpha)}}\, 1 \tag{4.33}$$

with $M_{\alpha,1}$ as in (4.23). We have the following:

**Theorem 4.16.** *The level one $\mathfrak{sl}_{r+1}$ graded characters* (4.33) *are eigenfunctions of the degenerate Macdonald difference operators $M_{\alpha,0} = M_\alpha$ of* (4.9)*, namely:*

$$M_{\alpha,0}\,\chi_{\mathbf{n}}(q^{-1},\mathbf{z}) = E_{\alpha,n}\,\chi_{\mathbf{n}}(q^{-1},\mathbf{z}), \quad E_{\alpha,n} = q^{\sum_\beta \mathrm{Min}(\alpha,\beta)n^{(\beta)}}$$

*Proof.* Starting from formula (4.33), we compute:

$$\begin{aligned} M_{\alpha,0}\,\chi_{\mathbf{n}}(q^{-1},\mathbf{z}) &= q^{-\frac{1}{2}\sum_{\alpha,\beta} n^{(\alpha)}\mathrm{Min}(\alpha,\beta)n^{(\beta)} + \frac{1}{2}\sum_\alpha \alpha n^{(\alpha)}} M_{\alpha,0} \prod_{\beta=1}^{r} (M_{\beta,1})^{n^{(\beta)}}\, 1 \\ &= q^{\sum_\beta \mathrm{Min}(\alpha,\beta)n^{(\beta)}}\,\chi_{\mathbf{n}}(q^{-1},\mathbf{z}) \end{aligned}$$

by use of the commutation relations (4.24), and the fact that $M_{\alpha,0}\,1 = 1$ by Lemma 4.12. □

Recall that the symmetric $A_r$ $(q,t)$-Macdonald polynomials[2] $P_\lambda^{q,t}(\mathbf{z})$ of the variables $\mathbf{z} = (z_1, \ldots, z_{r+1})$, indexed by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r+1} \geq 0)$, are defined as the unique family of common eigenvectors to the difference operators $M_\alpha^{q,t}$, $\alpha = 1, 2, \ldots, r$, and whose leading term is the symmetric monomial $m_\lambda = \prod_i z_i^{\lambda_i}$ + permutations. The Macdonald polynomials $P_\lambda^{q,t}(\mathbf{z})$ satisfy the following duality property [Mac95]:

$$P_\lambda^{q,t}(\mathbf{z}) = P_\lambda^{q^{-1},t^{-1}}(\mathbf{z}) \tag{4.34}$$

[2]Again, the superscript $q,t$ refers to the original notations of [Mac95]. The parameter $t$ is different from that used in our notations.

Comparing this with the result of Theorem 4.16, we conclude:

**Corollary 4.17.** *The level one $A_r$ graded characters $\chi_{\mathbf{n}}(q^{-1},\mathbf{z})$ are the following degenerate limits of the Macdonald polynomials:*

$$\chi_{\mathbf{n}}(q^{-1},\mathbf{z}) = \lim_{t\to\infty} P_\lambda^{q,t}(\mathbf{z}) = P_\lambda^{q^{-1},0}(\mathbf{z}) \tag{4.35}$$

*where the correspondance between* $\mathbf{n}$ *and* $\lambda$ *is via:*

$$\lambda_1 = n^{(1)} + n^{(2)} + \cdots + n^{(r)}, \quad \lambda_2 = n^{(2)} + \cdots + n^{(r)}, \quad \ldots \quad \lambda_r = n^{(r)}, \quad \lambda_{r+1} = 0$$

Note that we have picked $\lambda_{r+1} = 0$, as the variables $\mathbf{z}$ satisfy $z_1 z_2 \cdots z_{r+1} = 1$, so that $P^{q,t}_{\lambda_1,..,\lambda_r,\lambda_{r+1}}(\mathbf{z}) = (z_1 z_2 \cdots z_{r+1})^{\lambda_{r+1}} P^{q,t}_{\lambda_1,..,\lambda_r,0}(\mathbf{z})$ is independent of $\lambda_{r+1}$.

**Remark 4.18.** *From eq.*(4.35)*, we may identify the graded level one character $\chi_{\mathbf{n}}(q,\mathbf{z})$ with the Whittaker limit $t \to 0$ of the Macdonald polynomial $\lim_{t\to 0} P_\lambda^{q,t}(\mathbf{z})$. This shows in particular that $\chi_{\mathbf{n}}(q,\mathbf{z})$ is a polynomial of $q$.*

**Remark 4.19.** *The raising operators $M_{\alpha,1}$ coincide with the raising operators $K_\alpha^+$ for Macdonald polynomials introduced by Kirillov and Noumi* [KN99]*, in the limit $t \to \infty$, as well as with the dual raising operators $K_\alpha^-$ in the Whittaker limit $t \to 0$.*

## 5. The solution for $\mathfrak{sl}_2$

Here, we work out the consequences of our results in the case of $\mathfrak{sl}_2$. We also give an an alternative derivation of Theorem 2.10, which allows us to derive Cauchy conditions obeyed by the graded characters, thus determining the solution entirely.

We also show how one can use the expression for the solution in terms of the raising operators of Theorem 4.5. This gives a new proof of the expression in terms of raising operators, by showing that both forms of the characters satisfy the same difference equations and Cauchy conditions.

This gives an explicit expressions for the graded characters, which recovers the formulas of Feigin and Feigin [FF02] for the case of $\mathfrak{sl}_2$. In particularly simple cases, one may use an inductive limit of the graded tensor product to find the fermionic expressions level-$k$ characters [LP84, SF94].

### 5.1. Quantum Q-system, conserved quantity and linear recursion relations.

The quantum Q-system for $\mathfrak{sl}_2$ is simply

$$t\mathcal{Q}_{n+1}\,\mathcal{Q}_{n-1} = \mathcal{Q}_n^2 - 1, \qquad \mathcal{Q}_n\,\mathcal{Q}_{n+1} = t\mathcal{Q}_{n+1}\,\mathcal{Q}_n \qquad (n\in\mathbb{Z}) \tag{5.1}$$

and the only non-trivial conserved quantity is

$$C_1 = \mathcal{Q}_{n+1}\mathcal{Q}_n^{-1} - \mathcal{Q}_{n+1}^{-1}\mathcal{Q}_n^{-1} + \mathcal{Q}_{n+1}^{-1}\mathcal{Q}_n = \mathcal{Q}_{n+1}\mathcal{Q}_n^{-1} + \mathcal{Q}_n^{-1}\mathcal{Q}_{n-1} \tag{5.2}$$

We note also the linear recursion relation

$$t\mathcal{Q}_k - \mathcal{Q}_{k-1}C_1 - \mathcal{Q}_{k-2} = 0 \qquad (k\in\mathbb{Z}) \tag{5.3}$$

5.2. **Uniqueness of the solution to the difference equations.** Recall the definition of the generating function $G^{(k)}(\mathbf{y})$, with $\mathbf{y}=(y_1,...,y_k)$:

$$G^{(k)}(\mathbf{y}) = \sum_{n_1,\ldots,n_k\in\mathbb{Z}_+} \prod_{i=1}^k y_i^{n_i}\, G^{(k)}_{\mathbf{n}} \tag{5.4}$$

$$= \sum_{n_1,\ldots,n_k\in\mathbb{Z}_+} q^{\frac{1}{4}\mathbf{n}\cdot A\mathbf{n}+\frac{1}{2}\sum_{i=1}^k n_i} \prod_{i=1}^k y_i^{n_i}\, \chi_{\mathbf{n}}(q^{-1},z) \tag{5.5}$$

$$= \phi\left(\mathcal{Q}_1 \prod_{i=1}^k \frac{1}{1-y_i\,\mathcal{Q}_i}\,\tau(z)\right) \tag{5.6}$$

where the series $\tau(z)$ is given by:

$$\tau(z) = t^{\frac{1}{2}} \sum_{\ell\geq 0} \xi^{\ell+1} s_\ell(z) \tag{5.7}$$

where $s_\ell(z)=\frac{z^{\ell+1}-z^{-\ell-1}}{z-z^{-1}}$, and $\xi=t^{\frac{1}{2}}\lim_{n\to\infty}\mathcal{Q}_n\mathcal{Q}_{n+1}^{-1}$, expanded as a power series of $\mathcal{Q}_1^{-1}$ with coefficients in $\mathbb{Z}[\mathcal{Q}_0,\mathcal{Q}_0^{-1}]$.

The action on the tail $\tau(z)$ on the conserved quantity $C_1=t^{\frac{1}{2}}(\xi+\xi^{-1})$ is simply:

$$C_1\,\tau(z) = t^{\frac{1}{2}}\, e_1\,\tau(z)+t, \quad \text{where} \quad e_1=z+z^{-1} \tag{5.8}$$

**Definition 5.1.** *We introduce the shift operator $d_i$ acting on functions of $\mathbf{y}=(y_1,...,y_k)$ as:*

$$d_i f(\mathbf{y}) = f(y_1,\ldots,y_{i-1},ty_i,y_{i+1},\ldots,y_k) \tag{5.9}$$

**Definition 5.2.** *We introduce the divided difference operator $\delta_i$ at $0$ acting on power series of $\mathbf{y}$ as:*

$$\delta_i f(\mathbf{y}) = \frac{f(\mathbf{y})-f(\mathbf{y})|_{y_i=0}}{y_i} \tag{5.10}$$

*where we use the notation $f(\mathbf{y})|_{y_i=0}:=f(y_1,\ldots,y_{i-1},0,y_{i+1},\ldots,y_k)$.*

Note that $\delta_i$ acts very simply on the fraction $\frac{1}{1-y_i\mathcal{Q}_i}$ by insertion of $\mathcal{Q}_i$:

$$\delta_i\,\frac{1}{1-y_i\mathcal{Q}_i} = \frac{\mathcal{Q}_i}{1-y_i\mathcal{Q}_i} \tag{5.11}$$

Note finally the following commutation relations between $\delta_i$ and $d_j$, when acting on functions of $\mathbf{y}$:

$$\delta_i\, d_j = t^{\delta_{i,j}}\, d_j\,\delta_i \tag{5.12}$$

The following theorem gives an alternative derivation of the difference equation of Theorem 2.10 in the case of $A_1$, plus some initial Cauchy conditions that determine the solution $G^{(k)}(\mathbf{y})$ completely.

**Theorem 5.3.** *The generating series $G^{(k)}(\mathbf{y})$ for graded characters satisfies the following "second order" difference equation for all $k\geq 1$:*

$$H_k\, G^{(k)}(\mathbf{y})=0 \tag{5.13}$$

*where*

$$H_k=t^2\delta_k^2-1-t^{\frac{1}{2}}e_1 d_k^{-1}\delta_{k-1}\delta_k+(\delta_{k-1}d_k^{-1})^2 \tag{5.14}$$

*with the convention that $\delta_0=t^{-1}$. Moreover, we have the following initial Cauchy data for $G^{(k)}$ as a function of $y_k$ for all $k\geq 1$, that determine the solution $G^{(k)}(\mathbf{y})$ completely by induction on $k$:*

$$G^{(k)}(\mathbf{y})|_{y_k=0} \;=\; G^{(k-1)}(\mathbf{y}') \tag{5.15}$$

$$\frac{\partial G^{(k)}(\mathbf{y})}{\partial y_k}\Big|_{y_k=0} \;=\; t^{-1}\Big(t^{\frac{1}{2}}e_1\,\delta_{k-1}-\delta_{k-2}d_{k-1}^{-1}\Big)G^{(k-1)}(\mathbf{y}') \tag{5.16}$$

*where $\mathbf{y}=(y_1,\ldots,y_k)$, $\mathbf{y}'=(y_1,\ldots,y_{k-1})$, and with the convention that $G^{(0)}=1$ and $\delta_{-1}G^{(0)}=0$.*

*Proof.* For simplicity, let us denote by $\alpha_i=\frac{1}{1-y_i\mathcal{Q}_i}$. Using the linear recursion relation (5.3) with $m=k$, we compute in two ways the quantity:

$$\begin{aligned}
A &= \delta_{k-1}\delta_k\phi\left(\mathcal{Q}_1\alpha_1\alpha_2...\alpha_k C_1\tau(z)\right)\\
&= \phi\left(\mathcal{Q}_1\alpha_1...\alpha_{k-1}\mathcal{Q}_{k-1}\alpha_k\mathcal{Q}_k C_1\tau(z)\right)\\
&= \phi\left(\mathcal{Q}_1\alpha_1...\alpha_{k-1}d_k\alpha_k\mathcal{Q}_{k-1}\mathcal{Q}_k(\mathcal{Q}_k\mathcal{Q}_{k-1}^{-1}-\mathcal{Q}_k^{-1}\mathcal{Q}_{k-1}^{-1}+\mathcal{Q}_k^{-1}\mathcal{Q}_{k-1})\big)\tau(z)\right)\\
&= d_k\phi\left(\mathcal{Q}_1\alpha_1...\alpha_k(t^2\mathcal{Q}_k^2-1+\mathcal{Q}_{k-1}^2)\tau(z)\right)\\
&= d_k\Big(t^2\delta_k^2-1+d_k^{-2}\delta_{k-1}^2\Big)\phi\left(\mathcal{Q}_1\alpha_1...\alpha_k\tau(z)\right)
\end{aligned}$$

On the other hand, using (5.8), we also get:

$$A=\delta_{k-1}\delta_k\phi\left(\mathcal{Q}_1\alpha_1...\alpha_k t^{\frac{1}{2}}e_1\tau\right)$$

and (5.13) follows by identifying the two results.

Note that for $k=1$ the action of the operator $\delta_0$ on $G^{(k)}$ may be defined as the insertion of $\mathcal{Q}_0$ between the terms $\mathcal{Q}_1$ and $\alpha_1$ (and then commuting the $\mathcal{Q}_0$ through $\mathcal{Q}_1$ to the left gives the left evaluation $t^{-1}$), hence the above argument still holds for $k=1$, with the convention that $\delta_0=t^{-1}$.

The first Cauchy condition (5.15) follows immediately from the definition of $G^{(k)}$ for $k\geq 2$. For $k=1$, recalling that $\xi=t^{\frac{1}{2}}\theta$, with $\theta=\mathcal{Q}_0\mathcal{Q}_1^{-1}+O(\mathcal{Q}_1^{-3})$ and that $\tau(z)=t\theta+O(\mathcal{Q}_1^{-2})$, we find that $G^{(1)}(0) = \phi(\mathcal{Q}_1\mathcal{Q}_0^{-1}(\theta + O(\mathcal{Q}_1^{-2})) = \phi(\mathcal{Q}_1\mathcal{Q}_0^{-1}\mathcal{Q}_0\mathcal{Q}_1^{-1}) = 1$, which matches (5.15) with $k = 1$ and $\delta_0 = t^{-1}$.

The second condition (5.16) is proved as follows for $k \geq 2$:

$$\begin{aligned}
\frac{\partial}{\partial y_k} G^{(k)}(\mathbf{y})\big|_{y_k=0} &= \phi\left(\mathcal{Q}_1\alpha_1...\alpha_{k-1}\,\mathcal{Q}_k\,\tau\right) \\
&= t^{-1}\phi\left(\mathcal{Q}_1\alpha_1...\alpha_{k-1}(\mathcal{Q}_{k-1}C_1 - \mathcal{Q}_{k-2})\tau\right) \\
&= t^{-1}\phi\left(\mathcal{Q}_1\alpha_1...\alpha_{k-1}(e_1\mathcal{Q}_{k-1} - \mathcal{Q}_{k-2})\tau\right) \\
&= t^{-1}\Big(t^{\frac{1}{2}}e_1\delta_{k-1} - \delta_{k-2}D_{k-1}^{-1}\Big)G^{(k-1)}
\end{aligned}$$

where in the second line we have used the linear recursion relation (5.3), and in the third line we have used Lemma 5.8. For $k = 1$, writing $\mathcal{Q}_0^{-1}\mathcal{Q}_1 = t^{-1}(C_1 - \mathcal{Q}_1(\mathcal{Q}_0 - \mathcal{Q}_0^{-1}))$ and using (5.8), we easily get:

$$\begin{aligned}
\frac{d}{dy_1}G^{(1)}(0) &= \phi\Big(\mathcal{Q}_1\mathcal{Q}_0^{-1}\mathcal{Q}_1\tau(z)\Big) = \phi\Big(\mathcal{Q}_1 t^{-1}\big(C_1 - \mathcal{Q}_1^{-1}(\mathcal{Q}_0 - \mathcal{Q}_0^{-1})\big)\tau(z)\Big) \\
&= \phi\Big(\mathcal{Q}_1 t^{-\frac{1}{2}}e_1\tau(z)\Big) = t^{-\frac{3}{2}}e_1
\end{aligned}$$

where the second term contributes 0 in the left evaluation at $\mathcal{Q}_0 = 1$. This expression matches (5.16) for $k = 1$ with $\delta_0 = t^{-1}$ and $\delta_{-1} = 0$. □

The difference equation of Theorem 5.3 together with the initial data specified is equivalent to the following

**Corollary 5.4.** *For all $m > 0$, $G^{(m)}(\mathbf{y})$ is the unique solution to the following system of difference equations for $k = 1, 2, ..., m$:*

$$(5.17)\quad (t^2y_k^{-2} - t^{\frac{1}{2}}e_1\delta_{k-1}d_k^{-1}y_k^{-1} + (\delta_{k-1}d_k^{-1})^2 - 1)G^{(k)}(\mathbf{y}) = \Big(t^2y_k^{-2} - ty_k^{-1}\delta_{k-2}d_{k-1}^{-1}\Big)G^{(k-1)}(\mathbf{y}')$$

*with $\mathbf{y} = (y_1, ..., y_k)$, $\mathbf{y}' = (y_1, ..., y_{k-1})$, and with $G^{(0)} = 1$, $\delta_{-1} = 0$, $\delta_0 = t^{-1}$ as above.*

*Proof.* Using the definition of $\delta_k$ we have:

$$\begin{aligned}
0 &= H_k\, G^{(k)}(\mathbf{y}) = t^2 \frac{G^{(k)}(\mathbf{y}) - G^{(k)}(\mathbf{y})|_{y_k=0} - y_k \frac{\partial G^{(k)}(\mathbf{y})}{\partial y_k}\Big|_{y_k=0}}{y_k^2} \\
&\quad - t^{\frac{1}{2}} e_1 \delta_{k-1} D_k^{-1} \frac{G^{(k)}(\mathbf{y}) - G^{(k)}(\mathbf{y})|_{y_k=0}}{y_k} + ((\delta_{k-1}D_k^{-1})^2 - 1)G^{(k)}(\mathbf{y}) \\
&= (t^2 y_k^{-2} - t^{\frac{1}{2}} e_1 \delta_{k-1} d_k^{-1} y_k^{-1} + (\delta_{k-1} d_k^{-1})^2 - 1) G^{(k)}(\mathbf{y}) \\
&\quad - \Big( t^2 y_k^{-2} + t y_k^{-1} (t^{\frac{1}{2}} e_1 \delta_{k-1} - \delta_{k-2} d_{k-1}^{-1}) - t^{\frac{3}{2}} e_1 y_k^{-1} \delta_{k-1} d_k^{-1} \Big) G^{(k-1)}(\mathbf{y}') \\
&= (t^2 y_k^{-2} - t^{\frac{1}{2}} e_1 \delta_{k-1} d_k^{-1} y_k^{-1} + (\delta_{k-1} d_k^{-1})^2 - 1) G^{(k)}(\mathbf{y}) \\
&\quad - \Big( t^2 y_k^{-2} - t y_k^{-1} \delta_{k-2} d_{k-1}^{-1} \Big) G^{(k-1)}(\mathbf{y}')
\end{aligned}$$

where we have substituted the initial data of Theorem 5.3. □

The difference equation of Theorem 5.3 may also be rewritten for the coefficients $G_{\mathbf{n}}^{(k)}$ of (2.13), and therefore for the graded characters $\chi_{\mathbf{n}}(q^{-1}, z)$. For the vectors $\mathbf{n} = (n_1, n_2, ..., n_k)$, we consider shifts by $\pm 1$ in the variables $n_{k-1}$, $n_k$, which we write $\mathbf{n} \to \mathbf{n} \pm \epsilon_{k-1} \pm \epsilon_k$, where $\epsilon_{k-1} = (0, 0, ..., 0, 0, 1, 0)$ and $\epsilon_k = (0, 0, ..., 0, 1)$.

**Corollary 5.5.** *The coefficients $G_{\mathbf{n}}^{(k)}$ for $k \geq 2$ obey the following recursion relations:*

$$q^{-1} G^{(k)}_{\mathbf{n}-\epsilon_{k-1}+\epsilon_k} + q^{n_k - 1} G^{(k)}_{\mathbf{n}+\epsilon_{k-1}-\epsilon_k} - G^{(k)}_{\mathbf{n}-\epsilon_{k-1}-\epsilon_k} = q^{\frac{n_k}{2} - \frac{3}{4}} (z + z^{-1}) G^{(k)}_{\mathbf{n}} \tag{5.18}$$

*for $n_k, n_{k-1} \geq 1$, $n_1, n_2, ..., n_{k-2} \geq 0$. For $k = 1$, we have for $n \geq 1$:*

$$q^{-1} G^{(1)}_{n+1} + (q^n - 1) G^{(1)}_{n-1} = q^{\frac{n}{2} - \frac{1}{4}} (z + z^{-1}) G^{(1)}_n \tag{5.19}$$

*with the initial conditions for $k \geq 2$:*

$$G^{(k)}_{\mathbf{n},0} = G^{(k-1)}_{\mathbf{n}}, \qquad G^{(k)}_{\mathbf{n},1} = q^{\frac{1}{4}} (z + z^{-1}) G^{(k-1)}_{\mathbf{n}+\epsilon_{k-1}} - q^{\frac{n_{k-1}+1}{2}} G^{(k-1)}_{\mathbf{n}+\epsilon_{k-2}} \tag{5.20}$$

*whereas for $k = 1$ we have*

$$G^{(1)}_0 = 1, \qquad \text{and} \qquad G^{(1)}_1 = q^{\frac{3}{4}} (z + z^{-1}) \ . \tag{5.21}$$

*Proof.* Eq.(5.18) follows from (5.13-5.14) and the following identities (valid for $k \geq 2$):

$$\begin{aligned}
\delta_j G^{(k)}(\mathbf{y})|_{y_1^{n_1} \cdots y_k^{n_k}} &= G^{(k)}_{\mathbf{n}+\epsilon_j} \\
d_j^{-1} G^{(k)}(\mathbf{y})|_{y_1^{n_1} \cdots y_k^{n_k}} &= q^{\frac{n_j}{2}} G^{(k)}_{\mathbf{n}}
\end{aligned}$$

For $k = 1$, we must use instead $\delta_0 G^{(1)}(y_1) = t^{-1} G^{(1)}(y_1)$, which yields the slightly different equation (5.19). The initial conditions follow similarly from the Cauchy conditions (5.15-5.16). □

Finally, using the relation

$$G_{\mathbf{n}}^{(k)} = q^{\frac{1}{2}\sum_{i=1}^{k} n_i + \frac{1}{4}\sum_{i,j=1}^{k} n_i \mathrm{Min}(i,j) n_j}\, \chi_{\mathbf{n}}(q^{-1}, z) \tag{5.22}$$

we obtain:

**Corollary 5.6.** *For all $k \geq 1$ the graded characters $\chi_{\mathbf{n}} \equiv \chi_{\mathbf{n}}(q^{-1}, z)$ satisfy the difference equation (with the convention that $\epsilon_0 = 0$):*

$$\chi_{\mathbf{n}-\epsilon_{k-1}+\epsilon_k} + \chi_{\mathbf{n}+\epsilon_{k-1}-\epsilon_k} - q^{k-1-\sum_{i=1}^{k} i n_i}\, \chi_{\mathbf{n}-\epsilon_{k-1}-\epsilon_k} = (z + z^{-1})\chi_{\mathbf{n}} \tag{5.23}$$

*We also have the initial conditions for $k \geq 2$, $\mathbf{n} = (n_1, n_2, ..., n_{k-1})$:*

$$\chi_{\mathbf{n},0} = \chi_{\mathbf{n}}, \qquad \chi_{\mathbf{n},1} = (z + z^{-1})\chi_{\mathbf{n}+\epsilon_{k-1}} - \chi_{\mathbf{n}+\epsilon_{k-2}} \tag{5.24}$$

*and for $k = 1$: $\chi_{-1} = 0$ and $\chi_0 = 1$ so that $\chi_1 = z + z^{-1}$.*

The graded characters $\chi_{\mathbf{n}}$ are entirely determined by the relation (5.23) and the initial conditions (5.24).

**Example 5.7.** *For $k = 1$ we have:*

$$\chi_{n+1} + (1 - q^{-n})\chi_{n-1} = (z + z^{-1})\chi_n$$

*with the initial condition $\chi_{-1} = 0$ and $\chi_0 = 1$. This coincides with the $q$-Toda equation (3.10).*

**Example 5.8.** *For $k = 2$ we have:*

$$\chi_{n_1-1,n_2+1} + \chi_{n_1+1,n_2-1} - q^{1-n_1-2n_2}\chi_{n_1-1,n_2-1} = (z + z^{-1})\chi_{n_1,n_2}$$

*with the initial conditions:*

$$\chi_{n_1,0} = \chi_{n_1}, \quad \chi_{n_1,1} = (z + z^{-1})\chi_{n_1+1} - \chi_{n_1}$$

*and $\chi_{n_1}$ determined as in Example 5.7 above.*

5.3. **Solution in terms of raising operators.** We set $z_1 = z$ and $z_2 = z^{-1}$. The raising operators $\mathcal{D}_n := \mathcal{D}_{1,n}$ of Def.4.1 for $\mathfrak{sl}_2$ are:

$$\mathcal{D}_n = \frac{q^{\frac{n+2}{4}}}{z_1 - z_2}\left(z_1^{n+1} D_1 - z_2^{n+1} D_2\right) \tag{5.25}$$

with $D_1 f(z) = f(t^{-1}z)$, $D_2 f(z) = f(tz)$, whereas $\mathcal{D}_{0,n} = 1$ and $\mathcal{D}_{2,n} = (z_1 z_2)^n D_1 D_2 = 1$, These form a representation $\pi$ of the dual quantum $Q$-system acting on the space of polynomials of $z_1, z_2$ with coefficients in $\mathbb{Z}[t, t^{-1}]$:

$$t^{-1}\mathcal{D}_{n+1}\mathcal{D}_{n-1} = (\mathcal{D}_n)^2 - t^{-2}, \qquad \mathcal{D}_n\, \mathcal{D}_{n+1} = t^{-1}\mathcal{D}_{n+1}\, \mathcal{D}_n \tag{5.26}$$

The dual conserved quantity $\mathcal{C}_1$ is obtained from $C_1$ by the anti-homorphism $*$ and reads in the representation $\pi$:

$$\mathcal{C}_1 = \mathcal{D}_{n-1}\, \mathcal{D}_n^{-1} + \mathcal{D}_n^{-1}\, \mathcal{D}_{n+1}$$

By Theorem 4.10, it acts on functions $f(z_1, z_2)$ by multiplication by $t^{\frac{1}{2}} e_1 = t^{\frac{1}{2}}(z + z^{-1})$. As a consequence, we have the following linear recursion relation, obtained from (5.3) by applying $*$ and $\pi$:

$$t\mathcal{D}_k - t^{\frac{1}{2}}(z + z^{-1})\,\mathcal{D}_{k-1} - \mathcal{D}_{k-2} = 0 \qquad (k \in \mathbb{Z}) \tag{5.27}$$

Theorem 4.5 in the $A_1$ case reduces to the following identity:

$$G_{\mathbf{n}}^{(k)} = (\mathcal{D}_k)^{n_k}(\mathcal{D}_{k-1})^{n_{k-1}} \cdots (\mathcal{D}_1)^{n_1}\, 1 \tag{5.28}$$

while Corollary 4.6 becomes:

$$\chi_{\mathbf{n}}(q^{-1}, z) = q^{-\frac{1}{4}\sum_{i,j} n_i \mathrm{Min}(i,j) n_j - \frac{1}{2}\sum_i n_i}(\mathcal{D}_k)^{n_k}(\mathcal{D}_{k-1})^{n_{k-1}} \cdots (\mathcal{D}_1)^{n_1}\, 1 \tag{5.29}$$

Let us now give an alternative proof of Theorem 4.5 in the $A_1$ case. As $G_{\mathbf{n}}^{(k)}$ is entirely determined by the equations of Corollary 5.5, we simply have to show that the quantity

$$\Pi_{\mathbf{n}} := (\mathcal{D}_k)^{n_k}(\mathcal{D}_{k-1})^{n_{k-1}} \cdots (\mathcal{D}_1)^{n_1}\, 1 \tag{5.30}$$

obeys the same equations as $G_{\mathbf{n}}^{(k)}$. As before, the difference equations (5.18) and (5.19) are straightforwardly satisfied as a consequence of the dual $Q$-system relation (5.26), and the fact that $\mathcal{D}_{-1}\, 1 = 0$ and $\mathcal{D}_0\, 1 = q^{\frac{1}{2}} = t^{-1}$.

The Cauchy conditions (5.20) are also satisfied by the quantity $\Pi_{\mathbf{n}}$. First, we have trivially: $\Pi_{\mathbf{n},0} = \Pi_{\mathbf{n}}$ for any $\mathbf{n} = (n_1, ..., n_{k-1})$. The second part of (5.20) follows from the linear recursion relation (5.27). Indeed, letting the l.h.s. of the recursion relation act on $\Pi_{\mathbf{n}}$ for $\mathbf{n} = (n_1, ..., n_{k-1})$, we find:

$$\begin{aligned} 0 &= \Big(\mathcal{D}_k - t^{-\frac{1}{2}}(z + z^{-1})\,\mathcal{D}_{k-1} - t^{-1}\mathcal{D}_{k-2}\Big)\,\Pi_{\mathbf{n}} \\ &= \mathcal{D}_k\,\Pi_{\mathbf{n}} - t^{-\frac{1}{2}}(z + z^{-1})\,\Pi_{\mathbf{n}+\epsilon_{k-1}} - t^{-1}\mathcal{D}_{k-2}\mathcal{D}_{k-1}^{n_{k-1}}\mathcal{D}_{k-2}^{n_{k-2}} \cdots \mathcal{D}_1^{n_1}\, 1 \\ &= \Pi_{\mathbf{n},1} - q^{\frac{1}{4}}(z + z^{-1})\,\Pi_{\mathbf{n}+\epsilon_{k-1}} - q^{\frac{1+n_{k-1}}{2}}\,\Pi_{\mathbf{n}+\epsilon_{k-2}} \end{aligned}$$

The Theorem follows from uniqueness of the solution to the difference equations obeying the Cauchy conditions.

5.4. **Explicit expressions for characters.** We now use (5.28) to compute $G_{\mathbf{n}}^{(k)}$ and subsequently $\chi_{\mathbf{n}}(q^{-1}, z)$ as explicit functions of $q, z$. We need the preliminary:

**Lemma 5.9.** *The difference operators $\mathcal{D}_k$ satisfy the following relations:*

$$\mathcal{D}_k = q^{\frac{k+2}{4}} z^k D_1 + q^{\frac{1}{4}} z^{-1}\mathcal{D}_{k-1} \tag{5.31}$$

$$\mathcal{D}_k = q^{\frac{k+2}{4}} z^{-k} D_2 + q^{\frac{1}{4}} z\mathcal{D}_{k-1} \tag{5.32}$$

*Proof.* By direct calculation, using the formula (5.25). □

Explicit expressions turn out to be easier to derive by use of the generating function $G^{(k)}(\mathbf{y})$ (5.6). In particular, (5.28) implies that

$$\delta_k G^{(k)}(\mathbf{y}) = \sum_{\mathbf{n}} \prod_{i=1}^{k} y_i^{n_i} G^{(k)}_{\mathbf{n}+\epsilon_k} = \mathcal{D}_k\, G^{(k)}(\mathbf{y})$$

Using the relation (5.31) above, we easily get:

**Corollary 5.10.** *The identity* (5.31) *of Lemma 5.9 turns into the following "first order" difference equation for* $G^{(k)}$ *for all* $k \geq 0$*:*

$$\delta_k G^{(k)}(\mathbf{y}) = \Big((q^{\frac{1}{4}} z)^k q^{\frac{1}{2}} D_1 + q^{\frac{1}{4}} z^{-1} \delta_{k-1} d_k^{-1}\Big) G^{(k)}(\mathbf{y})$$

*Proof.* We write

$$\delta_k G^{(k)}(\mathbf{y}) = \mathcal{D}_k\, G^{(k)}(\mathbf{y}) = \Big(q^{\frac{k+2}{4}} z^k D_1 + q^{\frac{1}{4}} z^{-1} \mathcal{D}_{k-1}\Big)\, G^{(k)}(\mathbf{y})$$

and finally rewrite

$$\mathcal{D}_{k-1}\, G^{(k)}(\mathbf{y}) = \delta_{k-1} d_k^{-1}\, G^{(k)}(\mathbf{y})$$

□

Note that $\delta_k$ is now expressed in terms of the two *commuting* operators $D_1$ and $\delta_{k-1} d_k^{-1}$, as opposed to $\mathcal{D}_k$ being expressed in terms of $D_1$ and $\mathcal{D}_{k-1}$, whose commutation is more involved. Our strategy is now to iterate the result of Corollary 5.10, to get a compact expression for $G^{(k)}(\mathbf{y})$.

**Definition 5.11.** *We define the difference operators* $\mathcal{D}_{k,j}$ *for* $j, k \geq 1$ *as:*

$$\mathcal{D}_{k,j} = (q^{\frac{1}{4}} z)^k q^{\frac{1}{2}} D_1 + q^{\frac{2j-1}{4}} z^{-1} \delta_{k-1} d_k^{-1}$$

**Lemma 5.12.** *We have for all* $n \geq 0$*:*

$$(\delta_k)^n G^{(k)}(\mathbf{y}) = \mathcal{D}_{k,n} \mathcal{D}_{k,n-1} \cdots \mathcal{D}_{k,1} G^{(k)}(\mathbf{y}) \tag{5.33}$$

*Proof.* The result of Corollary 5.10 reads $\delta_k G^{(k)}(\mathbf{y}) = \mathcal{D}_{k,1} G^{(k)}(\mathbf{y})$. We then use iteratively the commutation $\delta_k \mathcal{D}_{k,j} = \mathcal{D}_{k,j+1} \delta_k$. □

**Definition 5.13.** *We define the operators:*

$$\delta_{k,n} = \mathcal{D}_{k,n} \mathcal{D}_{k,n-1} \cdots \mathcal{D}_{k,1}$$

The operator $\delta_{k,n}$ may be expanded as follows.

**Lemma 5.14.** *We have:*

$$\delta_{k,n} = \mathcal{D}_{k,n} \mathcal{D}_{k,n-1} \cdots \mathcal{D}_{k,1} = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} z^{kn-(k+1)j}\, q^{\frac{1}{4}\alpha^{(k)}_{n,j}}\, (\delta_{k-1} d_k^{-1})^j\, D_1^{n-j}$$

*where* $\begin{bmatrix} n \\ j \end{bmatrix}$ *stands for the q-binomial* $\begin{bmatrix} n \\ j \end{bmatrix} = \frac{(q)_n}{(q)_j(q)_{n-j}}$*, with* $(q)_k = \prod_{i=1}^k (1-q^i)$*, and*

$$\alpha_{n,j}^{(k)} = (n+1-2j)^2 + k(n-j)^2 - (n+1-j)^2 + 2n \qquad (0 \le j \le n)$$

*Proof.* By induction. The Lemma holds for $n=1$ by definition, with $\alpha_{1,0}^{(k)} = k+2$ and $\alpha_{1,1}^{(k)} = 1$. Assume it holds for $n-1$, we then compute:

$$\begin{aligned} & \mathcal{D}_{k,n} \sum_{\ell=0}^{n-1} \begin{bmatrix} n-1 \\ \ell \end{bmatrix} z^{k(n-1)-(k+1)\ell}\, q^{\frac{1}{4}\alpha_{n-1,\ell}^{(k)}}\, (\delta_{k-1} d_k^{-1})^\ell D_1^{n-\ell-1} \\ = & \ ((q^{\frac{1}{4}} z)^k q^{\frac{1}{2}} D_1 + q^{\frac{2n-1}{4}} z^{-1} \delta_{k-1} d_k^{-1}) \sum_{\ell=0}^{n-1} \begin{bmatrix} n-1 \\ \ell \end{bmatrix} z^{k(n-1)-(k+1)\ell}\, q^{\frac{1}{4}\alpha_{n-1,\ell}^{(k)}}\, (\delta_{k-1} d_k^{-1})^\ell D_1^{n-\ell-1} \\ = & \ \sum_{\ell=0}^{n-1} \left( q^{\frac{k+2}{4} + \frac{k(n-1)-(k+1)\ell}{2} + \frac{1}{4}\alpha_{n-1,\ell}^{(k)}} \begin{bmatrix} n-1 \\ \ell \end{bmatrix} + q^{\frac{2n-1}{4} + \frac{1}{4}\alpha_{n-1,\ell-1}^{(k)}} \begin{bmatrix} n-1 \\ \ell-1 \end{bmatrix} \right) \\ & \qquad \times z^{kn-(k+1)\ell} (\delta_{k-1} d_k^{-1})^\ell D_1^{n-\ell} \end{aligned}$$

and the Lemma follows from the relations

$$\alpha_{n,\ell}^{(k)} - \alpha_{n-1,\ell}^{(k)} = k+2+2k(n-1) - 2(k+1)\ell, \qquad \alpha_{n,\ell}^{(k)} - \alpha_{n-1,\ell-1}^{(k)} = 2n-1-4(n-j)$$

and the $q$-binomial identity:

$$\begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n-1 \\ j \end{bmatrix} + q^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} \tag{5.34}$$

□

Let us now compute the coefficients $G_{n_1,n_2,\ldots,n_k}^{(k)}$ of the generating series $G^{(k)}(\mathbf{y})$. Lemmas 5.12 and 5.14 lead immediately to the following inductive relation:

**Lemma 5.15.** *For all* $k \ge 1$*, the coefficient of* $y_k^{n_k}$ *in* $G^{(k)}$ *is given by:*

$$\begin{aligned} & \left( \delta_k^{n_k} G^{(k)}(y_1, \ldots, y_k) \right)|_{y_k=0} \\ = & \sum_{j_{k-1}=0}^{n_k} \begin{bmatrix} n_k \\ j_{k-1} \end{bmatrix} z^{kn_k-(k+1)j_{k-1}}\, q^{\frac{1}{4}\alpha_{n_k,j_{k-1}}^{(k)}}\, D_1^{n_k-j_{k-1}} \delta_{k-1}^{j_{k-1}} G^{(k-1)}(y_1, \ldots, y_{k-1}) \end{aligned}$$

Iterating the latter formula, we get the following final result:

**Theorem 5.16.** *For all* $k \ge 1$*, we have:*

$$G_{n_1,n_2,\ldots,n_k}^{(k)} = \sum_{j_{k-1}=0}^{n_k} \sum_{j_{k-2}=0}^{n_{k-1}+j_{k-1}} \cdots \sum_{j_0=0}^{n_1+j_1} z^{\mu_k(\mathbf{n};\mathbf{j})} q^{\frac{1}{4}\beta_k(\mathbf{n};\mathbf{j})} \begin{bmatrix} n_1+j_1 \\ j_0 \end{bmatrix} \begin{bmatrix} n_2+j_2 \\ j_1 \end{bmatrix} \cdots \begin{bmatrix} n_k+j_k \\ j_{k-1} \end{bmatrix} \tag{5.35}$$

*with the convention that $j_k = 0$, and where*

$$\mu_k(\mathbf{n};\mathbf{j}) = \sum_{i=1}^{k} i n_i - 2\sum_{i=0}^{k-1} j_i \tag{5.36}$$

$$\beta_k(\mathbf{n};\mathbf{j}) = \sum_{i=1}^{k} \Big(\sum_{\ell=i}^{k} n_\ell - 2j_{i-1}\Big)^2 + 2\sum_{i=1}^{k} n_i \tag{5.37}$$

*Proof.* By induction. The formula holds for $k = 1$. Indeed, using the formula of Lemma 5.15 for $k = 1$, with $G^{(0)} = 1$ and $\delta_0 = t^{-1}$, we identify $\mu_1(n_1; j_0) = n_1 - 2j_0$ and $\beta_1(n_1, j_0) = \alpha^{(1)}_{n_1,j_0} + 2j_0 = (n_1 - 2j_0)^2 + 2n_1$. Assume the formula holds for $k-1$, for some $k \geq 2$. We compute for $\mathbf{n} = (n_1, ..., n_{k-1})$:

$$G^{(k)}_{\mathbf{n},n_k} = \sum_{j_{k-1}=0}^{n_k} \begin{bmatrix} n_k \\ j_{k-1} \end{bmatrix} z^{kn_k-(k+1)j_{k-1}} \, q^{\frac{1}{4}\alpha^{(k)}_{n_k,j_{k-1}}} \, D_1^{n_k-j_{k-1}} \, G^{(k-1)}_{\mathbf{n}+j_{k-1}\epsilon_{k-1}}$$

where $D_1^m$ acts as usual on powers of $z^{\pm 1}$ as $z^{\pm p} \mapsto q^{\pm\frac{mp}{2}} z^{\pm p}$. Plugging in the value of $G^{(k-1)}$ from (5.35) for $k-1$, we find a formula of the form (5.35) for $k$, with

$$\begin{aligned} \mu_k(\mathbf{n};\mathbf{j}) &= \mu_{k-1}(n_1, ..., n_{k-2}, n_{k-1} + j_{k-1}; j_0, ..., j_{k-2}) + kn_k - (k+1)j_{k-1} \\ &= \sum_{i=1}^{k-1} i n_i + (k-1)j_{k-1} - 2\sum_{i=0}^{k-2} j_i + kn_k - (k+1)j_{k-1} \\ &= \sum_{i=1}^{k-1} i n_i - 2\sum_{i=0}^{k-1} j_i \end{aligned}$$

and

$$\begin{aligned}
\beta_k(\mathbf{n};\mathbf{j}) &= \beta_{k-1}(n_1,...,n_{k-2},n_{k-1}+j_{k-1};j_0,...,j_{k-2})+\alpha^{(k)}_{n_k,j_{k-1}}\\
&\qquad +2(n_k-j_{k-1})\mu_{k-1}(n_1,...,n_{k-2},n_{k-1}+j_{k-1};j_0,...,j_{k-2})\\
&= \sum_{i=1}^{k-1}\big(j_{k-1}+\sum_{\ell=i}^{k-1}n_\ell-2j_{i-1}\big)^2+2\sum_{i=1}^{k-1}n_i+2j_{k-1}+\alpha^{(k)}_{n_k,j_{k-1}}\\
&\qquad +2(n_k-j_{k-1})(\sum_{i=1}^{k-1}in_i+(k-1)j_{k-1}-2\sum_{i=0}^{k-2}j_i)\\
&= \sum_{i=1}^{k-1}\Big\{\big(\sum_{\ell=i}^{k}n_\ell-2j_{i-1}\big)^2-2(n_k-j_{k-1})(\sum_{\ell=i}^{k}n_\ell-2j_{i-1})+(n_k-j_{k-1})^2\Big\}\\
&\quad +2\sum_{i=1}^{k-1}n_i+2j_{k-1}+(n_k-2j_{k-1})^2+(k-1)(n_k-j_{k-1})^2+2(n_k-j_{k-1})\\
&\quad +2(n_k-j_{k-1})(\sum_{i=1}^{k-1}in_i+(k-1)j_{k-1}-2\sum_{i=0}^{k-2}j_i)\\
&= \sum_{i=1}^{k}\big(\sum_{\ell=i}^{k}n_\ell-2j_{i-1}\big)^2+2\sum_{i=1}^{k}n_i
\end{aligned}$$

where we have rewritten $\alpha^{(k)}_{n,j}=(n-2j)^2+(k-1)(n-j)^2+2(n-j)$. These give respectively (5.36) and (5.37), and the Theorem follows. $\square$

**Corollary 5.17.** *We have the following formula for the graded characters:*

$$\begin{aligned}
\chi_{\mathbf{n}}(q^{-1};z) &= \sum_{j_{k-1}=0}^{n_k}\sum_{j_{k-2}=0}^{n_{k-1}+j_{k-1}}\cdots\sum_{j_0=0}^{n_1+j_1} z^{\sum_{i=1}^k in_i-2\sum_{i=0}^{k-1}j_i}\\
&\qquad \times q^{\sum_{i=0}^{k-1}j_i(j_i-\sum_{\ell=i+1}^k n_i)}\begin{bmatrix}n_1+j_1\\ j_0\end{bmatrix}\begin{bmatrix}n_2+j_2\\ j_1\end{bmatrix}\cdots\begin{bmatrix}n_k+j_k\\ j_{k-1}\end{bmatrix}
\end{aligned}\tag{5.38}$$

*Proof.* Using the relation $G^{(k)}_{n_1,\dots,n_k}=q^{\frac{1}{2}\sum_i n_i+\frac{1}{4}\sum_{i,j}n_i\mathrm{Min}(i,j)n_j}\chi_{\mathbf{n}}(q^{-1};z)$ the Corollary follows by noticing that

$$\sum_{i=1}^k\Big(\sum_{\ell=i}^k n_\ell\Big)^2=\sum_{i,j=1}^k n_i\,\mathrm{Min}(i,j)\,n_j$$

for all $k\geq 1$. $\square$

**Remark 5.18.** *Corollary 5.17 is equivalent to Theorem 5.1 of* [FF02].

**Example 5.19.** *For $k=1$, we have respectively from Theorem 5.16 and Corollary 5.17:*

$$G_n^{(1)} = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} z^{n-2j}\, q^{\frac{(n-2j)^2}{4}+\frac{n}{2}} \tag{5.39}$$

$$\chi_n(q^{-1};z) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} z^{n-2j}\, q^{j(j-n)} = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{q^{-1}} z^{n-2j} \tag{5.40}$$

*where we have used the notation* $\begin{bmatrix} n \\ j \end{bmatrix}_{q^{-1}} = \frac{\prod_{i=n-j+1}^{n}(1-q^{-i})}{\prod_{i=1}^{j}(1-q^{-i})}$) *as well as the standard identity* $\begin{bmatrix} n \\ j \end{bmatrix} q^{j(j-n)} = \begin{bmatrix} n \\ j \end{bmatrix}_{q^{-1}}$.

**Example 5.20.** *For $k=2$, we have respectively from Theorem 5.16 and Corollary 5.17:*

$$G_{p,n}^{(2)} = \sum_{j=0}^{n}\sum_{k=0}^{p+j} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} p+j \\ k \end{bmatrix} z^{p+2n-2j-2k}\, q^{\frac{(n-2j)^2+(n+p-2k)^2}{4}+\frac{n+p}{2}} \tag{5.41}$$

$$\chi_{p,n}(q^{-1};z) = \sum_{j=0}^{n}\sum_{k=0}^{p+j} \begin{bmatrix} n \\ j \end{bmatrix} \begin{bmatrix} p+j \\ k \end{bmatrix} z^{p+2n-2j-2k}\, q^{j(j-n)+k(k-p-n)} \tag{5.42}$$

$$= \sum_{j=0}^{n}\sum_{k=0}^{p+j} \begin{bmatrix} n \\ j \end{bmatrix}_{q^{-1}} \begin{bmatrix} p+j \\ k \end{bmatrix}_{q^{-1}} z^{p+2n-2j-2k}\, q^{-k(n-j)} \tag{5.43}$$

5.5. **Special cases: Demazure characters and semi-infinite limits.** The formula of Theorem 5.16 makes it easy to derive the large $n_k = N$ limit of $\chi_{n_1,\ldots,n_k}(q^{-1};z)$ for fixed values of $n_1,\ldots,n_{k-1}$, and for $|q|<1$. We must distinguish the cases $N$ even and $N$ odd. Let us perform the following changes of summation variables $j_i \to a_i$ defined as follows:

$$\text{if } N \text{ even}: \quad j_i = \frac{N}{2} + a_i, \qquad (i=0,1,\ldots,k-1)$$
$$\text{if } N \text{ odd}: \quad j_i = \frac{N-1}{2} + a_i, \qquad (i=0,1,\ldots,k-1)$$

The new ranges of the variables $a_i$ are

$$\text{if } N \text{ even}: \quad \begin{cases} n_\ell + a_\ell \geq a_{\ell-1} \geq -\frac{N}{2} & (\ell=1,2,\ldots,k-1) \\ \frac{N}{2} \geq a_{k-1} \geq -\frac{N}{2} \end{cases}$$

$$\text{if } N \text{ odd}: \quad \begin{cases} n_\ell + a_\ell \geq a_{\ell-1} \geq -\frac{N-1}{2} & (\ell=1,2,\ldots,k-1) \\ \frac{N+1}{2} \geq a_{k-1} \geq -\frac{N-1}{2} \end{cases}$$

When $N \to \infty$ with fixed parity, both ranges become

$$a_i \in \mathbb{Z} \quad \text{and} \quad n_i + a_i \geq a_{i-1} \qquad (i = 1, 2, ..., k-1)$$

and the summand tends to a finite limit, as

$$\begin{bmatrix} N \\ \lfloor \frac{N}{2} \rfloor + a_{k-1} \end{bmatrix} \to \frac{1}{(q)_\infty} \quad \text{and} \quad \begin{bmatrix} n_i + \lfloor \frac{N}{2} \rfloor + a_i \\ \lfloor \frac{N}{2} \rfloor + a_{i-1} \end{bmatrix} \to \frac{1}{(q)_{n_i + a_i - a_{i-1}}}$$

while both powers of $z$ and $q$ become independent of $N$. (The notation $(q)_\infty$ stands for the infinite product $\lim_{k\to\infty}(q)_k = \prod_{i=1}^\infty (1-q^i)$.). This yields the following.

**Theorem 5.21.** *We have*

$$\lim_{\substack{N\to\infty \\ N \text{ even}}} q^{-\frac{N}{2}} G^{(k)}_{n_1,...,n_{k-1},N} = \frac{1}{(q)_\infty} \sum_{\substack{a_0,a_1,...,a_{k-1}\in\mathbb{Z} \\ n_i+a_i\geq a_{i-1},\, i=1,2,...,k-1}} \prod_{i=1}^{k-1} \frac{1}{(q)_{n_i+a_i-a_{i-1}}} \times z^{\sum_{i=1}^{k-1} i n_i - 2\sum_{i=0}^{k-1} a_i} q^{\frac{1}{4}\sum_{i=0}^{k-1}\left(\sum_{\ell=i+1}^{k-1} n_\ell - 2a_i\right)^2} \tag{5.44}$$

$$\lim_{\substack{N\to\infty \\ N \text{ odd}}} q^{-\frac{N}{2}} G^{(k)}_{n_1,...,n_{k-1},N} = \frac{1}{(q)_\infty} \sum_{\substack{a_0,a_1,...,a_{k-1}\in\mathbb{Z} \\ n_i+a_i\geq a_{i-1},\, i=1,2,...,k-1}} \prod_{i=1}^{k-1} \frac{1}{(q)_{n_i+a_i-a_{i-1}}} \times z^{k+\sum_{i=1}^{k-1} i n_i - 2\sum_{i=0}^{k-1} a_i} q^{\frac{1}{4}\sum_{i=0}^{k-1}\left(1+\sum_{\ell=i+1}^{k-1} n_\ell - 2a_i\right)^2} \tag{5.45}$$

**Definition 5.22.** *For $i \in \{0, 1\}$, we define:*

$$\chi_{i;n_1,n_2,...,n_{k-1}}(q^{-1}; z) = \lim_{\substack{N\to\infty \\ N = i \bmod 2}} q^{-\frac{N}{2}} G^{(k)}_{n_1,...,n_{k-1},N}$$

**Theorem 5.23.** *Denoting by $\mathbf{n}_i = (n_{1,i}, n_{2,i}, ..., n_{k-1,i})$ with $n_{j,i} = \delta_{j,i}$ for $j = 1, 2, ..., k-1$ and $i = 0, 1, ..., k-1$, we have the following identification with the level $k$ affine characters $\chi_{i,k}$ for the representation with highest weight $(k-i)\Lambda_0 + i\Lambda_1$ in the standard notation* [DFMS97]*:*

$$\begin{aligned} \chi_{0;\mathbf{n}_i}(q^{-1}; z) &= q^{-u_{i,k}} \chi_{i,k}(q^{-1}; z) \qquad (i = 0, 1, 2, ..., k-1) \\ \chi_{1;0,0,...,0}(q^{-1}; z) &= q^{-u_{k,k}} \chi_{k,k}(q^{-1}; z) \end{aligned}$$

*where $u_{i,k} = \frac{(i+1)^2}{4(k+2)} - \frac{1}{8}$ for $i = 0, 1, 2, .., k$.*

**Example 5.24.** *For $k=1$, we have:*

$$\chi_0(q^{-1},z) = \lim_{\substack{N\to\infty\\ N \text{ even}}} q^{-\frac{N}{2}}\, G_N^{(1)} = \frac{1}{(q)_\infty}\sum_{a\in\mathbb{Z}} z^{-2a} q^{a^2} \tag{5.46}$$

$$\chi_1(q^{-1},z) = \lim_{\substack{N\to\infty\\ N \text{ odd}}} q^{-\frac{N}{2}}\, G_N^{(1)} = \frac{1}{(q)_\infty}\sum_{a\in\mathbb{Z}} z^{1-2a} q^{\frac{(2a-1)^2}{4}} \tag{5.47}$$

*These are identified with level 1 affine characters as:*

$$\begin{aligned}\chi_0(q^{-1},z) &= q^{\frac{1}{24}}\,\chi_{0,1}(q^{-1},z)\\ \chi_1(q^{-1},z) &= q^{-\frac{5}{24}}\,\chi_{1,1}(q^{-1},z)\end{aligned}$$

**Example 5.25.** *For $k=2$, we have:*

$$\chi_{0;p}(q^{-1},z) = \frac{1}{(q)_\infty}\sum_{\substack{a,b\in\mathbb{Z}\\ p+a\geq b}} \frac{1}{(q)_{p+a-b}} z^{p-2a-2b} q^{\frac{(p-2b)^2}{4}+a^2} \tag{5.48}$$

$$\chi_{1;p}(q^{-1},z) = \frac{1}{(q)_\infty}\sum_{\substack{a,b\in\mathbb{Z}\\ p+a\geq b}} \frac{1}{(q)_{p+a-b}} z^{2+p-2a-2b} q^{\frac{(1+p-2b)^2+(2a-1)^2}{4}} \tag{5.49}$$

*Note that these only depend on $p$ mod 2, so we get four functions identified with the level 2 affine characters as follows:*

$$\begin{aligned}\chi_{0;0}(q^{-1},z) &= q^{\frac{1}{16}}\,\chi_{0,2}(q^{-1},z)\\ \chi_{1;1}(q^{-1},z)=\chi_{0;1}(q^{-1},z) &= q^{-\frac{1}{8}}\,\chi_{1,2}(q^{-1},z)\\ \chi_{1;0}(q^{-1},z) &= q^{-\frac{7}{16}}\,\chi_{2,2}(q^{-1},z)\end{aligned}$$

We can also use Theorem 5.21 to derive results on the affine character limits at level $k$. We have:

**Theorem 5.26.** *We have the following difference equations for the character limits of Def. 5.22:*

$$t\,\mathcal{D}_k\,\chi_{0;n_1,..,n_{k-1}}(q^{-1},z) = \chi_{1;n_1,..,n_{k-1}}(q^{-1},z) \tag{5.50}$$

$$t\,\mathcal{D}_k\,\chi_{1;n_1,..,n_{k-1}}(q^{-1},z) = \chi_{0;n_1,..,n_{k-1}}(q^{-1},z) \tag{5.51}$$

*In particular, we have for $i=0,1$:*

$$\big((t\,\mathcal{D}_k)^2-1\big)\,\chi_{i;n_1,..,n_{k-1}}(q^{-1},z)=0 \tag{5.52}$$

*Proof.* Formally, we may write

$$\chi_{i;n_1,..,n_{k-1}}(q^{-1},z) = \lim_{\substack{N\to\infty\\ N=i \bmod 2}} t^N(\mathcal{D}_k)^N(\mathcal{D}_{k-1})^{n_{k-1}}\cdots(\mathcal{D}_1)^{n_1}\,1$$

The limits clearly satisfy the desired equations. □

Note that (5.52) is also satisfied by the affine characters of level $k$.

## Appendix A. Proof of Lemmas 4.3 and 4.4

The proof of Lemmas 4.3 and 4.4 goes as follows. First we rewrite the statement of the Lemmas as a vanishing condition for the antisymmetrized version of some rational fraction of the $z$'s. Then we show that all residues at the poles of this antisymmetrized expression vanish. Finally we conclude that the result is proportional to the antisymmetrization of a polynomial of the $z$'s with a too small degree, which must therefore vanish.

A.1. **Antisymmetrization: general properties.** For any function $f(\mathbf{z})$ of the variables $\mathbf{z}=(z_1,...,z_N)$ we define the symmetrization ($S$) and antisymmetrization ($AS$) operators as:

$$S(f)(\mathbf{z}) = \frac{1}{N!}\sum_{\sigma\in S_N} f(z_{\sigma(1)},...,z_{\sigma(N)}) \tag{A.1}$$

$$AS(f)(\mathbf{z}) = \frac{1}{N!}\sum_{\sigma\in S_N} \mathrm{sgn}(\sigma)\, f(z_{\sigma(1)},...,z_{\sigma(N)}) \tag{A.2}$$

We have the following immediate result:

**Lemma A.1.** *For any function $f_{I,J}(\mathbf{z})$ of $\mathbf{z}$ indexed by two subsets $I,J$ of $[1,N]$ we have:*

$$\sum_{\substack{I,J\subset[1,N], I\cap J=\emptyset\\ |I|=a,\ |J|=N-a}} f_{I,J}(\mathbf{z}) = \binom{N}{a} S\left(f_{I_0,J_0}(\mathbf{z})\right)$$

*where $I_0=[1,a]$ and $J_0=[a+1,N]$.*

Lemma A.1 allows to rephrase the statements of Lemmas 4.3 and 4.4 as identities on symmetrized expressions.

For $\mathbf{z}=(z_1,...,z_N)$, we define the Vandermonde determinant $\Delta(\mathbf{z})=\prod_{1\leq i<j\leq N}(z_i-z_j)$ It is anti-symmetric, hence $AS(\Delta(\mathbf{z}))=\Delta(\mathbf{z})$, and moreover for any function $f(\mathbf{z})$ we have $AS(\Delta(\mathbf{z})f(\mathbf{z}))=\Delta(\mathbf{z})S(f(\mathbf{z}))$.

We have the following standard fact about anti-symmetric polynomials.

**Lemma A.2.** *The non-zero anti-symmetric polynomial $P$ of $\mathbf{z}$ of smallest total degree, namely such that $AS(P)=P$, is proportional to the Vandermonde determinant of the z's, up to a constant independent of the z's.*

This implies the following:

**Corollary A.3.** *For any polynomial $P(\mathbf{z})$ of total degree strictly less than $N(N-1)/2$, we have $AS(P)=0$.*

A.2. **Proof of Lemma 4.3.** For any integers $b \geq a \geq 0$, and $p \geq m \geq 0$, $I_0 = [1, a]$, $J_0 = [a+1, a+b]$, let us define

$$\varphi_{a,b}^{m,p}(\mathbf{z}) = S\left( z_{I_0}^m z_{J_0}^p \prod_{\substack{i \in I_0 \\ j \in J_0}} \frac{z_i}{z_i - z_j} \frac{z_j}{z_j - q z_i} \right)$$

We note that as $m \leq p$, then:

$$\varphi_{a,b}^{m,p}(\mathbf{z}) = (z_1 \cdots z_{a+b})^m \, \varphi_{a,b}^{0,p-m}(\mathbf{z})$$

We also define:

$$\psi_{a,b}^{m,p}(\mathbf{z}) := \varphi_{a,b}^{m,p}(\mathbf{z}) - q^{a(p-m)} \varphi_{b,a}^{p,m}(\mathbf{z}) \tag{A.3}$$

Using Lemma A.1, it is straightforward to show that the statement of Lemma 4.3 is equivalent to:

$$\psi_{a,b}^{m,p}(\mathbf{z}) = 0 \tag{A.4}$$

In the following, we use the notation $\Delta_I = \prod_{1 \leq k \leq \ell \leq n} (z_{i_k} - z_{i_\ell})$ for any *ordered* set $I = \{i_1, i_2, ..., i_n\}$. With $I_0, J_0$ as above, we now express:

$$\begin{aligned}
\Delta(\mathbf{z}) \varphi_{a,b}^{m,p}(\mathbf{z}) &= AS\left( \Delta(\mathbf{z}) z_{I_0}^m z_{J_0}^p \prod_{\substack{i \in I_0 \\ j \in J_0}} \frac{z_i}{z_i - z_j} \frac{z_j}{z_j - q z_i} \right) \\
&= AS\left( \Delta_{I_0} \Delta_{J_0} z_{I_0}^{m+b} z_{J_0}^p \prod_{\substack{i \in I_0 \\ j \in J_0}} \frac{z_j}{z_j - q z_i} \right)
\end{aligned}$$

The only possible poles of $\Delta(\mathbf{z}) \varphi_{a,b}^{p,q}(\mathbf{z})$ are for $z_i \to q z_j$ for $i \neq j$. Let us compute the residue at the pole $z_2 \to q z_1$ in $z_2$. Pick two ordered sets $I_0', J_0'$ with $I_0' \cap J_0' = \emptyset$, $I_0' \cup J_0' = [1, a+b]$, $|I_0'| = a$, $|J_0'| = b$, and such that 1 is the first element of $I_0' = \{1\} \cup I_1$ and 2 the last element of $J_0' = J_1 \cup \{2\}$. For any subset $L \subset [1, N]$, we denote by $AS_L$ the antisymmetrization over the set $\{z_i\}_{i \in L}$. We compute

$$\begin{aligned}
\mathrm{Res}_{z_2 \to q z_1} \Delta(\mathbf{z}) \varphi_{a,b}^{m,p}(\mathbf{z}) &= z_1^{m+b} (q z_1)^{p+1} AS_{[3,a+b]} \Bigg( \Delta_{I_1} \prod_{i \in I_1} (z_1 - z_i) \Delta_{J_1} \prod_{j \in J_1} (z_j - q z_1) \\
&\qquad z_{I_1}^{m+b} z_{J_1}^p \prod_{\substack{i \in I_1 \\ j \in J_1}} \frac{z_j}{z_j - q z_i} \prod_{i \in I_1} \frac{q z_1}{q z_1 - q z_i} \prod_{j \in J_1} \frac{z_j}{z_j - q z_1} \Bigg) \\
&= q^{p+1} z_1^{m+p+b+a} \Delta(\mathbf{z}') \varphi_{a-1,b-1}^{m+1,p+1}(\mathbf{z}')
\end{aligned}$$

where $\mathbf{z}' = (z_3, ..., z_{a+b})$. Using (A.3), we deduce that:

$$\begin{aligned}\mathrm{Res}_{z_2\to qz_1}\Delta(\mathbf{z})\psi^{m,p}_{a,b}(\mathbf{z}) &= \Delta(\mathbf{z}')\left\{q^{p+1}z_1^{m+p+b+a}\varphi^{m+1,p+1}_{a-1,b-1}(\mathbf{z}') - q^{a(p-m)}q^{m+1}z_1^{m+p+b+a}\varphi^{p+1,m+1}_{b-1,a-1}(\mathbf{z}')\right\}\\ &= q^{p+1}z_1^{m+p+b+a}\Delta(\mathbf{z}')\psi^{m+1,p+1}_{a-1,b-1}(\mathbf{z}')\end{aligned}$$

We now proceed by induction on $a$. For $a = 0$, we have:

$$\varphi^{m,p}_{0,b}(\mathbf{z}) = S\left((z_1\cdots z_b)^p\right) = (z_1\cdots z_b)^p = \varphi^{p,m}_{b,0}(\mathbf{z})$$

hence $\psi^{m,p}_{0,b}(\mathbf{z}) = 0$. Assuming that $\psi^{m+1,p+1}_{a-1,b-1}(\mathbf{z}') = 0$, we see that the residue at $z_2 \to qz_1$ of $\psi^{m,p}_{a,b}(\mathbf{z})$ vanishes, hence the is no pole of the form $1/(z_2 - qz_1)$ in the antisymmetrized expression. By symmetry, this holds for any pole $z_i \to qz_j$. We conclude that $\psi^{m,p}_{a,b}(\mathbf{z})$ is a polynomial. Using the antisymmetrization formula, we easily get:

$$\begin{aligned}\Delta(\mathbf{z})\varphi^{m,p}_{a,b}(\mathbf{z}) &= AS\left(\Delta_{I_0}\Delta_{J_0}z_{I_0}^{m+b}z_{J_0}^{p+a}\prod_{\substack{i\in I_0\\ j\in J_0}}\frac{1}{z_j - qz_i}\right)\\ &= (z_1\cdots z_{a+b})^{p+a}AS\left(\Delta_{I_0}\Delta_{J_0}z_{I_0}^{b-a-(p-m)}\prod_{\substack{i\in I_0\\ j\in J_0}}\frac{1}{z_j - qz_i}\right)\end{aligned}$$

Similarly:

$$\Delta(\mathbf{z})\varphi^{p,m}_{b,a}(\mathbf{z}) = (z_1\cdots z_{a+b})^{p+a}AS\left(\Delta_{I_0}\Delta_{J_0}z_{I_0}^{b-a-(p-m)}\prod_{\substack{i\in J_0\\ j\in I_0}}\frac{1}{z_j - qz_i}\right)$$

Finally, we have:

$$\frac{\Delta(\mathbf{z})\psi^{m,p}_{a,b}(\mathbf{z})}{(z_1\cdots z_{a+b})^{p+a-1}} = AS\left(\Delta_{I_0}\Delta_{J_0}z_{I_0}^{b-a+1-(p-m)}z_{J_0}\left\{\prod_{\substack{i\in I_0\\ j\in J_0}}\frac{1}{z_j - qz_i} - q^{a(p-m)}\prod_{\substack{i\in J_0\\ j\in I_0}}\frac{1}{z_j - qz_i}\right\}\right)$$

where the r.h.s. is a polynomial, as $b - a + 1 - (p - m) \geq 0$ and it has no poles at $z_i = qz_j$. Writing $N = a + b$, its total degree is:

$$\frac{N(N-1)}{2} + ma + pb - N(p + a - 1) = \frac{N(N-1)}{2} - (p - m)a - N(a - 1) < \frac{N(N-1)}{2}$$

for $a \geq 1$. The degree of the polynomial is therefore too small, and it must vanish by Corollary A.3. The Lemma 4.3 follows.

A.3. **Proof of Lemma 4.4.** We proceed analogously. For $I_0 = [1, a]$ and $J_0 = [a+1, 2a]$, we define

$$\theta_a(\mathbf{z}) = S\left(\prod_{\substack{i\in I_0\\ j\in J_0}} \frac{z_i}{z_i - z_j}\frac{z_j}{z_j - qz_i}\left(1 - q^a\frac{z_{I_0}}{z_{J_0}}\right)\right)$$

We also have:

$$\Delta(\mathbf{z})\theta_a(\mathbf{z}) = AS\left(\Delta_{I_0}\Delta_{J_0}z_{I_0}^a\prod_{\substack{i\in I_0\\ j\in J_0}}\frac{z_j}{z_j - qz_i}\left(1 - q^a\frac{z_{I_0}}{z_{J_0}}\right)\right)$$

Let us compute the residue of the pole of this expression at $z_2 \to qz_1$. As before, we pick two ordered sets $I_0'$ and $J_0'$ of cardinality $a$ such that $I_0' \cap J_0' = \emptyset$, $I_0' \cup J_0' = [1, 2a]$ and 1 is the first element of $I_0' = \{1\} \cup I_1$ and 2 the last element of $J_0' = J_1 \cup \{2\}$. We compute:

$$\begin{aligned}
\mathrm{Res}_{z_2\to qz_1}\Delta(\mathbf{z})\theta_a(\mathbf{z}) &= qz_1^{a+1}AS\Bigg(\Delta_{I_1}\prod_{i\in I_1}(z_1 - z_i)\Delta_{J_1}\prod_{j\in J_1}(z_j - qz_1)z_{I_1}^a \\
&\qquad \prod_{\substack{i\in I_1\\ j\in J_1}}\frac{z_j}{z_j - qz_i}\prod_{i\in I_1}\frac{qz_1}{qz_1 - qz_i}\prod_{j\in J_1}\frac{z_j}{z_j - qz_1}\left(1 - q^{a-1}\frac{z_{I_1}}{z_{J_1}}\right)\Bigg) \\
&= qz_1^{2a}(z_3\cdots z_{2a})AS\left(\Delta_{I_1}\Delta_{J_1}z_{I_1}^{a-1}\prod_{\substack{i\in I_1\\ j\in J_1}}\frac{z_j}{z_j - qz_i}\left(1 - q^{a-1}\frac{z_{I_1}}{z_{J_1}}\right)\right) \\
&= qz_1^{2a}(z_3\cdots z_{2a})\Delta(\mathbf{z}')\theta_{a-1}(\mathbf{z}')
\end{aligned}$$

where we denote by $\mathbf{z}' = (z_3, z_4, ..., z_{2a})$.

Likewise, we define for $I_2 = [1, a+1]$ and $J_2 = [a+2, 2a]$:

$$\varphi_a(\mathbf{z}) = S\left(\prod_{\substack{i\in I_2\\ j\in J_2}}\frac{z_i}{z_i - z_j}\frac{z_j}{z_j - qz_i}\right)$$

We also have:

$$\Delta(\mathbf{z})\varphi_a(\mathbf{z}) = AS\left(\Delta_{I_2}\Delta_{J_2}z_{I_2}^{a-1}\prod_{\substack{i\in I_2\\ j\in J_2}}\frac{z_j}{z_j - qz_i}\right)$$

Let us compute the residue of the pole of this expressions at $z_2 \to qz_1$. We pick two ordered sets $I_2'$ and $J_2'$ such that $I_2' \cap J_2' = \emptyset$, $I_2' \cup J_2' = [1, 2a]$, $|I_2'| = a+1$, $|J_2'| = a-1$, and 1 is the first element of $I'_2 = \{1\} \cup I_3$ and 2 the last element of $J'_2 = J_3 \cup \{2\}$. We compute:

$$\begin{aligned}
\mathrm{Res}_{z_2 \to qz_1} \Delta(\mathbf{z})\varphi_a(\mathbf{z}) &= qz_1^a AS \Bigg( \Delta_{I_3} \prod_{i \in I_3} (z_1 - z_i) \Delta_{J_3} \prod_{j \in J_3} (z_j - qz_1) z_{I_3}^{a-1} \\
&\qquad \prod_{\substack{i \in I_3 \\ j \in J_3}} \frac{z_j}{z_j - qz_i} \prod_{i \in I_3} \frac{qz_1}{qz_1 - qz_i} \prod_{j \in J_3} \frac{z_j}{z_j - qz_1} \Bigg) \\
&= qz_1^{2a} AS \left( \Delta_{I_3} \Delta_{J_3} z_{I_3}^{a-1} z_{J_3} \prod_{\substack{i \in I_3 \\ j \in J_3}} \frac{z_j}{z_j - qz_i} \right) \\
&= qz_1^{2a} (z_3 z_4 \cdots z_{2a}) \Delta(\mathbf{z}') \varphi_{a-1}(\mathbf{z}')
\end{aligned}$$

We conclude that

$$\mathrm{Res}_{z_2 \to qz_1} \Delta(\mathbf{z}) \left\{ \theta_a(\mathbf{z}) - \varphi_a(\mathbf{z}) \right\} = qz_1^{2a} (z_3 z_4 \cdots z_{2a}) \Delta(\mathbf{z}') \left\{ \theta_{a-1}(\mathbf{z}') - \varphi_{a-1}(\mathbf{z}') \right\}$$

We proceed by induction on $a$. For $a = 1$ we have

$$(z_1 - z_2)\theta_1(z_1, z_2) = AS \left( \frac{z_1 z_2}{z_2 - qz_1} (1 - q\frac{z_1}{z_2}) \right) = z_1 - z_2$$

Analogously, we find

$$\varphi_1(z_1, z_2) = S(1) = 1$$

hence $\theta_1(z_1, z_2) - \varphi_1(z_1, z_2) = 0$. Assuming that $\theta_{a-1}(\mathbf{z}') - \varphi_{a-1}(\mathbf{z}') = 0$, we deduce that $\Delta(\mathbf{z})(\theta_a(\mathbf{z}) - \varphi_a(\mathbf{z}))$ has no pole at $z_2 = qz_1$. By symmetry, it has no pole at any $z_i = qz_j$, hence it is a polynomial. Finally we write:

$$\begin{aligned}
\frac{\Delta(\mathbf{z})(\theta_a(\mathbf{z}) - \varphi_a(\mathbf{z}))}{(z_1 z_2 \cdots z_{2a})^{a-1}} &= AS \left( \Delta_{I_0} \Delta_{J_0} z_{I_0} \prod_{\substack{i \in I_0 \\ j \in J_0}} \frac{1}{z_j - qz_i} (z_{J_0} - q^a z_{I_0}) \right) \\
&\quad - AS \left( \Delta_{I_2} \Delta_{J_2} z_{J_2}^2 \prod_{\substack{i \in I_2 \\ j \in J_2}} \frac{1}{z_j - qz_i} \right)
\end{aligned}$$

where the r.h.s. is a polynomial of total degree $N(N-1)/2 - 2a(a-1) < N(N-1)/2$ for $a \geq 2$ and $N = 2a$. By Corollary A.3, the result must vanish, and the Lemma 4.4 follows.

## Appendix B. Proof of Lemma 4.12

Notations are as in Sect. A.1. Let us consider for $\alpha \in [1, N]$ and $p \in \mathbb{Z}$ the quantity

$$A_{\alpha,p}(\mathbf{z}) = \sum_{\substack{I \subset [1,N] \\ |I|=\alpha}} (z_I)^p \, a_I(\mathbf{z})$$

Picking the particular subset $I_\alpha = \{1, 2, ..., \alpha\}$, we may also write

$$A_{\alpha,p}(\mathbf{z}) = \binom{N}{\alpha} S\left((z_{I_\alpha})^p \, a_{I_\alpha}(\mathbf{z})\right)$$

We now wish to eliminate the denominators in this (symmetric) expression. We use that $AS(\Delta(\mathbf{z})f(\mathbf{z})) = \Delta(\mathbf{z})\, S\,(f(\mathbf{z}))$ for any $f$ to rewrite:

$$\Delta(\mathbf{z})\, A_{\alpha,p}(\mathbf{z}) = \binom{N}{\alpha} AS\left((z_{I_\alpha})^{p+N-\alpha} \Delta(z_1, ..., z_\alpha)\Delta(z_{\alpha+1}, ..., z_N)\right)$$

The function to be antisymmetrized is a polynomial if $p \geq \alpha - N$, and then it has total degree

$$\alpha(p + N - \alpha) + \frac{\alpha(\alpha-1)}{2} + \frac{(N-\alpha)(N-\alpha-1)}{2} = \alpha\, p + \frac{N(N-1)}{2}$$

By Corollary A.3, we deduce that for $p = -1, -2, ..., \alpha - N$ the antisymmetrized expression must vanish.

When $p = 0$, the degree is exactly $N(N-1)/2$ and therefore $\Delta(\mathbf{z})\, A_{\alpha,p}(\mathbf{z})$ is proportional to $\Delta(\mathbf{z})$. The proportionality constant is fixed by evaluating $A_{\alpha,0}$ in the successive limits $z_1 \to \infty, z_2 \to \infty, ..., z_\alpha \to \infty$, and we finally get $A_{\alpha,0} = 1$.

This completes the proof of Lemma 4.12.

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