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Abstract

In this paper, we investigate the problem of designing good non-binary LDPC codes for Slepian-Wolf coding. The design method is based on Density Evolution which gives the asymptotic error probability of the decoder for given code degree distributions. Density Evolution was originally introduced for channel coding under the assumption that the channel is symmetric. In Slepian-Wolf coding, the correlation channel is not necessarily symmetric and the source distribution has to be taken into account. In this paper, we express the non-binary Density Evolution recursion for Slepian-Wolf coding. From Density Evolution, we then perform code degree distribution optimization using an optimization algorithm called differential evolution. Both asymptotic performance evaluation and finite-length simulations show the gain at considering optimized degree distributions for SW coding.

I. INTRODUCTION

In this paper, we consider the lossless coding of a source X with the help of some side information Y available at the decoder only (see Figure 1). This setup is called asymmetric Slepian-Wolf (SW)

Fig. 1. Asymetric Slepian-Wolf coding

coding [28]. Here, for simplicity, it is referred to as SW coding. For this problem, it is well known that the infimum of achievable rates is given by $H(X|Y)$, the conditional entropy of X knowing Y and several practical coding schemes have been proposed [10], [24], [38]. Most of them are based on channel codes [9], [29], and particularly Low Density Parity Check (LDPC) codes [11], [20], [22]. In source coding, the source symbols are in general non-binary (for example the pixels or the quantized coefficients of the transformed blocks of an image). A usual coding solution is to transform the non-binary symbols into bits and to encode the bit planes independently with binary LDPC codes. To avoid a performance loss, the dependency between bit planes has to be taken into account at the decoder [18], [36], at the price of a complexity increase. In this paper, in order to avoid this operation, we consider directly non-binary LDPC codes [12].

Many efforts have been made in channel coding for the design of good LDPC codes. In particular, [25], [26] show that the performance of a code depends on its degree distributions. Codes of regular degrees were first considered, and [25] points out the gap between the performance of regular LDPC codes and the channel capacity. Then, codes constructed from optimized irregular degree distributions were shown to reduce the gap with the channel capacity. The degree distribution optimization can be realized from an optimization algorithm called differential evolution [31] and from Monte Carlo Simulations [15]. Although the issue of constructing properly the coding matrix at finite length remains [23], degree distribution optimization constitutes a good starting point for practical code design.

In the former optimization method, the code performance evaluation is realized from Density Evolution [25]. Denote $\mathcal{C}(\lambda,\rho)$ the ensemble of codes of variable and check node degree distributions $\lambda(x)$ and $\rho(x)$. From an asymptotic analysis, density evolution gives an evaluation of the average error rate over $\mathcal{C}(\lambda,\rho)$ for a given channel of input U and output W described by the conditional distribution $P(W|U)$ which is assumed to be symmetric. Density evolution techniques have been developed in channel coding both for binary [25], [26], [34] and non-binary [1], [19] codes.

In SW coding, from the joint probability distribution $P(X,Y)$, one could think of identifying the correlation channel $P(Y|X)$ and then simply applying the standard density evolution derived for channel coding. Unfortunately, as pointed out in [2], [5], a good LDPC code for channel coding is not necessarily good for SW coding. Two differences have to be taken into account. In SW coding, the source distribution is not necessarily uniform [33], and the correlation channel $P(Y|X)$ is not necessarily symmetric [6], [32]. The channel coding scheme and the SW coding scheme thus require codes of different rate and different code degree distributions.

In this paper, we adress the issue of optimizing non-binary LDPC code degree distributions for the SW coding problem. In particular we take into account the possibly non-uniform probability distribution of X and the fact that $P(Y|X)$ is not necessarily symmetric. The contributions of the papers are as follows.

1) In channel coding, we derive an analytical form of the density evolution recursion for symmetric channels. We also derive an analytical form of the density evolution in SW coding for any joint probability distribution $P(X,Y)$. From the two previous recursions, we restate the result of [4] which shows an equivalence between the two problems.
2) From the density evolution recusion, we optimize the code degree distributions for SW coding and show the asymptotic performance gain at considering optimized degree distributions.
3) We also illustrate at finite length the gain at considering optimized degree distributions compared to regular codes and the gain at considering non-binary LDPC codes.

The paper is organized as follows. Section II presents the related works. Section III introduces the notations and recalls some results on Galois Fields. Section IV restates the non-binary LDPC decoding algorithm for SW coding. Section V expresses the density evolution for channel and SW coding and restates the result of [4] on the equivalence between the two problems. Section VI presents the code degree distribution optimization. Section VII gives finite-length simulation results.

II. Related Works

Binary LDPC codes have been used for SW coding in [3], [7], [20], [22], [30] and references therein. In all of the above cases, the LDPC decoder consists of a message passing procedure referred to as the sum-product algorithm. In the same way, [35] proposes to use non-binary LDPC codes and derives the decoding algorithm expressions. Nevertheless these works do not provide a solution for the design of good non-binary LDPC codes for SW coding.

On the other hand, density evolution was initially introduced in [25] for binary symmetric channels and then used in [26] for irregular code optimization. The case of binary non-symmetric channels was further investigated in [34]. All these works give an analytic expression of the density evolution. Then, [5] considered density evolution for binary SW coding and non-symmetric channels. In [5], an equivalence between SW coding and channel coding under density evolution is derived.

For non-binary LDPC codes, the exact density evolution equations are only known for erasure channels [27]. Alternatively, approximation methods have been proposed, *e.g.*, the density evolution under Gaussian approximation, which can be applied for the AWGN channel model, for binary [8] as well as for non-binary LDPC codes [19]. Then, [1] considered density evolution for coset non-binary LDPC codes. In this case, the channels are not necessarily symmetric, because it is shown that the coset has a symmetrizing effect. As before, no analytic expression of the density evolution is given, except with the Gaussian approximation. Although SW codes can be seen as particular coset LDPC codes, [1] considers channel coding and consequently fixed input symbols distribution. To finish, [15] shows that, if the all-zero codeword assumption holds, density evolution in channel coding can be approximated through the use of Monte-Carlo methods (referred to as MC-DE).

III. Notations and preliminaries

In the following, upper case letters, *e.g.*, X, denote random variables whereas lower case letters, x, represent their realizations. Vectors, *e.g.*, $\mathbf{X} = \{X_k\}_{k=1}^n$, are in bold. When it is clear from the context that the distribution of a random variable X_k does not depend on k, the index k is omitted. The imaginary unit is denoted i. The Kronecker function is denoted $\delta(x)$, *i.e.*, $\delta(x) = 1$ if $x = 0$, $\delta(x) = 0$ otherwise. In the following, $\bigotimes$ stands for the convolution product (not to be confused with $\otimes$, the multiplicative operator in GF(q)) and $\circ$ is the composition operator. In SW coding (see Figure 1), the source X to be compressed and the SI Y available at the decoder produce sequences of independent and identically distributed (*i.i.d.*) discrete symbols $\{X_n\}_{n=1}^{+\infty}$ and $\{Y_n\}_{n=1}^{+\infty}$ respectively. The realizations of the random variable X belong to GF(q) with $q = \kappa^\alpha$ and κ is prime. The realizations of Y belong to a discrete alphabet $\mathcal{Y}$. Denote $P(X = x) = p_x$ where $0 < p_x < 1$ and assume $\forall(x, y)$, $0 < P(Y = y|X = x) < 1$.

A. *Operations in GF(q)*

In order to introduce some notations and conventions that will be used in the paper, we recall here some standard definitions related to Galois Fields. See [21, Chapter 4] for more details. Define $\mathbb{Z}_\kappa[D]$ as the set of polynomials with coefficients in $\mathbb{Z}/\kappa\mathbb{Z}$ and let $P(D) \in \mathrm{G}_\kappa[D]$ an irreducible polynomial of degree α. Define GF(q) $= \mathrm{G}_\kappa[D]/P(D)$. It follows that every element of GF(q) can be uniquely represented by a polynomial of degree less than α, *i.e*, $\forall P_a(D) \in$ GF(q),

$$P_a(D) = a_0 + a_1 D + \dots a_{\alpha-1} D^{\alpha-1}. \tag{1}$$

where $a_k \in \{0 \dots \kappa - 1\}$ As a consequence, one can define a one-to-one correspondence between $\{0, \dots, q-1\}$ and GF(q) by associating to each $P_a(D) \in$ GF(q) a value $a \in \{0, \dots, q-1\}$. Remarking that any $a \in \{0, \dots, q-1\}$ can be uniquely decomposed as

$$a = a_0 + a_1 \kappa + \dots a_{\alpha-1} \kappa^{\alpha-1}, \tag{2}$$

a is by convention associated to the polynomial $P_a(D)$ (1). In the following, $\oplus$, $\ominus$, $\otimes$, $\oslash$ are the usual operators in GF(q). By an abuse of notation, we will denote by a both its integer value and the corresponding element of GF(q). Thus, for any real or complex value x, x^a is evaluated from the integer version of a, but in the expression $x^{a\oplus b}$, $a \oplus b$ is performed in GF(q). Throughout the remaining of the paper, we denote by r the κ-th root of unity defined by $r = \exp(\frac{2\mathrm{i}\pi}{\kappa})$. With the above convention, one can show that $r^{a\oplus b} = r^a r^b$.

B. Probability evaluation in GF(q)

Let Z be a random variable with values in GF(q). Denote $\mathbf{p}$ the probability vector of size q with k-th component $p_k = P(Z = k)$ and $0 < p_k < 1$. Denote $\mathbf{m}$ the message vector of size q with k-th component $m_k = \log \frac{p_0}{p_k} = \log \frac{P(Z=0)}{P(Z=k)}$. From the previous expression, one has $p_k = \frac{e^{-m_k}}{\sum_{k'=0}^{q-1} e^{-m_{k'}}}$. As part of the LDPC decoder consists of the evaluation of the probability of linear combinations of random variables, we first express here the probabilities of $Z \oplus a$, $Z \otimes a$, where $a \in$ GF(q), and of $Z_1 \oplus Z_2$. Note that the operators we describe here to realize these evaluations were initially introduced in [1] and [19]. We restate them here to make the paper more self contained.

Denote $\mathbf{p}^{\times a}$ and $\mathbf{m}^{\times a}$ ($\forall a \in$ GF$(q)\backslash\{0\}$), $\mathbf{p}^{+a}$ and $\mathbf{m}^{+a}$ ($\forall a \in$ GF(q)) the probability and message vectors associated to $Z \otimes a$ and $Z \oplus a$. By definition, $\forall a \neq 0$, $p_k^{\times a} = P(Z \otimes a = k) = P(Z = k \oslash a)$ and

$$m_k^{\times a} = \log \frac{P(Z \otimes a = 0)}{P(Z \otimes a = k)} = \log \frac{P(Z = 0)}{P(Z = k \oslash a)} \ . \tag{3}$$

Let $W[a]$ be a $q \times q$ matrix such that $\forall k, j = 0, \ldots, q-1$, $W_{k,j}[a] = \delta(a \otimes j \ominus k)$. Then, $\mathbf{p}^{\times a} = W[a]\mathbf{p}$ and $\mathbf{m}^{\times a} = W[a]\mathbf{m}$. On the other hand, $p_k^{+a} = P(Z \oplus a = k) = P(Z = k \ominus a)$ and

$$m_k^{+a} = \log \frac{P(Z \oplus a = 0)}{P(Z \oplus a = k)} = \log \frac{P(Z = \ominus a)}{P(Z = k \ominus a)} \tag{4}$$

Denote $R[a]$ the $q \times q$ matrix such that $\forall k, j = 0, \ldots, q-1$, $R_{k,j}[a] = \delta(a \oplus k \ominus j)$. Denote $\mathcal{A}[a]$ the $q \times q$ matrix such that $\mathcal{A}_{0,0}[a] = 1$ and $\forall k, j = 0, \ldots, q-1$, $(k, j) \neq (0, 0)$, $\mathcal{A}_{k,j}[a] = \delta(a \oplus k \ominus j) - \delta(a \ominus j)$. Then, $\mathbf{p}^{+a} = R[a]\mathbf{p}$ and $\mathbf{m}^{+a} = \mathcal{A}[\ominus a]\mathbf{m}$. Here, two different transforms are needed because of the numerator in (4). The notations $\mathbf{m}^{\times a}$ and $\mathbf{m}^{+a}$ come from [1] while $W[a]$ and $\mathcal{A}[a]$ come from [19].

Now, let Z_1 and Z_2 be two random variables with realizations in GF(q) and probability vectors $\mathbf{p}_1$ and $\mathbf{p}_2$. Then,

$$P(Z_1 \oplus Z_2 = k) = \sum_{j=0}^{q-1} P(Z_1 = j) P(Z_1 \oplus Z_2 = k | Z_1 = j) = \sum_{j=0}^{q-1} p_{1,j} p_{2,k \ominus j} \tag{5}$$

$$:= (\mathbf{p}_1 \overline{\otimes} \mathbf{p}_2)_k \ . \tag{6}$$

The operator $\overline{\otimes}$ represents a discrete convolution product but *does not* correspond to the classical circular convolution product. Consequently, as pointed out in [16], the usual discrete Fourier Transform cannot be used for the evaluation of (5) and there is a need to define an adapted Fourier-like transform $\mathcal{F}$. Let $\mathbf{f} = \mathcal{F}(\mathbf{p})$ and $\mathbf{p} = \mathcal{F}^{-1}(\mathbf{f})$ with from [19],

$$f_j = \sum_{k=0}^{q-1} r^{k \otimes j} p_k \ , \quad p_k = \frac{1}{q} \sum_{j=0}^{q-1} r^{-k \otimes j} f_j \ . \tag{7}$$

Then

$$P(Z_1 \oplus Z_2 = k) = \left(\mathcal{F}^{-1} \left(\mathcal{F}(\mathbf{p}_1) \mathcal{F}(\mathbf{p}_2) \right) \right)_k \ . \tag{8}$$

This expression can easily be generalized to a sum of K elements. A message version of the Fourier-like transform can also be defined as $\mathbf{f} = \tilde{\mathcal{F}}(\mathbf{m})$ and $\mathbf{m} = \tilde{\mathcal{F}}^{-1}(\mathbf{f})$ with

$$f_j = \sum_{k=0}^{q-1} r^{k \otimes j} \frac{e^{-m_k}}{\sum_{k'=0}^{q-1} e^{-m_{k'}}} \ , \quad m_k = \log \frac{\sum_{j=0}^{q-1} f_j}{\sum_{j=0}^{q-1} r^{-k \otimes j} f_j} \ . \tag{9}$$

Note that if q is a power of 2, then $\mathcal{F}$ becomes the Hadamard transform [13].

IV. LDPC ENCODING AND DECODING

LDPC codes initially introduced for channel coding can also be used for SW coding, after adaptation of the coding process and the decoding algorithm [20], [22]. The SW coding of a source vector $\mathbf{x}$ of length n is performed by producing a vector $\mathbf{s} = H^T \mathbf{x}$ of length $m < n$. The matrix H is sparse, with coefficients in GF(q). In the bipartite graph representing the dependencies between the random variables of $\mathbf{X}$ and $\mathbf{S}$, the entries of $\mathbf{X}$ are represented by Variable Nodes (VN) and the entries of $\mathbf{S}$ are represented by Check Nodes (CN). The set of CN connected to a VN n is denoted $\mathcal{N}_{\mathrm{C}}(n)$ and the set of

VN connected to a CN m is denoted $\mathcal{N}_{\mathrm{V}}(m)$. The sparsity of H is determined by the edge-perspective VN degree distribution $\lambda(x)$ and CN degree distribution $\rho(x)$, where

$$\lambda(x) = \sum_{k\geq 2} \lambda_k x^{k-1}, \quad \rho(x) = \sum_{j\geq 2} \rho_j x^{j-1} \tag{10}$$

The constant $0 \leq \lambda_k \leq 1$ is the fraction of edges emanating from a VN of degree k and $0 \leq \rho_j \leq 1$ is the fraction of edges emanating from a CN of degree j. In SW coding, the coding efficiency $r(\lambda,\rho)$ of a code is given by $r(\lambda,\rho) = \frac{m}{n} = \frac{\sum_{j\geq 2}\rho_j/j}{\sum_{k\geq 2}\lambda_k/k}$. A code is said to be regular if the VN and CN have constant degrees d_v and d_c. In this case, $r(d_v,d_c) = \frac{d_v}{d_c}$.

The sum-product LDPC decoder performs an approximate Maximum *A Posteriori* (MAP) estimation of $\mathbf{x}$ from the received codeword $\mathbf{s}$ and the observed side information $\mathbf{y}$ by the mean of message exchange in the bipartite graph. In non-binary channel coding, the sum-product LDPC decoder is described in [19]. We expressed the SW version of the algorithm in [14] and restate it here for the sake of completeness. The initial message for a VN n is denoted $\mathbf{m}^{(0)}(n)$, and its k-th component is

$$m_k^{(0)}(n) = \log \frac{P(X_n = 0|Y_n = y_n)}{P(X_n = k|Y_n = y_n)}, \ k = 0 \ldots q-1 \ . \tag{11}$$

Note that, here, the messages are expressed as vectors of log-likelihood ratios (LLR). Although exchanged messages may alternatively be represented as vector of probabilities [37], it is more convenient for our purpose to assume that they are represented as vectors of LLR values. At iteration ℓ, the message $\mathbf{m}^{(\ell)}(m,n)$ from CN m to VN n is

$$\mathbf{m}^{(\ell)}(m,n) = \mathcal{A}[\overline{s}_m]\tilde{\mathcal{F}}^{-1}\left(\prod_{n'\in\mathcal{N}_{\mathrm{V}}(m)\backslash n} \tilde{\mathcal{F}}\left(W\left[\overline{g}_{n'm}\right]\mathbf{m}^{(\ell-1)}(n',m)\right)\right) \tag{12}$$

where the product is componentwise, $\bar{s}_m = \ominus s_m \oslash H_{n,m}$, and $\overline{g}_{n'm} = \ominus H_{n',m} \oslash H_{n,m}$. Note that $\mathcal{A}[\overline{s}_m]$ does not appear in the channel coding version of the algorithm and is specific to SW coding (since in channel coding the syndrome is zero). At a VN n, a message $\mathbf{m}^{(\ell)}(n,m)$ is sent to the CN m and an *a posteriori* message $\tilde{\mathbf{m}}^{(\ell)}(n)$ is computed. They both satisfy

$$\mathbf{m}^{(\ell)}(n,m) = \mathbf{m}^{(0)}(n) + \sum_{m' \in \mathcal{N}_{\mathrm{C}}(n) \backslash m} \mathbf{m}^{(\ell)}(m',n) \ , \tag{13}$$

$$\tilde{\mathbf{m}}^{(\ell)}(n) = \mathbf{m}^{(0)}(n) + \sum_{m' \in \mathcal{N}_{\mathrm{C}}(n)} \mathbf{m}^{(\ell)}(m',n) \ . \tag{14}$$

The channel version of the algorithm has the same VN message computation. From (14), each VN n produces an estimate $\widehat{x}_n^{(\ell)} = \arg\min_k \tilde{m}_k^{(\ell)}(n)$ of x_n. The algorithm ends if $H^{\mathrm{T}}\widehat{\mathbf{x}}^{(\ell)} = \mathbf{s}$ or if $\ell = L_{\max}$, the maximum number of iterations.

The CN message (12) is calculated from linear operators and a componentwise product. Since the probability density of these products may be difficult to derive, we introduce the following transform $\boldsymbol{\gamma}$. The function $\boldsymbol{\gamma}$ applies on vectors of size q and has $k-$th component $\gamma_k : \mathbb{C} \to \mathbb{R} \times [-\pi, \pi]$ with

$$\gamma_k(x_k + \mathrm{i} y_k) = (z_k, t_k) = \begin{cases} \left(\frac{1}{2}\log(x_k^2 + y_k^2), \arctan\frac{y_k}{x_k}\right) & \text{if } x_k \geq 0, y_k \neq 0 \\ \left(\frac{1}{2}\log(x_k^2 + y_k^2), \arctan\frac{y_k}{x_k} + \pi\right) & \text{if } x_k \leq 0 \ , \ y_k \geq 0 \\ \left(\frac{1}{2}\log(x_k^2 + y_k^2), \arctan\frac{y_k}{x_k} - \pi\right) & \text{if } x_k \leq 0 \ , \ y_k < 0 \ . \end{cases} \tag{15}$$

where x_k and y_k are real numbers. Note that there is an overlap between the first two lines of the definition of γ_k when $x_k = 0$ and $y_k \neq 0$. However, both lines lead to $\gamma_k(x_k + \mathrm{i}y_k) = (\frac{1}{2}\log(y_k^2), \frac{\pi}{2})$ when $x_k = 0$ and $y_k \neq 0$. Also note that γ_k can also be seen as a function from $\mathbb{R}^2$ to $\mathbb{R} \times [-\pi, \pi]$. We complete the definition of γ_k by assuming that when $x_k + \mathrm{i}y_k = 0$, the value of t_k is given by the realization of a random variable Θ taking its values in $[0, 2\pi]$ and with probability density function $f_\Theta(\theta) = \frac{1}{2\pi}$. The inverse function $\boldsymbol{\gamma}^{-1}$ applies on vectors of size q and has $j-$th component $\gamma_j^{-1} : \mathbb{R} \times [-\pi, \pi] \to \mathbb{C}$ with

$$\gamma_j^{-1}(z_j, t_j) = \exp(z_j)\cos t_j + \mathrm{i}\exp(z_j)\sin t_j \ . \tag{16}$$

The CN to VN equation (12) can then be restated as

$$\mathbf{m}^{(\ell)}(m,n) = \mathcal{A}[\overline{s}_m]\tilde{\mathcal{F}}^{-1}\left(\boldsymbol{\gamma}^{-1}\left(\sum_{n' \in \mathcal{N}(m) \backslash n} \boldsymbol{\gamma}\left(\tilde{\mathcal{F}}\left(W\left[\overline{g}_{n'm}\right]\mathbf{m}^{(\ell-1)}(n',m)\right)\right)\right)\right) \ . \tag{17}$$

Density evolution consists in the evaluation of the probability densities of the messages at each iteration. The decoding error probability can then be calculated from the probability of the messages giving false estimates $\hat{x}_k$. In this way, the probability densities of $\mathbf{m}^{(\ell)}(n,m)$ and $\tilde{\mathbf{m}}^{(\ell)}(n)$ in (13), (14), are easy to evaluate from the probability densities of the $\mathbf{m}^{(\ell)}(m',n)$ (assuming the $\mathbf{m}^{(\ell)}(m',n)$ are realizations of independent random variables). On the opposite, the probability density of $\mathbf{m}^{(\ell)}(m,n)$ in (12) is difficult to derive because of the componentwise product. That is why we introduced the function γ that transforms the product in (12) into a sum.

V. Density evolution

This section evaluates the probability densities of the messages in channel coding and in SW coding. The messages exchanged in the graph during the decoding can be seen as random variables. From the density of the initial messages (11), we want to calculate recursively the probability density of the messages at iteration ℓ, exploiting (13) and (17). For this, several simplifying assumptions can be performed. First, it is assumed that the messages arriving at a node at iteration ℓ are independent. The so-called independence assumption was originally discussed in [26] and proved formally to be reasonable in [34]. The main idea is that the messages are independent if they have been calculated on independent subtrees of the bipartite graph. It is called the cycle-free case. In [34], it is shown that this cycle-free case happens with probability arbitrarily closed to 1 when $n \to \infty$.

The second simplifying assumption is called the all-zero codeword assumption. In channel coding, the all-zero codeword assumption applies if both the decoder and the channel are symmetric. The BP decoder described in Section IV fullfils the symmetry conditions of [25] in the case of channel coding and of SW coding. However, in SW coding, the correlation channel may not be symmetric. Thus, before explaining the all-zero codeword assumption, we restate the definition of a symmetric channel.

Definition 1. [19] *Let $P(W|U)$ be a channel with q-ary input U and arbitrary output W. Denote $\mathcal{I}[a]$ the $(q-1)\times(q-1)$ diagonal matrix with $\mathcal{I}[a]_{i,i} = r^{i\otimes a}$, $i = 1,\ldots,(q-1)$. The channel $P(W|U)$ is said to be q-*ary input symmetric-output *if the possible values of W can be relabeled into length $(q-1)$*

complex-valued vectors $\tilde{\mathbf{W}}$ *such that*

$$\forall a \in \{0 \ldots (q-1)\}, P(\tilde{\mathbf{W}} = \tilde{\mathbf{w}}|U = a) = P(\tilde{\mathbf{W}} = \mathcal{I}[a]\tilde{\mathbf{w}}|U = 0) \ . \tag{18}$$

As this definition is not intuitive, we also derive the following equivalent definition when both U and W take their values in GF(q).

Proposition 1. *Let* U *and* W *be two random variables taking their values in GF(q). Then the channel* $P(W|U)$ *is symmetric if and only if there exists a bijective function* h : *GF*$(q) \rightarrow$ *GF*(q) *such that*

$$P(W = w|U = u) = P(W = h^{-1}(h(w) \oplus u)|U = 0). \tag{19}$$

For the proof, see Appendix A.

As a consequence, from (19), at most q parameters are needed to describe the channel. These parameters correspond to the transition probabilities for the input $U = 0$. Then, the transition probabilities for any other input $U = i$ are simply the permuted transition probabilities for $U = 0$. The permutation is defined by the function h. In channel coding, [19, Proposition 2] shows that for symmetric channels, the error probability of the decoding algorithm is independent of the transmitted codeword. Consequently, the recursion on the probability density is calculated assuming the all-zero codeword was transmitted. In SW coding, this result applies only if X is distributed uniformly and $P(Y|X)$ is symmetric In this case, density evolution for channel coding can be performed directly with $P(Y|X)$.

In the following, we first express recursions on the probability densities of the messages in the case of channel coding for symmetric channels. Then, we express the recursion for SW coding, for any channel.

A. Density evolution in channel coding for symmetric channels

In the case of a symmetric channel, the probability densities of the messages exchanged in the graph do not depend on the transmitted codeword [19]. Consequently, we assume that the all-zero codeword was transmitted and express the density evolution with this assumption. First, denote $\tilde{P}^{(\ell)}$ the probability density of the *a posteriori* messages (14) at iteration ℓ under the all-zero codeword assumption. It is shown in [19] that the error probability of the sum-product LDPC decoder at iteration ℓ can be calculated as

$$p_e^{(\ell)} = 1 - \int_{\mathbf{m}\in\mathbb{R}_+^q} \tilde{P}^{(\ell)}(\mathbf{m})d\mathbf{m} \tag{20}$$

where $\mathbb{R}_+^q$ is the set of length q real-valued vectors with positive components only. It thus suffices to express $\tilde{P}^{(\ell)}$ at each iteration to obtain the error probability. For the purpose of the paper, we need an analytical form of DE for non-binary channel coding. As [18] (and any other paper, to the best of our knowledge) does not provide such an analytical form, we state it in the following proposition.

Proposition 2. *Consider a q-ary input symmetric-output channel $P(W|U)$, a code ensemble $\mathcal{C}(\lambda,\rho)$, and sum-product LDPC decoding for channel coding. Assume that the decoding graph is cycle-free and that the all-zero codeword is transmitted. At iteration ℓ, denote $P^{(\ell)}$ the probability density of the messages from VN to CN, $Q^{(\ell)}$ the probability density of the messages from CN to VN, and $\tilde{P}^{(\ell)}$ the probability density of the* a posteriori *messages. Then*

$$Q^{(\ell)}(\mathbf{m}) = \Gamma_d^{-1}\left(\frac{1}{q-1}\sum_{g=1}^{q}\rho\left(\Gamma_c^g(P^{(\ell-1)})\right)\right)(\mathbf{m}) \tag{21}$$

$$P^{(\ell)}(\mathbf{m}) = P^{(0)}\bigotimes\lambda\left(Q^{(\ell)}\right)(\mathbf{m}) \tag{22}$$

where Γ_d^{-1} and Γ_c^h are density transform operators defined in Appendix B. Consequently,

$$P^{(\ell)}(\mathbf{m}) = P^{(0)}\bigotimes\lambda\left(\Gamma_d^{-1}\left(\frac{1}{q-1}\sum_{g=1}^{q}\rho\left(\Gamma_c^g\left(P^{(\ell-1)}\right)\right)\right)\right)(\mathbf{m}) \tag{23}$$

$$\tilde{P}^{(\ell)}(\mathbf{m}) = P^{(0)}\bigotimes\tilde{\lambda}\left(\Gamma_d^{-1}\left(\frac{1}{q-1}\sum_{g=1}^{q}\rho\left(\Gamma_c^g\left(P^{(\ell-1)}\right)\right)\right)\right)(\mathbf{m}) \tag{24}$$

where $\tilde{\lambda}(x) = \sum_{k\geq 2}\tilde{\lambda}_k x^k$.

Proof. The channel coding version of the message computation from VN to CN is given by (13). Consequently, (21) is obtained directly from (13) (sum of *i.i.d.* random variables of probability distribution $P^{(\ell-1)}$ and marginalization according to the VN degree distribution). The channel version of the message computation from CN to VN is given removing $\mathcal{A}[\overline{s}_m]$ in (17). Denote $\bar{G}$ a random variables taking its values in GF(q). For any message $\mathbf{m}$, the density $\Gamma_W^{\bar{G}}$ of $W[\bar{G}]\mathbf{m}$ can be obtained by marginalizing with respect to $\bar{G}$. From the density transform operator obtained in Appendix B1, it is

$$\Gamma_W^{\bar{G}}(\mathbf{m}) = \frac{1}{q-1}\sum_{\bar{g}=1}^{q-1}\Gamma_W^{\bar{g}}(P^{(\ell-1)})(\mathbf{m}). \tag{25}$$

Furthermore, denote $\Gamma_{\mathbf{m}}$, $\Gamma_{\mathcal{F}}$, $\Gamma_{\boldsymbol{\gamma}}$ the density transform operators obtained respectively for the transform of $\mathbf{m}$ into $\mathbf{p}$ (see Appendix B2), for the Fourier Transform (Appendix B3), and for $\boldsymbol{\gamma}$ (Appendix B4) and denote $\Gamma_c^{\bar{g}} = \Gamma_{\boldsymbol{\gamma}}\Gamma_{\mathcal{F}}\Gamma_{\mathbf{m}}\Gamma_W^{\bar{g}}$. The density $\Gamma_{\gamma}^{\bar{G}}$ of $\boldsymbol{\gamma}\left(\tilde{\mathcal{F}}\left(W\left[\overline{G}\right]\mathbf{m}\right)\right)$ is given by

$$\Gamma_{\gamma}^{\bar{G}}(\mathbf{m}) = \frac{1}{q-1}\sum_{\bar{g}=1}^{q-1}\Gamma_c^{\bar{g}}(P^{(\ell-1)})(\mathbf{m}) \tag{26}$$

by the linearity of the density transform operators. To finish, from the density transform operators $\Gamma_{\mathbf{p}}$, $\Gamma_{\mathcal{F}^{-1}}$, $\Gamma_{\boldsymbol{\gamma}^{-1}}$ obtained respectively for the transformation of $\mathbf{p}$ into $\mathbf{m}$ (see Appendix B2), for the inverse Fourier Transform (see Appendix B3), and for $\boldsymbol{\gamma}^{-1}$ (see Appendix B4), we get (22) where $\Gamma_d^{-1} = \Gamma_{\mathbf{p}}\Gamma_{\boldsymbol{\gamma}^{-1}}\Gamma_{\mathcal{F}^{-1}}$. Finally combining (21) and (22) gives (23). To finish, (24) directly derives from (23). $\square$

The initial $P^{(0)}$ is obtained by evaluating the probability density of (11) conditioned on the fact that $U = 0$. Note that (23) is not convenient for practical density evolution (see the expressions of the operators in Appendix B). The objective here is only to express a recursion in order to show that a similar form is obtained in SW coding.

B. Density evolution in SW coding

In SW coding, the all-zero codeword transmission cannot be assumed anymore, even if the correlation channel $P(Y|X)$ is itself symmetric, because of the source distribution. Denote respectively $P_k^{(\ell)}$, $Q_k^{(\ell)}$, and $\tilde{P}_k^{(\ell)}$ the probability densities of the messages from VN to CN, from CN to VN, and of the *a posteriori* messages conditioned on the fact that $X = k$. Note that $P_k^{(\ell)}$, $Q_k^{(\ell)}$, and $\tilde{P}_k^{(\ell)}$ are probability densities conditioned on the fact that $X = k$ but *do not* correspond to an all-k codeword assumption. In fact, *e.g.*, $P_k^{(\ell)}$ can be expressed by marginalizing according to the node neighbor values and thus depend on all the $P_j^{(\ell-1)}$, $j = 0, \ldots, (q-1)$. The following proposition gives the expression of the error probability of the sum-product LDPC decoder in case of SW coding.

Proposition 3. *Consider a joint distribution $P(X, Y)$, where X and Y take their values in GF(q) and $\mathcal{Y}$ respectively, a code ensemble $\mathcal{C}(\lambda, \rho)$, and sum-product LDPC decoding. Let $\tilde{P}_k^{(\ell)}$ be the probability density of the* a posteriori *messages conditioned on the fact that $X = k$ and define*

$$\langle \tilde{P}^{(\ell)} \rangle(\mathbf{m}) = \sum_{k=0}^{q-1} P(X = k) \tilde{P}_k^{(\ell)} \circ \mathcal{A}[\ominus k](\mathbf{m}) \tag{27}$$

Then, in SW coding, the error probability of the LDPC decoder at iteration ℓ is given by

$$p_e^{(\ell)} = 1 - \int_{\mathbf{m} \in \mathbb{R}_+^q} \langle \tilde{P}^{(\ell)} \rangle(\mathbf{m}) d\mathbf{m}. \tag{28}$$

See Appendix C1 for the proof.

Proposition 3 can be interpreted as follows. For a randomly selected variable node of the bipartite graph (see Section IV), $p_e^{(\ell)}$, the probability of error at iteration ℓ, is the probability for an *a posteriori* message to produce a false estimate of the symbol value at the variable node. For example, in the binary case, if $X = 0$ but the scalar message $m^{(\ell)} < 0$, a false estimate of X is produced. Consequently, in the non-binary case, the error probability can be obtained by marginalizing according to $k = 0, \ldots, (q-1)$ and, for each k, by integrating $\tilde{P}_k^{(\ell)}$ over the set of messages producing an error. For $X = k$, this corresponds to the set of messages $\mathbf{m}$ such that there exists $i \neq k$ such that $m_i < m_k$. The marginalization operation appears in (27). Moreover, the operators $\mathcal{A}[\ominus k]$ realize the projection of the space $\mathbb{R}_-^q$ on the set of messages producing an error, thus giving (28).

The following proposition gives the expression of $\langle \tilde{P}^{(\ell)} \rangle$ obtained in SW coding.

Proposition 4. *Consider a joint distribution $P(X, Y)$, where X and Y takes their values in GF(q) and $\mathcal{Y}$ respectively, a code ensemble $\mathcal{C}(\lambda, \rho)$, and sum-product LDPC decoding. Assume that the decoding graph is cycle-free. Denote $P_k^{(\ell)}$ and $\tilde{P}_k^{(\ell)}$ the respective probability densities of the messages from VN to CN and of the* a posteriori *messages at iteration ℓ conditioned on the fact that $X = k$. Denote also*

$\langle P^{(\ell)}\rangle(\mathbf{m}) = \sum_{k=0}^{q-1} P(X=k) P_k^{(\ell)} \circ \mathcal{A}[\ominus k](\mathbf{m})$ and $\langle \tilde{P}^{(\ell)}\rangle(\mathbf{m}) = \sum_{k=0}^{q-1} P(X=k) \tilde{P}_k^{(\ell)} \circ \mathcal{A}[\ominus k](\mathbf{m})$. In SW coding, the following expressions holds

$$\langle P^{(\ell)}\rangle(\mathbf{m}) = \langle P^{(0)}\rangle \bigotimes \lambda\left(\Gamma_d^{-1}\left(\frac{1}{q-1}\sum_{g=1}^{q} \rho\left(\Gamma_c^g\left(\langle P^{(\ell-1)}\rangle\right)\right)\right)\right)(\mathbf{m}) \tag{29}$$

$$\langle \tilde{P}^{(\ell)}\rangle(\mathbf{m}) = \langle P^{(0)}\rangle \bigotimes \tilde{\lambda}\left(\Gamma_d^{-1}\left(\frac{1}{q-1}\sum_{g=1}^{q} \rho\left(\Gamma_c^g\left(\langle P^{(\ell-1)}\rangle\right)\right)\right)\right)(\mathbf{m}) \tag{30}$$

where Γ_d^{-1} and Γ_c^g are density transform operators defined in Appendix B and $\tilde{\lambda}(x) = \sum_{k\geq 2} \tilde{\lambda}_k x^k$.

See Appendix C for the proof. The initial density is given by

$$\langle P^{(0)}\rangle = \sum_{k=0}^{q-1} P(X=k) P_k^{(0)} \circ \mathcal{A}[\ominus k](\mathbf{m}) \tag{31}$$

where $P_k^{(0)}$ is calculated $\forall k = 0, \dots q-1$ from the expression of the initial messages (11).

We see that the recursion in SW coding is exactly that obtained in channel coding, except that it now applies on $\langle P^{(\ell)}\rangle$. Consequently, the only difference is on the initial $\langle P^{(0)}\rangle$ which, as expected, takes into account the probability distribution of X. Consequently, we see that if two joint probability distributions $P(X,Y)$ and $P(U,W)$ have the same initial probability densities respectively $\langle P^{(0)}\rangle$ and $P^{(0)}$, *i.e.*, $\langle P^{(0)}\rangle = P^{(0)}$, then they have the same density evolution equations. The result of [4] on the equivalence between channel coding and SW coding can be restated from this remark.

From the DE equations, we now explain how to optimize the code degree distributions.

VI. Asymptotic analysis

In this section, we perform code degree optimization from the DE recursion. We consider two particular correlation channels $P(Y|X)$ and various input probability distributions $P(X)$. One of the considered correlation channels is symmetric, while the other is not. For each of the considered source models, we perform code degree distribution optimization based on density evolution for the equivalent channel, using a differential evolution algorithm [31].

The results of Proposition 4 show that the probability distributions of the messages can be obtained recursively. However, no convenient closed-form expression of the density evolution is known for this model. Thus, here, an approximate $P_e^{(\ell)}(\lambda, \rho)$ will be obtained from an MCMC-based density evolution method called MC-DE [15]. From this, and assuming that the distribution of X is fixed, we get an approximate *threshold* of the code, that is the largest parameter p for which $P_e^{(\ell)}(\lambda, \rho)$ goes to 0 when ℓ goes to infinity.

Now, we want to fix the rate r of the code, and find degree distributions $(\lambda(x), \rho(x))$ of rate r that maximizes the threshold. This optimization can be realized using a genetic algorithm called differential evolution [31]. Here, the code degree optimization will be on the VN degree distribution $\lambda(x)$ only. The CN degree distribution $\rho(x)$ can then be calculated from $\lambda(x)$ and r.

In the following optimization runs, we always perform MC-DE on 1000 samples and 100 iterations. This parameters are shown in [15] to be sufficient to obtain good error probability approximations. For the differential evolution, we consider populations of size 500, with 100 iterations, a crossover probability of 1, and a mutation factor of 0.85 (see [31]). The optimization is then performed for a given maximum VN degree value. For each considered setup and maximum VN degree, the following tables give the best obtained threshold p and the corresponding entropy $H(p) = H(X|Y)$ [1]. The obtained threshold values are also compared to the threshold for a regular code. In the following, $\overline{p}$ denotes the approximate maximum parameter that can be coded with a code of rate r (*i.e.* for which $H(X|Y) \leq r$). The following setups are considered.

A. Symmetric Correlation Channel

We first consider a symmetric correlation channel. Consider a source X taking its values in GF(q) and such that $P(X = x) = p_x$. Here, the correlation channel between X and Y is described by a q-ary

[1] The optimized degree distributions leading to these threshold values are available online at http://www.elsa-dupraz.fr/documents/degrees.dat

symmetric channel in GF(q) with

$$P(Y = x|X = x) = 1 - p \tag{32}$$

$$\forall y \neq x, \; P(Y = y|X = x) = \frac{p}{q-1}$$

where $0 < p < 1$. For given source parameters p_x and p, density evolution gives the error probability $P_e^{(\ell)}(\lambda, \rho)$ of an LDPC code of degree distributions $(\lambda(x), \rho(x))$. We now consider particular choices of source distributions and give optimization results in the considered cases.

a) GF(4), $X \sim [0.25, 0.25, 0.25, 0.25]$, $r = 3/4$, $\overline{p} = 0.355$: In this case, the input probability distribution is uniform and density evolution is the same for channel coding and SW coding. The following results are obtained.

Max VN deg.	7	10	15	Reg $(3,4)$
p	0.340	0.346	0.347	0.278
$H(p)$	0.731	0.739	0.740	0.647

We see that code degree optimization enables to obtain codes with higher threshold values. Also, when the maximum possible variable node degree is increased, the threshold value is also increased. This result is expected, because increasing the number of variable node degrees increases the number of degrees of freedom for the optimization. Comparison with binary case

b) GF(4), $X \sim [0.5, 0.25, 0.125, 0.125]$, $r = 1/2$, $\overline{p} = 0.225$: Now, the input probability distribution is not uniform anymore and density evolution for SW coding differs from density evolution for channel coding.

Max VN deg.	7	10	15	Reg $(3,6)$
p	0.214	0.220	0.221	0.175
$H(p)$	0.483	0.492	0.494	0.421

The same conclusions are obtained.

c) GF(16), $X \sim [0.4, 0.04, \dots, 0.04]$, $r = 1/2$, $\overline{p} = 0.367$: Here, the input probability distribution is not uniform, and we consider a bigger Galois field.

Max VN deg.	10	15	21	Reg $(3,6)$
p	0.321	0.325	0.325	0.294
$H(p)$	0.454	0.458	0.458	0.426

In all cases, increasing the maximum VN degree enables to increase the performance of the code. Moreover, the obtained codes perform much better than the regular code.

B. Non-Symmetric Correlation Channel

We now consider a correlation channel that is no more symmetric. The correlation channel between X and Y is now described by

$$\begin{aligned} P(Y=0|X=1) &= 1-p, & \forall y \neq 0,\ P(Y=y|X=1) &= \frac{p}{q-1} \\ \forall x \neq 1,\ P(Y=x|X=x) &= 1-p, & \forall x \neq 1, \forall y \neq x,\ P(Y=y|X=x) &= \frac{p}{q-1} \end{aligned} \tag{33}$$

where $0 < p < 1$. The optimization process is the same as before.

d) GF(4), $X \sim [0.25, 0.25, 0.25, 0.25]$, $r = 1/2$, $\overline{p} = 0.114$:

Max VN deg.	7	10	15	Reg $(3,6)$
p	0.089	0.094	0.097	0.091
$H(p)$	0.456	0.465	0.470	0.460

e) GF(4), $X \sim [0.5, 0.25, 0.125, 0.125]$, $r = 3/4$, $\overline{p} = 0.360$:

Max VN deg.	7	10	15	Reg $(3,6)$
p	0.306	0.316	0.317	0.257
$H(p)$	0.714	0.721	0.722	0.677

f) GF(16), $X \sim [0.4, 0.04, \ldots, 0.04]$, $r = 1/2$, $\overline{p} = 0.367$:

Max VN deg.	10	15	20	Reg $(3,6)$
p	0.341	0.345	0.346	0.281
$H(p)$	0.494	0.498	0.499	0.436

We get the same conclusions as for the symmetric case.

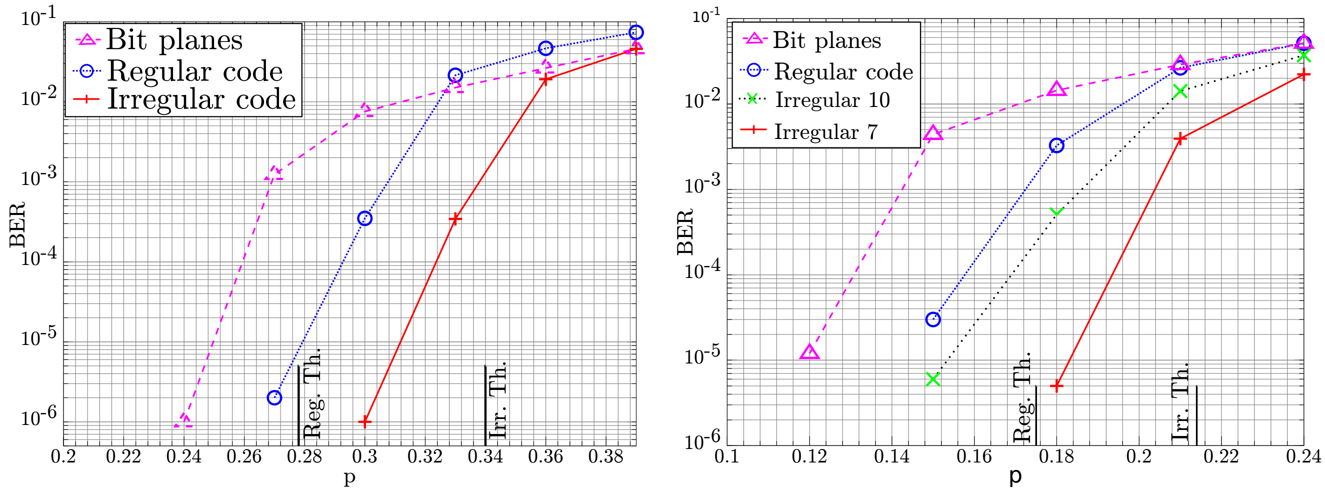
Now that optimized degree distributions are obtained, the finite-length code construction can be performed with an LDPC PEG (Progressive Edge Growth) algorithm [17]. Once the code is constructed, one has to deal with potentially harmful local structures (mainly short cycles) in order to obtain low error floors [23]. However, as illustrated in the following section, degree distribution optimization with density evolution can be seen as a good departure point at the code design process.

VII. Finite-Length Results

In this section, we analyze the performance of finite-length LDPC codes constructed from regular and optimized irregular degree distributions. The finite-length construction is performed with an LDPC PEG (Progressive Edge Growth) algorithm [17]. We consider a codeword length $N = 10000$, 50 decoding iterations, and source symbols in GF(4). Two setups are evaluated.

A. q-ary Symmetric Channel with Uniform Source Distribution

We consider the case of the q-ary symmetric channel with uniform source distribution. For performance comparison, three codes are constructed. The first one is the regular $(3,4)$-code with threshold value $\bar{p} = 0.278$. The second one is the optimized irregular code with maximum VN degree 7 obtained in the previous section. It has threshold value $\bar{p} = 0.340$. In order to evaluate the gain at considering non-binary LDPC codes, we also construct a $(3,4)$ binary LDPC code. It will be applied on the bit planes obtained from the non-binary symbols.



(a) (b)

Fig. 2. BER with respect to correlation channel parameter p, (a) Uniform Source distribution, regular $(3, 4)$-code and optimized irregular code with maximum VN degree 7 (b) Non-Uniform source distribution, regular $(3, 6)$-code and optimized irregular codes with maximum VN degrees 7 and 10

Fig. 2 (a) represents the obtained Bit Error Rates (BER) with respect to the correlation channel parameter p. First, we see that the BER performance of the regular code is well predicted by the threshold value given by density evolution. On the other hand, there is a gap between the BER performance of the irregular code and the threshold value for the irregular code. The gap comes from the finite-length construction. In fact, the girth of the code constructed from the LDPC PEG algorithm is 12 for the regular code and 10 for the irregular code, which penalizes the irregular code. The girth difference is due to higher degrees in the irregular code. Moreover, at finite length, the decimal coefficients of the degree distribution are in fact truncated which may result in a performance loss compared to the threshold value. However, despite the loss due to finite-length construction, we see that there is a clear performance gain at considering optimized irregular codes.

To finish, we see that the binary regular code performs poorly compared to the non-binary regular code. This shows the gain at considering non-binary symbols instead of bit planes.

B. q-ary Symmetric Channel with Non-Uniform Source Distribution

We now consider the q-ary symmetric channel with source distribution $X \sim [0.5, 0.25, 0.125, 0.125]$. Here, four codes are constructed. The first code is the regular $(3, 6)$-code with threshold value $\bar{p} = 0.175$. The second and third codes are the irregular codes optimized with maximum VN degrees 7 and 10, respectively. They have threshold values $\bar{p} = 0.214$ and $\bar{p} = 0.220$, respectively. The last code is the regular $(3, 6)$-code for binary symbols. It is applied on bit planes obtained from the non-binary symbols.

Fig 2 (b) gives the BERs with respect to p. We obtain the same results as before on the gap between the threshold value and the BER performance for regular and irregular codes. We also observe that the optimized irregular code of maximum VN degree 10 performs worst than the irregular code of maximum VN degree 7. As before, this is due to finite-length construction which penalyzes the code with higher degree. We also observe an important loss at considering bit plane coding instead of non-binary LDPC coding.

To conclude, the simulations illustrate the gain at finite-length at considering optimized irregular code degrees. They show that the gap between the threshold value and the BER performance is higher for irregular codes than for regular codes. As a consequence, there is some space to improve the BER performance of irregular code at finite-length. The simulation results also show the BER gain at considering non-binary codes instead of binary codes applied on bit-planes.

VIII. Conclusion

In this paper, we derived the Density Evolution recursion for non-binary LDPC codes for SW coding. From this recursion, we performed code degree optimization from the differential evolution algorithm. Asymptotic analysis and finite-length simulations illustrated the performance gain at considering optimized degree distributions. Future work will be related to the finite-length code design and to the extension to the non-symmetric SW coding setup.

APPENDIX

A. Symmetry

First, from Definition 1, to each value $w \in$ GF(q), one has to associate a vector $\tilde{\mathbf{w}}(w) \in \mathbb{C}^{q-1}$. Denote $\tilde{\Omega} = \{\tilde{\mathbf{w}}(0), \ldots, \tilde{\mathbf{w}}(q-1)\}$.

From (18),

$$\text{if } \tilde{\mathbf{w}} \in \tilde{\Omega}, \text{ then } \forall u = 0, \ldots q-1, I[u]\tilde{\mathbf{w}} \in \tilde{\Omega}. \tag{34}$$

Consequently, from the expressions of $I[u]$ and r, every non-zero component of $\tilde{\mathbf{w}}$ can take at least κ different values. On the other side, from (18),

$$\forall \tilde{\mathbf{w}} \in \tilde{\Omega}, \{I[u]\tilde{\mathbf{w}}\}_{u=0,\ldots,q-1} = \tilde{\Omega}. \tag{35}$$

Consequently, each non-zero component of $\tilde{\mathbf{w}}$ can take at most κ different values. Thus each non-zero component of $\tilde{\mathbf{w}}$ takes exactly κ different values and any vector $\tilde{\mathbf{w}}$ has exactly α non-zero independent components. We restrict the analysis to these α components of interest and assume without loss of generality that the other components are always equal to 0.

From the previous restriction, we now assume that $\tilde{\mathbf{w}}(w) \in \mathbb{C}^{\alpha}$ and denote

$$\forall k = 1 \ldots \alpha, \tilde{w}_k(w) = a_k(w) \exp(\mathrm{i} b_k(w)) \tag{36}$$

where $a_k(w), b_k(w) \in \mathbb{R}$. From (34) and (35), $a_k(w)$ does not depend on w. Consequently, without loss of generality, we take $\forall w \in$ GF(q), $\forall k = 1, \ldots, \alpha$, $a_k(w) = 1$. In the same way, we show that the $b_k(w)$ can be decomposed into

$$b_k(w) = c_k + \frac{2\pi}{\kappa} d_k(w) \tag{37}$$

where $c_k \in \mathbb{R}$ and $d_k(w) \in \{0, \ldots, \kappa-1\}$. As before, without loss of generality, we denote $c_k = 0$, $\forall k = 1, \ldots, \alpha$. Finally, one has $\tilde{w}_k = \exp\left(\mathrm{i}\frac{2\pi}{\kappa} d_k(w)\right)$.

Define

$$\mathbf{d} : \text{GF}(q) \to \{0, \dots, \kappa - 1\}^\alpha$$
$$w \mapsto (d_1(w), \dots, d_\alpha(w)). \tag{38}$$

$\mathbf{d}$ is necessarily bijective because every value of GF(q) has to be represented differently. Consequently, there exists a function $\mathbf{d}^{-1} : \{0, \dots, \kappa - 1\}^\alpha \to \text{GF}(q)$. Then

$$\left(I[u]\tilde{w}(w)\right)_k = r^{\mathrm{i}\otimes u} \exp\left(\mathrm{i}\frac{2\pi}{\kappa} d_k(w)\right) = \exp\left(\mathrm{i}\frac{2\pi}{\kappa}\left(d_k(w) \oplus k \otimes u\right)\right) \tag{39}$$

and from (18),

$$P(W = w|U = u) = P\left(W = \mathbf{d}^{-1}\left(\mathbf{d}(w) \oplus [1, \dots, \alpha] \otimes u\right)|U = 0\right) \tag{40}$$

in which the operations $\oplus$ and $\otimes$ are componentwise. Further denote $\mathbf{h}(w) = [1, \dots, \alpha] \otimes \mathbf{d}(w)$ ($\mathbf{h}$ is necessarily bijective). Define an invertible mapping from $\{0, \dots, \kappa - 1\}^\alpha$ to GF(q) and denote h : GF(q) $\to$ GF(q) the composition of $\mathbf{h}$ and of the invertible mapping. We get

$$P(W = w|U = u) = P(W = h^{-1}(h(w) \oplus u)|U = 0). \tag{41}$$

B. Recursion for channel coding

We look for recursive expressions of $Q^{(\ell)}$ from $P^{(\ell)}$ from (13) and (17). For this, we express the probability density transformations of the operators involved in (17).

1) $\mathcal{W}[g]$ and $R[s]$: In the following, $g \in \text{GF}(q)\backslash\{0\}$ and $s \in \text{GF}(q)$. Let $\mathbf{m}$ be a real-valued vector of size q and $\boldsymbol{\ell} = W[g]\mathbf{m}$. Denote $P_{\mathbf{M}}$ and $P_{\mathbf{L}}$ their respective probability densities and define $\varphi(\boldsymbol{\ell}) = W[g^{-1}]\boldsymbol{\ell}$. The function φ is invertible, and both φ and its inverse φ^{-1} are $\mathcal{C}^1$. The Jacobian matrix of φ is $J_\varphi = W[g^{-1}]$ and $\det(J_\varphi) \neq 0$. Consequently, φ is a $\mathcal{C}^1$-diffeomorphism. By expressing $E[f(\mathbf{L})]$ for any $\mathcal{L}^1$ function f and by variable change we get

$$P_{\mathbf{L}}(\boldsymbol{\ell}) = \det(J_\varphi) P_{\mathbf{M}}(W[g^{-1}]\boldsymbol{\ell}) = \Gamma_W^g(P_{\mathbf{M}})(\boldsymbol{\ell}) \tag{42}$$

where Γ_W^g is the density transform operator.

Using a similar derivative, a density transform operator Γ_R^s can be obtained for $R[s]$.

2) From LLR to probability representation: Define $\mathcal{P}$ as the set of vectors of q components such that $\forall k = 0 \dots q-1$, $0 < p_k < 1$ and $\sum_{k=0}^{q-1} p_k = 1$. Let $\mathbf{m} \in \{0\} \times \mathbb{R}^{q-1}$ and $\mathbf{p} \in \mathcal{P}$ be vectors of size q. The probability densities of $\mathbf{m}$ and $\mathbf{p}$ are denoted respectively $P_{\mathbf{M}}$ and $P_{\mathbf{P}}$. Define the function $\varphi : \{0\} \times \mathbb{R}^{q-1} \to \mathcal{P}$ with $\varphi(\mathbf{m}) = (\varphi_0(\mathbf{m}), \dots, \varphi_{q-1}(\mathbf{m}))$ and $\forall k = 0 \dots q-1$,

$$\varphi_k(\mathbf{m}) = \frac{\exp(-m_k)}{\sum_{k'=0}^{q-1} \exp(-m'_k)} \ . \tag{43}$$

The function φ is invertible with inverse $\varphi^{-1} : \mathcal{P} \to \{0\} \times \mathbb{R}^{q-1}$ with $\varphi^{-1}(\mathbf{p}) = (\phi_0(\mathbf{p}), \dots, \phi_{q-1}(\mathbf{p}))$ and $\forall j = 0 \dots q-1$,

$$\phi_j(\mathbf{p}) = \log \frac{1 - \sum_{j'=1}^{q-1} p'_j}{p_j} \ . \tag{44}$$

Both φ and φ^{-1} are $\mathcal{C}^1$. The Jacobian matrix J_φ of φ is given by

$$\begin{aligned} (J_\varphi(\mathbf{m}))_{k,k} &= -\exp(-m_k) \left(\sum_{k'=0, k' \neq k}^{q-1} \exp(-m'_k) \right) / \left(\sum_{k=0}^{q-1} \exp(-m'_k) \right)^2 \\ j \neq k : (J_\varphi(\mathbf{m}))_{j,k} &= \exp(-m_k) \exp(-m_j) / \left(\sum_{k=0}^{q-1} \exp(-m'_k) \right)^2 \end{aligned} \tag{45}$$

and $\det(J_\varphi(\mathbf{m})) \neq 0$. Consequently φ is a $\mathcal{C}^1$-diffeomorphism and by variable change in $E\left[f(\mathbf{M})\right]$ for every $\mathcal{L}^1$ function f,

$$P_{\mathbf{M}}(\mathbf{m}) = \det(J_\varphi(\mathbf{m})) P_{\mathbf{P}}(\varphi_1(\mathbf{m}) \dots \varphi_{q-1}(\mathbf{m})) = \Gamma_{\mathbf{m}}(P_{\mathbf{P}})(\mathbf{m}) \tag{46}$$

where $\Gamma_{\mathbf{m}}$ is the density transform operator. On the other hand, the Jacobian matrix $J_{\varphi^{-1}}$ of φ^{-1} is given by

$$\forall j \neq 0 : (J_{\varphi}^{-1}(\mathbf{p}))_{j,j} = -\frac{1}{p_j} - \frac{1}{\sum_{j'=1}^{q-1} p'_j} \tag{47}$$

$$\forall j \neq 0 : (J_{\varphi}^{-1}(\mathbf{p}))_{j,0} = 0 \tag{48}$$

$$\forall j \neq 0 : (J_{\varphi}^{-1}(\mathbf{p}))_{0,j} = -\frac{1}{\sum_{j'=1}^{q-1} p'_j} \tag{49}$$

$$\forall j, k \neq 0 : (J_{\varphi}^{-1}(\mathbf{p}))_{j,k} = -\frac{1}{\sum_{j'=1}^{q-1} p'_j} \tag{50}$$

$$(J_{\varphi}^{-1}(\mathbf{p}))_{0,0} = -\frac{1}{\sum_{j'=1}^{q-1} p'_j} \tag{51}$$

$$\tag{52}$$

Thus $\det(J_{\varphi}^{-1}(\mathbf{p})) \neq 0$ and from the same arguments as before, a density transform operator $\Gamma_{\mathbf{p}}$ can be obtained for the transformation of $\mathbf{m}$ into $\mathbf{p}$.

3) Fourier Transform and inverse Fourier Transform: We consider the Fourier Transform $\mathbf{f} = \mathcal{F}(\mathbf{p})$ of a vector $\mathbf{p}$. As $\mathcal{F}$ is an invertible linear application, by variable change and from the arguments of Appendix B1, we show that

$$P_{\mathbf{F}}(\mathbf{f}) = \det(J_{\mathcal{F}^{-1}}) P_{\mathbf{P}}(\mathcal{F}^{-1}(\mathbf{f})) = \Gamma_{\mathcal{F}}(P_{\mathbf{P}})(\mathbf{f}) \tag{53}$$

where $J_{\mathcal{F}^{-1}}$ is the Jacobian of $\mathcal{F}^{-1}$ and $\Gamma_{\mathcal{F}}$ is the defined density transform operator. A density transform operator $\Gamma_{\mathcal{F}^{-1}}$ can also be obtained from the inverse Fourier transform $\mathbf{p} = \mathcal{F}^{-1}(\mathbf{f})$.

4) γ transform: Define the restricted equivalent function $\tilde{\gamma} : \mathbb{R}^2 \backslash \{0, 0\} \to \mathbb{R} \times [-\pi, \pi]$ and

$$\tilde{\gamma}(x, y) = \begin{cases} \left(\frac{1}{2}\log(x^2 + y^2), \arctan\frac{y}{x}\right) & \text{if } x \geq 0 \\ \left(\frac{1}{2}\log(x^2 + y^2), \arctan\frac{y}{x} + \pi\right) & \text{if } x < 0 \ , \ y \geq 0 \\ \left(\frac{1}{2}\log(x^2 + y^2), \arctan\frac{y}{x} - \pi\right) & \text{if } x < 0 \ , \ y < 0 \ . \end{cases} \tag{54}$$

We show that $\tilde{\gamma}$ is $\mathcal{C}^1$ over its interval of definition even in the particular points $(x, 0)$ $\forall x \neq 0$ and $(0, y)$ $\forall y \neq 0$. Its inverse application is $\tilde{\gamma}^{-1} : \mathbb{R} \times [-\pi, \pi] \to \mathbb{R}^2 \backslash \{0, 0\}$ and $\gamma^{-1}(z, t) = (\exp(z) \cos t, \exp(z) \sin t)$.

The determinants of the Jacobian matrices $J_{\tilde{\gamma}}$ of $\tilde{\gamma}$ and $J_{\tilde{\gamma}^{-1}}$ of $\tilde{\gamma}^{-1}$ are given by

$$\det(J_{\tilde{\gamma}}(x,y)) = \frac{1}{x^2+y^2} > 0 \quad , \quad \det(J_{\tilde{\gamma}^{-1}}(z,t)) = \exp(2z) > 0 \ . \tag{55}$$

Consequently, $\tilde{\gamma}$ and $\tilde{\gamma}^{-1}$ are $\mathcal{C}^1-$diffeomorphisms. Denote $P_{X,Y}$ and $P_{Z,T}$ the probability densities associated to random variables (X,Y) and (Z,T). By expressing $E[f(X,Y)]$ and $E[f(Z,T)]$ for every $\mathcal{L}^1$ function f and by variable change, we show that density transform operators can be obtained $\forall (x,y) \in \mathbb{R}^2 \backslash \{0,0\}$ and $\forall (z,t) \in \mathbb{R} \times [-\pi,\pi]$ as

$$\tilde{P}_{X,Y}(x,y) = \Gamma_{\gamma}(P_{Z,T})(x,y) = \frac{1}{x^2+y^2} P_{Z,T} \circ \tilde{\gamma}(x,y) \tag{56}$$

$$\tilde{P}_{Z,T}(z,t) = \Gamma_{\gamma^{-1}}(P_{X,Y})(z,t) = \exp(z) P_{X,Y} \circ \tilde{\gamma}^{-1}(z,t) \ . \tag{57}$$

The density cannot be obtained in $(0,0)$ by the same method because γ is not continuous in $(0,0)$. However, the probability density functions have to be completed. We get

$$\lim_{z \to -\infty} \tilde{P}_{Z,T}(z,t) = \frac{1}{2\pi} P_{X,Y}(0,0) \tag{58}$$

$$\tilde{P}_{X,Y}(0,0) = \lim_{z \to -\infty} P_{Z,T}(z,t) = \lim_{z \to -\infty} P_Z(z) \tag{59}$$

where $\lim_{z \to -\infty} P_{Z,T}(z,t)$ does not depend on t and P_Z is the marginal density of the random variable Z.

Note that in (15), a transform $\boldsymbol{\gamma}$ involving vectors of size $q-1$ is defined. Its components γ_j, $j = 1 \ldots q-1$ apply independently on the components of the input vector (not necessarily composed by independent random variables). Consequently, the transforms defined in (56) can be directly generalized to the vector version.

C. Recursion for Slepian-Wolf coding

1) Expression of the error probability: The error probability $p_e^{(\ell)}$ can be expressed as

$$p_e^{(\ell)} = 1 - \sum_{k=0}^{q-1} P(X=k) \int_{\mathbf{m} \in \Omega_k} \tilde{P}_k^{(\ell)}(\mathbf{m}) d\mathbf{m} \tag{60}$$

where $\Omega_k = \{\mathbf{m} \in \mathbb{R}^q : \forall k' \neq k : m_{k'} > m_k\}$ is the set of messages giving the right value of X. The function $\tilde{\mathbf{m}} \to \mathcal{A}[\ominus k]\tilde{\mathbf{m}}$ is invertible, $\mathcal{C}_1$, and its inverse is also $\mathcal{C}_1$. The Jacobian of the application is $\mathcal{A}[\ominus k]$ and $\det(\mathcal{A}[\ominus k]) \neq 0$. Thus the application is a $\mathcal{C}_1$-diffeomorphism. By change of variable,

$$p_e^{(\ell)} = 1 - \sum_{k=0}^{q-1} P(X=k) \int_{\tilde{\mathbf{m}} \in \mathbb{R}_+^q} \tilde{P}_k^{(\ell)}(\mathcal{A}[\ominus k]\tilde{\mathbf{m}}) d\tilde{\mathbf{m}} \tag{61}$$

To finish (and by replacing $\tilde{\mathbf{m}}$ by $\mathbf{m}$),

$$p_e^{(\ell)} = 1 - \int_{\mathbf{m} \in \mathbb{R}_+^q} \langle \tilde{P}^{(\ell)} \rangle (\mathbf{m}) d\mathbf{m}. \tag{62}$$

2) Multinomial formula: The multinomial formula is restated here because it will be useful for the proof of the recursion. Let $(x_1 \dots x_m)$ be m scalar values. The multinomial formula gives

$$\left(\sum_{k=1}^{m} x_k \right)^n = \sum_{k_1 + \cdots + k_m = n} \binom{n}{k_1, \dots, k_m} \prod_{i=1}^{m} x_i^{k_i} \tag{63}$$

where $\binom{n}{k_1,\dots,k_m} = \frac{n!}{k_1! \dots k_m!}$ is the multinomial coefficient. On the other hand, denote $\mathcal{S}_x = \{x_1, \dots, x_m\}$. One can show that the multinomial formula (63) gives also

$$\left(\sum_{k=1}^{m} x_k \right)^n = \sum_{(x'_1 \dots x'_n) \in \mathcal{S}_x^n} \prod_{i=1}^{n} x'_i \,. \tag{64}$$

3) Recursion: For the sake of simplicity, the code is assumed regular with degrees d_v and d_c. The irregular version of the recursion is directly obtained by marginalization according to the degree distributions.

The expression of the density $P_x^{(\ell)}$ is directly obtained from (13) (sum of random variables) as

$$P_x^{(\ell)}(\mathbf{m}) = P_x^{(0)} \bigotimes (Q_x^{(\ell-1)})^{\otimes (d_v - 1)}(\mathbf{m}). \tag{65}$$

On the other hand, $Q_x^{(\ell)}(\mathbf{m})$ can be developed as

$$Q_x^{(\ell)}(\mathbf{m}) = \sum_{\bar{g}_1 \dots \bar{g}_{d_c-1}} \sum_{x_1 \dots x_{d_c-1}} \left(\prod_{i=1}^{d_c-1} \frac{p_{x_i}}{q-1} \right) P(\mathbf{m} | x, x_1 \dots x_{d_c-1}, \bar{g}_1 \dots \bar{g}_{d_c-1}) \tag{66}$$

$$P(\mathbf{m} | x, x_1 \dots x_{d_c-1}, \bar{g}_1 \dots \bar{g}_{d_c-1}) = \Gamma_d^{-1} \left(\bigotimes_{i=1}^{d_c-1} \Gamma_c^{\bar{g}_i} (P_{x_i}^{(\ell-1)}) \right) \circ \mathcal{A}[\ominus \bar{s}](\mathbf{m}) \tag{67}$$

where $\bar{s} = x + \sum_{i=1}^{d_c-1} \bar{g}_i x_i$ and (67) is obtained from (22) completed with $\mathcal{A}$ and from the multinomial formula. Furthermore, $\mathcal{A}[c \oplus b]\mathbf{m} = \mathcal{A}[c]\mathcal{A}[b]\mathbf{m}$ and from (66),

$$Q_a^{(\ell)}(\mathbf{m}) = Q_b^{(\ell)} \circ \mathcal{A}[a \ominus b](\mathbf{m}) \tag{68}$$

Moreover,

$$\begin{aligned} Q_0^{(\ell)}(\mathbf{m}) &= \sum_{\bar{g}_1 \ldots \bar{g}_{d_c-1}} \sum_{x_1 \ldots x_{d_c-1}} \left(\prod_{i=1}^{d_c-1} \frac{p_{x_i}}{q-1} \right) \Gamma_d^{-1} \left(\bigotimes_{i=1}^{d_c-1} \Gamma_c^{\bar{g}_i} \left(P_{x_i}^{(\ell-1)} \circ \mathcal{A}[\ominus x_i] \right) \right) (\mathbf{m}) && (69) \\ &= \Gamma_d^{-1} \left(\left(\sum_{\bar{g}=1}^{q-1} \sum_{x=0}^{q-1} \frac{p_x}{q-1} \Gamma_c^{\bar{g}} \left(P_x^{(\ell-1)} \circ \mathcal{A}[\ominus x] \right) \right)^{\bigotimes (d_c-1)} \right) (\mathbf{m}) && (70) \end{aligned}$$

by the multinomial formula. Finally, by linearity of the density transform operators

$$\begin{aligned} Q_0^{(\ell)}(\mathbf{m}) &= \Gamma_d^{-1} \left(\left(\frac{1}{q-1} \sum_{\bar{g}=1}^{q-1} \Gamma_c^{\bar{g}} \left(\sum_{x=0}^{q-1} p_x P_x^{(\ell-1)} \circ \mathcal{A}[\ominus x] \right) \right)^{\bigotimes (d_c-1)} \right) (\mathbf{m}) && (71) \\ &= \Gamma_d^{-1} \left(\left(\frac{1}{q-1} \sum_{\bar{g}=1}^{q-1} \Gamma_c^{\bar{g}} \left(\langle P^{(\ell-1)} \rangle \right) \right)^{\bigotimes (d_c-1)} \right) (\mathbf{m}). && (72) \end{aligned}$$

Then from (65)

$$\begin{aligned} \langle P^{(\ell)} \rangle (\mathbf{m}) &= \sum_{x=0}^{q-1} p_x \left(P_x^{(0)} \bigotimes \left(Q_x^{(\ell-1)} \right)^{\bigotimes (d_v-1)} \right) \circ \mathcal{A}[\ominus x](\mathbf{m}) && (73) \\ &= \sum_{x=0}^{q-1} p_x \left(P_x^{(0)} \circ \mathcal{A}[\ominus x] \right) \bigotimes \left(Q_x^{(\ell-1)} \circ \mathcal{A}[\ominus x] \right)^{\bigotimes (d_v-1)} (\mathbf{m}) && (74) \end{aligned}$$

by property of the convolution product. Furthermore, from (68),

$$\langle P^{(\ell)} \rangle = \langle P^{(0)} \rangle \bigotimes \left(Q_0^{(\ell-1)} \right)^{\bigotimes (d_v-1)} (\mathbf{m}). \tag{75}$$

To finish, replacing $Q_0^{(\ell-1)}$ from (72) gives (29) and (30) derives directly from (29).

showing the entropy equality.

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