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A Clifford algebra gauge invariant Lagrangian for gravity. Part 2 : compatibility with General Relativity tests.

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Introduction

In the last years, there has been a large number of papers discussing possible extensions of the Einstein-Hilbert Lagrangian for gravity : $L = R$, where R is the scalar space-time curvature. There has been mainly 2 kinds of theories discussed, the so-called $f(R)$ theories, where f is a function of the scalar curvature [1], and theories based on quadratic terms built on the curvature tensor [2], more or less like the Lagrangian of gauge fields, or even more general ones [3] including torsion. The existence of torsion has also often been discussed with possible effects for the early universe [4].

In the present note we consider a gauge theory based on the Clifford algebra represented by the Dirac matrices. This leads to a Lagrangian containing four terms : the Einstein-Hilbert one, a cosmological constant term, a quadratic Lagrangian term and a torsion one. Torsion is naturally introduced because the connexion coefficients and the basis vectors are independent fields.

The goal of this note is to show that there is no contradiction with the experimental observations. The consequences of such an extended Lagrangian would appear only in strong gravitational fields, but we have not yet look at them.

The construction is independent of the number of space-time dimensions. This is discussed in a separate note [5]. The reduction from an n -dimensional space-time to the 4-dimensional space-time gives the gravitational Lagrangian described above and the usual quadratic Lagrangian of gauge fields.

The next section sets the notations and recalls very briefly some basic geometrical equations. Section 2 explains the construction of the Lagrangian, and the equations of motion are deduced in section 3. The following sections look at the consequences for, respectively, the static spherical symmetric case, the expanding isotropic universe, and the gravitational field of a rotating body far from it. The appendix A details the consequences of the universe isotropy on

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the tensor components. The present note, called part 2 in the title, can be read independently of the note on higher dimensions, called part 1.

1 Notations and basic geometrical equations.

The space-time coordinates $\{x^\alpha\}$ of a point x are labelled with Greek letters : $\alpha, \beta, \gamma, \dots, 0 \leq \alpha, \beta, \gamma, \dots < n$. The time coordinate is : x^0 and, when it is necessary to distinguish spatial coordinates from the time coordinate, the letters : $\mu, \nu, \rho, \eta, \dots$ are used. The vectors of the local natural frame are written : $\vec{e}_\alpha, \vec{e}_\beta, \dots$. When tensors are expressed with respect to local orthonormal frames they are labelled with Latin letters : a, b, c, \dots . The orthonormal local frame basis vectors are called : \vec{h}_a , and we set : $\vec{h}_a = h_a^\alpha \vec{e}_\alpha$. The metric tensor is $g_{\alpha\beta}$, and $g^{\alpha\beta}$ is its inverse. The signature of the metric is : $(+ - - -)$. In the case of local orthonormal frames, the metric tensor is written : η_{ab} and its diagonal terms are : $\eta_{aa} = (+1, -1, -1, -1)$. The commutator of the vectors $\{\vec{h}_a\}$ is : $[h_a, h_b]^\gamma = h_a^\alpha \partial_\alpha h_b^\gamma - h_b^\alpha \partial_\alpha h_a^\gamma = C_{ab}^\gamma h_c^\gamma$ (1.1)

In the neighborhood of a given point, the local coordinates, with respect to the local orthonormal frame attached to this point, are given by the 1-forms : $\omega^a = h_a^\alpha dx^\alpha$, which satisfy the structure equations :

$$d\omega^a + \omega^a_{.b} \wedge \omega^b = \Sigma^a \quad (1.2)$$

where : $\omega^a_{.b} = \omega^a_{.b\gamma} dx^\gamma$ are the connexion 1-forms and Σ^a is the torsion 2-form. We shall also write: $\omega^a_{.b} = \omega^a_{.bc} \omega^c \leftrightarrow \omega^a_{.bc} = \omega^a_{.b\gamma} h_c^\gamma$. The connexion 1-forms are related to the connexion coefficients by :

$$\omega^a_{.b\gamma} = \Gamma^a_{.b\gamma} h_a^\alpha h_b^\beta + h_\delta^\alpha \partial_\gamma h_b^\delta \quad (1.3)$$

The connexion coefficients are the sum of two terms :

$$\Gamma^a_{.b\gamma} = \tilde{\Gamma}^a_{.b\gamma} + \bar{S}^a_{.b\gamma} \quad (1.4)$$

where the first term is the Christoffel symbol and the second is the contorsion tensor. The contorsion is anti symmetric with respect to the two first indices : $\bar{S}^a_{.b\gamma} + \bar{S}^a_{.\gamma b} = 0$. The torsion tensor is :

$$S^a_{.b\gamma} = \frac{1}{2} (\Gamma^a_{.b\gamma} - \Gamma^a_{.\gamma b}) = \frac{1}{2} (\bar{S}^a_{.b\gamma} - \bar{S}^a_{.\gamma b}) \quad (1.5a)$$

and inversely: $\bar{S}^a_{.b\gamma} = S^a_{.b\gamma} - S^a_{.\gamma b} - S^a_{.c\gamma} \omega^c_{.b} ; S^a_{.\gamma b} = g^{\alpha\delta} S_{\beta\delta\gamma}$ (1.5b)

The torsion 2-form is: $\Sigma^a = \Sigma^a_{.bc} \omega^b \wedge \omega^c = -h_a^\alpha S^a_{.b\gamma} dx^\beta \wedge dx^\gamma$ (1.6)

Using (1.3) we set : $\tilde{\Gamma}^a_{.b\gamma} = \tilde{\Gamma}^a_{.b\gamma} h_a^\alpha h_b^\beta + h_\delta^\alpha \partial_\gamma h_b^\delta$ (1.7)

The curvature 2-form is defined by :

$$\Omega^a_{.b} = d\omega^a_{.b} + \omega^a_{.c} \wedge \omega^c_{.b} = R^a_{.bcd} \omega^c \wedge \omega^d \quad (1.8)$$

2 The gravitational field as a gauge field.

Let us consider the Dirac matrices γ^a which represent the basis elements $\{e_a\}$ of the Clifford algebra: $e_a \cdot e_b + e_b \cdot e_a = 2\eta_{ab}$

The commutators: $R^{ab} = \frac{1}{4} [\gamma^a, \gamma^b]$ represent the generators of the rotation group. They satisfy:

$$[R^{ab}, R^{cd}] = -\eta^{ad} R^{cb} - \eta^{ac} R^{bd} - \eta^{bd} R^{ac} - \eta^{bc} R^{da} \quad (2.1a)$$

$$[\gamma^a, \gamma^b] = 4 R^{ab} \quad (2.1b)$$

$$[\gamma^a, R^{cd}] = \eta^{ac}\gamma^d - \eta^{ad}\gamma^c \quad (2.1c)$$

This set of relations is a graded Lie algebra which satisfies Jacobis identities. The γ^a and R^{ab} matrices are the elements of a representation Γ of an algebra defined by the relations (2.1) and named $\Gamma(X_x)$, where X_x are the basis elements of this algebra. To this algebra we associate a gauge field:

$$W = \alpha \omega_{ab} R^{ab} + \beta \omega_a \gamma^a \quad (2.2)$$

where: $\omega_{ab} = -\omega_{ba}$ and ω_a are differential forms of degree 1:

$$\omega_{ab} = \omega_{ab\alpha} dx^\alpha \quad \omega_a = \eta_{ab} \omega^b = \eta_{ab} h_\alpha^b dx^\alpha \quad (2.3)$$

and: α, β are arbitrary constants.

The meaning of these gauge fields is the following: the fields ω_{ab} are the connexion coefficients defined with respect to a family of local orthonormal frames, and the 1-forms $\omega^a(x) = \eta^{ab}\omega_b(x)$ are the coordinates, in the neighborhood of a given point x , with respect to these frames. This can be understood by computing the curvature 2-form (1.8) :

$$G = dW + W \wedge W \quad (2.4)$$

$$G = (\alpha d\omega_{ab} R^{ab} + \beta d\omega_a \gamma^a) + (\alpha \omega_{cd} R^{cd} + \beta \omega_c \gamma^c) \wedge (\alpha \omega_{ef} R^{ef} + \beta \omega_e \gamma^e)$$

$$G = \alpha d\omega_{ab} R^{ab} + \frac{\alpha^2}{2} (\omega_{cd} \wedge \omega_{ef} R^{cd} R^{ef} + \omega_{ef} \wedge \omega_{cd} R^{ef} R^{cd}) \\ + \beta d\omega_a \gamma^a + \alpha \beta (\omega_{cd} \wedge \omega_e R^{cd} \gamma^e + \omega_e \wedge \omega_{cd} \wedge \gamma^e R^{cd}) + \beta^2 \omega_c \wedge \omega_e \gamma^c \gamma^e$$

and since the gauge fields (2.3) are 1-forms, one has:

$$G = \alpha d\omega_{ab} R^{ab} + \frac{\alpha^2}{2} \omega_{cd} \wedge \omega_{ef} [R^{cd}, R^{ef}] + \beta d\omega_a \gamma^a \\ + \alpha \beta \omega_{cd} \wedge \omega_e [R^{cd}, \gamma^e] + 2\beta^2 \omega_a \wedge \omega_b R^{ab}$$

With the relations(2.1) this becomes:

$$G = [\alpha d\omega_{ab} + 2\alpha^2 \omega_{af} \wedge \omega_{.b}^f + 2\beta^2 \omega_a \wedge \omega_b] R^{ab} \\ + \beta [d\omega_a + 2\alpha \omega_{.a}^e \wedge \omega_e] \gamma^a \quad (2.5)$$

$$\text{We can set :} \quad \alpha = \frac{1}{2} \quad (2.6)$$

$$\text{Then :} \quad G = \frac{1}{2} [d\omega_{ab} + \omega_{af} \wedge \omega_{.b}^f + 4\beta^2 \omega_a \wedge \omega_b] R^{ab} \\ + \beta [d\omega_a + \omega_{.a}^e \wedge \omega_e] \gamma^a \quad (2.7a)$$

With the above interpretation of the gauge fields, the first two terms in the first brackets correspond to the usual curvature 2-form : $\Omega_{.b}^a = d\omega_{.b}^a + \omega_{.c}^a \wedge \omega_{.b}^c$, and the second brackets contains the structure equations : $d\omega^a + \omega_{.b}^a \wedge \omega^b = \Sigma^a$, where Σ^a represents the torsion 2-form (1.6) :

$$G = \frac{1}{2} [\Omega_{ab} + 4\beta^2 \omega_a \wedge \omega_b] R^{ab} + \beta \Sigma_a \gamma^a \quad (2.7b)$$

The standard minimum gauge field Lagrangian is :

$$L = Tr(G \wedge * G) \quad (2.8)$$

where: $*$ is the Hodges star operator. Taking into account the γ^a matrices properties:

$$Tr(R^{ab} R^{cd}) = \frac{N}{4} (\eta^{ad} \eta^{bc} - \eta^{ac} \eta^{bd}) \\ Tr(R^{ab} \gamma^c) = 0 \quad Tr(\gamma^a \gamma^b) = N \eta^{ab} \quad (2.9)$$

where: N is the spinor dimension, one has :

$$L/N = -\beta^2 \Omega^{ab} \wedge * (\omega_a \wedge \omega_b) - \frac{1}{8} \Omega^{ab} \wedge * \Omega_{ab} \quad (2.10a)$$

$$-2\beta^4 (\omega^a \wedge \omega^b) \wedge * (\omega_a \wedge \omega_b) + \beta^2 \eta_{ab} (d\omega^a + \omega_{.c}^a \wedge \omega^c) \wedge * (d\omega^b + \omega_{.d}^b \wedge \omega^d)$$

The first term correspond to the Einstein-Hilbert Lagrangian of General Relativity. The third term represents the contribution of a cosmological constant, since this term is proportional to the volume element. The second term is quadratic and has the form of standard gauge field Lagrangian. The equation (2.10a) can also be re-written:

$$L/N = -\beta^2 \Omega^{ab} \wedge * (\omega_a \wedge \omega_b) - \frac{1}{8} \Omega^{ab} \wedge * \Omega_{ab} \quad (2.10b)$$

$$-2\beta^4 (\omega^a \wedge \omega^b) \wedge * (\omega_a \wedge \omega_b) + \beta^2 \Sigma^a \wedge * \Sigma_a + \mu (d\omega^a + \omega_{.b}^a \wedge \omega^b - \Sigma^a)$$

where: μ is a Lagrange multiplier.

In (2.10) the torsion is introduced naturally, not as an extra field, this is a direct consequence of the definition (2.2) where ω_a and ω_{ab} are independent fields. The equations of motion are calculated in the next chapter.

Until now, we have discussed the interpretation of the gauge fields introduced in (2.2), but it remains to see how these fields transform. Taking into account the algebra (2.1), one considers the infinitesimal transformations : $S = I + i\varepsilon_a \gamma^a + \varepsilon_{ab} R^{ab}$, where: $\varepsilon_{ab} = -\varepsilon_{ba}$ and: $|\varepsilon_a|, |\varepsilon_{ab}| \ll 1$. The transformation law of gauge field is: $W' = S^{-1} W S + S^{-1} dS$, which gives : $G' = S^{-1} G S$ and makes the Lagrangian (2.8) invariant. With, at first order : $S^{-1} = I - i\varepsilon_a \gamma^a - \varepsilon_{ab} R^{ab}$, one has, still at first order:

$$W' = W + i\alpha \omega_{ab} \varepsilon_e [R^{ab}, \gamma^e] + \alpha \omega_{ab} \varepsilon_{ef} [R^{ab}, R^{ef}] + i\beta \omega_a \varepsilon_e [\gamma^a, \gamma^e] + \beta \omega_a \varepsilon_{ef} [\gamma^a, R^{ef}] + i d\varepsilon_e \gamma^e + d\varepsilon_{ef} R^{ef}$$

then, with the algebra (2.1):

$$\omega'_a = \omega_a + \varepsilon_{ea} \omega^e - \varepsilon_{ae} \omega^e + \frac{i}{\beta} (d\varepsilon_a + \alpha \omega_a^e \varepsilon_e - \alpha \omega^e_a \varepsilon_e)$$

and with (2.6) : $\omega'_a = \omega_a + \varepsilon_{ea} \omega^e - \varepsilon_{ae} \omega^e + \frac{i}{\beta} D\varepsilon_a$

If: $D\varepsilon_a = 0$, ω^a transforms like a vector under an (infinitesimal) rotation whose coefficients are the ε_{ef} . In that case: $\varepsilon_a = 0$ and the field ω_{ab} transforms like a gauge field with respect to rotations. The restriction of the gauge transformation makes ω^a transform like a vector. The 1-forms: $\omega^a = h^\alpha_a dx^\alpha$ are directly related to the basis vectors of the local frames.

In the above description ω^a and ω_{ab} are independent gauge fields. How does that modifies the Lagrangian of matter fields? We shall suppose that ordinary matter fields are spinor fields, and therefore we shall consider the covariant derivative of such fields.

Let ψ be a spinor field. The covariant derivative of a spinor field with gauge fields is :

$$D\psi = d\psi + \frac{1}{4} \omega_{cd} \gamma^c \gamma^d \psi + \beta \omega_a \gamma^a \psi \\ \rightarrow D_\alpha \psi = \partial_\alpha \psi + \frac{1}{4} \omega_{cd\alpha} \gamma^c \gamma^d \psi + \beta h^b_\alpha \eta_{bc} \gamma^c \psi$$

The last term is not present in the usual covariant derivative of a spinor field. The Lagrangian of such a field is : $L = \bar{\psi} h^\alpha_a \gamma^a i D_\alpha \psi + h.c.$

where: *h.c.* means: Hermitic conjugate. If one uses the above covariant derivative, the contribution of the unwanted terms is (if β is real) :

$$i\beta \bar{\psi} h^\alpha_a \gamma^a h^b_\alpha \eta_{bc} \gamma^c \psi + h.c. = i\beta \bar{\psi} \gamma^a \eta_{ac} \gamma^c \psi + h.c. \sim i\beta \bar{\psi} \psi + h.c. = 0$$

Therefore, the gauge field (2.2) gives the usual spinor field Lagrangian built with the usual covariant derivative : $D_\alpha \psi = \partial_\alpha \psi + \frac{1}{4} \omega_{cd\alpha} \gamma^c \gamma^d \psi$

Remark : the above Lagrangian of a spinor field is built using the generator γ^a of the algebra (2.1). Why not use the generators of type R^{ab} instead? A possible Lagrangian could be: $L = \overline{D_a \psi} R^{ab} D_b \psi + h.c.$, which gives the second order Dirac equation. In conclusion, the gauge field (2.2) does not introduces unwanted terms.

The above calculations are independent of the space-time dimension which hereafter is 4. We show, in another note [5], that, following the Kaluza-Klein program, one can recover both the quadratic Lagrangian of gauge fields and the Lagrangian (2.10) for the gravitation.

Summary of this section.

The gauge field (2.2) associated to the algebra (2.1) leads to a gravitational Lagrangian which is more general than the Einstein-Hilbert one, introducing naturally a quadratic term and a cosmological constant. The 1-form fields ω^a and ω_{ab} are independent of each other, and as a consequence, torsion may exist as an independent field.

3 The Equations of motion.

We now re write the Lagrangian (2.10) in a slightly more general form which puts the Einstein-Hilbert Lagrangian as the main term :

$$-\frac{L}{\beta^2 N} = \Omega^{ab} \wedge *(\omega_a \wedge \omega_b) + \eta \Omega^{ab} \wedge * \Omega_{ab} + \lambda dV + \mu \Sigma^a \wedge * \Sigma_a \quad (3.1)$$

with the constraint : $\Sigma^a = d\omega^a + \omega^a_b \wedge \omega^b$, and where : $dV = \omega_a \wedge *\omega^a$.

The parameters η , λ , μ are now free, although they should be linked by :

$$\eta = 1/(8\beta^2) \quad \lambda = 2\beta^2 \quad \mu = -1 \quad (3.2)$$

The first term of (3.1) is the Einstein-Hilbert Lagrangian, it is : $L_{EH} = R dV$.

The second term is : $\eta L_Q = \eta \Omega^{ab} \wedge * \Omega_{ab} = \eta R^{ab} \wedge * R_{ab}$.

The variation of the first term is :

$$\begin{aligned} \delta L_{EH} = & -\delta\omega^{ab} \wedge \varepsilon_{abcd} \omega^c \wedge (\Sigma^d_{ef} \omega^e \wedge \omega^f) \\ & + \Omega^{ab} \wedge \varepsilon_{abcd} \omega^c \wedge \delta\omega^d + d(\frac{1}{2} \delta\omega^{ab} \wedge \varepsilon_{abcd} \omega^c \wedge \omega^d) \end{aligned} \quad (3.3)$$

The variation of the quadratic term is :

$$\delta L_Q = 2\delta\omega_{ab} \wedge D * \Omega^{ab} + 2d(\delta\omega_{ab} \wedge * \Omega^{ab}) \quad (3.4)$$

where : $\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b = R^a_b \wedge \omega^c \wedge \omega^d$ is the curvature 2-form. This equation can be transformed into :

$$\begin{aligned} \delta L_Q = & 2d(\delta\omega_{ab} \wedge * \Omega^{ab}) + \\ & 4\delta\omega_{ab} \wedge \left\{ \partial_d R^{abcd} + \Gamma^a_{ed} R^{ebcd} + \Gamma^b_{ed} R^{aecd} + \tilde{\Gamma}^c_{ed} R^{abed} + \tilde{\Gamma}^d_{ed} R^{abce} \right\} \wedge \\ & *\omega_c \end{aligned} \quad (3.5)$$

The variation of the last term of equation (3.1) : $L_\Sigma = \Sigma^a \wedge * \Sigma_a$ is :

$$\delta L_\Sigma = 2(d\delta\omega^a + \delta\omega^a_c \wedge \omega^c + \omega^a_c \wedge \delta\omega^c) \wedge * \Sigma_a$$

then, the variation with respect to the connexion is :

$$\delta_\Gamma L_\Sigma = 2\delta\omega_{ac} \wedge \omega^c \wedge *\Sigma^a_{ef} \omega^e \wedge \omega^f = 2\delta\omega_{ac\gamma} h_c^\gamma (\bar{S}^{acb} - \bar{S}^{abc}) dV \quad (3.6)$$

and the variation with respect to the ω^a is :

$$\delta_h L_\Sigma = 2\delta\omega^a \wedge D(*\Sigma_a) + 2d(\delta\omega^a \wedge *\Sigma_a) \quad (3.7)$$

where : $D(*\Sigma_a) = d(*\Sigma_a) - \omega^b_a \wedge *\Sigma_b$

After some manipulations, one obtains :

$$\begin{aligned} \delta_h L_\Sigma = & 8\delta h^\alpha_\alpha h_g^\alpha [D_c \Sigma_a^{gc} + 2\Sigma_a^{gc} \Sigma^d_{cd} + \Sigma_a^{ec} \Sigma^g_{ec}] dV \\ & + exact\ dif f. \end{aligned} \quad (3.8)$$

The next task is to transform these variational equations into equations involving tensors written with their components. For that purpose we use (1.3).

The connexion coefficients are (1.4) : $\Gamma^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\beta\gamma} + \bar{S}^\alpha_{\beta\gamma}$ where the first term is the Christoffel symbol and the second is the contorsion tensor, therefore :

$$\omega^a_{b\gamma} = h^\alpha_\delta \tilde{D}_\gamma h_b^\delta + \bar{S}^\alpha_{\beta\gamma} h^\alpha_a h_b^\beta = h^\alpha_\delta \tilde{D}_\gamma h_b^\delta + \bar{S}^\alpha_{b\gamma}$$

where : \tilde{D} is the covariant derivative built with the Christoffel symbol only.

The covariant derivative of a tensor is :

$$\begin{aligned} DT^{\alpha_1 \dots \alpha_p} = & dT^{\alpha_1 \dots \alpha_p} + \Gamma^{\alpha_i}_{\beta_i \gamma} dx^\gamma T^{\alpha_1 \dots \beta_i \dots \alpha_p} \\ = & \tilde{D}T^{\alpha_1 \dots \alpha_p} + \bar{S}^{\alpha_i}_{\beta_i \gamma} dx^\gamma T^{\alpha_1 \dots \beta_i \dots \alpha_p} \end{aligned}$$

Since $DT^{\alpha_1 \dots \alpha_p}$ and the last term are tensors, $\tilde{D}T^{\alpha_1 \dots \alpha_p}$ is also a tensor. This

allows to express the various quantities indifferently in the orthonormal local frames or with coordinate indices.

In the following, we separate the terms involving the Christoffel symbols only from those involving the contorsion. Please note that, the variable used is the contorsion tensor, not the torsion. First, the curvature tensor is decomposed as :

$$2 R_{abfg} = h_f^\alpha \partial_\alpha \tilde{\Gamma}_{abg} - h_g^\alpha \partial_\alpha \tilde{\Gamma}_{abf} + \tilde{\Gamma}_{aef} \tilde{\Gamma}_{.bg}^e - \tilde{\Gamma}_{aeg} \tilde{\Gamma}_{.bf}^e - \tilde{\Gamma}_{abe} C_{.fg}^e \\ + h_f^\alpha D_\alpha \bar{S}_{abg} - h_g^\alpha D_\alpha \bar{S}_{abf} + \bar{S}_{aef} \bar{S}_{.bg}^e - \bar{S}_{aeg} \bar{S}_{.bf}^e$$

where the commutation coefficients $C_{.fg}^e$ have been defined in (1.1) . This relation is simply :

$$2 R_{abfg} = h_a^\alpha h_b^\beta h_f^\varphi h_g^\gamma \\ \left[2 \tilde{R}_{\alpha\beta\varphi\gamma} + \tilde{D}_\varphi \bar{S}_{\alpha\beta\gamma} - \tilde{D}_\gamma \bar{S}_{\alpha\beta\varphi} + \bar{S}_{\alpha\delta\varphi} \bar{S}_{. \beta\gamma}^\delta - \bar{S}_{\alpha\delta\gamma} \bar{S}_{. \beta\varphi}^\delta \right]$$

where : \tilde{D} is, as above, the covariant derivative involving Christoffel symbols only. We set :

$$R_{\alpha\beta\varphi\gamma} = \tilde{R}_{\alpha\beta\varphi\gamma} + K_{\alpha\beta\varphi\gamma} \quad (3.9a)$$

where : $2 K_{\alpha\beta\varphi\gamma} = \tilde{D}_\varphi \bar{S}_{\alpha\beta\gamma} - \tilde{D}_\gamma \bar{S}_{\alpha\beta\varphi} + \bar{S}_{\alpha\delta\varphi} \bar{S}_{. \beta\gamma}^\delta - \bar{S}_{\alpha\delta\gamma} \bar{S}_{. \beta\varphi}^\delta$ (3.9b)

which satisfies the same symmetry relations as the curvature tensor :

$$K_{\alpha\beta\varphi\gamma} = -K_{\beta\alpha\varphi\gamma} \quad K_{\alpha\beta\varphi\gamma} = -K_{\alpha\beta\gamma\varphi} \quad (3.9c)$$

Going back to equation (3.5), we define :

$$D_R^{abc} = \partial_d R^{abcd} + \Gamma_{.ed}^\alpha R^{bcd} + \Gamma_{.ed}^b R^{aec} + \tilde{\Gamma}_{.ed}^c R^{abed} + \tilde{\Gamma}_{.ed}^d R^{abce}$$

which is also :

$$D_R^{abc} = h_a^\alpha h_b^\beta h_c^\gamma \\ \left[\partial_\delta R^{\alpha\beta\gamma\delta} + \Gamma_{. \varepsilon\delta}^\alpha R^{\varepsilon\beta\gamma\delta} + \Gamma_{. \varepsilon\delta}^\beta R^{\alpha\varepsilon\gamma\delta} + \tilde{\Gamma}_{. \varepsilon\delta}^\gamma R^{\alpha\beta\varepsilon\delta} + \tilde{\Gamma}_{. \varepsilon\delta}^\delta R^{\alpha\beta\gamma\varepsilon} \right]$$

and finally :

$$D_R^{\alpha\beta\gamma} = \tilde{D}_\delta \tilde{R}^{\alpha\beta\gamma\delta} + \tilde{D}_\delta K^{\alpha\beta\gamma\delta} + \bar{S}_{. \varepsilon\delta}^\alpha \tilde{R}^{\varepsilon\beta\gamma\delta} + \bar{S}_{. \varepsilon\delta}^\beta \tilde{R}^{\alpha\varepsilon\gamma\delta} + \bar{S}_{. \varepsilon\delta}^\gamma K^{\varepsilon\beta\gamma\delta} \\ + \bar{S}_{. \varepsilon\delta}^\delta K^{\alpha\varepsilon\gamma\delta} \quad (3.10)$$

This tensor satisfies the anti-symmetry relation : $D_R^{\alpha\beta\gamma} = -D_R^{\beta\alpha\gamma}$ as expected.

The equations of motion can now be written with tensor components. The variation of the Lagrangian with respect to the connexion is :

$$-\frac{1}{N\beta^2} \frac{1}{dV} \frac{\delta L}{\delta \omega_{ab\gamma}} = 4\eta D_R^{\alpha\beta\gamma} h_\alpha^a h_\beta^b + h_c^\gamma (\eta^{bc} \bar{S}^a - \eta^{ac} \bar{S}^b) \\ + h_c^\gamma (2\mu \bar{S}^{abc} + (\mu - 1) (\bar{S}^{cab} - \bar{S}^{cba})) \quad (3.11a)$$

where : $\bar{S}^a = \bar{S}_{. \cdot d}^{da}$. Or, equivalently :

$$4\eta D_R^{\alpha\beta\gamma} + (g^{\beta\gamma} \bar{S}^\alpha - g^{\alpha\gamma} \bar{S}^\beta) + 2\mu \bar{S}^{\alpha\beta\gamma} \\ + (\mu - 1) (\bar{S}^{\gamma\alpha\beta} - \bar{S}^{\gamma\beta\alpha}) \sim S_\psi^{\alpha\beta\gamma} \quad (3.11b)$$

where : $S_\psi^{\alpha\beta\gamma}$ is the spin tensor of the matter fields.

The variation of the Lagrangian with respect to the fields $\{h_a^\alpha\}$ is :

$$\frac{1}{2N\beta^2} \frac{1}{dV} \delta_h L = h_a^\alpha \delta h_\gamma^a \left\{ \tilde{G}^\gamma_\alpha + K^\gamma_\alpha - \frac{K}{2} \delta_\alpha^\gamma - \lambda \delta_\alpha^\gamma - 2\mu (\tilde{D}_\delta \bar{S}_{\alpha \cdot \cdot}^{\delta\gamma} - \tilde{D}_\delta \bar{S}_{\alpha \cdot \cdot}^{\gamma\delta}) \right. \\ \left. - 2\mu (\bar{S}_{\alpha\varepsilon\delta} \bar{S}^{\varepsilon\delta\gamma} - \bar{S}_{\alpha\varepsilon\delta} \bar{S}^{\varepsilon\gamma\delta}) \right\} \quad (3.12)$$

where : $\tilde{G}^\gamma_\alpha = \tilde{R}^\gamma_\alpha - \frac{R}{2} \delta_\alpha^\gamma$ is the Einstein tensor obtained with the Christoffel symbol only, and : $K_{\gamma\delta} = K_{. \gamma\varepsilon\delta}^\varepsilon$.

The equations (3.11) and (3.12) are nonlinear, as are gauge theory equations. In the next two sections we shall look at these equations for three particular cases : the central symmetric gravitational field, the expanding universe at present

time and the asymptotic gravitational field induced by a rotating body, in order to show that they do not contradict the present observations. If the quadratic part of the Lagrangian (3.1) does not exist ($\eta = 0$), then the equations (3.11) are satisfied if the torsion is null, therefore the equations (3.12) are simply the usual Einstein equations with a cosmological constant. In the case of the Einstein-Hilbert Lagrangian, the torsion is coupled to the spin density directly via a contact term and does not propagate. Here, equation (3.11b) shows that torsion can propagate.

4 The static central symmetric case.

The solution of the Einstein equations in the static spherical symmetric case is given by the Schwarzschild linear element. Besides torsion terms, the Lagrangian (2.10) contains also a term corresponding to a cosmological constant. We shall proceed as follows : we first consider the linear element which is solution of the Einstein equations with cosmological constant. Then we compute the term : $\tilde{D}_R^{\alpha\beta\gamma} = \tilde{D}_\delta \tilde{R}^{\alpha\beta\gamma\delta}$ of (3.11), and look at the consequences for the torsion.

We start from the following linear element (see for instance [6]) :

$$ds^2 = e^\nu dt^2 - e^\sigma dr^2 - r^2 (d\theta^2 + \sin^2(\theta) d\varphi^2) \quad (4.1)$$

where: ν and σ are functions of the radius : r . We set :

$$c = \cos(\theta) \quad s = \sin(\theta) \quad c_\varphi = \cos(\varphi) \quad s_\varphi = \sin(\varphi)$$

The Einstein equations with cosmological constant are satisfied if :

$$\nu = -\sigma = \ln\left(1 + \frac{\alpha}{r} - \frac{\lambda}{3} r^2\right) \quad (4.2)$$

(for a discussion of this expression, see for instance [7]).

The non-zero components of : $\tilde{D}_R^{\alpha\beta\gamma}$ (up to the anti-symmetry in the two first indices) are :

$$\tilde{D}_R^{010} = e^{-\nu-2\sigma} \left\{ \partial_r \tilde{R}^0_{\cdot 101} + \left(\frac{2}{r} - \sigma'\right) \tilde{R}^0_{\cdot 101} + \frac{\nu'}{r^2} \right\} \quad (4.3)$$

where the sign prime means the derivation with respect to r , and where :

$$\tilde{R}^0_{\cdot 101} = -\frac{\nu''}{2} + \frac{\nu'}{4} (\sigma' - \nu') \quad (4.4)$$

$$\tilde{D}_R^{122} = \frac{e^{-2\sigma}}{2r^3} \left\{ \sigma'' + \frac{\nu'^2}{r^2} + \frac{\nu'\sigma'}{2} - \sigma'^2 + \frac{2}{r^2} \right\} - \frac{e^{-\sigma}}{r^5} \quad (4.5)$$

$$\tilde{D}_R^{133} = \frac{1}{s^2} \tilde{D}_R^{122}$$

These non-zero components are consistent with the constraints (A.7). Now, using the expression (4.2), one gets : $\tilde{D}_R^{\alpha\beta\gamma} = \tilde{D}_\delta \tilde{R}^{\alpha\beta\gamma\delta} = 0$ (4.6)

The equations (3.11) are satisfied if the contorsion tensor (the torsion tensor) is null. This means that the equations (3.12) reduce to the ordinary Einstein equations, and that (4.1) with (4.2) is a solution of the problem, as in the ordinary case. However, we do not claim that it is the unique solution to equations (3.11) and (3.12).

5 The expanding isotropic universe.

In this section we consider that the global structure of the universe is described by an isotropic space and that the space-time linear element is [6] :

$$ds^2 = a^2(\eta) (d\eta^2 + \gamma_{\mu\nu} dx^\mu dx^\nu) \quad (5.1)$$

where : $\eta = x^0$ is the conformal time. We use the standard notation for it,

although this symbol has already been used as the coefficient of the quadratic term in the Lagrangian, but there should be no confusion. The metric coefficients : $\gamma_{\mu\nu}$ are assumed to be functions of the space coordinates only, and are given, up to a sign, by (A2) and (A5) for the hyperbolic and spherical cases respectively. As usually done, we set : $H = \frac{\dot{a}}{a}$, where the dot above the letter means the derivative with respect to η .

The first step is to compute the term : $D_R^{\alpha\beta\gamma}$ of (3.11) using (3.10), as in the previous section. The only non-zero components of $D_R^{\alpha\beta\gamma}$ are :

$$\tilde{D}_R^{0\mu\nu} = \frac{1}{a^6} \left[\ddot{H} - 2H(1+H^2) \right] \gamma^{\mu\nu} \quad (5.2)$$

in agreement with the general form (A9).

Then we have to compute : $K^{\alpha\beta}_{\cdot\cdot\gamma\delta}$. According to (A9) the only non-zero components of the contorsion tensor are : $\bar{S}^{0\mu\nu} = q(\eta) \gamma^{\mu\nu}$ (5.3)

up to the anti-symmetry on the first two indices. In (5.1), η and x^μ have no dimension, then $\tilde{\Gamma}^{\alpha}_{\cdot\beta\gamma}$ has no dimension, therefore q must have the dimension of a length to the power of -4. Using (5.3), the non-zero components of : $K^{\alpha\beta}_{\cdot\cdot\gamma\delta}$ are (no sum over μ) :

$$\begin{aligned} K^{0\mu}_{\cdot\cdot 0\mu} &= a^2 (\dot{q} + 4Hq) \quad \forall \mu \\ K^{\mu\nu}_{\cdot\cdot \rho\eta} &= a^2 q (2H - a^4 q) (\delta_\rho^\mu \delta_\eta^\nu - \delta_\eta^\mu \delta_\rho^\nu) \end{aligned} \quad (5.4)$$

and finally, the only components of $\tilde{D}_\delta K^{\alpha\beta\gamma\delta} = \tilde{D}_K^{\alpha\beta\gamma}$ left are :

$$\tilde{D}_K^{0\mu\nu} = -\frac{1}{a^4} \left[\partial_\eta^2 (a^2 q) + 4H \partial_\eta (a^2 q) + 2\dot{H} a^2 q + 2H a^6 q^2 \right] \gamma^{\mu\nu} \quad (5.5)$$

Gathering all the terms of (3.11b), one obtains :

$$D_R^{0\mu\nu} = \tilde{D}_R^{0\mu\nu} + \tilde{D}_K^{0\mu\nu} + q a^4 \gamma_{\rho\eta} \tilde{R}^{\rho\mu\nu\eta} + q a^4 \gamma_{\rho\eta} K^{\rho\mu\nu\eta}$$

with : $\tilde{R}^{\rho\mu\nu\eta} = \frac{1}{a^6} (H^2 - \epsilon) (\gamma^{\mu\nu} \gamma^{\rho\eta} - \gamma^{\rho\nu} \gamma^{\mu\eta})$

and where : $\epsilon = 1$ for a spherical space, $\epsilon = -1$ for the hyperbolic case and $\epsilon = 0$ for an Euclidean space.

With all the above expressions, and assuming that there is no macroscopic spin density, equation (3.11b) becomes :

$$\begin{aligned} 4\eta \left\{ \tilde{D}_R^{0\mu\nu} + \tilde{D}_K^{0\mu\nu} + 2 \frac{q}{a^2} (H^2 - \epsilon) \gamma^{\mu\nu} - 2 a^2 q^2 (2H - a^4 q) \gamma^{\mu\nu} \right\} \\ + (\mu - 2) q \gamma^{\mu\nu} = 0 \end{aligned} \quad (5.6)$$

In order to discuss the Einstein equations we need the following results :

$$\begin{aligned} a^2 \tilde{G}^0_{\cdot 0} &= 3H^2 + 3\epsilon & \tilde{G}^0_{\cdot \mu} &= 0 & a^2 \tilde{G}^\mu_{\cdot \nu} &= (\epsilon + H^2 + 2\dot{H}) \delta_\nu^\mu \\ K^0_{\cdot 0} - \frac{K}{2} &= 3a^2 q (a^4 q - 2H) \\ K^\mu_{\cdot \mu} - \frac{K}{2} &= a^2 q (a^4 q - 10H - 2 \frac{\dot{q}}{q}) \quad \forall \mu \end{aligned}$$

In equation (3.12) one has to compute : $E^{\alpha\gamma} = \tilde{D}_\delta \Omega^{\alpha\delta\gamma} - \tilde{D}_\delta \Omega^{\alpha\gamma\delta}$ for the different cases. We obtain :

$$E^{00} = -3Hq \quad E^{0\mu} = 0 \quad E^{\mu\nu} = -(\dot{q} + 5Hq) \gamma^{\mu\nu}$$

Finally, calling T^α_γ the energy-momentum tensor, the Einstein equations (3.12)

are : $\tilde{R}^0_{\cdot 0} - \frac{\tilde{R}}{2} - \lambda - 12H a^2 q + 9a^6 q^2 = T^0_{\cdot 0}$
 $\left[\tilde{R}^\mu_{\cdot \nu} - \frac{\tilde{R}}{2} \delta_\nu^\mu - \lambda \delta_\nu^\mu \right] - [20H a^2 q + 4a^2 \dot{q} - a^6 q^2] \delta_\nu^\mu = T^\mu_{\cdot \nu}$ (5.7)

and equation (3.12) is null if the indices are : $\alpha = 0$, $\gamma \equiv \mu$ or the converse.

Are equations (5.6) and (5.7) compatible with the present day observable universe ? The only length in the problem is $a(\eta)$ in (5.1), therefore we set :

$q = r a^{-4}$, which gives : $\tilde{D}_K^{0\mu\nu} = -\frac{1}{a^6} [\ddot{r} - 4H^2 r + 2Hr^2] \gamma^{\mu\nu}$

Neglecting the terms \ddot{H} and H^2 in (5.2), equation (5.6) becomes :

$$\frac{4\eta}{a^2} \{2H + \ddot{r} - 4H^2 r + 2Hr^2 - 2r(H^2 - \varepsilon) + 2r^2(2H - r)\} + (\mu - 2)r = 0$$

When : $a \rightarrow \infty$, this equation implies : $r \rightarrow 0$. Neglecting again the terms H^2 and r^2 , one has, at lowest order, when $a \rightarrow \infty$: $r \sim -\frac{4\eta H}{a^2}$

This asymptotic value brings, in (5.7), corrections of order a^{-2} with respect to the usual Einstein equations without torsion. Therefore, when the universe has become large, the torsion is invisible.

6 The asymptotic gravitational field of a rotating body.

The space-time linear element induced by a rotating body, far from it, is written as [8] :

$$ds^2 = a dt^2 - b dr^2 - r^2 (d\theta^2 + \sin^2(\theta) d\varphi^2) - \frac{2l}{r} s^2 dt d\varphi \quad (6.1)$$

where l is proportional to the source body angular momentum, and where :

$$g_{00} = a = 1 + \frac{\alpha}{r} + O(r^{-2}) \quad -g_{11} = b = \frac{1}{a} + O(r^{-2})$$

with α a constant. In this section, the cosmological constant is set to 0. The non zero Christoffel symbols at leading order and at lowest order in l are :

$$\begin{aligned} \tilde{\Gamma}^0_{.01} &\approx \frac{a'}{2a} & \tilde{\Gamma}^0_{.13} &\approx \frac{3ls^2}{2ar^2} & \tilde{\Gamma}^1_{.11} &= \frac{b'}{2b} & \tilde{\Gamma}^1_{.00} &= \frac{a'}{2b} & \tilde{\Gamma}^1_{.03} &= \frac{ls^2}{2br^2} \\ \tilde{\Gamma}^1_{.22} &= -\frac{r}{b} & \tilde{\Gamma}^1_{.33} &= -\frac{rs^2}{b} & \tilde{\Gamma}^2_{.03} &= -\frac{ls^2c}{r^3} & \tilde{\Gamma}^2_{.12} &= \frac{1}{r} & & \\ \tilde{\Gamma}^2_{.33} &= -sc & \tilde{\Gamma}^3_{.01} &\approx -\frac{l}{2r^3} \left(\frac{a'}{a} + \frac{1}{r} \right) & & & & & & \\ \tilde{\Gamma}^3_{.02} &\approx \frac{lc}{r^3s} & \tilde{\Gamma}^3_{.13} &\approx \frac{1}{r} & \tilde{\Gamma}^3_{.23} &= \frac{c}{s} \end{aligned} \quad (6.2)$$

The components of the Ricci tensor built with the Christoffel symbols are :

$$\begin{aligned} \tilde{R}^0_0 &= \frac{1}{2b} \left[\frac{a''}{a} - \frac{a'^2}{2a^2} + \frac{a'}{a} \left(\frac{a'}{r} - \frac{b'}{2b} \right) \right] + O(r^{-6}) \\ \tilde{R}^1_1 &= \frac{1}{2b} \left[-\frac{a''}{a} + \frac{a'^2}{2a^2} + \frac{a'b'}{2ab} \right] + \frac{b'}{br} + O(r^{-6}) \\ \tilde{R}^2_2 &= \tilde{R}^3_3 = \frac{1}{2br} \left[\frac{a'}{a} - \frac{b'}{b} \right] - \frac{1}{r^2} \left(1 - \frac{1}{b} \right) + O\left(\frac{l^2}{r^6}\right) \\ R_{01} &= R_{02} = R_{13} = R_{23} = 0 & R_{12} &= O\left(\frac{l}{r^5}\right) \\ \tilde{R}^0_{03} &= \tilde{R}^3_{30} = \frac{ls^2}{r^3} \left[1 - \frac{1}{b} - \frac{5a'r}{4ab} - \frac{b'r}{4b^2} \right] = O\left(\frac{l}{r^4}\right) \end{aligned}$$

The important point is that the diagonal terms of the Ricci tensor components due to the Christoffel symbols are unchanged with respect to the spherical symmetric case at lowest order, and that these diagonal terms are of order : r^{-3} .

As before the next step is to compute the quantities : $\tilde{D}_R^{\alpha\beta\gamma} = \tilde{D}_\delta \tilde{R}^{\alpha\beta\gamma\delta}$ which appear in equations (3.10) and (3.11). The calculations are done at lowest order only, which means that we write : $\tilde{\Gamma}^{\alpha}_{. \beta\gamma} = \tilde{\Gamma}^{\alpha}_{0. \beta\gamma} + \tilde{\Gamma}^{\alpha}_{1. \beta\gamma}$, where the first term on the right corresponds to the case : $l = 0$, and where the second term are of order : l . Likewise, we write : $\tilde{R}^{\alpha\beta\gamma\delta} = \tilde{R}^{\alpha\beta\gamma\delta}_0 + \tilde{R}^{\alpha\beta\gamma\delta}_1$ with the same meaning. In the calculations of the covariant derivatives, terms like : $\tilde{\Gamma}^{\alpha}_{1. \varepsilon\delta} \tilde{R}^{\varepsilon\beta\gamma\delta}_1$ are neglected. The result is that the non-zero terms are :

$$\begin{aligned} \tilde{D}_R^{013} &\approx \frac{3l}{br^6} \left(\frac{c^2}{s^2} - \frac{1}{2} \right) + O\left(\frac{l}{r^7}\right) & \tilde{D}_R^{031} &\approx O\left(\frac{l}{r^6}\right) & \tilde{D}_R^{130} &\approx \frac{3l}{br^6} \left(2 - \frac{c^2}{2s^2} \right) + O\left(\frac{l}{r^7}\right) \\ \tilde{D}_R^{023} &\approx \frac{lc}{r^6s} \left(\frac{9}{r} - \frac{a'}{a} \right) & \tilde{D}_R^{032} &\approx -\tilde{D}_R^{023} & \tilde{D}_R^{230} &\approx -2\tilde{D}_R^{023} \end{aligned} \quad (6.3)$$

This is consistent with the fact that : $\tilde{D}_R^{\alpha\beta\gamma} = 0$ when : $l = 0$ (section 4).

From equation (3.11) this means that the contorsion has non-zero components.

In each of these non-zero terms the indices are all different. In other words, the quantities : $\tilde{D}_R^{\alpha\beta\gamma}$ are null if two of the indices are equal. If we look back at the Christoffel symbols, the non-zero terms in (6.3) bear the same triplet of indices as the Christoffel symbols of order l . This suggests to try a solution of the form

$$\begin{aligned} : \quad \bar{S}_{\alpha\beta\gamma} &= \frac{\omega_{\alpha\beta\gamma}(r,\theta)}{r^p} & \text{if :} & \quad \{\alpha, \beta, \gamma\} = \{0, 1, 3\} \\ \bar{S}_{\alpha\beta\gamma} &= \frac{\omega_{\alpha\beta\gamma}(r,\theta)}{r^q} & \text{if :} & \quad \{\alpha, \beta, \gamma\} = \{0, 2, 3\} \end{aligned} \quad (6.4)$$

In the following calculations, the indices are raised or lowered using only the diagonal terms of the metric tensor, because the non-diagonal terms introduce higher order contributions.

The hypotheses (6.4) simplify the evolution equations. Firstly, since all the indices are different in (6.4), one has : $\bar{S}^\alpha = 0$. Secondly, when two of the indices of the triplet $\{\alpha, \beta, \gamma\}$ are equal, equation (3.11b) (in vacuum) reduces to the condition : $D_R^{\alpha\beta\gamma} = 0$.

Now we consider the fifth term of Einstein equation (3.12) :

$$\begin{aligned} \tilde{D}_\delta \bar{S}_{\alpha\dots}^{\delta\gamma} - \tilde{D}_\delta \bar{S}_{\alpha\dots}^{\gamma\delta} &= \partial_\delta \bar{S}_{\alpha\dots}^{\delta\gamma} - \partial_\delta \bar{S}_{\alpha\dots}^{\gamma\delta} - \tilde{\Gamma}_{\alpha\delta}^\epsilon (\bar{S}_{\alpha\dots}^{\delta\gamma} - \bar{S}_{\alpha\dots}^{\gamma\delta}) \\ &+ \tilde{\Gamma}_{\epsilon\delta}^\delta (\bar{S}_{\alpha\dots}^{\epsilon\gamma} - \bar{S}_{\alpha\dots}^{\gamma\epsilon}) + \tilde{\Gamma}_{\epsilon\delta}^\gamma (\bar{S}_{\alpha\dots}^{\delta\epsilon} - \bar{S}_{\alpha\dots}^{\epsilon\delta}) \end{aligned}$$

If : $\alpha = \gamma$, this term is null, either because : $\bar{S}_{\alpha\dots}^{\beta\gamma} = 0$, or because it contains the products of the symmetric Christoffel symbol times an anti-symmetric term. The hypotheses (6.4) provide a solution to the equation of motion. The calculations are long and will not be detailed, only the conclusions are given. First we consider equation (3.11b) when 2 indices of the triplet $\{\alpha, \beta, \gamma\}$ are equals. We have seen that (3.11b) reduces to : $D_R^{\alpha\beta\gamma} = 0$. The contorsion terms of (3.10) can compensate for the second order terms of g_{00} and g_{11} if : $p \geq 1$ and : $q \geq 0$. Then, when the triplet $\{\alpha, \beta, \gamma\}$ is equal to the sets defined in (6.4), the contorsion terms of (3.10) are of the same order of those of the expressions (6.3) if : $p \geq 2$ and : $q \geq 1$. But, the left side last two terms of (3.11b) are of the same order of the terms (6.3) if : $p \geq 4$ and : $q \geq 3$. The six functions of (6.4) provides enough freedom to cancel the six terms (6.3).

Conclusion : the equations of motion (3.11) and (3.12) can be satisfied if :

$$\bar{S}_{013} , \bar{S}_{031} , \bar{S}_{130} \sim O\left(\frac{1}{r^4}\right) \quad \bar{S}_{023} , \bar{S}_{032} , \bar{S}_{230} \sim O\left(\frac{1}{r^3}\right) \quad (6.5)$$

Going back to the Christoffel symbols (6.2), these contorsion components bring second order corrections to them and would have no effects on the trajectories of test particles. At last, the contorsion terms (at lowest order) are proportional to the angular momentum as expected.

Appendix A . Spaces of constant curvature, isotropy and invariances.

Spaces of constant curvature [9] are isotropic spaces and symmetric spaces. This appendix recalls the definition of hyperbolic and spherical spaces and their invariance properties. The 3-dimensional spaces H^3 and S^3 are respectively hyper-spheres in Minkowski space M^4 and Euclidean space E^4 . The invariance with respect to the rotation group in these larger spaces implies local rotational invariance with respect to a point and invariance by transvections [9] which are defined below.

6.1 Hyperbolic spaces.

The hyperbolic n dimensional space H^n is defined as the upper part of the sphere of radius $\sqrt{|K|}$ in the Minkowski space M^{n+1} . More precisely, if $\{x^\alpha\}$ are Cartesian coordinates in M^{n+1} with origin O_M , H^n is the surface defined by:

$$\sum_{\alpha=0}^{n-1} x^\alpha x^\alpha - x^n x^n = K$$

where: $K < 0$ and $x^n \geq \sqrt{|K|}$. We set: $R = \sqrt{|K|}$ and use spherical coordinates. For H^3 in M^4 : $x^0 = R \operatorname{sh} \chi c$, $c = \cos(\theta)$, $s = \sin(\theta)$

$$\begin{aligned} x^1 &= R \operatorname{sh} \chi s c_\varphi & ; & \quad c_\varphi = \cos(\varphi) \quad , \quad s_\varphi = \sin(\varphi) \\ x^2 &= R \operatorname{sh} \chi s s_\varphi & \quad x^3 &= R \operatorname{ch} \chi \\ \chi &\geq 0 & \quad \theta &\in [0, \pi] \quad \quad \varphi \in [0, 2\pi] \end{aligned} \quad (\text{A.1})$$

Then the linear element of H^3 is:

$$\begin{aligned} ds^2 &= (dx^0)^2 + (dx^1)^2 + (dx^2)^2 - (dx^3)^2 \\ &= R^2 \left[d\chi^2 + \operatorname{sh}^2 \chi ((d\theta)^2 + s^2 (d\varphi)^2) \right] \end{aligned} \quad (\text{A.2})$$

The coordinates (χ, θ, φ) are the Riemann normal (spherical) coordinates with origin at $\chi = 0$, which corresponds to the point $(0, 0, 0, R)$ in M^4 . The curvature tensor is: $R_{\alpha\beta\gamma\delta} = -\frac{1}{R^2} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma})$, the Ricci tensor is: $R_{\alpha\beta} = -\frac{2}{R^2} g_{\alpha\beta}$, and the scalar curvature: $R_H = -\frac{6}{R^2}$. R is a scale factor, in the following it is set to 1.

H^n , which is a space of constant curvature, is a symmetric space. A transvection in a symmetric space is an isometry which generalizes the notion of translation in Euclidean space. It is defined as the product of two successive symmetries with respect to two different points A and B . The geodesic going through these two points is invariant and is called the base geodesic. In H^3 one can perform a rotation around this base geodesic, it commutes with the transvection and the base geodesic is invariant. The base geodesic of a transvection γ is given by the intersection of the invariant plane, associated to the real eigenvalues of the $SO(3,1)$ element representing γ in M^4 , with H^3 . If we call L the length of the transvection, which is twice the distance between A and B , a point p whose spherical coordinates are: (χ, θ, φ) is transformed, by a transvection along Oz , into a point q of coordinates: $(\chi_q, \theta_q, \varphi_q)$ given by:

$$\begin{aligned} \operatorname{ch}(\chi_q) &= \operatorname{ch}(\chi) \operatorname{ch}(L) + \operatorname{sh}(\chi) \operatorname{sh}(L) c \\ c_q &= (\operatorname{ch}(\chi) \operatorname{sh}(L) + c \operatorname{sh}(\chi) \operatorname{ch}(L)) / \operatorname{sh}(\chi_q) \\ \varphi_q &= \varphi \end{aligned} \quad (\text{A.3})$$

where: $c_q = \cos(\theta_q)$.

6.2 Spherical spaces.

The spherical n dimensional space S^n is defined as the sphere of radius R in the Euclidean space E^{n+1} . For S^3 in E^4 : $x^0 = R \sin \chi c$

$$\begin{aligned} x^1 &= R \sin \chi s c_\varphi & \quad x^2 &= R \sin \chi s s_\varphi & \quad x^3 &= R \cos \chi \\ \chi, \theta &\in [0, \pi] & \quad \varphi &\in [0, 2\pi] \end{aligned} \quad (\text{A.4})$$

Then the linear element of S^3 is:

$$\begin{aligned} ds^2 &= (dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \\ &= R^2 \left[d\chi^2 + \sin^2 \chi ((d\theta)^2 + s^2 (d\varphi)^2) \right] \end{aligned} \quad (\text{A.5})$$

As above R is set to 1. With the same notations as in the hyperbolic case, a point of coordinates: (χ, θ, φ) is transformed, by a transvection along Oz ,

into a point whose coordinates are given by :

$$\begin{aligned} \cos(\chi_q) &= \cos(\chi) \cos(L) - \sin(\chi) \sin(L) c \\ c_q &= (\cos(\chi) \sin(L) + c \sin(\chi) \cos(L)) / \sin(\chi_q) \\ \varphi_q &= \varphi \end{aligned} \quad (\text{A.6})$$

6.3 Rotational invariance.

In order to obtain the transformation law of tensors, we need to know how local frames transform. With the above Riemann normal coordinates the problem is the same as in Euclidean space. We shall use local orthonormal frames whose basis vectors are aligned with the vectors of the natural frame associated with the usual spherical coordinates in the 3 dimensional Euclidean space. These vectors, named $\vec{h}_1(x)$, $\vec{h}_2(x)$, $\vec{h}_3(x)$, are respectively aligned with $\vec{e}_r(x)$, $\vec{e}_\theta(x)$, $\vec{e}_\varphi(x)$. The Riemann normal coordinates are defined by :

$$x = \chi s c_\varphi \quad y = \chi s s_\varphi \quad z = \chi c$$

Rotation around the Oz axis: $\chi_q = \chi$, $\theta_q = \theta$, $\varphi_q = \varphi + \alpha$

In that case the transformed local frame of point p coincides with the local frame at q : $\vec{f}_a(q) = \vec{h}_a(q)$. Invariance with respect to these rotations does not say much except that tensors do not depend on φ .

Rotation around the Ox axis: the action of a rotation of angle α is written using Cartesian coordinates :

$$\begin{aligned} x_q = x &\rightarrow s_q c_{q\varphi} = s c_\varphi & y_q = c_\alpha y + s_\alpha z &\rightarrow s_q s_{q\varphi} = c_\alpha s s_\varphi + s_\alpha c \\ z_q = -s_\alpha y + c_\alpha z &\rightarrow c_q = -s_\alpha s s_\varphi + c_\alpha c \\ s_q d\theta_q &= (s_\alpha c s_\varphi + c_\alpha s) d\theta + s_\alpha s c_\varphi d\varphi \\ s_q^2 d\varphi_q &= -s_\alpha c_\varphi d\theta + (c_\alpha s^2 + s_\alpha s c s_\varphi) d\varphi \end{aligned}$$

We get : $\vec{f}_1(q) = \vec{h}_1(q)$, $\vec{f}_2(q) = a \vec{h}_2(q) + b \vec{h}_3(q)$, $\vec{f}_3(q) = a \vec{h}_3(q) - b \vec{h}_2(q)$ where : $a = (c_\alpha s + s_\alpha c s_\varphi) / s_q$, $b = -s_\alpha c_\varphi / s_q$

The contorsion tensor satisfies: $\overline{S}^{\alpha\beta\gamma} = -\overline{S}^{\beta\alpha\gamma}$. With these transformation laws, we deduce that the only nonzero components of the contorsion tensor are:

$$\begin{aligned} \overline{S}_h^{010} &= -\overline{S}_h^{100} & \overline{S}_h^{101} &= -\overline{S}_h^{011} \\ \overline{S}_h^{202} &= -\overline{S}_h^{022} = \overline{S}_h^{303} = -\overline{S}_h^{033} \\ \overline{S}_h^{212} &= -\overline{S}_h^{122} = \overline{S}_h^{313} = -\overline{S}_h^{133} \end{aligned} \quad (\text{A.7})$$

where the subscript h says that the components are taken with respect to local orthonormal frames. These relations are also valid for any tensor of rank 3 anti-symmetric with respect to the two first indices. Rotations around the Oy axis can be obtained from the product of the two previous rotations, and will therefore be not considered.

6.4 Invariance with respect to transvections.

It is sufficient to consider transvections along the Oz axis only, since transvections in other directions can be deduced by rotation. In hyperbolic space, from the relations (A.3), we have :

$$\begin{aligned} sh^2(\chi_q) &= (sh(\chi) ch(L) + ch(\chi) sh(L) c)^2 + sh^2(L) s^2 \\ sh(\chi_q) d\chi_q &= (ch(L) sh(\chi) + c sh(L) ch(\chi)) d\chi + sh(L) sh(\chi) dc \\ sh(\chi_q) dc_q &= -\frac{sh(L) sh(\chi) s^2}{sh^2(\chi_q)} d\chi + \frac{sh^2(\chi)}{sh^2(\chi_q)} (sh^2(\chi_q) - s^2 sh^2(L)) \frac{1}{2} dc \end{aligned}$$

from which we get :

$$\begin{aligned}
\frac{\partial \chi_q}{\partial \chi} &= \frac{(sh(\chi) ch(L) + ch(\chi) sh(L) c)}{sh(\chi_q)} = \sqrt{1 - \frac{sh^2(L) s^2}{sh^2(\chi_q)}} \\
\frac{\partial \theta_q}{\partial \theta} &= \frac{sh(\chi)}{sh(\chi_q)} \frac{\partial \chi_q}{\partial \chi} \\
\frac{\partial \chi_q}{\partial \theta} &= -s \frac{sh(L) sh(\chi)}{sh(\chi_q)} \\
\frac{\partial \theta_q}{\partial \chi} &= s \frac{sh(L)}{sh^2(\chi_q)} \\
\frac{\partial \varphi_q}{\partial \chi} &= \frac{\partial \varphi_q}{\partial \theta} = 0 \quad \frac{\partial \varphi_q}{\partial \varphi} = 1
\end{aligned} \tag{A.8}$$

With these transformation relations we find that the only nonzero space-time components of a rank 3 tensor, anti-symmetric in the first two indices : $T^{\alpha\beta\gamma} = -T^{\beta\alpha\gamma}$, are of the form :

$$T^{0\mu\nu} = -T^{\mu 0\nu} = q(\eta) \gamma^{\mu\nu} \tag{A.9}$$

The same calculations can be done in the spherical case, using the relations (A.6) instead of (A.3), leading to equation (A.9).

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