THE LEVEL SET FLOW OF A HYPERSURFACE IN $\mathbb{R}^4$ OF LOW ENTROPY DOES NOT DISCONNECT

JACOB BERNSTEIN AND SHENGWEN WANG

ABSTRACT. We show that if $\Sigma \subset \mathbb{R}^4$ is a closed, connected hypersurface with entropy $\lambda(\Sigma) \leq \lambda(\mathbb{S}^2 \times \mathbb{R})$, then the level set flow of Σ never disconnects. We also obtain a sharp version of the forward clearing out lemma for non-fattening flows in $\mathbb{R}^4$ of low entropy.

1. INTRODUCTION

A family of hypersurfaces $\Sigma_t \subset \mathbb{R}^{n+1}$ evolves by mean curvature flow (MCF) if it satisfies

$$(1.1) \qquad \left(\frac{\partial}{\partial t} \mathbf{x}_{\Sigma_t} \right)^{\perp} = \mathbf{H}_{\Sigma_t}$$

here a hypersurface is a smooth submanifold of codimension one and $\mathbf{x}_{\Sigma_t}$ is the position vector, $\mathbf{H}_{\Sigma_t}$ is the mean curvature vector and $\perp$ is the projection onto the normal of Σ_t. A fundamental property of MCF is that the flow of a closed hypersurface must develop a singularity in finite time. If one considers the level set flow (see Chen-Giga-Goto [5] and Evans-Spruck [6–9]), then one obtains a canonical set theoretic weak mean curvature flow that persists through singularities and, for closed initial data, vanishes in finite time. By definition, as long as the flow is smooth, then the topology of the hypersurfaces does not change, however this need not be the case for the level set flow after the first singularity.

When $n = 1$, it follows from Gage-Hamilton [10] and Grayson [11] that the flow disappears when it becomes singular. In particular, the flow remains connected until it disappears. In contrast, when $n > 1$, non-degenerate neck-pinch examples show that there are flows that become singular without disappearing. In these examples, the level set flow disconnects after the neck-pinch singularity. In [2], the first author and L. Wang showed that, when $n = 2$ and the initial entropy is small (see (2.1) below), then the flow disappears at its first singularity. This result makes use of a classification of singularity models in $\mathbb{R}^3$ of low entropy from [2] and whether such a classification exists in higher dimension is unknown. In the present note we show that when $n = 3$ and the initial hypersurface is closed, connected and of low entropy, then even if the flow forms a singularity before it disappears, its level set flow remains connected until its extinction time.

Theorem 1.1. *Let $\Sigma \subset \mathbb{R}^4$ be a closed, connected hypersurface and let $\{\Gamma_t\}_{t \in [0,T]}$ be the level set flow with initial condition $\Gamma_0 = \Sigma$ and extinction time T. If $\lambda(\Sigma) \leq \lambda(\mathbb{S}^2 \times \mathbb{R})$, then, for all $t \in [0,T]$, Γ_t is connected. Moreover, if $W[t] = \mathbb{R}^4 \setminus \Gamma_t$, then $W[t]$ has at most two connected components for all $t \in [0,T]$.*

A technical feature of the level set flow is that it may "fatten", i.e., develop non-empty interior. If this occurs in Theorem 1.1, then there will be a $T_0 \in [0,T)$ so that $W[t]$ has

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two components for $t \in [0, T_0)$ and one component for $t \in [T_0, T]$ – see Theorem 4.1 for a proof of this fact. Roughly speaking, the idea is that if the flow fattens, then there are two natural flows starting from Σ, the innermost flow and the outermost flow and the level set flow lies between these two flows. In this situation, T_0 is the extinction time of the inner flow and T is the extinction time of the outer flow.

In [17], the second author showed, for flows of low entropy, a forward in time analog of the clearing out lemma. Specifically he showed that if such a flow reaches the point x_0 at time t_0, then the flow remains near x_0 after t_0 until it disappears. This is a forward in time analog of the standard, unconditional, clearing out lemma – e.g., [8, Theorem 3.1] – that says that if the flow reaches x_0 at time t_0, then the flow must be near x_0 at earlier times. Theorem 1.1 allows us to sharpen the result from [17] and prove the forward clearing out lemma in $\mathbb{R}^4$ with the optimal upper bound on the entropy.

Corollary 1.2. *Given $\epsilon > 0$, there exist uniform constants $C = C(\epsilon) > 0$ and $\eta = \eta(\epsilon) > 0$, so that if $\{M_t\}_{t \in [0,T]}$ is a a non-fattening level set flow in $\mathbb{R}^4$ that starts from a smooth closed hypersurface $M_0 \subset \mathbb{R}^4$ with $\lambda(M_0) \leq \lambda(\mathbb{S}^2 \times \mathbb{R}) - \epsilon$, $x_0 \in M_{t_0}$ and $M_{t_0 + R^2} \neq \emptyset$, then for all $\rho \in (0, \frac{R}{2C})$,*

$$\mathcal{H}^3(B_\rho(x_0) \cap M_{t_0 + C^2 \rho^2}) \geq \eta \rho^3.$$

Here $\mathcal{H}^3$ denotes three-dimensional Hausdorff measure.

Remark 1.3. The entropy assumption can be seen to be sharp by considering the translating bowl soliton in $\mathbb{R}^4$ of larger and larger speed, and, in the closed setting, by considering a sequence of unit spheres at increasing distance from one another and joined by a thin tube.

2. NOTATION AND BACKGOUND

Let $B_R(x_0)$ be the open ball in $\mathbb{R}^{n+1}$ centered at x_0 and, for a set $K \subset \mathbb{R}^{n+1}$, let

$$T_r(K) = \bigcup_{x \in K} B_r(x)$$

be the r-tubular neighborhood of K. For any $\rho > 0$, $x_0 \in \mathbb{R}^{n+1}$ and subset $\Omega \subset \mathbb{R}^{n+1}$, set

$$\Omega + x_0 = \{x + x_0 \in \mathbb{R}^{n+1} : x \in \Omega\} \text{ and } \rho\Omega = \{\rho x : x \in \Omega\}.$$

Following [4], the entropy of a closed hypersurface, Σ, is defined by

$$(2.1) \qquad\qquad \lambda(\Sigma) = \sup_{(\mathbf{y}, \rho) \in \mathbb{R}^{n+1} \times \mathbb{R}} F(\rho\Sigma + \mathbf{y})$$

where F is the Gaussian area of Σ given by

$$(2.2) \qquad\qquad F(\Sigma) = (4\pi)^{-\frac{n}{2}} \int_\Sigma e^{-\frac{|x|^2}{4}} \, d\mathcal{H}^n.$$

The entropy and Gaussian area readily extend to the less regular objects studied in geometric measure theory. Clearly, $\lambda(\mathbb{R}^n) = 1$. If $\mathbb{S}^n$ is the unit n-sphere in $\mathbb{R}^{n+1}$, then

$$\Lambda_k = \lambda(\mathbb{S}^k) = \lambda(\mathbb{S}^k \times \mathbb{R}^{n-k}) = F(\sqrt{2k}\mathbb{S}^k)$$

and so, by a computation of Stone [16],

$$(2.3) \qquad\qquad 2 > \Lambda_1 > \frac{3}{2} > \Lambda_2 > \ldots > \Lambda_n > \ldots \to \sqrt{2}.$$

Let us now briefly recall some background results in the theory of (weak) mean curvature flow – our primary sources are [6–9] and [13]. We begin with the level set flow, whose mathematical theory was developed by Chen-Giga-Goto [5] and Evans-Spruck [6–9].

Let Γ be a non-empty compact subset of $\mathbb{R}^{n+1}$. Select a Lipschitz function u_0 so that $\Gamma = \{x : u_0(x) = 0\}$ and so that $u_0(x) = -C$ when $|x| \geq R$ for some constants $C, R > 0$. For such a u_0, $\{u_0 \geq a > -C\}$ is compact. In [6], Evans-Spruck established the existence and uniqueness of viscosity solutions to the initial value problem:

$$(2.4) \qquad \begin{cases} u_t = \Sigma_{i,j=1}^{n+1}(\delta_{ij} - u_{x_i}u_{x_j}|Du|^{-2})u_{x_i x_j} & \text{on } \mathbb{R}^{n+1} \times (0, \infty) \\ u = u_0 & \text{on } \mathbb{R}^{n+1} \times \{0\}. \end{cases}$$

Setting $\Gamma_t = \{x : u(x,t) = 0\}$, define $\{\Gamma_t\}_{t \geq 0}$ to be the level set flow of $\Gamma = \Gamma_0$. As shown in [6], the Γ_t depend only on Γ and are independent of the choice of u_0. The level set flow has a uniqueness property and satisfies an avoidance principle. As such, for any closed initial set, the level set flow vanishes after a finite amount of time. Furthermore, as long as the initial set is a closed hypersurface, the level set flow agrees with the classical solution to (1.1) as long as the latter exists. A technical feature of the level set flow is that some time slices may develop non-trivial interior – a phenomena called "fattening". Importantly, initial sets are generically non-fattening – see for instance [13, Theorem 11.3]

In addition to the level set flow, we will also need to consider the measure theoretic version of MCF introduced by Brakke. An n-*dimensional Brakke flow* (or *Brakke motion*), $\mathcal{K}$, in $\mathbb{R}^{n+1}$ is a family of Radon measures $\mathcal{K} = \{\mu_t\}_{t \in I}$, that satisfies (1.1) in the sense of being a negative gradient flow, see [13] for the precise definition. The Brakke flow is *integral* if for almost every $t \in I$, $\mu_t \in \mathbf{IM}_n(\mathbb{R}^{n+1})$, that is, μ_t is an integer n-rectifiable Radon measure. The Hausdorff n-measure, $\mathcal{H}^n$ restricted to any classical solution of (1.1) is an integral Brakke flow.

Denote the parabolic rescaling and translation of a Brakke flow $\mathcal{K} = \{\mu_t\}$ by

$$D_\rho \mathcal{K} = \left\{ \mu_{\rho^{-2}t}^{\rho,0} \right\} \text{ and } \mathcal{K} - (x_0, t_0) = \left\{ \mu_{t+t_0}^{1,x_0} \right\}$$

where

$$\mu^{\rho,x_0}(A) = \rho^n \mu(\rho^{-1}A + x_0).$$

It follows from the Brakke's compactness theorem [13, 7.1] and the Huisken monotonicity formula [12, 14] that given an integral Brakke flow $\mathcal{K} = \{\mu_t\}_{t \in I}$ with uniformly bounded area ratios, for any $t_0 > \inf I$ and $x_0 \in \mathbb{R}^{n+1}$ and any sequence $\rho_i \to \infty$ there exists a subsequence $\rho_{i_j} \to \infty$ so that $D_{\rho_{i_j}}(\mathcal{K} - (x_0, t_0))$ converges (in the sense of Brakke flows – see [13]) to a Brakke flow $\mathcal{T} = \{\nu_t\}_{t \in \mathbb{R}}$. We call such a flow a *tangent flow* to $\mathcal{K}$ at (x_0, t_0) and denote the set of all possible limits (for different sequences of scalings) by $\mathrm{Tan}_{(x_0,t_0)}\mathcal{K}$. By Huisken's monotonicity formula, $\mathcal{T} \in \mathrm{Tan}_{(x_0,t_0)}\mathcal{K}$ is backwardly self-similar. If $\nu_{-1} = \mathcal{H}^n \llcorner \Upsilon$ for a smooth hypersurface Υ, then Υ satisfies the equation

$$(2.5) \qquad \qquad \mathbf{H}_\Upsilon + \frac{\mathbf{x}^\perp}{2} = 0.$$

Any hypersurface, Υ, that satisfies (2.5) is called a *self-shrinker* and is *asymptotically conical* if $\lim_{\rho \to 0} \rho \Upsilon = C$ in $C_{loc}^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ for some regular cone C. For instance, any hyperplane through the origin is an asymptotically conical self-shrinker.

A feature of Brakke flows is that they may suddenly vanish. In order to handle technical issues that arise from this possibility we will need Ilmanen's enhanced motions [13, 8.1] [18]. Following the formulation in [18], a pair $(\tau, \mathcal{K})$ is an *enhanced motion*, if $\tau \in \mathbf{I}_{n+1}^{loc}(\mathbb{R}^{n+1} \times \mathbb{R})$ is a locally $(n+1)$-dimensional integral current in space-time and $\mathcal{K} = \{\mu_t\}_{t \in \mathbb{R}}$ is a Brakke flow that together satisfy

(1) $\partial \tau = 0$ and $\partial(\tau_{t \geq s}) = \tau_s$ and $\tau_t \in \mathbf{I}_n(\mathbb{R}^{n+1})$ for each time slice t
(2) $\partial \tau_t = 0$ for all t

(3) $t \mapsto \tau_t$ is continuous in the flat topology
(4) $\mu_{\tau_t} \leq \mu_t$ for all t
(5) $V_{\mu_t} = V_{\tau_t} + 2W_t$ for some integral varifold W_t for a.e. t. In other words, they are compatible for a.e. t as defined in [18].

Here τ is the called the *undercurrent* and $\mathcal{K}$ is the *overflow*. Likewise $(\tau, \mathcal{K})$ is an *enhanced motion* with initial condition $\tau_0 \in \mathbf{I}_n(\mathbb{R}^{n+1} \times \{t_0\})$ if the above holds for all $t \geq t_0$ and $\partial \tau = \tau_0$. An enhanced motion $(\tau, \mathcal{K})$ is a *matching motion* if $\mu_{\tau_t} = \mu_t$ for a.e. t for which this makes sense.

Associated to each $E \subset \mathbb{R}^{n+1} \times \mathbb{R}$ of locally finite perimeter, there is a unique $(n + 2)$-dimensional integral current $[E] \in \mathbf{I}_{n+2}^{loc}(\mathbb{R}^{n+1} \times \mathbb{R})$. Similarly, given an oriented dimension-k submanifold $\Sigma \subset \mathbb{R}^{n+1} \times \mathbb{R}$ there is a unique $[\Sigma] \in \mathbf{I}_k^{loc}(\mathbb{R}^{n+1} \times \mathbb{R})$. If $\partial^* E$ is the reduced boundary of E, then $[\partial^* E] = \partial[E] \in \mathbf{I}_{n+1}^{loc}(\mathbb{R}^{n+1} \times \mathbb{R})$. As such, there is an integer $(n + 1)$-rectifiable Radon measure $\mathcal{H}^n \llcorner \partial^* E$ – see [13] for details. In what follows when we refer to a set of finite perimeter, we mean a specific set, E, that has finite perimeter, not an equivalence class of sets. In particular, we may *a priori* have $\overline{\partial^* E} \neq \partial E$.

We extend the notion of canonical boundary motion from [1] – see also [3, 13]. These flows are special cases of flows introduced by Ilmanen in [13] that synthesize the level set flow and Brakke flow in a natural way and are key to our approach.

Definition 2.1. A *canonical boundary motion* is a triple $(E_0, E, \mathcal{K})$ consisting of an open bounded set $E_0 \subset \mathbb{R}^{n+1} \times \{0\}$ with ∂E_0 a smooth closed hypersurface, an open bounded set $E \subset \mathbb{R}^{n+1} \times [0, \infty)$ of finite perimeter and a Brakke flow $\mathcal{K} = \{\mu_t\}_{t \geq 0}$ so:

(1) $E = \{(x, t) : u(x, t) > 0\}$, where u solves equation (2.4) with u_0 chosen so $E_0 = \{x : u_0(x) > 0\}$ and $\partial E_0 = \{x : u_0(x) = 0\}$;
(2) The level set flow of ∂E_0 is non-fattening;
(3) For $t \geq 0$, each $E_t = \{x : (x, t) \in E\}$ is of finite perimeter and $\mu_t = \mathcal{H}^n \llcorner \partial^* E_t$.

If, in addition,

(4) $\{u = 0\} = \overline{\partial^* E}$ in $\mathbb{R}^{n+1} \times (0, \infty)$,

where u is from Item (1), then $(E_0, E, \mathcal{K})$ is a *strong canonical boundary motion*.

Remark 2.2. Observe, $\{u > 0\} = E \subset \bar{E} \subset \{u \geq 0\}$ for a canonical boundary motion and $\bar{E} = \{u \geq 0\}$ for a strong canonical boundary motion. If $\Gamma_t = \{x \in \mathbb{R}^{n+1} | u(x, t) = 0\}$, then $\{\Gamma_t\}_{t \geq 0}$ is the level set flow of $\Gamma_0 = \Sigma$ and is non-fattening. Clearly, $\partial E_t \subset \Gamma_t$, but equality need not hold – even for strong canonical boundary motions. For instance, at the extinction time of any compact flow one does not have equality.

By [13, 11.4], for a E_0 with the property that the level set flow of ∂E_0 is non-fattening, there are E and $\mathcal{K}$ so $(E_0, E, \mathcal{K})$ is a canonical boundary motion. In general, the non-fattening condition is not enough to ensure the existence of a strong canonical boundary motion, however, in [13, 12.11], Ilmanen shows such existence for "generic" E_0.

Finally, we introduce the following notation for a level set flow $\{\Gamma_t\}_{t \geq 0}$ in $\mathbb{R}^{n+1}$, $n \geq 1$,

$$W[t] = \mathbb{R}^{n+1} \setminus \Gamma_t$$

$$W[s, r] = \{(x, t) | x \in (\mathbb{R}^{n+1} \setminus \Gamma_t), s \leq t \leq r\} = \bigcup_{t \in [s, r]} W[t]$$

$$n(t) = \#\{\text{connected components of } W[t]\} \in \mathbb{N} \cup \{\infty\}.$$

As Γ_t is compact and $n \geq 1$, there is exactly one unbounded component of $W[t]$, denoted by $W^-[t]$. Let $W^+[t] = W[t] \backslash W^-[t]$ be the bounded components and set

$$W^\pm[s, r] = \bigcup_{t \in [s,r]} W^\pm[t].$$

3. PROOF OF THEOREM 1.1 FOR STRONG CANONICAL BOUNDARY MOTIONS

In this section we show Theorem 1.1 for flows that are strong canonical boundary motions. We begin with several preliminary results. The first is an elementary topological result – we include a proof for the sake of completeness.

Lemma 3.1. *Let $\Gamma \subset \mathbb{R}^{n+1}$ be a compact set. If $\mathbb{R}^{n+1} \backslash \Gamma$ has exactly two components, $W^\pm$, and $\Gamma = \partial W^\pm$, then Γ is connected.*

Proof. Suppose that Γ is not connected. Let K be one component of Γ and $K' = \Gamma \backslash K \neq \emptyset$. Observe that both K and K' are compact and so there is a $r > 0$ so that $T_r(K) \cap T_r(K') = \emptyset$ and, hence, $T_r(\Gamma)$ is not connected. Let $\hat{W}^\pm = W^\pm \cup T_r(\Gamma)$. Clearly, $\hat{W}^\pm$ are open sets with $\hat{W}^+ \cap \hat{W}^- = T_r(\Gamma)$. For each $x \in \Gamma$, $W^\pm \cap B_r(x) \neq \emptyset$ as $\Gamma = \partial W^\pm$. As the union of intersecting connected sets is connected, $W^\pm \cup B_r(x)$ is connected. It readily follows that both $\hat{W}^-$ and $\hat{W}^+$ are connected. Finally, by the Mayer-Vietoris long exact sequence for reduced homology, as $\mathbb{R}^{n+1} = \hat{W}^+ \cup \hat{W}^-$ is simply connected and both $\hat{W}^\pm$ are connected, $T_r(\Gamma) = \hat{W}^+ \cap \hat{W}^-$ must be connected. This contradicts our choice of r and proves the lemma. $\square$

Another elementary fact is that the level set flow remains connected up to and including its first disconnection time.

Lemma 3.2. *Let $\{\Gamma_t\}_{t \in [0,T]}$ be a level set flow of compact sets in $\mathbb{R}^{n+1}$. If Γ_t is connected for $t \in [0, t_0)$, then Γ_{t_0} is connected.*

Proof. By the definition and basic properties of level set flow $\lim_{t \to t_0^-} \Gamma_t = \Gamma_{t_0}$ in Hausdorff distance. Indeed, on the one hand, by the avoidance principle,

$$\Gamma_{t_0} \subset T_{\sqrt{4n(t_0-t)}}(\Gamma_t).$$

On the other, as the space-time track of the level set flow, $\mathbb{R}^{n+1} \times [0, T] \backslash W[0, T]$, is closed and Γ_{t_0} is compact, for every $\epsilon > 0$, there is a $\delta > 0$ so that if $0 < t_0 - t < \delta$, then $\Gamma_t \subset T_\epsilon(\Gamma_{t_0})$. Hence, if Γ_{t_0} is disconnected, then for $t < t_0$ close enough to t_0, Γ_t is disconnected, proving the claim. $\square$

The next result summarizes and extends [3] and provides a description of the regularity properties of strong canonical boundary motions flows in $\mathbb{R}^4$ of low entropy.

Proposition 3.3. *Let $\left(E_0, E, \mathcal{K} = \{\mu_t\}_{t \geq 0}\right)$ be a strong canonical boundary motion in $\mathbb{R}^4$. Suppose the flow has extinction time T and $\Sigma_0 = \partial E_0$ satisfies $\lambda(\Sigma_0) < \Lambda_2$.*

(1) For each $t \in [0, T)$, there are a finite, possibly empty, set of points $x_1, \ldots, x_{m(t)} \in \mathbb{R}^4$ so that $\mu_t = \mathcal{H}^3 \llcorner \Sigma_t$ where Σ_t is a hypersurface in $\mathbb{R}^4 \backslash \{x_1, \ldots, x_m\}$.

(2) For an open dense subset $I \subset [0, T]$, if $t \in I$, then $\mu_t = \mathcal{H}^3 \llcorner \Sigma_t$ where Σ_t is a closed hypersurface. That is, $m(t) = 0$.

(3) Let $(x_0, t_0) \in \mathbb{R}^4 \times (0, T]$ be a point at which $\mathcal{K}$ has positive Gaussian density. If $\{\nu_t\}_{t \in \mathbb{R}} = \mathcal{T} \in \text{Tan}_{(x_0, t_0)} \mathcal{K}$, then $\nu_{-1} = \mathcal{H}^3 \llcorner \Upsilon$ where Υ is a smooth self-shrinker and either Υ is closed or it is asymptotically conical. Moreover, whichever holds depends only on (x_0, t_0) and not on the choice of tangent flow.

(4) *For each* $(x_0, t_0) \in \mathbb{R}^4 \times (0, T]$ *for which* $\mathrm{Tan}_{(x_0, t_0)} \mathcal{K}$ *contains an asymptotically conical shrinker, there is an* $R_0 = R_0(x_0, t_0, \partial E_0) > 0$ *so that for all* $R \in (0, R_0]$

$$\Sigma_{t_0}(x_0, R) = \mathrm{spt}(\mu_{t_0}) \cap B_R^*(x_0) = \Sigma_{t_0} \cap B_R^*(x_0) = \partial E_{t_0} \cap B_R^*(x_0) = \partial^* E_{t_0} \cap B_R^*(x_0),$$

is a connected hypersurface that divides $B_R^*(x_0)$ *into two components, one contained in* E_t *and one disjoint from it. Here* $B_R^*(x_0) = B_R(x_0) \setminus \{x_0\}$.

Proof. Note first that as $(E_0, E, \mathcal{K})$ is a strong canonical boundary motion, $(E, \mathcal{K})$ is a canonical boundary motion in the sense of [3] – see Theorem 2.3 and the discussion at the beginning of Section 4 of [3]. As such, Items (1) and (2) are both immediate consequences of [3, Theorem 4.3] – see [3, Corollary 4.4] and the proof of [3, Theorem 4.5] for details. Item (3) follows from [3, Proposition 4.1 and Lemma 4.2].

It remains to show Item (4). First, set $\epsilon_0 = \Lambda_2 - \lambda(\partial E_0) > 0$. Next observe that if (x_0, t_0) is a singular point of $\mathcal{K}$, then, by hypothesis, it is a non-compact singularity and so, by [3, Theorem 4.2(2)], there is a $\alpha = \alpha(\epsilon_0) > 0$ and a $\rho_0 = \rho_0(x_0, t_0) > 0$ so that for all $(\rho, t) \in (0, \rho_0) \times (t_0 - \rho^2, t_0 + \rho^2)$,

$$A_t(x_0, t_0, \rho) = \Sigma_t \cap \left(B_{2\alpha\rho}(x_0) \setminus \bar{B}_{\frac{1}{2}\alpha\rho}(x_0) \right) = \mathrm{spt}(\mu_t) \cap \left(B_{2\alpha\rho}(x_0) \setminus \bar{B}_{\frac{1}{2}\alpha\rho}(x_0) \right)$$

is a connected non-empty hypersurface that is proper in $B_{2\alpha\rho}(x_0) \setminus \bar{B}_{\frac{1}{2}\alpha\rho}(x_0)$. The same is true if (x_0, t_0) is not a singular point as then $\mathrm{Tan}_{(x_0, t_0)} \mathcal{K}$ consists of a static hyperplane. For $\rho \in (0, \rho_0)$, let

$$A(x_0, t_0, \rho) = \bigcup_{t \in (t_0 - \rho^2, t_0 + \rho^2)} A_t(x_0, t_0, \rho) \times \{t\} \subset \mathbb{R}^4 \times \mathbb{R} = \mathbb{R}^5$$

this is a connected non-empty hypersurface that is proper in the hollow space-time cylinder

$$C(x_0, t_0, \rho) = \left(B_{2\alpha\rho}(x_0) \setminus \bar{B}_{\frac{1}{2}\alpha\rho}(x_0) \right) \times (t_0 - \rho^2, t_0 + \rho^2).$$

Clearly, $A_t(x_t, t_0, \rho) \times \{t\} = A(x_0, t_0, \rho) \cap \left(\mathbb{R}^4 \times \{t\} \right)$ and this intersection is transverse.

By Item (3) of the definition of canonical boundary motion, $\mathrm{spt}(\mu_t) = \overline{\partial^* E_t}$, and so

$$A(x_0, t_0, \rho) = \overline{\partial^* E} \cap C(x_0, t_0, \rho).$$

As $A(x_0, t_0, \rho)$ is smooth, every point is in the reduced boundary and so

$$A(x_0, t_0, \rho) = \partial^* E \cap C(x_0, t_0, \rho).$$

Hence, by Item (4) of the definition of a strong canonical boundary motion,

$$A(x_0, t_0, \rho) = \partial^* E \cap C(x_0, t_0, \rho) = \overline{\partial^* E} \cap C(x_0, t_0, \rho) = \partial E \cap C(x_0, t_0, \rho).$$

Together with the fact that that $A(x_0, t_0, \rho)$ meets $\mathbb{R}^4 \times \{t_0\}$ transversally, this means

$$A_{t_0}(x_0, t_0, \rho) = \partial^* E_{t_0} \cap \left(B_{2\alpha\rho}(x_0) \setminus \bar{B}_{\frac{1}{2}\alpha\rho}(x_0) \right) = \partial E_{t_0} \cap \left(B_{2\alpha\rho}(x_0) \setminus \bar{B}_{\frac{1}{2}\alpha\rho}(x_0) \right).$$

Set $R_0 = 2\alpha\rho_0$ and, for any $R \in (0, R_0)$, let

$$\Sigma_{t_0}(x_0, R) = \bigcup_{i=0}^{\infty} A_{t_0}(x_0, t_0, 2^{-i} R).$$

By the above, $\Sigma_{t_0}(x_0, R)$ is a connected non-empty hypersurface proper in $B_R^*(x_0)$ and, moreover,

$$\Sigma_{t_0}(x_0, R) = \partial^* E_{t_0} \cap B_R^*(x_0) = \partial E_{t_0} \cap B_R^*(x_0) = \mathrm{spt}(\mu_{t_0}) \cap B_R^*(x_0) = \Sigma_{t_0} \cap B_R^*(x_0).$$

Finally, as $\Sigma_{t_0}(x_0, R)$ is connected, non-empty and proper in $B_R^*(x_0)$, $B_R^*(x_0) \backslash \Sigma_{t_0}(x_0, R)$ has two components. On the one hand, $\Sigma_{t_0}(x_0, R) \subset \partial E_{t_0}$ implies at least one of these is a subset of E_t. On the other, $\Sigma_{t_0}(x_0, R) \subset \partial^* E_{t_0}$ means the other is disjoint from E_{t_0}. □

Next we use the above regularity properties to relate the level set flow and its interior for strong canonical boundary motions of low entropy – compare with Remark 2.2.

Proposition 3.4. *Let* $(E_0, E, \mathcal{K} = \{\mu_t\}_{t \geq 0})$ *be a strong canonical boundary motion in* $\mathbb{R}^4$ *with* $\lambda(\partial E_0) < \Lambda_2$ *and let* $\{\Gamma_t\}_{t \in [0,T]}$ *be the level set flow with* $\Gamma_0 = \partial E_0$. *For any* $s \in (0, T]$ *there are a finite, possible empty, set of isolated points of* Γ_s, $p_1, \ldots, p_{M(s)}$, *so that* $\mathcal{K}$ *has a closed singularity at* (p_i, s) *and*

$$\Gamma_s \setminus \{p_1, \ldots, p_{M(s)}\} = \mathrm{spt}(\mu_s) = \partial E_s \setminus \{p_1, \ldots, p_{M(s)}\} = \partial(\mathbb{R}^4 \backslash \bar{E}_s).$$

If, in addition, Γ_s *is connected and not a point, then* $E_s = W^+[s]$ *and* $\Gamma_s = \partial W^\pm[s]$.

Proof. As the level set flow is the biggest flow, $\mathrm{spt}(\mu_t) \subset \Gamma_t$ – see [13, 10.7]. Pick a $s \in (0, T]$, the entropy assumption ensures that there are at most a finite set of points $p_1, \ldots, p_{M(s)} \in \mathbb{R}^4$ so $\mathcal{K}$ has a closed singularity at (p_i, s) – see [3, Theorem 4.3 and Corollary 4.4]. Moreover, there are radii $r_i > 0$ so that the Gaussian density of $\mathcal{K}$ at any (p, s) with $p \in B_{r_i}(p_i) \setminus \{p_i\}$ is zero. Hence, $p_i \notin \mathrm{spt}(\mu_s)$ and so $\mathrm{spt}(\mu_s) \subset \Gamma_s \setminus \{p_1, \ldots, p_{M(s)}\}$.

Now pick a $x_0 \in \Gamma_s \setminus \{p_1, \ldots, p_{M(s)}\}$. Let $\mathcal{T} \in \mathrm{Tan}_{(x_0, s)} \mathcal{K}$ be a tangent flow to $\mathcal{K}$ at the point (x_0, s). By Item (4) of the definition of strong canonical boundary motion, $(x_0, s) \in \overline{\partial^* E}$. Hence, there is a sequence $(x_i, s_i) \in \partial^* E$ with $s_i > 0$ and $\lim_{i \to \infty}(x_i, s_i) = (x_0, s)$. As $(x_i, s_i) \in \partial^* E$, the Gaussian density of $\mathcal{K}$ at (x_i, s_i) is at least 1 and so, by the upper semicontinuity property of Gaussian density, the Gaussian density of $\mathcal{K}$ at (x_0, s) is positive and so $\mathcal{T}$ is non-trivial. Hence, by Item (3) of Proposition 3.3 and the fact that $x_0 \neq p_i$ for any $1 \leq i \leq M(s)$, $\mathcal{T} = \{\nu_t\}_{t \in \mathbb{R}}$ is asymptotically conical. A further consequence is that the p_i are isolated points of Γ_s.

Thus, Item (4) of Proposition 3.3 implies that there is a $R_0 > 0$ so for all $R \in (0, R_0)$, $\mathrm{spt}(\mu_s) \cap B_R^*(x_0)$ is non-trivial. As $\mathrm{spt}(\mu_s)$ is closed, this means that $x_0 \in \mathrm{spt}(\mu_s)$ and hence, $\mathrm{spt}(\mu_s) = \Gamma_t \setminus \{p_1, \ldots, p_{M(s)}\}$ for all $s \in (0, T]$ proving the first equality. To see the second equality, first note that, by definition, $\partial E_s \subset \Gamma_s$. Now suppose that $x_0 \in \Gamma_s \setminus \{p_1, \ldots, p_{M(s)}\}$. By what we have already shown, $x_0 \in \mathrm{spt}(\mu_s)$ and Item (4) of Proposition 3.3 both hold at (x_0, s). Hence, there is a $R_0 > 0$ so for all $R \in (0, R_0)$,

$$\mathrm{spt}(\mu_s) \cap B_R^*(x_0) = \partial E_s \cap B_R^*(x_0)$$

and this intersection is non-empty. As the topological boundary of a set is closed, $x_0 \in \partial E_s$ and so $\Gamma_s \setminus \{p_1, \ldots, p_{M(s)}\} \subset \partial E_s$, completing the proof of the second equality. As $\mathrm{spt}(\mu_s) = \partial E_s \setminus \{p_1, \ldots, p_{M(s)}\}$, Item (4) of Proposition 3.3 and the above argument implies that $\partial E_s \setminus \{p_1, \ldots, p_{M(s)}\} \subset \partial(\mathbb{R}^4 \setminus \bar{E}_s)$. Clearly, any $p_i \in \bar{E}_s$ is an interior point and so $p_i \notin \partial \bar{E}_s$. Hence, as $\partial(\mathbb{R}^4 \setminus \bar{E}_s) = \partial \bar{E}_s \subset \partial E_s$, the third equality follows.

To complete the proof, first observe that, if Γ_s is connected and not a single point, then $M(s) = 0$ – i.e., there are no closed singularities at time s and $\Gamma_s = \partial E_s$. By definition, $E_s \subset W^+[s]$ and $\partial E_s \subset \partial W^+[s] \subset \Gamma_s$. As $\partial E_s = \Gamma_s$, this immediately implies $\Gamma_s = \partial W^+[s]$. Similarly, by definition $\partial W^-[s] \subset \Gamma_s$ and so, for any $x \in \partial W^-[s]$, Item (4) of Proposition 3.3 implies that there is an $R > 0$ so that $B_R^*(x) \cap \Gamma_s$ divides $B_R^*(x)$ into exactly two components, $U^\pm(x)$, with $\partial U^\pm(x) \cap B_R^*(x) = \Gamma_s \cap B_R^*(x)$.

Moreover, up to relabeling, $U^+(x) \subset E_s$ and $U^-(x) \cap E_s = \emptyset$. As $x \in \partial W^-[s]$ and $W^-[s] \cap E_s = \emptyset$, $U^-(x) \subset W^-[s]$ and so $\partial W^-[s] \cap B_R^*(x) = \Gamma_s \cap B_R^*(x)$. Hence, as

$x \in \partial W^-[s] \subset \Gamma_s$, $B_R(x) \cap \Gamma_s \subset \partial W^-[s]$ and so $\partial W^-[s]$ is an open non-empty subset of Γ_s. As $\partial W^-[s]$ is also closed and Γ_s is assumed to be connected, $\Gamma_s = \partial W^-[s]$.

Finally, let $\Omega = W^+[s] \backslash E_s$. As $\partial E_s = \Gamma_s = \partial W^+[s]$, $\partial \Omega \subset \Gamma_s$. For each $x \in \Gamma_s$, Item (4) of Proposition 3.3, implies that, for R sufficiently small, $B_R(x) \backslash \Gamma_s$ consists of two components one disjoint from E_s and one contained in E_s. As $B_R(x) \cap W^-[s] \neq \emptyset$ the component disjoint from E_s is contained in $W^-[s]$ and so is disjoint from Ω. Likewise, the component contained in E_s is disjoint from Ω by construction. Hence, $\Omega \cap B_R(x) = \emptyset$ and so $x \notin \partial \Omega$. As x was arbitrary, this means $\partial \Omega = \emptyset$ which implies $\Omega = \emptyset$. That is, $E_s = W^+[s]$. $\square$

We use the preceding results and ideas from [19] to show that strong canonical boundary motions remain connected until they disappear. That is, we show Theorem 1.1 for strong canonical boundary motions.

Proposition 3.5. *Let $(E_0, E, \mathcal{K} = \{\mu_t\}_{t \geq 0})$ be a strong canonical boundary motion in $\mathbb{R}^4$ with ∂E_0 connected and $\lambda[\partial E_0] < \Lambda_2$. If $\{\Gamma_t\}_{t \in [0,T]}$ is the level set flow with $\Gamma_0 = \partial E_0$ and extinction time T, then Γ_t is connected and $n(t) = 2$ for all $t \in [0, T)$.*

Proof. As ∂E_0 is connected, bounded and $\partial E_0 = \Sigma$ is compact, $W^+[0] = E_0$. As Σ is a connected hypersurface, there is a $\delta > 0$ so that Γ_t is a smooth flow for $t \in [0, \delta]$ and so Γ_t is connected, $n(t) = 2$ and $W^+[t] = E_t$ for $t \in [0, \delta]$. Let

$$t_{dis} = \sup\{t \in (0, T) | n(s) = 2 \text{ and } \Gamma_s \text{ is connected for all } 0 \leq s < t\}$$

be the first possible disconnection time. Clearly, $t_{dis} > \delta$ and if $t_{dis} = T$, then we are done. In what follows we suppose $t_{dis} < T$ and derive a contradiction.

First, observe that, by construction, t_{dis} must be a singular time, but not the extinction time of the flow. As such, for any $(x, t_0) \in \mathbb{R}^4 \times (0, t_{dis}]$, for which $\mathcal{K}$ has positive Gaussian density all tangent flows to $\mathcal{K}$ at (x, t_0) are asymptotically conical. Indeed, by Proposition 3.3, if a tangent flow at (x, t_0) was closed, then, as Γ_t was connected for $t < t_0 \leq t_{dis}$, for $t < t_0$ and t close enough to t_0, $\mathrm{spt}(\mu_t)$ would also be a closed connected hypersurface. This would imply that the whole flow becomes extinct at t_0, contradicting the fact that $t_{dis} < T$ is not the extinction time.

By Lemma 3.2 and the definition of t_{dis}, Γ_t is connected for all $t \in [0, t_{dis}]$. Hence, by Proposition 3.4, for all $t \in [0, t_{dis}]$, $\Gamma_t = \mathrm{spt}(\mu_t) = \partial W^\pm[t]$ and $W^+[t] = E_t$. We conclude that $n(t_{dis}) = 2$. Indeed, if $n(t_{dis}) \geq 3$, then, as $W^-[t_{dis}]$ is connected, there is a component, Ω, of $W^+[t_{dis}]$ so $\Omega' = W^+[t_{dis}] \backslash \Omega$ is non-empty. As $E_{t_{dis}} = W^+[t_{dis}] = \Omega \cup \Omega'$, $\Omega \cap \Omega' = \emptyset$ and Ω, Ω' are both open, $\Gamma_{t_{dis}} = \partial E_{t_{dis}} = \partial\Omega \cup \partial\Omega'$. Hence, as $\Gamma_{t_{dis}}$ is connected, there is an $x \in \partial\Omega \cap \partial\Omega'$. By Item (4) of Proposition 3.3, there is an $R > 0$ so that $B_R^*(x) \cap E_{t_{dis}}$ has exactly one non-empty component, namely, $B_R^*(x) \cap \Omega = B_R^*(x) \cap \Omega'$. This contradicts $\Omega \cap \Omega' = \emptyset$ and implies $n(t_{dis}) = 2$.

We claim there is a $t_1 \in (t_{dis}, T)$ so $n(t_1) > 2$. If not, then, for all $t \in (t_{dis}, T)$, $n(t) = 2$ and there would be no compact singularities at time t as otherwise the flow would become extinct at $t < T$. Moreover, $W^+[t] = E_t$ for all $t \in (t_{dis}, T)$. This is because there is always exactly one unbounded component, $W^-[t]$, and so E_t would have to be the unique component of $W^+[t]$. As there are no compact singularities in $[0, T)$, Proposition 3.4 implies $\Gamma_t = \partial E_t = \partial W^+[t]$ and $\Gamma_t = \partial(\mathbb{R}^4 \backslash \bar{E}_t) = \partial W^-[t]$ and so Γ_t is connected by Lemma 3.1. That is, $t_{dis} = T$ which contradicts our assumption.

For each $t \in [0, T]$, let $\mathcal{C}[t]$ be the set of components of $W[t]$. By [19, Theorem 5.2], for any $0 \leq t < s \leq T$, there is a well-defined map $\pi_{s,t} : \mathcal{C}[s] \to \mathcal{C}[t]$ given by $\pi_{s,t}(\Omega_s) = \Omega_t$ if and only if there is a time-like continuous path in $W[t,s]$, connecting a point in $\Omega_s \times \{s\}$

to a point in $\Omega_t \times \{t\}$. As already observed, $n(t_1) > 2$, while $n(t_{dis}) = 2$. Hence, the pigeonhole principle implies that there are two distinct components $\Omega_1, \Omega_2 \in \mathcal{C}[t_1]$ so that $\pi_{t_1, t_{dis}}(\Omega_1) = \pi_{t_1, t_{dis}}(\Omega_2) = \Omega_0 \in \mathcal{C}[t_{dis}]$ As $n(t_{dis}) = 2$, either $\Omega_0 = W^+[t_{dis}] = E_{t_{dis}}$ or $\Omega_0 = W^-[t_{dis}]$. In the former case, $\Omega_1, \Omega_2 \subset E_{t_1}$ and in the latter Ω_1 and Ω_2 are both disjoint from E_{t_1}.

Pick $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$. By definition, $(x_1, t_1), (x_2, t_1)$ are each connected via time-like paths in $W[t_{dis}, t_1]$ to the same component, Ω_0, of $W[t_{dis}] \times \{t_{dis}\}$. Label the two paths, $p_1(s), p_2(s)$, so that $p_1(1) = (x_1, t_1), p_2(1) = (x_2, t_1)$. As $p_1(0), p_2(0)$ are in the same component of $W[t_{dis}] \times \{t_{dis}\}$, there is a path p_3 in $W[t_{dis}]$ so that $(p_3(0), t_{dis}) = p_1(0), (p_3(1), t_{dis}) = p_2(0)$. By the avoidance principle, there is a universal constant $C > 0$ so that if $B_r(y) \cap \Gamma_{t_{dis}} = \emptyset$, then $(y, t) \subset W[t]$ for any $t \in [t_{dis}, t_{dis} + Cr^2]$. As $p_3([0, 1])$ is compact, we can choose $0 < r_0 < \text{dist}(p_3[0, 1], \Gamma_{t_{dis}})$. Hence,

$$p_3([0, 1]) \times [t_{dis}, t_{dis} + Cr_0^2] \subset W[t_{dis}, t_{dis} + Cr_0^2]$$

As such, if $\delta_1 = \min\left\{\frac{t_1 - t_{dis}}{2}, Cr_0^2\right\}$, then for any $t \in (t_{dis}, t_{dis} + \delta_1), (x_1, t_1), (x_2, t_1)$ can also be connected via time-like paths in $W[t, t']$ to the same components of $W[t]$. That is, $\pi_{t_1, t}(\Omega_1) = \pi_{t_1, t}(\Omega_2)$.

Now let

$$I = \left\{s \in [t_{dis}, t_1] : \pi_{t_1, s}(\Omega_1) \neq \pi_{t_1, s}(\Omega_2)\right\}.$$

Clearly, $t_1 \in I$ and, by what we just established $[t_{dis}, t_{dis} + \delta_1) \cap I = \emptyset$. Let $t_* = \inf(I)$. So $t_{dis} + \delta_1 \leq t_* \leq t_1$ and t_* is a singular time of the flow. Moreover, by the openness argument of the previous paragraph, $t_* \in I$.

Let $\Omega_1^* = \pi_{t_1, t_*}(\Omega_1)$ and $\Omega_2^* = \pi_{t_1, t_*}(\Omega_2)$ be distinct components of $W[t_*]$. If $\mathcal{C}^* = \pi_{t_*, t_{dis}}^{-1}(\Omega_0)$, then Ω_1^* and Ω_2^* are elements of this set and all elements of $\mathcal{C}^*$ are either subsets of E_{t_*} or all are disjoint from E_{t_*}. In fact, as $n(t_{dis}) = 2$, either

$$E_{t_*} = \bigcup_{\Omega^* \in \mathcal{C}^*} \Omega^* \quad \text{or} \quad W[t_*] \setminus E_{t_*} = \bigcup_{\Omega^* \in \mathcal{C}^*} \Omega^*.$$

As there is only one unbounded component of $W[t_*]$, we may, by relabeling, assume that Ω_1^* is bounded.

Claim 3.6. *There is a point $p \in \partial \Omega_1^*$ so that, for any $R > 0$, there exists an element $\Omega^* \in \mathcal{C}^*$ distinct from Ω_1^* so that $B_R(p) \cap \Omega^* \neq \emptyset$. Observe, it is possible $\Omega^* \neq \Omega_2^*$.*

To prove the claim, we only need to prove for the case both $\Omega_1^*, \Omega_2^* \subset E_{t_*}$, the case that they are both disjoint from E_{t_*} follows from the same argument. If the claim is false, then for any $p \in \partial \Omega_1^*$, there is $R_p > 0$ such that for any $\Omega^* \in C^*, \Omega^* \neq \Omega_1^*$, one has $B_{R_p}(p) \cap \Omega^* = \emptyset$. As Ω_1^* is assumed bounded, $\partial \Omega_1^*$ is compact, and so there is a uniform R_0 such that $\text{dist}(\partial \Omega_1^*, \cup_{\Omega^* \in \mathcal{C}^*, \Omega^* \neq \Omega_1^*} \partial \Omega^*) > R_0 > 0$. As $E_{t_*} = \bigcup_{\Omega^* \in \mathcal{C}^*} \Omega^*$,

$$Z_{R_0} = \left\{x : \frac{1}{4}R_0 \leq \text{dist}(x, \Omega_1^*) \leq \frac{3}{4}R_0\right\} \cap \bar{E}_{t_*} = \emptyset.$$

Here Z_{R_0} is compact. By Proposition 3.4, as $\Gamma_{t_*} \setminus \bar{E}_{t_*}$ consists of a finite set of isolated points, one may shrink R_0 so

(3.1) $$Z_{R_0} \cap \left(\bar{E}_{t_*} \cup \Gamma_{t_*}\right) = \emptyset.$$

As we are considering a strong canonical boundary motion, this implies

$$Z_{R_0} \times \{t_*\} \cap \bar{E} = \emptyset.$$

Hence, as $\bar{E}$ is a closed set and Z_{R_0} is a compact set, there is a $\delta_* > 0$ so

$$(3.2) \qquad\qquad Z_{R_0} \times [t_* - \delta_*, t_*] \cap \bar{E} = \emptyset$$

An immediate consequence of this is that, $\pi_{t_*,s}(\Omega_1^*)$ is disjoint from $\pi_{t_*,s}(\Omega^*)$ for any $\Omega^* \in \mathcal{C}^*$ not equal to Ω_1^* and all $s \in [t_* - \delta_*, t_*]$. Indeed, otherwise there would be a continuous space-time curve connecting Ω_1^* and some distinct component, Ω^*, of $\mathcal{C}^*$ that lies entirely in $E \cap W[t_* - \delta_*, t_*]$. However, such a curve would have to intersect Z_{R_0}, contradicting (3.2). Hence, $\pi_{t_1,s}(\Omega_1) \neq \pi_{t_1,s}(\Omega_2)$ for all $s \in [t_* - \delta_*, t_*]$, contradicting the definition of t_*.

To complete the proof, observe that, for the point p given by the claim, one has that, for any small R, either:

  (1) $B_R^*(p) \backslash \partial E_{t^*}$ contains at least three components;
  (2) $B_R^*(p) \backslash \partial E_{t^*}$ contains two components of E_{t_*};
  (3) $B_R^*(p) \backslash \partial E_{t^*}$ contains two components both disjoint from E_{t_*}.

In any case, take a tangent flow at $P = (p, t_*)$. The point P cannot be a compact singularity by the choice of p and so the tangent flow is asymptotically conical. By Item (4) of Proposition 3.3, for small enough $R > 0$, the ball $B_R^*(p) \backslash \partial E_{t^*}$ has only two connected components, one contained in E_{t_*} and one disjoint from E_{t_*}, so none of the above three situations can happen. This contradiction completes the proof. $\qquad\qquad\square$

4. Proof of Theorem 1.1

In this section, we will show Theorem 1.1. In fact, we will show a stronger result from which Theorem 1.1 is an immediate consequence.

Theorem 4.1. *Let Σ be a smooth closed connected hypersurface in $\mathbb{R}^4$ with $\lambda[\Sigma] \leq \Lambda_2$. If $\{\Gamma_t\}_{t \in [0,T]}$ is the level set flow with $\Gamma_0 = \Sigma$ and extinction time T, then, for all $t \in [0, T]$, Γ_t is connected and $n(t) \leq 2$. Moreover, if*

$$E^+ = W^+[0, T] \text{ and } E^- = W^-[0, T] \cup \left(\mathbb{R}^4 \times (T, \infty) \right),$$

then $E^\pm$ are both sets of locally finite perimeter in $\mathbb{R}^4 \times [0, \infty)$ and there are Brakke flows $\mathcal{K}^\pm$ so that

$$\left(\tau^\pm = \pm \left(\partial [E^\pm] + [W^\pm[0] \times \{0\}] \right), \mathcal{K}^\pm \right)$$

are both matching motions with initial condition $[\Sigma \times \{0\}]$. Finally,

$$\overline{\partial^* E^\pm} = \partial E^\pm$$

in $\mathbb{R}^4 \times (0, \infty)$.

Proof. First observe that we may assume $\lambda(\Sigma) < \Lambda_2$. Indeed, suppose that $\lambda(\Sigma) = \Lambda_2$ and consider, $\{\Sigma_t\}_{t \in [0, \delta]}$, the classical solution to (1.1) with $\Sigma_0 = \Sigma$. As Σ is closed, $\lambda(\Sigma) = F[\rho^{-1}(\Sigma - x)]$ for some $\rho > 0$ and $x \in \mathbb{R}^{n+1}$. Hence, by the Huisken monotonicity formula, either $\lambda[\Sigma_\delta] < \Lambda_2$ or $\Sigma = \rho \Upsilon + x$ where Υ is a closed self-shrinker. In the latter case, the theorem is immediate (as the flow will remain smooth until disappearing), while in the former, one can prove the result for Σ_δ and then use the fact that the flow was smooth to conclude it also for Σ.

As Σ is a closed connected hypersurface in $\mathbb{R}^4$, standard topological results, e.g., [15], imply that there is a connected bounded domain $E_0 \subset \mathbb{R}^4$ with $\partial E_0 = \Sigma$. Let $\mathbf{n}$ be the unit normal to Σ that points into E_0. As Σ is smooth, there is an $\epsilon > 0$ so for $|s| < \epsilon$

$$\Sigma_s = \{p + s\mathbf{n}(p) | p \in \Sigma\}$$

is a foliation of $T_\epsilon(\Sigma)$ by hypersurfaces . By shrinking ϵ, if needed, we can also ensure that $\lambda(\Sigma_s) < \Lambda_2$ for $|s| < \epsilon$. Pick a Lipschitz function $u_0 : \mathbb{R}^4 \to \mathbb{R}$ with the property that

   (1) $\{u_0 = s\} = \Sigma_s$ for $|s| < \epsilon$,
   (2) $\{u_0 \leq -\epsilon\}$ is the unbounded component of $\mathbb{R}^4 \backslash T_\epsilon(\Sigma)$; and
   (3) $\{u_0 \geq \epsilon\}$ is the bounded component of $\mathbb{R}^4 \backslash T_\epsilon(\Sigma)$.

Let u be the solution to 2.4 with initial data u_0. As such, if $\Gamma_t^s = \{x | u(t,x) = s\}$, then for $|s| < \epsilon$, $\{\Gamma_t^s\}_{t \geq 0}$ is the level set flow with $\Gamma_0^s = \Sigma_s$. For each $i \geq 1$, pick $s_{\pm i} \in (-\epsilon, \epsilon)$ so that $s_{-i} < s_{-i-1} < 0 < s_{i+1} < s_i$ and $\lim_{i \to \pm \infty} s_i = 0$. Let $E_0^i = \{u_0 > s_i\}$ and $E^i = \{u > s_i\}$. By [13, 12.11], one can choose the s_i so that for $i \neq 0$, there are Brakke flows $\mathcal{K}^i$ so that $\left(E_0^i, E^i, \mathcal{K}^i\right)$ are all strong canonical boundary motion.

By Proposition 3.5, each $\Gamma_t^i = \Gamma_t^{s_i} = \{u = s_i\}$ is connected and for $t \in [0, T_i)$, where T_i is the extinction time of the flow, divides $\mathbb{R}^4$ into two components $W_i^\pm[t]$ which satisfy $\Gamma_t^i = \partial W_i^\pm[t]$ and $W_i^+[t] = E_t^i = \{x | u(t,x) > s_i\}$. Consider the open sets

$$U^+[t] = \bigcup_{i=1}^\infty W_i^+[t] = \{x | u(x,t) > 0\} \text{ and } U^-[t] = \bigcup_{i=1}^\infty W_{-i}^-[t] = \{x | u(x,t) < 0\}.$$

As each $W_i^\pm[t]$ is connected and $U^\pm[t]$ is their nested union, it follows that both the $U^\pm[t]$ are also connected. Moreover, as

$$\Gamma_t = \{x | u(x,t) = 0\} = \mathbb{R}^4 \backslash \left(U^+[t] \cup U^-[t]\right),$$

$W^\pm[t] = U^\pm[t]$. For $i \geq 1$ let,

$$G_i[t] = \mathbb{R}^4 \backslash \left(W_i^+[t] \cup W_{-i}^-[t]\right) = \{x | s_{-i} \leq u(x,t) \leq s_i\}$$

and observe that each $G_i[t]$ is a compact set, $G_{i+1}[t] \subset G_i[t]$ and $\bigcap_{i=1}^\infty G_i[t] = \Gamma_t$. For $t \in [0, T]$, each $G_i[t]$ is connected. Indeed, T_{-i}, the extinction time of $\left\{\Gamma_t^{-i}\right\}_{t \geq 0}$ must satisfy $T_{-i} > T$ and so, when $t \leq T$, Γ_t^{-i} and $W_{-i}^\pm[t]$ are both non-empty and connected. In particular, there is exactly one component, $G_i^-[t]$, of $G_i[t]$ that contains $\Gamma_t^{-i} = \partial W_{-i}^\pm[t]$. Let $G_i^+[t] = G_i[t] \backslash G_i^-[t]$, so $G_i^+[t]$ is closed and disjoint from $G_i^-[t]$. Observe that $W_{-i}^-[t] \cup G_i^-[t]$ is a closed non-empty subset of $\overline{W_i^-[t]} = W_i^-[t] \cup \Gamma_t^i = \{u \leq s_i\}$ that is disjoint from $G_i^+[t]$. As $G_i^+[t]$ is also a closed subset of $\overline{W_i^-[t]}$, $\overline{W_i^-[t]} = W_{-i}^-[t] \cup G_i^-[t] \cup G_i^+[t]$ and the closure of a connected set is connected, $G_i^+[t] = \emptyset$, and so $G_i[t]$ is connected. As the nested intersection of compact connected sets is connected, it follows that Γ_t is connected and so we've proved the first part of the theorem.

To prove the second part of the theorem we observe that for $i \geq 1$, $E^i = W_i^+[0, T]$ is a set of finite perimeter while

$$F^{-i} = \{u < s_{-i}\} = \mathbb{R}^4 \times [0, \infty) \backslash \bar{E}^{-i} = W_{-i}^-[0, T] \cup (\mathbb{R}^{n+1} \times (T, \infty)),$$

is a set of locally finite perimeter. Moreover, there are matching motions

$$\left(\tau^i = \partial[E_i] + [W_i^+[0]], \mathcal{K}^i\right) \text{ and } \left(\tau^{-i} = -\left(\partial[F^{-i}] + [W_{-i}^-[0]]\right), \mathcal{K}^{-i}\right)$$

with initial conditions $[\Sigma_{s_{\pm i}} \times \{0\}]$. As $\lambda(\Sigma_{s_{\pm i}}) < \Lambda_2 < 2$, [17, Theorem 3.4] implies that, up to passing to a subsequence, the two sequences of matching motions converge to matching motions $(\tau^+, \mathcal{K}^+)$ and $(\tau^-, \mathcal{K}^-)$ both with initial condition $[\Sigma \times \{0\}]$. It further follows, from standard compactness results for sets of locally finite perimeter, that the E^i converge, as sets of finite perimeter, to

$$E^+ = W^+[0, T] = \bigcup_{t \in [0, T]} U^+[t] = \{u > 0\}$$

which is also a set of finite perimeter. Likewise, the F^{-i} converge, as sets of locally finite perimeter, to F^- where

$$F^- = W^-[0,T] \cup \left(\mathbb{R}^4 \times (T, \infty) \right) = \left(\bigcup_{t \in [0,T]} U^-[t] \right) \cup \left(\mathbb{R}^4 \times (T, \infty) \right) = \{ u < 0 \}.$$

Set $E^- = F^-$ and observe that $\tau^{\pm} = \pm \left(\partial [E^{\pm}] + [W^{\pm}[0]] \right)$ follows from the continuity of the boundary operator.

It remains only to verify the claim about the reduced boundary. To that end observe that in $\mathbb{R}^4 \times (0, \infty)$

$$\overline{\partial^* E^+} \subset \partial E^+.$$

We now suppose that $(x,t) \in \partial E^+$ and $t > 0$. By definition, for any $r > 0$, $B_r(x,t) \cap E^+ \neq \emptyset$. In particular, for i sufficiently large $B_r(x,t) \cap W_i^+[0,T] \neq \emptyset$. As $x \in \Gamma_t$, we have $x \notin W_i^+[0,T]$ and so there is some point $(y_r, t_r) \in B_r(x,t) \cap \partial W_i^+[0,T]$. As $\left(E_0^i, E^i, \mathcal{K}^i \right)$ is a strong canonical boundary motion, it has only one compact singularity (at the terminal time $T_i < T$) and we can assume $t_r < T_i$. Hence, by Proposition 3.4 that $y_r \in \mathrm{spt}(\mu_{t_r}^i)$ and so (y_r, t_r) has positive Gaussian density for $\mathcal{K}^i$. As $\mathcal{K}^i$ converges to $\mathcal{K}^+$, the upper semicontinuity of Gaussian density implies that (x,t) is a point of positive Gaussian density for $\mathcal{K}^+$. As $(\tau^+, \mathcal{K}^+)$ is a matching motion starting from Σ and τ^+ is the reduced boundary of a set of finite perimeter, $(x,t) \in \overline{\partial^* E^+}$. That is, $\overline{\partial^* E} = \partial E^+$ in $\mathbb{R}^4 \times (0, \infty)$. Arguing in exactly the same way shows that $\overline{\partial^* E^-} = \partial E^-$ in $\mathbb{R}^4 \times (0, \infty)$. $\qquad \square$

Corollary 4.2. *Let Σ be a smooth closed connected hypersurface in $\mathbb{R}^4$ with $\lambda[\Sigma] \leq \Lambda_2$. If $\{\Gamma_t\}_{t \in [0,T]}$, the level set flow of Σ with extinction time T, is non-fattening, then there is a unique strong canonical boundary motion $(E_0, E, \mathcal{K})$, with $\partial E_0 = \Sigma$.*

5. FORWARD CLEARING OUT

In this section apply Theorem 1.1 to prove Corollary 1.2.

Proof of Corollary 1.2. If the Corollary is not true, then there exist $C_i \to 0, \eta_i > 0, R_i > 0, 0 < \rho_i < \frac{R_i}{2C_i}$ satisfying $\frac{\eta_i}{C_i^3} \to 0$ and a sequence of non-fattening level set flows $\{M_{i,t}\}_{t \geq 0}$ with $M_{i,0}$, closed hypersurfaces with $\lambda(M_{i,0}) \leq \Lambda_2 - \epsilon$, $M_{i,t} \neq \emptyset$ for $t \in (t_0, t_0 + R_i^2)$ and so that the flows reach the space-time point (x_0, t_0), but satisfy

$$\mathcal{H}^3(B_{\rho_i}(x_0) \cap M_{t_0 + C_i^2 \rho_i^2}) < \eta_i \rho_i^3.$$

By Theorem 4.1 and Corollary 4.2, the $M_{i,t}$ agree with the slices of a strong canonical boundary motion $(E_{i,0}, E_i, \mathcal{K}_i = \{\mu_{i,t}\})$ for $t \in (t_0, t_0 + R_i^2)$. In particular, in this time interval, by Proposition 3.4,

$$\mu_{i,t} = \mathcal{H}^3 \llcorner M_{i,t}$$

and so $\mu_{i,t}(B_{\rho_i}(x_0)) < \eta_i \rho_i^3$.

Rescale the flows to get a new flow $\tilde{K}_i = D_{\frac{1}{C_i \rho_i}}(\mathcal{K}_i - (x_0, t_0))$ and let $\{\tilde{M}_{i,t}\}$ be the corresponding rescaling of the level set flow $\{M_t\}$. By Brakke's compactness theorem [13, 7.1], up to passing to a subsequence, $\tilde{K}_i$ converges to a limit flow $\tilde{K} = \{\tilde{\mu}_t\}$, and moreover, by [17, Theorem 3.5], $(T_i, \mathcal{K}_i)$ converge to a matching motion $(\tilde{T}, \tilde{K})$. Clearly, $\lambda[\tilde{\mu}_t] \leq \Lambda_2 - \epsilon$. We also have

$$\tilde{\mu}_{i,1}\left(B_{\frac{1}{C_i}}(0) \right) < \frac{\eta_i}{(C_i)^3} \to 0$$

That is, $\tilde{\mu}_1(\mathbb{R}^4) = 0$ and so the limit flow $\tilde{K}$ must be extinct before $t = 1$. As $(\tilde{T}, \tilde{K})$ is a matching motion, this means that $\tilde{K}$ must develop a collapsed singularity at some $t_e \leq 1$. The entropy bound and the classification of singularities given in Proposition 3.3 imply that this singularity has compact support. Hence, by Brakke's regularity theorem, for large enough i, the flow $\{\tilde{M}_{i,t}\}$ must develop a compact singularity at some time $\tilde{t}_i < 2$, and hence $\{M_{i,t}\}$ develops a compact singularity at some time $t_i < t_0 + 2C_i^2 \rho_i^2 < t_0 + \frac{2R_i^2}{4} < t_0 + R_i^2$. As $M_{i,t_0+R_i^2} \neq \emptyset$ and there is a compact singularity before the extinction time, the flow must disconnect before time $t_0 + R_i^2$, contradicting Theorem 1.1. $\qquad\square$

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DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218, USA
Email address: `bernstein@math.jhu.edu`

DEPARTMENT OF MATHEMATICS, BINGHAMTON UNIVERSITY, VESTAL, NY 13850, USA
Email address: `swang@math.binghamton.edu`