Conservative Semi-Lagrangian solvers on mapped meshes

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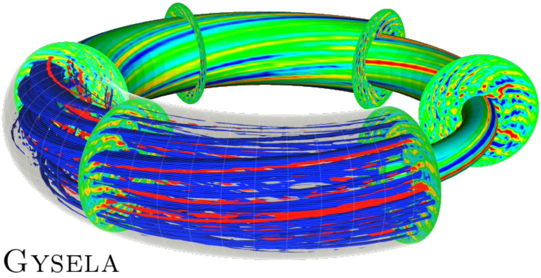
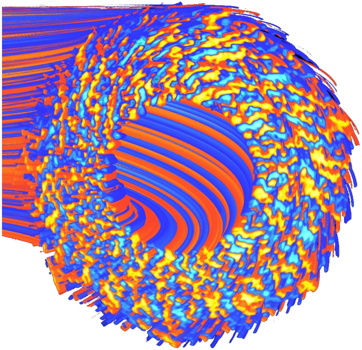
Outline

- Introduction
- Review of the methods on cartesian grids
- BSL, FSL and CSL

GYSELA

GYSELA code (GYrokinetic SEmi LAgrangian), CEA Cadarache





(Courtesy V. Grandgirard)
$5D$ mesh of $272 \cdot 10^9$ points. 31 days on 8192 processors

SeLaLib

SEmi LAgrangian LIbrary

Goal

modular library for the gyrokinetic simulation model by a semi-Lagrangian method

Support

- Large scale Initiative Fusion of INRIA
- ANR Project GYPSI (2010-2014)
- INRIA CALVI Project
- Collaboration with CEA Cadarache

Vlasov equation

Distribution function $f(t, x, v)$ solution of the Vlasov equation
$f(t, x, v)\mathrm{d}x\mathrm{d}v$ represents the probability of finding particules in a volume element $\mathrm{d}x\mathrm{d}v$ at time t at point (x, v) (position, velocity)

$$\partial_t f + v \cdot \nabla_x f + F(t, x) \cdot \nabla_v f = 0$$

• Transport equation
• Non linearity through the field F which depends on f (Poisson, Maxwell)
• Description of the dynamic of charged particles in a plasma

Vlasov-Poisson (1D $\times$ 1D)

Vlasov-Poisson system

$$\partial_t f(t,x,v) + v\partial_x f(t,x,v) + E(t,x)\partial_v f(t,x,v) = 0,$$

where the field E is solution of the Poisson equation

$$\partial_x E(t,x) = \int_{\mathbb{R}} f(t,x,v)dv - 1$$

with zero mean condition ($\int_0^L E(t,x)dx = 0$)

$\Rightarrow$ Simplified model ; first plasmas test cases
$\Rightarrow$ Smooth solution but development of small scales

Guiding center model (1D $\times$ 1D)

The guiding center model

$$\partial_t f(t,x,y) + \partial_x(E_y(t,x,y) f(t,x,v) + \partial_y(-E_x(t,x,y) f(t,x,y)) = 0,$$

where the field $E = (E_x, E_y) = -\nabla\Phi$ is solution of the Poisson equation

$$-\Delta\Phi = f(t,x,y)$$

GYSELA model 5D

1. The variables
$$f = f(t, r, \theta, \phi, v_{//}, \mu)$$
2. Transport equation in $r, \theta, \phi, v_{//}$, loop over μ
3. Quasi neutral equation, similar to a 3D Poisson equation
4. Gyroaverage operator : average on circle of radius depending on μ

Need of curvilinear meshes

1. Geometry of the tokamak
2. Mesh along field lines, invariants
3. Design of robust methods

Difficulties

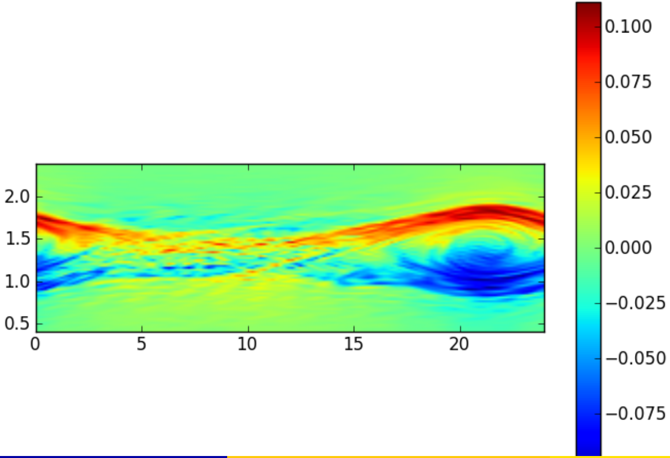
1. Definition of the mapping
2. Generalization of methods first defined on cartesian meshes
3. Keep if possible good properties valid on cartesian meshes

The constant advection case

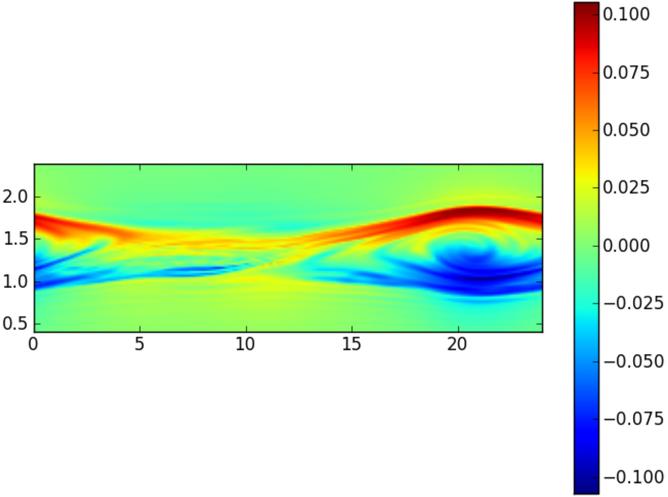
1. Typically for Strang splitting of Vlasov-Poisson
2. High order in space needed
3. Equivalence of conservative and advective form
4. No CFL restriction
5. Higher order splitting in time possible for Vlasov-Poisson
6. FFT type implementation possible
7. Little diffusion better than dispersion
8. Slope limiters possible (useful ?)

Keen testcase with SPL(7)

Some numerical results on the Keen code (Afeyan, Crouseilles, Sonnendrücker, 2012)



Keen testcase with LAG(17)



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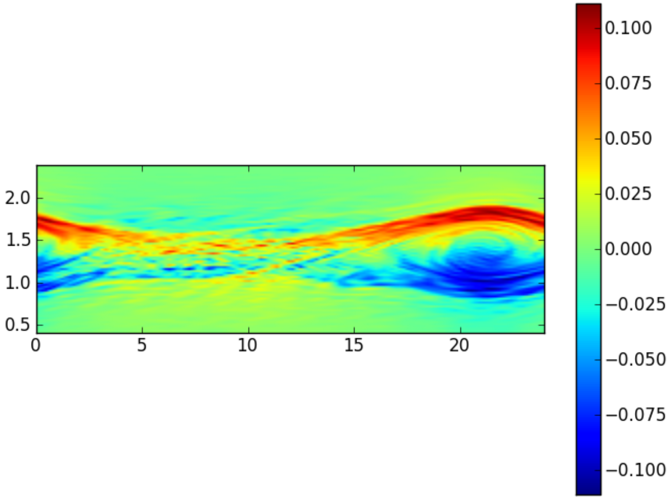
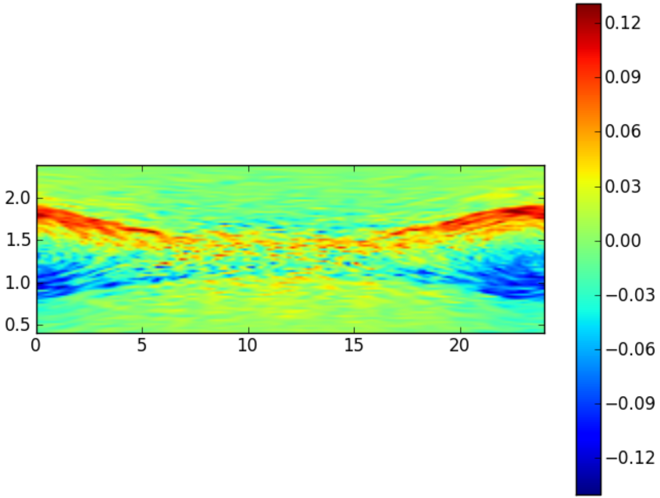
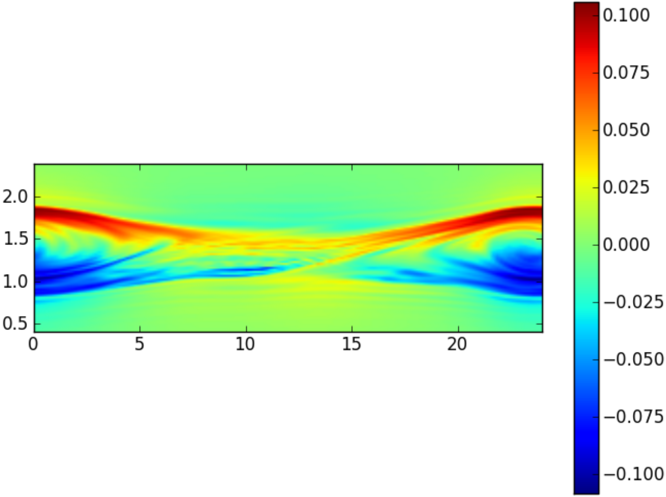


Keen testcase with SPL(7) $\Delta t/10$



Keen testcase with LAG(17) $\Delta t/10$



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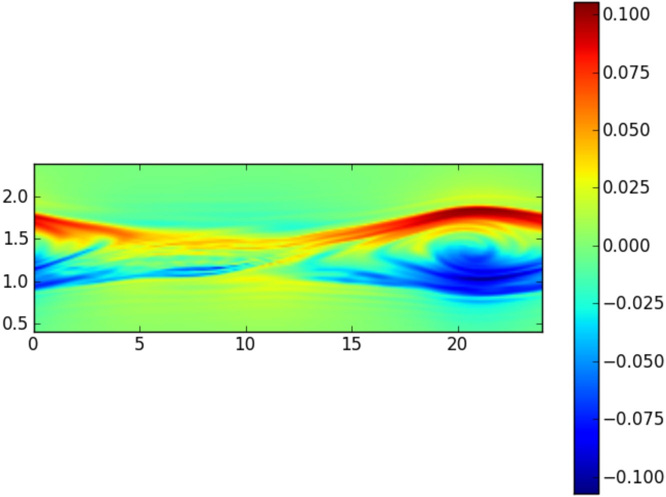


The non constant advection case

1. Typically for the guiding center model
2. History in the team
    - BSL [Sonnendrücker et al., 98]
    - CSL with 1d splitting on conservative form [Filbet et al., 2001, Crouseilles et al., 2010]
    - FSL [Respaud et al., 2010]

5. Mass conservation vs divergence free property
6. Distinction between point values and cell averages

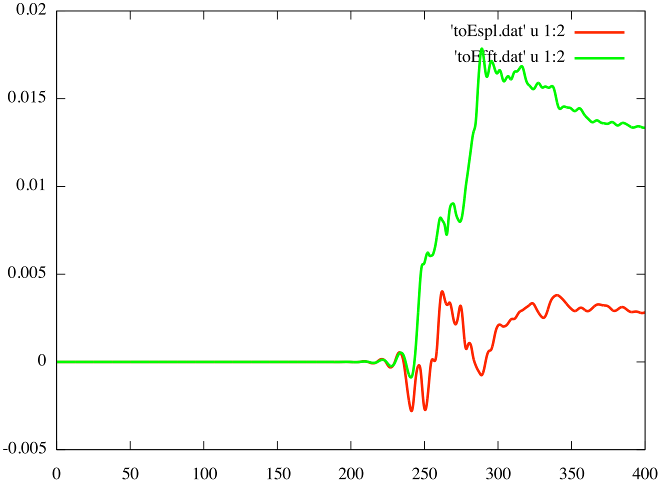


FIGURE: conservation of mass for Fourier (green) and spline (red) $dt = 0.1$ (middle) and $dt = 0.05$(right) versus time. $Nx = Ny = 64$, a guiding center testcase

Exemple in GYSELA

PSM [Braeunig et al., 2010]

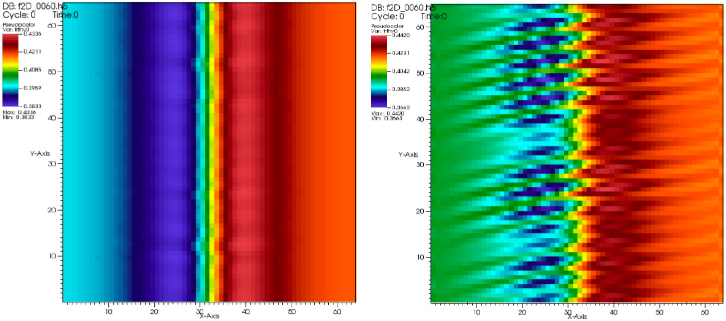
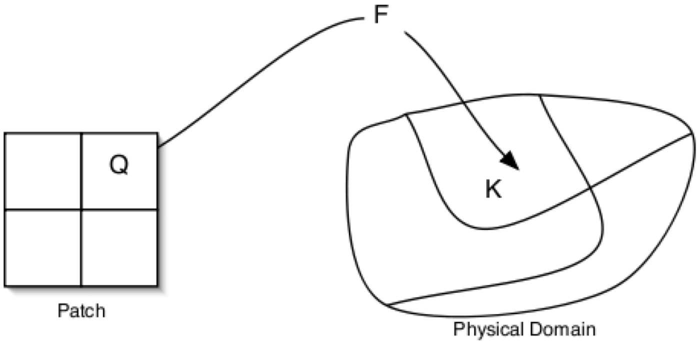


FIGURE: CSL with or without first order divergence free compatibility of the electric field

Properties to fulfill

1. High order accuracy ; problematic of cell average and point values
2. Mass conservation
3. Divergence free condition



Mapping leads to non constant advection case

$\Rightarrow$ similar problems

FSL and BSL in the curvilinear case

Conservative BSL Method

Decomposed function

$$\tilde{f}^n(\eta^1,\eta^2) \approx \sum_{k,l} \tilde{\omega}^n_{k,l}\, S_k(\eta^1)\, S_l(\eta^2)$$

Updating step (interpolation)

$$\tilde{f}^{n+1}_{i,j} = \sum_{k,l} \tilde{\omega}^n_{k,l} S_k(\eta_i^{1^*}) S_l(\eta_j^{2^*})$$

Conservative FSL Method

Decomposed function

$$\bar{f}^n(\eta^1,\eta^2) \approx \sum_{k,l} \bar{\omega}^n_{k,l}\, S_k(\eta^1)\, S_l(\eta^2)$$

Updating step (deposition)

$$\tilde{f}^{n+1}_{i,j} = \frac{1}{J_{ij}} \sum_{k,l} \bar{\omega}^n_{k,l} S_i(\eta_k^{1^*}) S_j(\eta_l^{2^*})$$

Non-Conservative BSL Method

Decomposed function

$$\bar{f}^n(\eta^1,\eta^2) \approx \sum_{k,l} \bar{\omega}^n_{k,l}\, S_k(\eta^1)\, S_l(\eta^2)$$

Updating step (interpolation)

$$\tilde{f}^{n+1}_{i,j} = \frac{1}{J_{ij}} \sum_{k,l} \bar{\omega}^n_{k,l} S_k(\eta_i^{1^*}) S_l(\eta_j^{2^*})$$

Non-Conservative FSL Method

Decomposed function

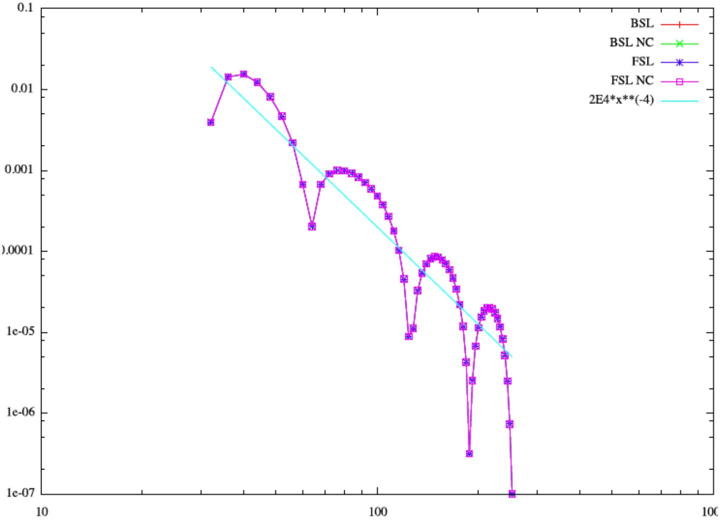
$$\tilde{f}^n(\eta^1,\eta^2) \approx \sum_{k,l} \tilde{\omega}^n_{k,l}\, S_k(\eta^1)\, S_l(\eta^2)$$

Updating step (deposition)

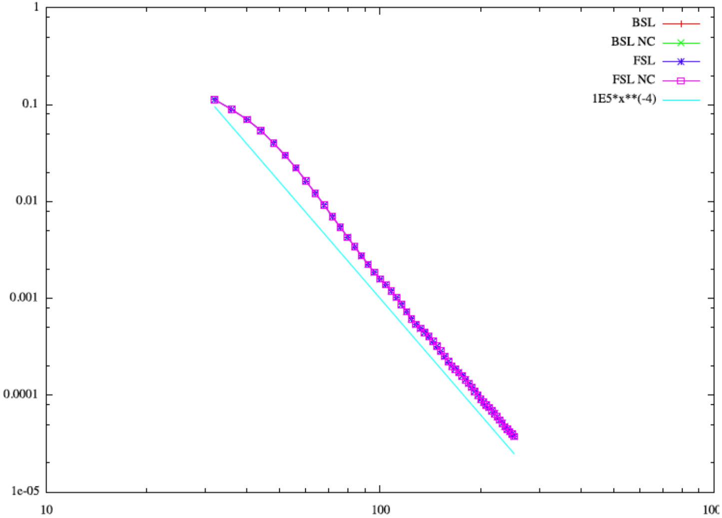
$$f^{n+1}_{i,j} = \sum_{k,l} \tilde{\omega}^n_{k,l} S_i(\eta_k^{1^*}) S_j(\eta_l^{2^*})$$

where $S_q(\eta_p) = S(\eta_p - \eta_q)$.

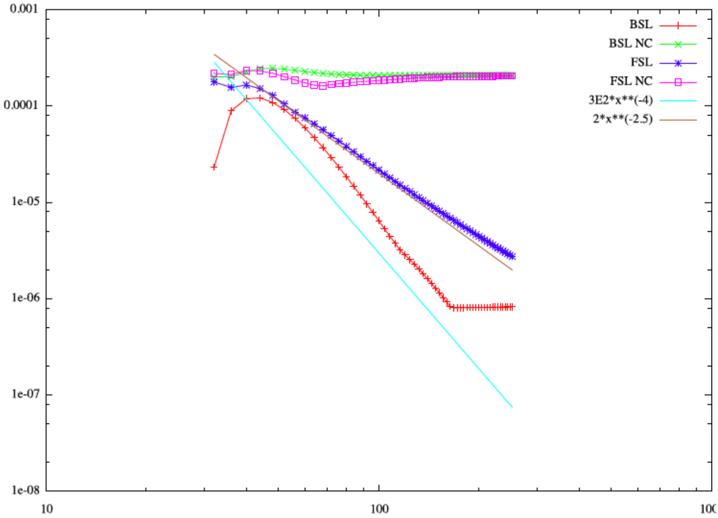
Case of rotation polar coordinates $\Delta t = 0.1\Delta x$



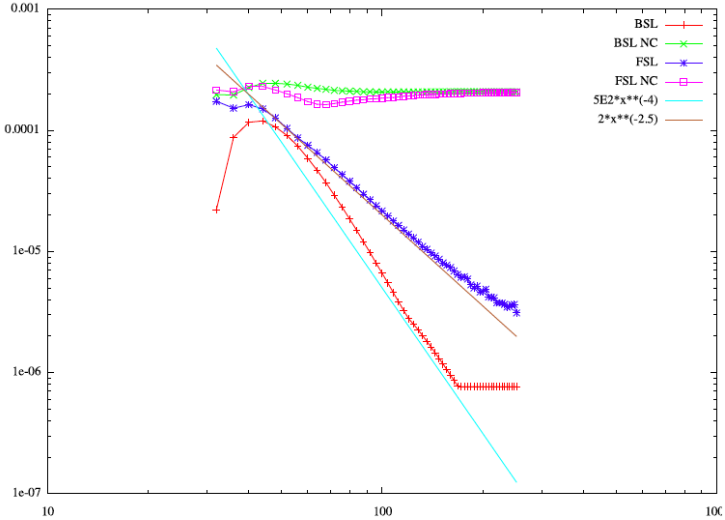
Case of rotation polar coordinates $\Delta t = 0.1$



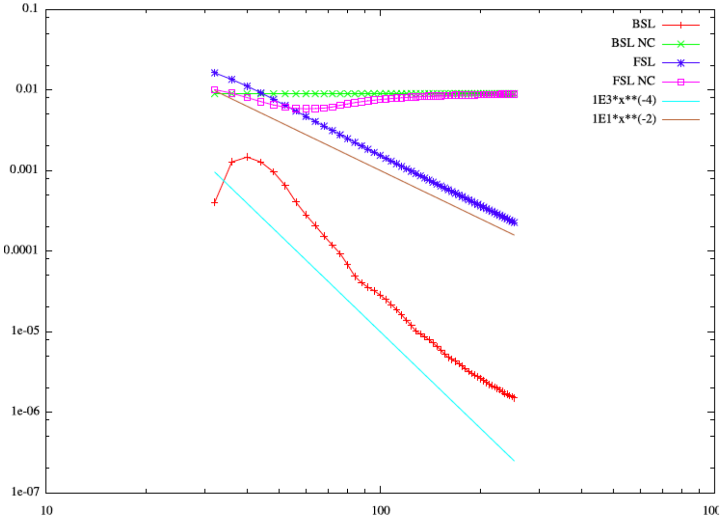
Case of translation polar coordinates $\Delta t = 0.1\Delta x$



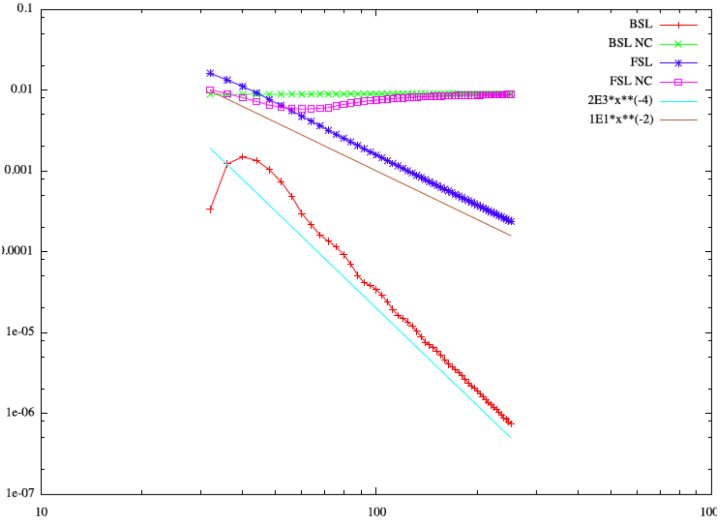
Case of translation polar coordinates $\Delta t = 0.1$



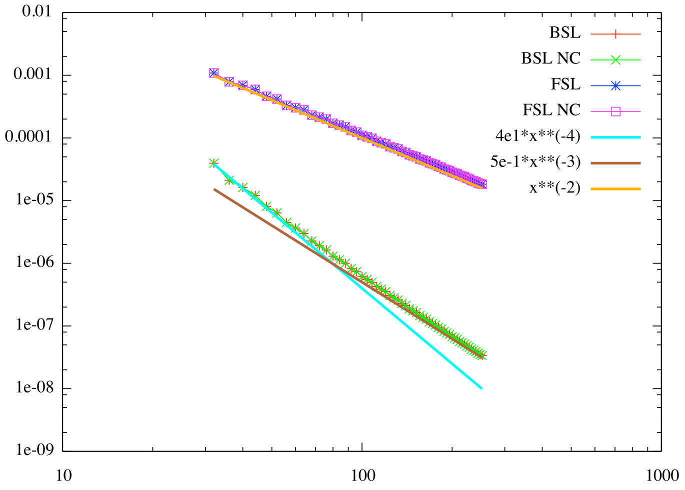
Case of complex field polar coordinates $\Delta t = 0.1\Delta x$



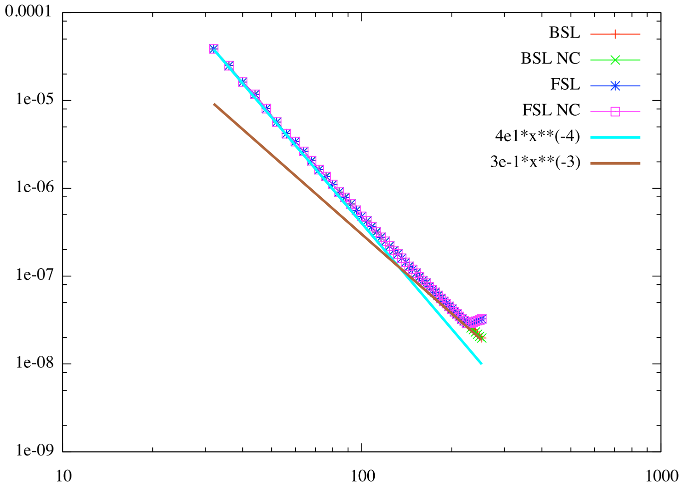
Case of complex field polar coordinates $\Delta t = 0.1$



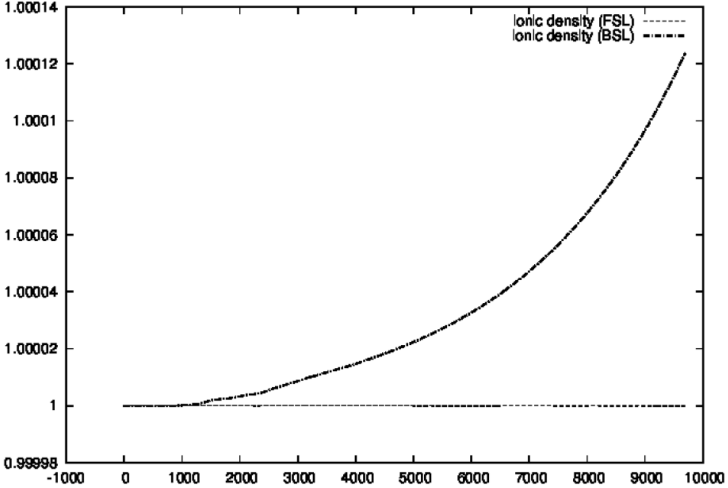
Cartesian case anisotropic rotation
$\Phi = 4(x - x0)^2 + (y - y0)^2 \; dy = dx$



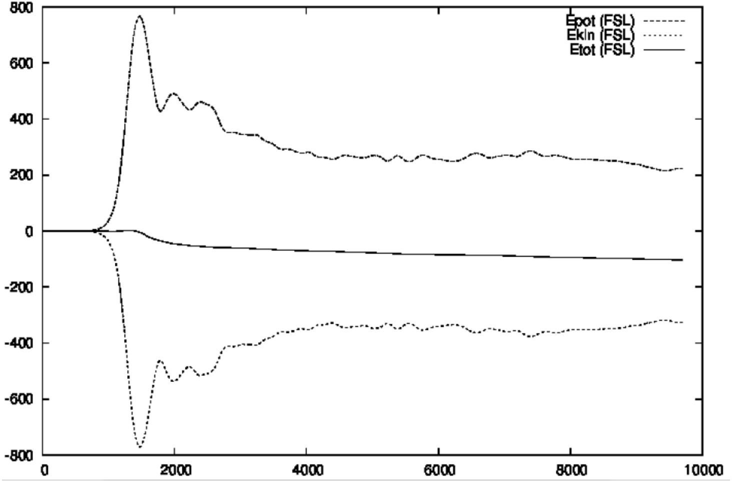
Cartesian case anisotropic rotation $\Phi = 4(x - x0)^2 + (y - y0)^2 \; dy = 2dx$



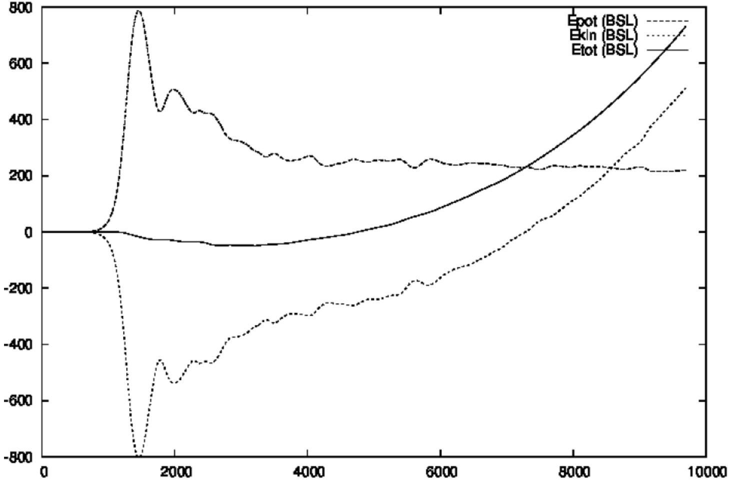
GYSELA 4D toric case $\mu = 0$ mass BSL/FSL



GYSELA 4D toric case $\mu = 0$ energy FSL



GYSELA 4D toric case $\mu = 0$ energy BSL



The equation

$$\partial_t \bar{f} + \partial_{\eta^1}\left(\frac{\partial_{\eta^2}\Psi}{\sqrt{g}}\bar{f}\right) + \partial_{\eta^2}\left(-\frac{\partial_{\eta^1}\Psi}{\sqrt{g}}\bar{f}\right) = 0,$$

with $\bar{f} = \sqrt{g}\tilde{f}$, $\tilde{f}(\eta^1,\eta^2) = f(x_1(\eta^1,\eta^2), x_2(\eta^1,\eta^2))$ and

$$\sqrt{g} = \partial_{\eta^1}x_1\partial_{\eta^2}x_2 - \partial_{\eta^2}x_1\partial_{\eta^1}x_2.$$

The CSL method

1. Split by direction
2. Start from

$$\frac{1}{\Delta\eta^1}\int_{\eta_i^1}^{\eta_{i+1}^1}\bar{f}_j^n(\eta^1)d\eta^1=\frac{1}{\Delta\eta^1}\int_{\xi_i^1}^{\xi_{i+1}^1}\tilde{f}_j^n(\eta^1(\xi^1))d\xi^1$$

3. Update through

$$\frac{1}{\Delta\eta^1}\int_{\xi_i^1}^{\xi_{i+1}^1}\tilde{f}_j^{n+1}(\eta^1)d\eta^1=\frac{1}{\Delta\eta^1}\int_{\xi_{i,j}^{1*}}^{\xi_{i+1,j}^{1*}}\tilde{f}_j^n(\eta^1(\xi^1))d\xi^1$$

4. The (ξ_i^1) mesh is non uniform ; use of cubic splines on non uniform mesh

Fondamental property

For

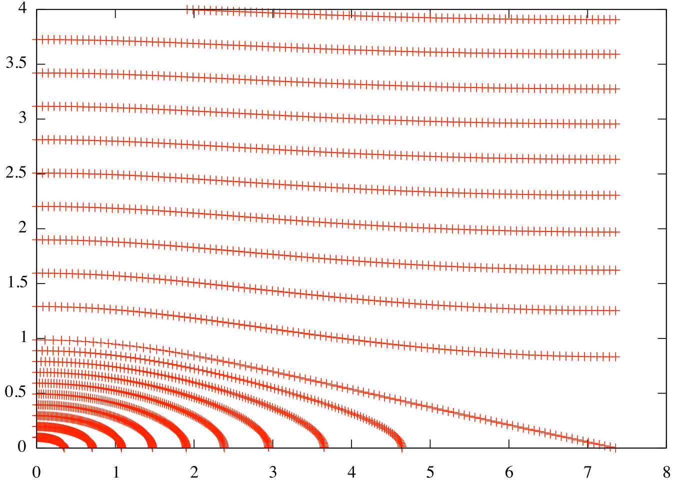
$$\tilde{\Psi}(t, \eta_1, \eta_2) = H = \eta_1,$$

we get

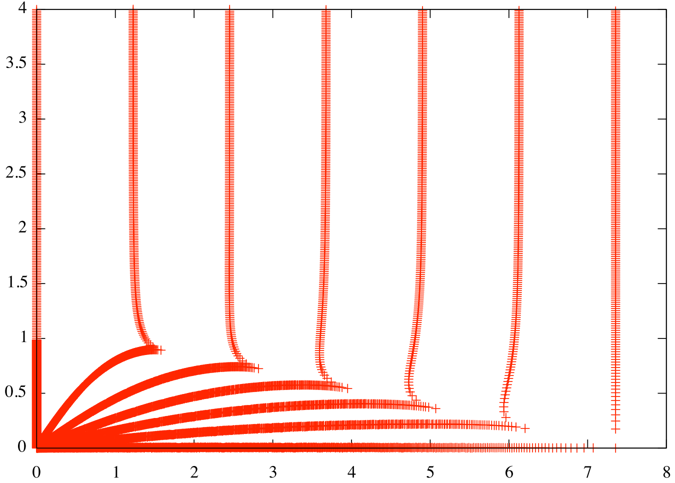
$$\xi_{i,j}^{1*} = \xi_i^1, \ \xi_{j,i}^{2*} = \xi_j^2 + \Delta t$$

A function constant in η_1 remains constant in the numerical scheme

Example of mesh



Example of mesh



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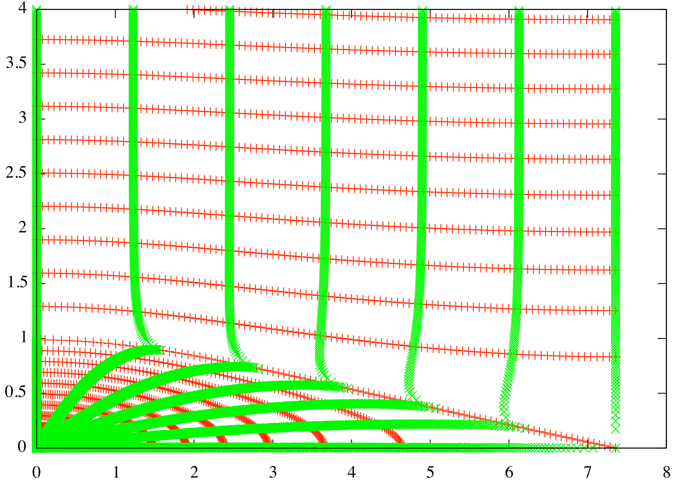
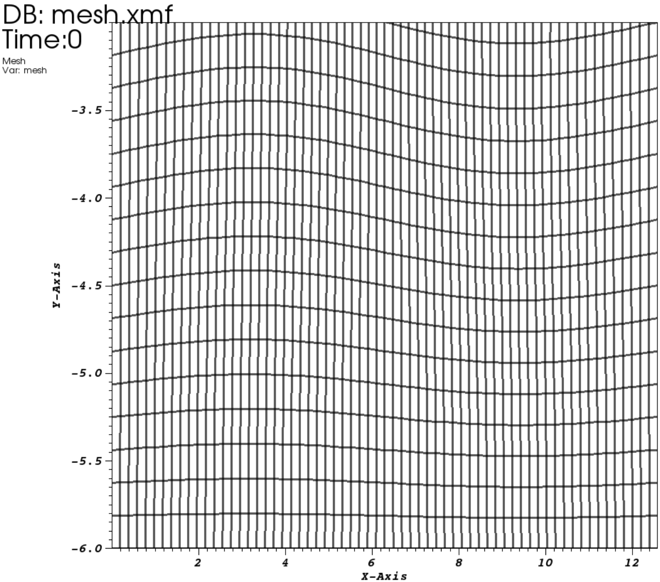


First tests on Colella mesh $\alpha = 1e-2$



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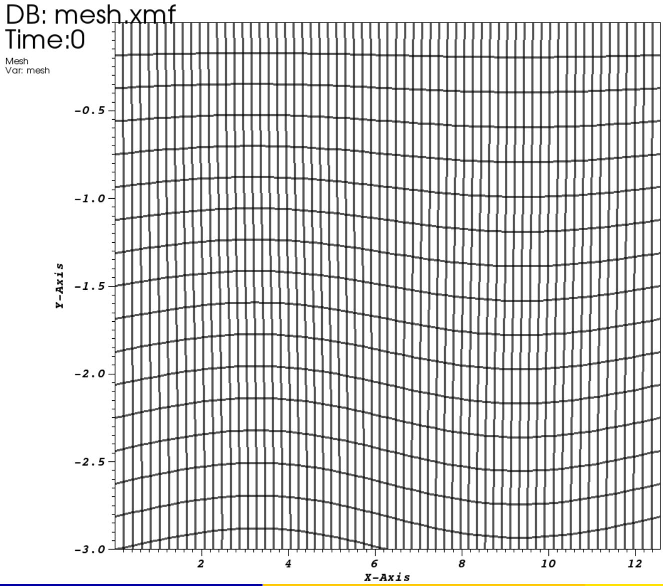
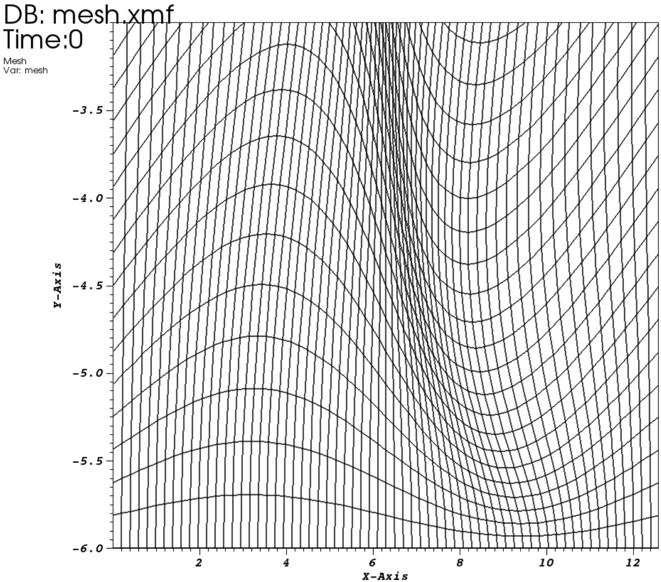
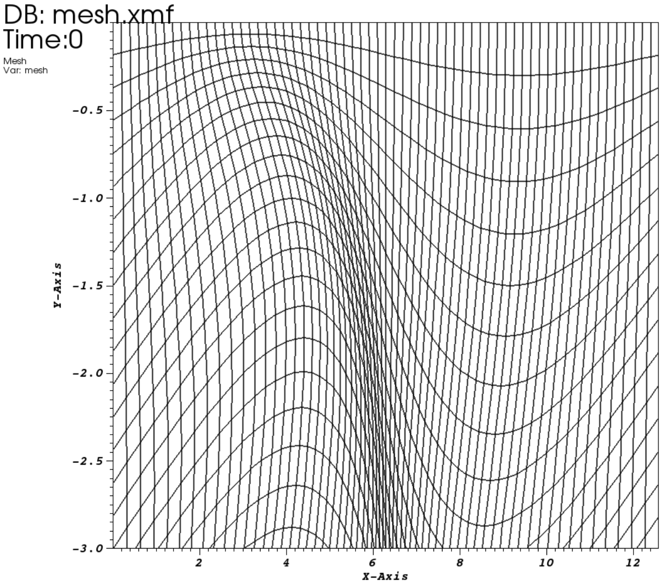


First tests on Colella mesh $\alpha = 1e - 1$



First tests on Colella mesh $\alpha = 1e-1$

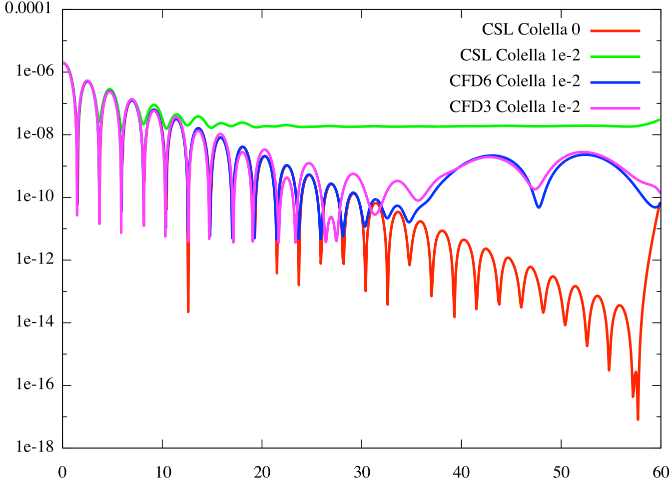


Conservative Finite Difference method

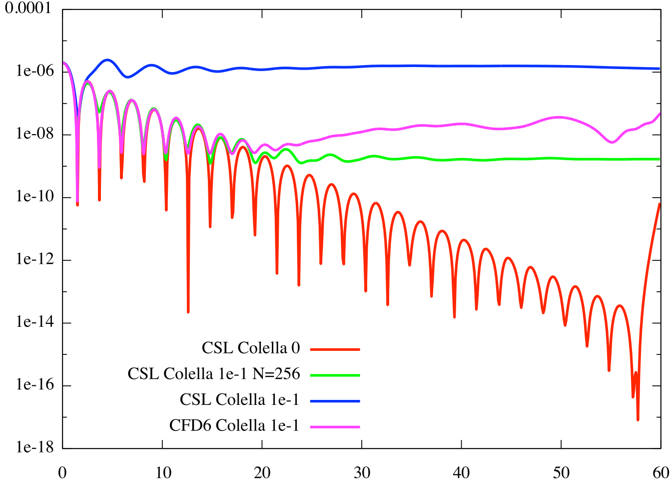
Comparison with a conservative finite difference method (CFD)

1. High order reconstruction
2. CFL restriction
3. divergence free preservation, for centered reconstruction

Linear Landau damping $N = 64$



Linear Landau damping $N = 64$



Conclusion/Perspectives

1. Highlighting of fondamental properties : mass, divergence free, high order
2. Difficulties to have everything in the Semi-Lagrangian framework
3. Mixing strategies : decouple known linear displacement and small non linear displacement