Properties of isoscalar-pair condensates

P. Van Isacker, A.O. Macchiavelli, P. Fallon, S. Zerguine

Properties of isoscalar-pair condensates

P. Van Isacker
Grand Accélérateur National d'Ions Lourds, CEA/DRF–CNRS/IN2P3, Bvd Henri Becquerel, F-14076 Caen, France

A. O. Macchiavelli and P. Fallon
Nuclear Science Division, Lawrence Berkeley National Laboratory, Berkeley, California 94720

S. Zerguine
Department of Physics, PRIMALAB Laboratory, University of Batna, Avenue Boukhelouf M El Hadi, 05000 Batna, Algeria

(Dated: July 22, 2016)

It is pointed out that the ground state of n neutrons and n protons in a single-j shell, interacting through an isoscalar ($T = 0$) pairing force, is not paired, $J = 0$, but rather spin-aligned, $J = n$. This observation is explained in the context of a model of isoscalar P ($J = 1$) pairs, which is mapped onto a system of p bosons, leading to an approximate analytic solution of the isoscalar-pairing limit in jj coupling.

PACS numbers: 21.60.Cs, 21.60.Fw, 03.75.Fi

I. INTRODUCTION

In 1958, Bohr, Mottelson, and Pines [1] suggested a possible analogy between the excitation spectra of nuclei and those of the superconducting metallic state. Since then, a wealth of experimental data have been accumulated, supporting the important role played by pairing correlations in defining properties of atomic nuclei, such as deformation, moments of inertia, alignments, etc. [2, 3]. Today, the study of pairing correlations continues to be a subject of active research in nuclear physics, with an emphasis in exotic nuclei. Of particular interest is the understanding of the role played by the isoscalar ($T = 0$) and isovector ($T = 1$) pairing forces [4] in the structure of $N \approx Z$ nuclei.

Given the charge independence of the nuclear force, $T = 1$ pairing is on an equal footing between the $T_z = 0$ neutron-proton (np) and $|T_z| = 1$ neutron-neutron and proton-proton (nn and pp) components. In addition, we have the unique possibility of studying the formation of a condensate of $T = 0$ np pairs, thus implying the possible co-existence of "Cooper" pairs of isoscalar and isovector type. Although the nuclear force is stronger in the $T = 0$ channel, it is still not clear how effective the (*in-medium*) $T = 0$ correlations are in giving rise to a ground-state isoscalar condensate [4].

In this paper we consider some interesting properties of the isoscalar condensate in the jj coupling scheme, in particular with regards to its angular momentum. Our motivation starts by studying the numerical results of a shell-model calculation, within the space of single-particle spin-orbit partners, showing that when the isoscalar component is dominant, the ground-state is not paired to $J^\pi = 0^+$ but, rather, it behaves as a state of aligned 1^+ *quasi-deuterons*. To gain further insight into the peculiar structure of these condensates, we develop a boson mapping of the shell model, leading to an approximate analytic solution. Group-theoretical solutions of the pairing problem are known in the isoscalar and isovector limits of LS coupling [5–7] and in the isovector limit of jj coupling [8, 9] but, to our knowledge, not in the isoscalar limit of jj coupling. Our results therefore provide, for the first time, approximate analytic formulae for the energies of the lowest states in that case. While we are of course aware that this limit is not applicable to real nuclei, these states might exist close to the ground state in specific regions of the $N = Z$ line [10, 11] and, more interestingly perhaps, could be realized in atomic traps.

II. SINGLE-j SHELL MODEL

Single-shell models that capture the main ingredients of the problem provide a useful framework to understand the competition of isovector and isoscalar pairing interactions. Here we start by considering the scattering of ($L = 0, S = 0, T = 1$) nn, np, and pp pairs as well as ($L = 0, S = 1, T = 0$) np pairs, describing the large spatial overlap of the nucleons' wave function in a relative $L = 0$ state. In the jj coupling scheme the spin-orbit splitting $v_{\ell s}$ increases the energy required to form the ($L = 0, S = 1, T = 0$) state, thus favoring a ($J = 1, T = 0$) *quasi-deuteron* configuration. It then seems of interest to consider a more realistic case, namely that of a single-j shell that incorporates the jj coupling scheme, more appropriate in heavier nuclei. The difference between these simple LS and jj models has been discussed in terms of the BCS approximation [12].

Our approach to study this problem is to use the shell-model code OXBASH [13] with an effective two-body force of the form

$$\hat{V}(g, x) = -xg\hat{V}_{J=0,T=1} - (1 - x)g\hat{V}_{J=1,T=0}, \quad (1)$$

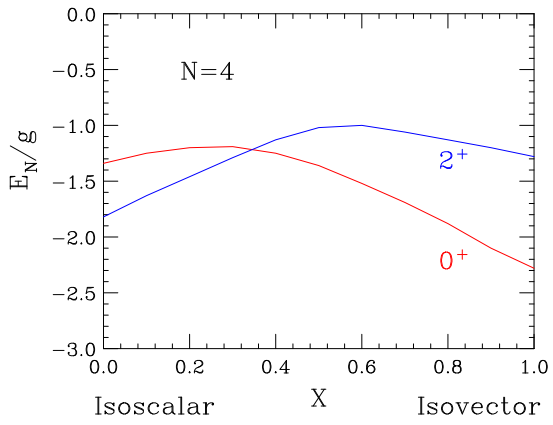

FIG. 1: Energies (in units of the pairing strength g) of the lowest two $T = 0$ states for $N = 4$ particles in an $f_{7/2}$ shell as a function of the relative mixture x of isovector and isoscalar pairing.

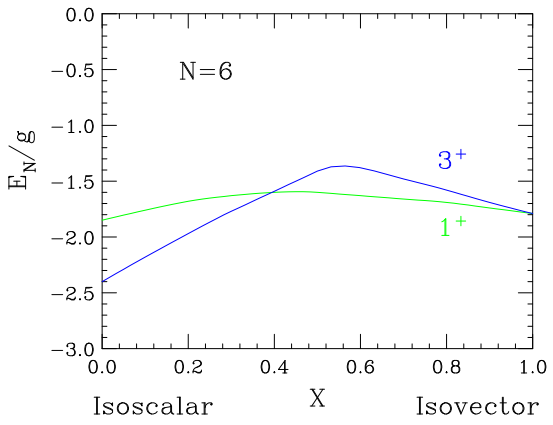

FIG. 2: Energies (in units of the pairing strength g) of the lowest two $T = 0$ states for $N = 6$ particles in the $f_{7/2}$ shell as a function of the relative mixture x of isovector and isoscalar pairing.

with

$$\hat{V}_{J,T} = \frac{1}{2}(a^\dagger_{jt} \times a^\dagger_{jt})^{(J,T)} \cdot (\tilde{a}_{jt} \times \tilde{a}_{jt})^{(J,T)}, \qquad (2)$$

where $a^\dagger_{jm_jtm_t}$ creates a nucleon with angular momentum j and projection m_j, isospin $t = \frac{1}{2}$ and projection m_t, and with $\tilde{a}_{jm_jtm_t} = (-)^{j+m_j+t+m_t} a_{j-m_jt-m_t}$. The notation $\times$ implies the coupling to angular momentum J and isospin T, and the dot $\cdot$ denotes a scalar product in angular momentum and isospin. The Hamiltonian (1) models the mixture of the two types of competing pairing interactions by the parameter x, with $x = 0$ corresponding to the isoscalar and $x = 1$ to the isovector limits respectively. The sign convention in Eq. (1) is such that g is positive for an attractive interaction $\hat{V}(g, x)$. We consider two spin-orbit partners $f_{7/2}$ and $f_{5/2}$ and study the low-lying spectra obtained as a function of the splitting $v_{\ell s}$. In the limit $v_{\ell s} = 0$ we recover the results of the LS coupling scheme. The results for $v_{\ell s} \gg \langle \hat{V} \rangle$ agree with those obtained for a single $f_{7/2}$ level only.

The intriguing phenomenon that motivated this study is seen in Fig. 1, showing the evolution of the two lowest states in the $N = 4$ particle system as a function of x. For an appreciable amount of isoscalar pairing ($x \lesssim 0.4$) the ground state changes from the expected 0^+ to a 2^+ state. Moreover, as seen in Fig. 2, the ground state for $N = 6$ is a 3^+ and not 1^+, and so on for more particles. Considering that, for two particles interacting with the force (1), $x \sim 0$ favors deuteron-like pairing with angular momentum $J = 1$, it appears that the ground state of the many-particle system prefers the aligned configuration of the $n = N/2$ pairs, *i.e.* the configuration with $J = n$.

We can trace back the change in the properties of the ground state to the spin-orbit splitting. In Fig. 3 we show the results for the $N = 4$ system and a pure isoscalar force. The energies of the 0^+ and 2^+ states are plotted as a function of the spin-orbit splitting $v_{\ell s}$. The two states cross, with a 0^+ ground state in LS coupling, which becomes a 2^+ in jj coupling, as we saw above.

To obtain an estimate of the critical value $v^*_{\ell s}$ at which the switch occurs, we consider the case of the $N = 2$ system, also shown in Fig. 3. Taking the limit of large j, to simplify the LS-jj re-coupling coefficients, we have in jj coupling $E_{jj}(1^+) = -g$ and in LS coupling $E_{LS}(1^+) \approx -6g$.

The $^3S_1$ state can be written in terms of the jj-coupled wave functions as [14]

$$|^3S_1\rangle \approx \frac{1}{\sqrt{6}}|j^2_>\rangle + \frac{2}{\sqrt{6}}|j_>j_<\rangle - \frac{1}{\sqrt{6}}|j^2_<\rangle, \qquad (3)$$

from which we can treat perturbatively the effect of the spin-orbit splitting $v_{\ell s}$. This gives for an intermediate coupling

$$E_{\rm IC}(1^+) \approx E_{LS}(1^+) + \frac{1}{6}2v_{\ell s} + \frac{4}{6}v_{\ell s} = E_{LS}(1^+) + v_{\ell s}. \quad (4)$$

The critical value is obtained when the energy above equals that of the jj-coupling limit (dashed lines in Fig. 3)

$$E_{jj}(1^+) = E_{LS}(1^+) + v^*_{\ell s} \qquad (5)$$

and we find (in the large-j limit)

$$\frac{v^*_{\ell s}}{g} \approx 5. \qquad (6)$$

For the particular case of the $f_{7/2}$-$f_{5/2}$ pair (finite-j) we find a value of ~ 3.5, in agreement with the estimate shown in Fig. 3 (shaded area).

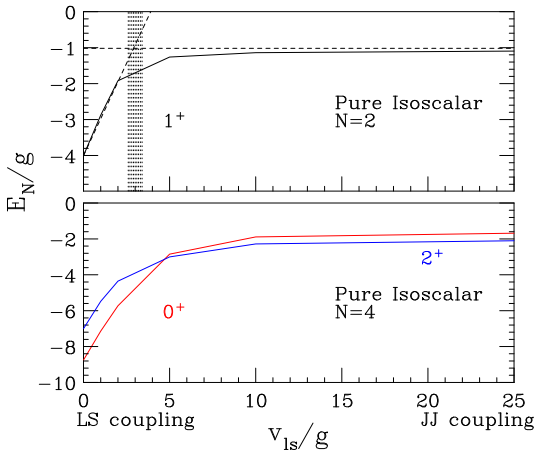

FIG. 3: Energies (in units of the pairing strength g) of the $T = 0$ ground state for $N = 2$ particles (top) and of the lowest two $T = 0$ states for $N = 4$ particles (bottom) as a function of the spin-orbit splitting between the $f_{7/2}$ and $f_{5/2}$ orbits and for pure isoscalar pairing. The shaded area in the top panel indicates the critical value of the spin-orbit splitting, $v^*_{\ell s}$, at which the jj coupling takes on. (See text for details).

To shed further light on the properties of the isoscalar condensate discussed above, we develop in the next section a description based on a mapping to interacting p bosons of angular momentum $J = 1$ and isospin $T = 0$. Based on the results above, and on the fermionic nature of the problem, we anticipate that the residual interaction between these bosons favors their aligned coupling.

III. ISOSCALAR PAIRING BETWEEN FERMIONS IN A SINGLE j SHELL

Consider a system of N particles, n neutrons and n protons, in a single-j shell, interacting through an isoscalar pairing interaction with angular momentum $J = 1$ and isospin $T = 0$, corresponding to the $x = 0$ limit of Eq. (1), $\hat{V}(g, x = 0) = -g\hat{V}_{10}$.

As discussed in the previous section, a possible strategy for simplifying the problem starts from the observation that, by definition of the interaction, the dominant pair in the two-particle system has $J = 1$ and $T = 0$. We attempt to represent a subset of the $2n$-particle eigenstates of this interaction, including hopefully those at lowest energies, in terms of a single state $|P\rangle \equiv P^\dagger|\mathrm{o}\rangle$ (with $|\mathrm{o}\rangle$ the vacuum), which has $J = 1$ and $T = 0$,

$$P^\dagger_{M_J} \equiv (a^\dagger_{jt} \times a^\dagger_{jt})^{(J=1,T=0)}_{M_J,M_T=0}\,. \tag{7}$$

The natural framework to test this idea is provided by the nucleon-pair shell model (NPSM), which assumes a basis constructed from nucleon pairs [15–18]. In this approximation the full $T = 0$ shell-model space is truncated to one constructed out of P pairs with basis states $|P^n J_2 \ldots J_{n-1} J\rangle$ that are proportional to

$$\Big(\cdots\Big(\big(P^\dagger \times P^\dagger\big)^{(J_2)} \times P^\dagger\Big)^{(J_3)} \times \cdots \times P^\dagger\Big)^{(J)} |\mathrm{o}\rangle. \tag{8}$$

This $2n$-particle state is characterized by the set of intermediate angular momenta $\{J_2, \ldots, J_{n-1}\}$, with $J_1 = 1$ and $J_n = J$, the total angular momentum of the state. All pairs have $T = 0$ and the coupling in isospin need not be considered. In principle, several intermediate couplings $\{J_2, \ldots, J_{n-1}\}$ are possible for a given total angular momentum J. Such is the case for arbitrary pairs but not for P pairs since the number of independent states with angular momentum J constructed out of n P pairs cannot exceed the corresponding number constructed out of n p bosons, which is 1 if $n-J$ is non-negative and even, and 0 otherwise. We conclude therefore that, for a given J, at most one state $|P^n J_2 \ldots J_{n-1} J\rangle$ exists, for which the intermediate angular momenta can be chosen as

$$\begin{cases} J_i = i \bmod 2, & 1 \le i \le n - J, \\ J_i = i - n + J, & n - J \le i \le n, \end{cases} \tag{9}$$

where it is implicitly assumed (as will be from now on) that $n - J$ is non-negative even. We denote normalized states as $|P^n J\rangle$, tacitly assuming the intermediate coupling convention (9). In this convention the paired and spin-aligned states of particular interest here correspond to the choice

$$\begin{array}{lll} \text{paired} & : J_i = i \bmod 2, & 1 \le i \le n, \\ \text{spin-aligned} & : J_i = i, & 1 \le i \le n. \end{array} \tag{10}$$

As long as $n \le (2j + 1)/2$ all states (9) exist. This is no longer necessarily true if the shell is more than half filled, in which case it is advantageous to reconsider the problem in terms of holes. We then construct basis states $|\tilde{P}^{2j+1-n} J_2 \ldots J_{n-1} J\rangle$ that are proportional to

$$\Big(\cdots\Big(\big(\tilde{P} \times \tilde{P}\big)^{(J_2)} \times \tilde{P}\Big)^{(J_3)} \times \cdots \times \tilde{P}\Big)^{(J)} |\tilde{\mathrm{o}}\rangle, \tag{11}$$

where $|\tilde{\mathrm{o}}\rangle$ represents a full shell and $\tilde{P}$ annihilates a P pair,

$$\tilde{P}_{M_J} \equiv (\tilde{a}_{jt} \times \tilde{a}_{jt})^{(J=1,T=0)}_{M_J,M_T=0}\,. \tag{12}$$

The angular momentum and anti-symmetry considerations concerning the states (8) and (11) are the same, and consequently the latter lead to the same allowed basis states (9) with n replaced by $\bar{n} \equiv 2j + 1 - n$. We denote such states as $|\tilde{P}^{\bar{n}} J\rangle$.

In general, $|P^n J\rangle$ and $|\tilde{P}^{\bar{n}} J\rangle$ are *not* the same state,

$$|P^n J\rangle \neq |\tilde{P}^{\bar{n}} J\rangle, \tag{13}$$

and it is possible that the state on the left-hand side exists while the one on the right-hand side does not (or *vice*

versa). Only if the shell-model state with a given J and $T = 0$ is unique, do the particle and hole representations become equivalent, as is the case, for example, for the states

$$|P^{2j+1}J = 0\rangle = |\tilde{\mathrm{o}}\rangle, \quad |P^{2j}J = 1\rangle = |\tilde{P}J = 1\rangle. \tag{14}$$

The choice $|P^nJ\rangle$ if $n \leq (2j+1)/2$ and $|\tilde{P}^{\bar{n}}J\rangle$ if $n \geq (2j+1)/2$, apart from being computationally simpler, gives the best approximation of shell-model states in terms of P pairs.

The summary of the above discussion is that the truncated shell-model basis constructed out of P pairs is spanned by the states $|P^nJ\rangle$ if $n \leq (2j+1)/2$ and by the states $|\tilde{P}^{\bar{n}}J\rangle$ if $n \geq (2j+1)/2$. These basis states exist (provided $n-J$ or $\bar{n}-J$ is non-negative even) and are unique for a given n and J, so that no additional labels are needed. Therefore, in the P-pair approximation of the NPSM, the correlation energy due to isoscalar pairing in the state with n neutrons and n protons, coupled to total angular momentum J and isospin $T = 0$, is[1]

$$\tilde{E}_{\mathrm{f}}(n, J) \equiv \langle P^nJ| - g\hat{V}_{10}|P^nJ\rangle, \tag{15}$$

for $n \leq (2j+1)/2$, and by

$$\tilde{E}_{\mathrm{f}}(\bar{n}, J) \equiv \langle \tilde{P}^{\bar{n}}J| - g\hat{V}_{10}|\tilde{P}^{\bar{n}}J\rangle, \tag{16}$$

for $n \geq (2j+1)/2$. The computation of the matrix elements of an arbitrary interaction between nucleon-pair states is possible with the recurrence relation devised by Chen [16]. In the general formulation of the NPSM care should be taken of the over-completeness and non-orthogonality of the pair basis. This is not an issue in the present application since basis states are unique for a given n and J. It should be stressed that Eqs. (15) and (16) yield an approximation to the exact isoscalar-pairing correlation energy.

The energy $E_{\mathrm{f}}(n, J)$ of a particle state is calculated with respect to the vacuum $|\mathrm{o}\rangle$ while that of a hole state, $E_{\mathrm{f}}(\bar{n}, J)$, is with respect to the full shell $|\tilde{\mathrm{o}}\rangle$. The particle-hole transformation gives a relation between both quantities, which is exact in the full shell-model space. For our particular case of isoscalar pairing this relation is

$$E_{\mathrm{f}}(n, J) = -\frac{3(2j+1-2\bar{n})}{2j+1}g + E_{\mathrm{f}}(\bar{n}, J). \tag{17}$$

We use the same equation to relate the approximate energies $\tilde{E}_{\mathrm{f}}(n, J)$ and $\tilde{E}_{\mathrm{f}}(\bar{n}, J)$. In the following absolute energies are quoted with respect to the vacuum $|\mathrm{o}\rangle$. For a particle state they are obtained directly while for a hole state they follow from Eq. (17).

[1] We reserve the notation $E_{\mathrm{f}}(n, J)$ and $E_{\mathrm{f}}(\bar{n}, J)$ (*i.e.*, expressions without tilde) for the exact correlation energy of the yrast state with angular momentum J calculated in the full shell-model space.

A further approximation is to replace the P pairs by p bosons, with single-boson energies and boson-boson interactions derived from the two-particle and four-particle systems, respectively.

With use of the OAI mapping [19] a p-boson Hamiltonian $\hat{H}_{\mathrm{b}}$ is obtained, which can be written as

$$\hat{H}_{\mathrm{b}} = \epsilon_p\, p^\dagger \cdot \tilde{p} + \frac{1}{2}\sum_{\lambda=0,2} v^{\mathrm{b}}_\lambda (p^\dagger \times p^\dagger)^{(\lambda)} \cdot (\tilde{p} \times \tilde{p})^{(\lambda)}, \tag{18}$$

where ϵ_p is the p-boson energy and v^{b}_λ are the two-body interaction matrix elements *between the p bosons*. The definition of the adjoint operator $\tilde{p}_m \equiv (-)^{1-m}p_{-m}$ ensures that $\tilde{p}_m$ is an annihilation operator with transformation properties under rotations that are the same as those for the creation operator $p^\dagger_m$ [20]. With the above definitions we have that $p^\dagger \cdot \tilde{p} = \sum_m p^\dagger_m p_m$ is the number operator $\hat{n}_p$.

The single-boson energy is

$$\epsilon_p \equiv \langle p|\hat{H}_{\mathrm{b}}|p\rangle \doteq \langle P| - g\hat{V}_{10}|P\rangle = -g. \tag{19}$$

where the notation $\doteq$ is used to indicate that the equality holds by virtue of the mapping procedure. The two-body boson matrix elements with $\lambda = 0, 2$ are

$$\begin{aligned} v^{\mathrm{b}}_\lambda &\equiv \langle p^2\lambda|\hat{H}_{\mathrm{b}}|p^2\lambda\rangle - 2\epsilon_p \\ &\doteq -g\left(\langle j^4[10,10]\lambda 0|\hat{V}_{10}|j^4[10,10]\lambda 0\rangle - 2\right), \end{aligned} \tag{20}$$

where the bra and ket represent normalized, anti-symmetric two-pair states,

$$|j^4[J_1T_1, J_2T_2]JT\rangle \propto \mathcal{A}|j^2(J_1T_1)j^2(J_2T_2)JT\rangle. \tag{21}$$

The notation in square brackets $[J_1T_1, J_2T_2]$ implies that the state (21) is constructed from a parent state with intermediate angular momenta and isospins J_1T_1 and J_2T_2. The anti-symmetrized states $|j^4[J_1T_1, J_2T_2]JT\rangle$ can be expanded in terms of the two-pair states $|j^2(J_1T_1)j^2(J_2T_2)JT\rangle$ by means of four-to-two-particle coefficients of fractional parentage (CFPs) [14],

$$[j^2(J_aT_a)j^2(J_bT_b)JT|\}j^4[J_1T_1, J_2T_2]JT], \tag{22}$$

which are known in closed form.

From the general expression for the matrix element (20) the following results are obtained:

$$\begin{aligned} v^{\mathrm{b}}_0/g =& -6[j^2(10)j^2(10)00|\}j^4[10,10]00]^2 + 2, \\ v^{\mathrm{b}}_2/g =& -6[j^2(10)j^2(10)20|\}j^4[10,10]20]^2 \\ & -6[j^2(30)j^2(10)20|\}j^4[10,10]20]^2 + 2, \end{aligned} \tag{23}$$

which, with the help of

$$\begin{aligned} &[j^2(10)j^2(10)00|\}j^4[10,10]00]^2 = \frac{2j^3-2j+3}{3j(j+1)(2j+1)}, \\ &[j^2(10)j^2(10)20|\}j^4[10,10]20]^2 = \frac{10j^3+9j^2-j-3}{15j(j+1)(2j+1)}, \\ &[j^2(30)j^2(10)20|\}j^4[10,10]20]^2 \\ &\qquad = \frac{9(j-1)(j+2)(2j+3)}{10j(j+1)(2j+1)(5j^2+7j+3)}, \end{aligned} \tag{24}$$

lead to the following expressions for the p-boson matrix elements:

$$v_0^{\rm b} = \frac{6(j^2+j-1)}{j(j+1)(2j+1)} g \stackrel{j\to\infty}{\longrightarrow} \left[\frac{3}{j} + \mathcal{O}\left(\frac{1}{j^2}\right)\right] g,$$
$$v_2^{\rm b} = \frac{3(4j^4+6j^3+j^2+7j+12)}{j(j+1)(2j+1)(5j^2+7j+3)} g \stackrel{j\to\infty}{\longrightarrow} \left[\frac{6}{5j} + \mathcal{O}\left(\frac{1}{j^2}\right)\right] g. \quad (25)$$

As anticipated, for an attractive isoscalar pairing interaction the boson-boson matrix elements are repulsive. This is a finite-space effect, due to the Pauli principle, since the matrix elements vanish in the large-j limit. A difference between the $\lambda = 0$ and $\lambda = 2$ matrix elements also arises due to Pauli effects, and it is seen that $v_2^{\rm b}$ is less repulsive. This favors the spin-aligned ground state, not only for two but also for more bosons as a result of the following argument.

Since a system of n interacting identical p bosons is solvable by virtue of a U(3) $\supset$ SO(3) dynamical symmetry [20], the eigenvalues of the Hamiltonian (18) are known in closed form,

$$E_{\rm b}(n,J) = n\epsilon_p + \frac{n(n+1)-J(J+1)}{6} v_0^{\rm b} + \frac{2n(n-2)+J(J+1)}{6} v_2^{\rm b}, \quad (26)$$

where the allowed angular momenta are $J = n, n-2, \ldots, 1$ or 0. The only possible ground states of a p-boson system are either paired or spin-aligned [21, 22]. The paired state has $J = 0$ or $J = 1$ with energies

$$E_{\rm b}(n, J=0) = n\epsilon_p + \frac{n(n+1)}{6} v_0^{\rm b} + \frac{n(n-2)}{3} v_2^{\rm b}, \quad (27)$$
$$E_{\rm b}(n, J=1) = n\epsilon_p + \frac{(n-1)(n+2)}{6} v_0^{\rm b} + \frac{(n-1)^2}{3} v_2^{\rm b},$$

depending on whether n is even or odd, respectively. The spin-aligned state has $J = n$ with energy

$$E_{\rm b}(n, J=n) = n\epsilon_p + \frac{n(n-1)}{2} v_2^{\rm b}. \quad (28)$$

The breaking of the rotational invariance in *gauge space* [23] leads to the emergence of isoscalar pairing rotational bands, as seen in the quadratic dependence of the energies as a function of the number of pairs n, Eqs. (27) and (28).

The difference in energy between the paired and the spin-aligned states can be written as

$$\Delta_{\rm b}(n) = \frac{(n-n_2)(n+1+n_2)}{6}(v_0^{\rm b} - v_2^{\rm b}) \approx g\frac{3(n-n_2)(n+1+n_2)}{10j}, \quad (29)$$

where n_2 is 0 for even n and 1 for odd n, $n_2 \equiv n \bmod 2$. This shows that for all n the difference in energy between the paired and the spin-aligned states is positive for an attractive pairing interaction, that is, the spin-aligned configuration is the ground state.

We recall that the preceding results, valid for an isoscalar pairing interaction in a single-j shell, are derived under the following simplifying assumptions:

TABLE I: Exact energies $E_{\rm f}(n,J)$ of paired ($J = 0$ or 1) and aligned ($J = n$) states with $T = 0$ of a system of n neutrons and n protons in a single-j shell interacting through an isoscalar pairing force, in units of the strength g, and the corresponding energies $\tilde{E}_{\rm f}(n,J)$ and $E_{\rm b}(n,J)$ obtained in the P-pair and p-boson approximations. A dash — means that a P-pair state does not exist while the absence of an entry indicates that the numerical result could not be obtained.

$E(n,J)$	$j=7/2$	$j=9/2$	$j=11/2$	$j=13/2$	$j=15/2$
$E_{\rm f}(2,0)$	−1.298	−1.424	−1.514	−1.580	−1.631
$\tilde{E}_{\rm f}(2,0)$	−1.298	−1.424	−1.514	−1.580	−1.631
$E_{\rm b}(2,0)$	−1.298	−1.424	−1.514	−1.580	−1.631
$E_{\rm f}(2,2)$	−1.793	−1.825	−1.847	−1.865	−1.879
$\tilde{E}_{\rm f}(2,2)$	−1.757	−1.799	−1.828	−1.850	−1.866
$E_{\rm b}(2,2)$	−1.757	−1.799	−1.828	−1.850	−1.866
$E_{\rm f}(3,1)$	−1.793	−1.953	−2.086	−2.192	−2.277
$\tilde{E}_{\rm f}(3,1)$	−1.636	−1.848	−2.010	−2.135	−2.233
$E_{\rm b}(3,1)$	−1.505	−1.772	−1.961	−2.100	−2.207
$E_{\rm f}(3,3)$	−2.365	−2.466	−2.537	−2.591	−2.634
$\tilde{E}_{\rm f}(3,3)$	−2.279	−2.403	−2.488	−2.552	−2.601
$E_{\rm b}(3,3)$	−2.271	−2.397	−2.484	−2.549	−2.599
$E_{\rm f}(4,0)$	−2.080	−2.251	−2.424		
$\tilde{E}_{\rm f}(4,0)$	−1.628	−1.887	−2.141	−2.353	−2.526
$E_{\rm b}(4,0)$	−1.010	−1.545	−1.921	−2.200	−2.413
$E_{\rm f}(4,4)$	−2.767	−2.925			
$\tilde{E}_{\rm f}(4,4)$	−2.577	−2.818	−2.985	−3.110	−3.207
$E_{\rm b}(4,4)$	−2.541	−2.794	−2.968	−3.098	−3.198
$E_{\rm f}(5,1)$	−2.543				
$\tilde{E}_{\rm f}(5,1)$	−2.386	−1.975	−2.284	−2.566	−2.806
$E_{\rm b}(5,1)$	−2.255	−1.241	−1.815	−2.239	−2.564
$E_{\rm f}(5,5)$	—				
$\tilde{E}_{\rm f}(5,5)$	—	−3.052	−3.324	−3.528	−3.687
$E_{\rm b}(5,5)$	—	−2.990	−3.281	−3.496	−3.663
$E_{\rm f}(6,0)$	−2.798	−2.851			
$\tilde{E}_{\rm f}(6,0)$	−2.798	−2.487	−2.206	−2.539	−2.847
$E_{\rm b}(6,0)$	−2.798	−2.145	−1.222	−1.858	−2.347
$E_{\rm f}(6,6)$	—	—			
$\tilde{E}_{\rm f}(6,6)$	—	—	−3.511	−3.810	−4.045
$E_{\rm b}(6,6)$	—	—	−3.421	−3.744	−3.994
$E_{\rm f}(7,1)$	−3.250	−3.153			
$\tilde{E}_{\rm f}(7,1)$	−3.250	−3.048	−2.784	−2.534	−2.892
$E_{\rm b}(7,1)$	−3.250	−2.972	−2.315	−1.417	−2.074
$E_{\rm f}(7,7)$	—	—	—		
$\tilde{E}_{\rm f}(7,7)$	—	—	—	−3.962	−4.284
$E_{\rm b}(7,7)$	—	—	—	−3.842	−4.192
$E_{\rm f}(8,0)$	−3.000	−3.224	−3.424		
$\tilde{E}_{\rm f}(8,0)$	−3.000	−3.224	−3.141	−2.968	−2.776
$E_{\rm b}(8,0)$	−3.000	−3.224	−2.921	−2.287	−1.431
$E_{\rm f}(8,8)$	—	—	—	—	
$\tilde{E}_{\rm f}(8,8)$	—	—	—	—	−4.408
$E_{\rm b}(8,8)$	—	—	—	—	−4.256

1. The full shell-model space is truncated to one constructed out of P pairs. The expectation value of the isoscalar pairing Hamiltonian $-g\hat{V}_{10}$ in the (unique) P-pair state takes fully account of the Pauli principle and leads to the approximate correlation energy $\tilde{E}_{\rm f}(n,J)$.

2. The fermionic Hilbert space constructed out of P pairs is mapped onto a corresponding bosonic Hilbert space constructed out of p bosons. The mapping of the Hamiltonian is carried out in the two- and four-nucleon spaces and leads to a boson Hamiltonian with up to two-body interactions.

3. The boson Hamiltonian is used to calculate the energies $E_{\rm b}(n,J)$ of n-boson states.

To gauge the adequacy of the different approximations, we show in Table I the exact energies $E_{\rm f}(n,J)$ (wherever they can be calculated) and the corresponding approximations $\tilde{E}_{\rm f}(n,J)$ and $E_{\rm b}(n,J)$ for $7/2 \leq j \leq 15/2$. Several comments are in order. First of all, we observe the identity

$$\tilde{E}_{\rm f}(n=2,J) = E_{\rm b}(n=2,J), \tag{30}$$

that is, the P-pair spectrum of the four-particle system coincides with that obtained for two p bosons. This is a generic property of the mapping and follows from the fact that up to two-body interactions between the bosons are considered. In fact, if up to q-body interactions are considered, the identity (30) remains valid up to the $n = q$. Secondly, we observe the identity

$$E_{\rm f}(n=2,J=0) = \tilde{E}_{\rm f}(n=2,J=0). \tag{31}$$

This is not a generic property but is valid for the isoscalar pairing interaction, for which $|P^2 J=0\rangle$ decouples from the rest of the shell-model space. This property of the isoscalar pairing interaction was already pointed out by Fu *et al.* [24] on the basis of analytic expressions for four-nucleon overlaps. Furthermore, we observe from Table I the following hierarchy:

$$E_{\rm f}(n,J) \leq \tilde{E}_{\rm f}(n,J) \leq E_{\rm b}(n,J), \tag{32}$$

valid for any j, n, and J. The first inequality results from the fact that the lowest eigenvalue of any Hamiltonian in a certain Hilbert space is lower than the lowest eigenvalue of the same Hamiltonian in a truncated subspace. We remark that an equality $E_{\rm f}(n=4,J) = \tilde{E}_{\rm f}(n=4,J)$ can be obtained by constructing *effective* operators in the truncated space, which is not done in the present application. The second inequality in Eq. (32) is a consequence of performing the mapping in the four-particle systems with an unnormalized (*i.e.*, not an effective) Hamiltonian. For a variety of bosonic systems (p, sd, sdp, etc.) we have consistently found that the boson Hamiltonian, as it is derived here from the four-particle system, gives an upper limit for the fermionic interaction energy of the n-particle system.

It is seen from Table I that the quality of the approximation varies with j, n, and J. Two effects are rather obvious: the approximation becomes (i) better with increasing j and (ii) worse with increasing n [as long as $n \leq (2j+1)/2$]. These effects result from the increasing importance of Pauli corrections that are neglected (*i.e.*, beyond two-body interactions between the bosons). A more subtle effect is the dependence on J. It is seen that the approximation for the aligned state $J = n$ is adequate, even close to mid shell, $n \approx (2j+1)/2$, and for low j. On the other hand, it is often rather poor for the paired state with $J = 0$ or 1. It can be conjectured that this is a generic property of phonon approximations in fermionic systems: while such descriptions are good for high-angular-momentum states, they become highly anharmonic at low angular momenta.

Despite the varying quality of the boson approximation, depending on j, n, and J, the overall conclusion is that the predicted feature of the lower energy of the aligned state as compared to the paired state is confirmed by the exact fermion calculation.

IV. CONCLUSION

We have considered some intriguing properties of a $T = 0$ isoscalar condensate in single j-shell, in particular with regards to its angular momentum coupling. We developed a description based on a mapping of the shell model to interacting p bosons of angular momentum $J = 1$ and isospin $T = 0$, providing for the first time approximate analytic formulae for the energies of the lowest states. Our results show that, due to the Pauli principle, the residual interaction between these bosons favors (*a priori* unexpected) the aligned configuration of $n = N/2$ *quasi-deuteron* pairs, *i.e.* that with $J = n$.

While we realize this limit may not be applicable to real nuclei, these states might exist close to the paired ground states in specific regions close to the $N = Z$ line. In fact, it was shown recently in Refs. [10, 11], using a phenomenological Hamiltonian within the framework of the Hartree-Fock-Bogoliubov theory, that the spin-triplet phase is favored over the spin-singlet one in the mass region $A \approx 130$ with $Z \approx 64$ (for example ^{132}Gd). This is found to depend on the occupation of specific low-j orbitals near the Fermi energy for which the spin-orbit splitting is small. More interestingly perhaps, it is envisioned that these condensates might be realized by tunable spin-orbit coupling in ultracold atomic traps [25], whereby the control parameter x in Eq. (1) could be adjusted to drive the system from diamagnetic to magnetic.

A full extension of the present formalism including the effect of the spin-orbit splitting will be the subject of a future publication.

V. ACKOWLEDGEMENTS

This work was supported in part by the FUSTIPEN (French-U.S. Theory Institute for Physics with Exotic Nuclei) under U.S. DOE grant No. DE-FG02-10ER41700, and by the U.S. DOE contract No. DE-AC02-05CH11231 (LBNL).

[1] A. Bohr, B. R. Mottelson, and D. Pines, Phys. Rev. **110**, 936 (1958).
[2] D. Brink and R. A. Broglia, *Nuclear Superfluidity: Pairing in Finite Systems* (Cambridge University Press, Cambridge, 2005).
[3] *Fifty Years of Nuclear BCS*, R. A. Broglia and V. Zelevinsky, Eds. (World Scientific, Singapore, 2013).
[4] S. Frauendorf and A. O. Macchiavelli, Progr. Part. Nucl. Phys. **78**, 24 (2014), and references therein.
[5] B. H. Flowers and S. Szpikowski, Proc. Phys. Soc. **84**, 673 (1964).
[6] S. Pang, Nucl. Phys. A **128**, 497 (1969).
[7] J. A. Evans, G. G. Dussel, E. E. Maqueda, and R. P. J. Perazzo, Nucl. Phys. A **367**, 77 (1981).
[8] G. Racah, in *L. Farkas Memorial Volume* (Research Council of Israel, Jerusalem, 1952) p. 294.
[9] A. K. Kerman, Ann. Phys. (NY) **12**, 300 (1961).
[10] A. Gezerlis, G. F. Bertsch, and Y. L. Luo, Phys. Rev. Lett. **106**, 252502 (2011).
[11] B. Bulthuis and A. Gezerlis, Phys. Rev. C **93**, 014312 (2016).
[12] D. R. Bes, O. Civitarese, E. E. Maqueda, and N. N. Scoccola, Phys. Rev. C **61**, 024315 (2000).
[13] B. A. Brown, A. Etchegoyen, and W. D. M. Rae, MSU-NSCL Report **524** (1988).
[14] I. Talmi, *Simple Models of Complex Nuclei. The Shell Model and Interacting Boson Model* (Harwood, Academic, Chur, Switzerland, 1993).
[15] J.-Q. Chen, Nucl. Phys. A **562**, 218 (1993).
[16] J.-Q. Chen, Nucl. Phys. A **626**, 686 (1997).
[17] Y. M. Zhao and A. Arima, Phys. Reports **545**, 1 (2014).
[18] G. J. Fu, Y. Lei, Y. M. Zhao, S. Pittel, and A. Arima, Phys. Rev. C **87**, 044310 (2013).
[19] T. Otsuka, A. Arima, and F. Iachello, Nucl. Phys. A **309**, 1 (1978).
[20] F. Iachello, *Lie algebras and applications* (Springer, Berlin Heidelberg, 2006).
[21] C. K. Law, H. Pu, and N. P. Bigelow, Phys. Rev. Lett. **81**, 5257 (1998).
[22] P. Van Isacker and S. Heinze, J. Phys. A: Math. Theor. **40**, 14811 (2007).
[23] R. A. Broglia, J. Terasaki, and N. Giovanardi, Phys. Rep. **335**, 1 (2000).
[24] G. J. Fu, Y. M. Zhao, and A. Arima, Phys. Rev. C **90**, 054333 (2014).
[25] Y.-J. Lin, K. Jiménez-García, and I. B. Spielman, Nature **471**, 83 (2011).