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journal or publication title	SICE Journal of Control, Measurement, and System Integration
volume	14
number	1
page range	97-106
year	2021-05-02
URL	http://hdl.handle.net/2298/00045872

doi: 10.1080/18824889.2021.1918372

SICE Journal of Control, Measurement, and System Integration, Vol. 4, No. 1, pp. 001–008, January 2011

Estimation of Robust Invariant Set for Switched Linear Systems using Recursive State Updating and Robust Invariant Ellipsoid

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Abstract : This paper provides an analysis method of a robust invariant set for discrete-time linear switched systems with peak-bounded disturbances. In the case of a switched linear system, it is challenging to analyze the robust invariant set accurately than that of the linear time-invariant system. We propose a novel method to estimate a robust invariant set using a combination of a recursive state updating and an invariant ellipsoid for a common Lyapunov function. The effectiveness of the estimation accuracy by the proposed method is illustrated using numerical examples.

Key Words : Robust invariant set, Ellipsoid, Peak bounded disturbance, Switched linear system

1. Introduction

Switched linear systems have been widely studied in the past few decades[1]–[6].Typically, a switched linear system has a number of subsystems, and there is a switching signal determining which subsystem is active. They have been integrated into many practical systems, such as automobiles, power systems, and aircraft.

In this paper, we consider the problem of estimating a robust invariant set for discrete-time switched linear systems. A set is called an invariant set if the state of a system always stays in the given region in state space. In general, the invariant set is defined for autonomous systems. Moreover, invariant sets of state are considered for a system with a bounded disturbance, and such cases are called ”robust invariant set”[7]. Many studies have been developed[8]–[14]. Here, we analyze the invariant set for arbitrary switching signals and disturbances with a constrained peak value. In the case of robust invariant sets in linear systems, time-variant systems, and switching systems, the set of similarity expansion an invariant set is also an invariant set, and there is a countless number of invariant sets. To precisely estimate the impact of a disturbance on a state in terms of the size of the invariant set, it is important to estimate a smaller invariant set. Estimating robust invariant set can be used for power systems[15], constrained systems[16], quantized control systems[17], for example. If the system is a linear time-invariant system, a reachable set of the state is the smallest robust invariant set. The reachable set of states for a discrete-time linear time-invariant system is, in principle, obtained by setting the initial value of the state to zero, updating the state set with the state equation as the update rule, and then advancing the update rule to time infinity. Of course, this is impractical, so methods for estimating the reachable set by finite steps have been studied, such as [9],[10]. On the other hand, it is complicated for a switched linear system to estimate the reachable set because the number of parameters to be calculated is much larger than that of a time-invariant system. In addition, the reachable set for the switched linear system is not always become a convex set.

On the other hand, there are methods for estimating a Lyapunov function-based robust invariant set as an ellipsoid[9]. In this method, a robust invariant set for the linear time-invariant system is estimated as an ellipsoid region. In addition, a robust invariant set is estimated as an ellipsoidal region using Lyapunov function-based methods for the case with a switching system[18]. It is possible to compute a robust invariant set using a common-Lyapunov matrix for each subsystem of a switching system. However, considering the existence of a size gap between the reachable set and the invariant set estimated as an ellipsoid region and the use of a common-Lyapunov matrix, we expect a highly conservative result. For improving the conservativeness of the estimated set, a method with multiple Lyapunov strategies was proposed in [19].

This paper proposes a novel method for estimating a robust invariant set that combines a Lyapunov function-based ellipsoid estimation method with an updating the state equation. We apply the proposed method to the switched linear systems to estimate the robust invariant set of the switched linear system for peak-bounded disturbances accurately.

This paper is organized as follows: In Section 2, we first introduce methods for estimating reachable set by recursive computation using the discrete-time state equation and robust invariant ellipsoid by Lyapunov functions for linear time-invariant systems. In Section 3, we set up the problem of estimating a robust invariant set for switched linear systems. Furthermore, we estimate the robust invariant set of the switched linear system based on previous studies and investigate the analytical performance accuracy. In Section 4, we present a method for estimating robust invariant set using recursive state updating and robust invariant ellipsoid as the main result. The effectiveness of the proposed method is further verified by using numerical examples in Section 5.

In the conference paper of SICE Annual Conference[20], a robust invariant set is analyzed for linear time-invariant systems. This paper is an advanced version of [20]. In particular, robust invariant set analysis for the switching systems are

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(Received xxx 00, 2011)
(Revised xxx 00, 2011)

discussed.

2. Preparation

2.1 Robust invariant set and state reachable set

In this section, we describe robust invariant set and state reachable set for linear time-invariant systems. First, consider the following linear time-invariant systems

$$x_{k+1} = Ax_k + Bw_k \quad (1)$$

where $x_k \in \mathbb{R}^n$ is the state vector, $w_k \in \mathcal{W}$ is the disturbance, and the bounded set $\mathcal{W} \subset \{w \in \mathbb{R}^m\}$ is a convex polyhedron with an origin. In addition, let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and the matrix pair (A, B) is a controllable and A is a Schur stable matrix.

For the control system (1), a robust invariant set of states is defined as follows[9].

Definition 1 For a pair of discrete-time linear time-invariant systems (A, B) and bounded disturbances $w_k \in \mathcal{W}$, if

$$x_k \in \mathcal{X} \rightarrow Ax_k + Bw_k \in \mathcal{X} \quad (2)$$

holds at any time k, the set $\mathcal{X}$ is a robust invariant set. ∎

Although invariant set is generally defined for autonomous systems, the invariant set defined here is the invariant set for the application of a peak-constrained disturbance w_k, which is called a robust invariant set to indicate it. For a given robust invariant set, the control system guarantees that the state will not deviate from the set for any disturbance that satisfies the constraints. In addition, given a robust invariant set, the set containing it is also a robust invariant set, and there are countless such sets.

Then, we consider the reachable set with the origin as the initial state. The reachable set is a region that can be taken by a state x_k for a disturbance w_k and is defined as follows[10].

Definition 2 For a pair of discrete-time linear time-invariant systems (A, B) and a bounded disturbance $w_k \in \mathcal{W}$, the state reachable set $\mathfrak{R}_\infty$ is given by

$$\begin{aligned}\mathfrak{R}_\infty = \{&x_k \in \mathbb{R}^n \mid x_0 = 0, x_k \textbf{ satisfies } (1),\\ &w_k \in \mathcal{W}, k \geq 0\}.\end{aligned} \quad (3)$$

∎

By Definition 2, $\mathfrak{R}_\infty$ can be re-written as the following term in case of the linear time-invariant systems:

$$\begin{aligned}\mathfrak{R}_\infty = \{&x_k \in \mathbb{R}^n \mid x_0 = 0, \exists w_k \in \mathcal{W}, \exists t \geq 0,\\ &x_k = \sum_{i=0}^{t-1} A^{t-1-i} B w_i\}.\end{aligned} \quad (4)$$

The reachable set $\mathfrak{R}_\infty$ represents the state region where the state vector $x(\infty)$ is reachable from $x(0) = 0$ with appropriate choice of $\sigma(k)$. It can be considered the smallest robust invariant set. In another view, $\mathcal{X}$ is the set covering $\mathfrak{R}_\infty$ from the outside, and the relation

$$\mathfrak{R}_\infty \subset \mathcal{X} \quad (5)$$

holds. Therefore, to estimate $\mathfrak{R}_\infty$, it is important to estimate how small $\mathcal{X}$ is.

$\mathfrak{R}_\infty$ is also used to guarantee the safety of the system, and there are various studies on how to estimate this set. In this section, we introduce two methods for approximating $\mathfrak{R}_\infty$. One is to approximate $\mathfrak{R}_\infty$ by a convex polyhedral set, which is obtained by recursive state updating with the initial state, and the other is to approximate $\mathfrak{R}_\infty$ from the outside by a robust invariant set estimated as an ellipsoid region.

2.2 Estimation of state reachable set by recursive state updating

In this section, We explain how to caluculate the region where state $x(k)$ is reachable by time k and thereby approximate the state reachable set $\mathfrak{R}_\infty$. Specifically, we consider bounded disturbances and calculate the state region (convex polyhedron set $\mathcal{L}_k$) by recursively updating the state equation (1). First, we set up the set of disturbances $\mathcal{W}_p$ as a convex polyhedron as follows:

$$\mathcal{W}_p = \{w \in \mathbb{R}^m : M_{\mathcal{W}_p} w \leq m_{\mathcal{W}_p}\} \quad (6)$$

where $M_{\mathcal{W}_p} \in \mathbb{R}^{S_{\mathcal{W}_p}\times m}$, $m_{\mathcal{W}_p} \in \mathbb{R}^{S_{\mathcal{W}_p}}$ are the matrices representing the linear constraint equations that characterize $\mathcal{W}_p$, and $0 \in \mathcal{W}_p$ is assumed to hold. In this case, the convex polyhedron set $\mathcal{L}_k$ at time k can be calculated as follows:

$$\begin{aligned}&\mathcal{L}_0 = \{0\},\\ &\mathcal{L}_k = \{x \in \mathbb{R}^n \mid x = Az + Bw, z \in \mathcal{L}_{k-1},\\ &\qquad w \in \mathcal{W}_p\}, k \geq 1.\end{aligned} \quad (7)$$

where $\mathcal{L}_k$ denotes the region where the state of the linear time-invariant system can be reached from the origin by k steps and can be used as a set to approximate $\mathfrak{R}_\infty$. This $\mathcal{L}_k$ can be specifically constructed by using the Fourier-Motzkin algorithm[21] to remove the disturbance term and make it a linear constraint on the state term only.

In this case, the relationship between $\mathcal{L}_k$ and $\mathfrak{R}_\infty$ is as follow:

$$\mathcal{L}_0 \subset \cdots \subset \mathcal{L}_{k-1} \subset \mathcal{L}_k \subset \mathcal{L}_{k+1} \subset \mathfrak{R}_\infty. \quad (8)$$

Also, theoretically, $\lim_{k\to\infty} \mathcal{L}_k = \mathfrak{R}_\infty$ is valid by computing up to $k = \infty$. From the above, $\mathcal{L}_k$ becomes closer to $\mathfrak{R}_\infty$ as k is set larger, and setting the value of k appropriately gives a good approximation of $\mathfrak{R}_\infty$ for $\mathcal{L}_k$. Thus, we can approximate $\mathfrak{R}_\infty$ from the inside by recursive state updating.

For example, Hirata[10] proposed a method to estimate the set $\mathcal{L}_{k+}$ satisfying $\mathfrak{R}_\infty \subset \mathcal{L}_{k+}$ by expanding it in a ratio that depends on the accuracy of the estimate of $\mathcal{L}_k$ obtained for a linear time-invariant system. In this method, $\mathfrak{R}_\infty$ is estimated from the outside.

2.3 Estimation of state reachable set by robust invariant ellipsoid

In this section, we explain how to derive a robust invariant set of states in the form of an ellipsoid. First, we set up the set of disturbances $\mathcal{W}_c$ as follow:

$$\mathcal{W}_c = \{w \in \mathbb{R}^m \mid w^T w \leq 1\}. \quad (9)$$

Then, consider the set of states of an ellipsoid using a positive-definite matrix P that satisfies the following:

$$\mathcal{E}(P) = \{x \in \mathbb{R}^n \mid x^T P x \leq 1\}. \quad (10)$$

In this case, if

$$(Ax + Bw)^T P(Ax + Bw) \leq 1 \tag{11}$$

holds for the disturbance $w \in \mathcal{W}_c$, the ellipsoid $\mathcal{E}(P)$ is one of the robust invariant set from the Definition 1. In this connection, the following theorem holds where $\rho(A)$ is the spectral radius of the matrix A and represents the absolute value of the maximum eigenvalue.

Theorem 1 Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m,}$ be given in a pair of discrete-time linear time-invariant systems (A, B). Furthermore, we assume that $w_k \in \mathcal{W}_c$ holds for any k. The necessary and sufficient condition for an ellipsoid $\mathcal{E}(P) = \{x \in \mathbb{R}^n \mid x^T Px \leq 1\}$ to be a robust invariant set of pair (A, B) is that there exists an $\alpha \in [0, 1 - \rho(A)^2]$ satisfying

$$\begin{bmatrix} A^T PA - (1-\alpha)P & A^T PB \\ B^T PA & B^T PB - \alpha I \end{bmatrix} \leq 0. \tag{12}$$

∎

By finding P that satisfies the condition of the Theorem 1, the ellipsoid $\mathcal{E}(P)$ becomes a robust invariant set of systems in (1), and from (5) we have the following relation:

$$\mathfrak{R}_\infty \subset \mathcal{E}(\mathcal{P}). \tag{13}$$

Therefore, if we use robust invariant ellipsoid, we can estimate $\mathfrak{R}_\infty$ from the outside and thus guarantee the performance of the analysis results. Besides, there are innumerable robust invariant ellipsoids since there are innumerable P that satisfy the Theorem 1.

3. Problem Formulation

3.1 Discrete-time switched linear system

In this paper, we discuss robust invariant set estimation for switched linear systems by using linear time-invariant systems. Consider the following discrete-time switched linear system.

$$x_{k+1} = A_{\sigma(k)} x_k + B_{\sigma(k)} w_k \tag{14}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $w_k \in \mathcal{W}_p$ is the disturbance. The set of disturbance signals $\mathcal{W}_p$ is given by (6). The switching signal $\sigma(k) \in \{1, \cdots, N\}$ is a piecewise constant function of the time k. where $N \geq 1$ is the total number of subsystems and let $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $i \in \{1, \cdots, N\}$. In addition, the matrix pair (A_i, B_i) is a controllable and A_i is Schur-stable for all i.

In the same way as in Definition 1, we define the following for the switched linear system (14).

Definition 3 For a pair of discrete-time switched linear systems $(A_{\sigma(k)}, B_{\sigma(k)})$ with any switching signal $\sigma(k)$ and bounded disturbance $w_k \in \mathcal{W}_p$, if the following equation:

$$x_k \in \mathcal{Y}, \rightarrow A_{\sigma(k)} x_k + B_{\sigma(k)} w_k \in \mathcal{Y} \tag{15}$$

holds at any time k, the set $\mathcal{Y}$ is a robust invariant set. ∎

Moreover, In the same way as in Definition 2, we define the following for the switched linear system (14).

Definition 4 For a pair of discrete-time switched linear systems $(A_{\sigma(k)}, B_{\sigma(k)})$ with any switching signal $\sigma(k)$ and bounded disturbance $w_k \in \mathcal{W}_p$, the state reachable set $\mathcal{R}_\infty$ is given by

$$\begin{aligned} \mathcal{R}_\infty = \{&x_k \in \mathbb{R}^n \mid x_0 = 0, x_k \textbf{ satisfies } (14) \\ &w_k \in \mathcal{W}_p,\ k \geq 0\}. \end{aligned} \tag{16}$$

∎

The state reachable set $\mathcal{R}_\infty$ indicates the state region from the origin and can be considered as the set that gives the smallest estimate among the robust invariant sets. In another view, $\mathcal{Y}$ is the set covering $\mathcal{R}_\infty$ from the outside, and the relation

$$\mathcal{R}_\infty \subset \mathcal{Y} \tag{17}$$

holds. Therefore, in order to estimate $\mathcal{R}_\infty$, it is important to estimate how small $\mathcal{Y}$ is.

3.2 Estimation of reachable set from inside by recursive state updating

In this section, we apply Section 2.2 to approximate the reachable set $\mathcal{R}_\infty$ of the switched linear system from the inside. As in Section 2.2, the region of states that can be reached from the origin by each time is calculated recursively. However, it should be noted that the switched linear system parameters are time-varying. The region that the state can be reached by each time for the switched systems may not be a convex polyhedron set but a set of polyhedra with some concave parts. Therefore, the treatment is different from that of the reachable set estimation method for linear time-invariant systems (Section 2.2).

If the switching occurs at an arbitrary time k, the polyhedral set of the switched linear system $\mathcal{V}_k$ can be calculated as follows:

$$\begin{aligned} \mathcal{V}_{0,1} &= \{0\}, \\ \mathcal{V}_{k,N(j_{k-1}-1)+i} &= \{x \in \mathbb{R}^n \mid x = A_i z + B_i w, z \in \mathcal{V}_{k-1,j_{k-1}}, \\ &\quad w \in \mathcal{W}_p\}, \\ &\quad i \in \{1, \cdots, N\},\ j_k \in \{1, \cdots, N^k\}, \\ \mathcal{V}_k &= \bigcup_{j_k=1}^{N^k} \mathcal{V}_{k,j_k},\ k \geq 1. \end{aligned} \tag{18}$$

Then, $\mathcal{V}_{k,j_k}$ is determined by (18) for any $j_k \in \{1, \cdots, N^k\}$. j_k is the natural number that represents all combinations of switches. Note that since we assume arbitrary switching at arbitrary times in this paper, the combination of switching becomes exponentially larger when k becomes large. The specific flow of the calculation is as follows. For all polyhedral sets $\mathcal{V}_{k,j_k}$ at time k with the origin as the initial state set, calculate the polyhedron set $\mathcal{V}_{k,j_k}$ for all switch signals, respectively. Then calculate the sum set of all j_k for $\mathcal{V}_{k,j_k}$ at time k. By doing so, we can obtain the polyhedral set $\mathcal{V}_k$ of the switched linear system.

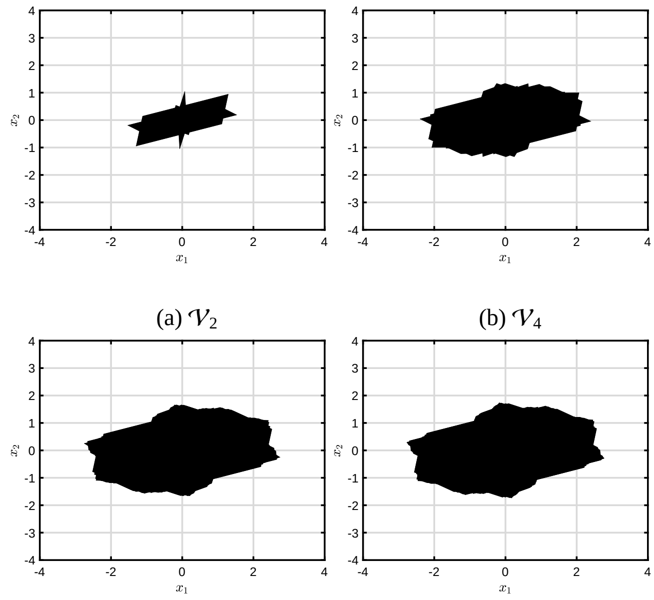
$\mathcal{V}_{k,j_k}$ can be constructed by removing the disturbance term using the Fourier-Motzkin algorithm[21] and making it a linear constraint of the state term only. This $\mathcal{V}_k$ indicates the region where the state of the switched linear system, where switching occurs at an arbitrary time, can be reached from the origin by the k step. Also, since $\mathcal{V}_k$ assumes all the switches for k, the state region increases as follows:

$$\mathcal{V}_0 \subset \cdots \subset \mathcal{V}_{k-1} \subset \mathcal{V}_k \subset \mathcal{V}_{k+1} \subset \mathcal{R}_\infty. \tag{19}$$

Also, theoretically, $\lim_{k\to\infty} \mathcal{V}_k = \mathcal{R}_\infty$ is valid by computing up to $k = \infty$. Therefore, the larger k is set for $\mathcal{V}_k$, the better the estimation results for $\mathcal{R}_\infty$. However, in practice, it is not possible to compute $k = \infty$ due to the enormous computational complexity, and it is not possible to find a set that is $\mathcal{V}_k = \mathcal{R}_\infty$. In addition, unlike linear time-invariant systems, when the total number of subsystems N is large, for example, the computational complexity increases exponentially with time k, as in the case of the summation set operation in (18). In such a case, it is complicated to calculate $\mathcal{V}_k$ for a massive k, and we have to stop the calculation for a lesser k.

We illustrate the results of estimating $\mathcal{R}_\infty$ by recursive state updating using numerical examples. We considered a switched linear system with $N = 2$ subsystems and set up the matrix as follows:

$$A_1 = \begin{bmatrix} 0.375 & -0.9 \\ 0.3 & 0.45 \end{bmatrix}, B_1 = \begin{bmatrix} 1.2 \\ 0.4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.225 & -0.75 \\ 0.45 & 0.225 \end{bmatrix}, B_2 = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}. \tag{20}$$



(a) $\mathcal{V}_2$ (b) $\mathcal{V}_4$

(c) $\mathcal{V}_7$ (d) $\mathcal{V}_8$

Fig. 1: State reachable set $\mathcal{R}_\infty$ estimation by polyhedral set $\mathcal{V}_k$

Table 1: The computation time t of $\mathcal{V}_k$

$\mathcal{V}_k$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$	$\mathcal{V}_6$	$\mathcal{V}_7$	$\mathcal{V}_8$
t (sec.)	3.24	9.89	25.73	65.98	168.63	425.8

Given a disturbance $w_k \in \mathcal{W}_p$ (where $M_W = [1\ -1]^T$, $m_W = [1\ 1]^T$) and an arbitrary switching signal, Fig. 1 shows the calculation results of $\mathcal{V}_k$ by recursive state updating.

As we can see from these comparisons, $\mathcal{V}_k$ is getting larger as k increases, which corresponds to (19). Since the volume of the $\mathcal{V}_7$, $\mathcal{V}_8$ is almost the same, we can estimate $\mathcal{R}_\infty$ with sufficient accuracy for this system from the inside of $\mathcal{R}_\infty$. In addition, the elapsed time from the start of computation is shown in Table 1 (Not including the drawing time of $\mathcal{V}_k$). Comparing Table 1 and Fig. 1, we can see that the computation time increases with the improvement of the estimation accuracy.

Here $\mathcal{V}_k$ is the inner set of $\mathcal{R}_\infty$ as indicated by (19), since it stops at a finite k. If we analyze the system using the inner set of $\mathcal{R}_\infty$, the actual states may reach outside that set. Therefore, if the system is analyzed using $\mathcal{V}_k$ computed by recursive state updating, it does not guarantee the performance of the analysis results. In addition, the estimation accuracy of the reachable set becomes worse when the order of the system and the total number of subsystems are large. Therefore, recursive state updating is not sufficient for the estimation of the reachable set of state for switched linear systems.

3.3 Estimation of state reachable set by robust invariant ellipsoid

We consider a method to approximate $\mathcal{R}_\infty$ using a robust invariant set of an ellipsoid (robust invariant ellipsoid). For linear time-invariant systems, the method introduced in Section 2.3 has been proposed, and in this paper, we consider a set of systems that extend it to the switched linear system.

In order to construct a robust invariant ellipsoid for a switched linear system that occurs at any given time, we consider a common Lyapunov function for all subsystems.

First, we extend the equation (9) and set the set of disturbances W as follow:

$$\overline{\mathcal{W}_c} = \{w \in \mathbb{R}^m \mid w^T w \le \overline{w}^2\}. \tag{21}$$

where $\mathcal{W}_c$ is the set containing $\mathcal{W}_p$ given by Section 3.1 and is set to satisfy ($\mathcal{W}_p \subset \mathcal{W}_c$). Then, consider the set of states of an ellipsoid that satisfies

$$\mathcal{E}(P) = \{x \in \mathbb{R}^n \mid x^T P x \le 1\}. \tag{22}$$

The ellipsoid $E(P)$ is a robust invariant set from Definition 3 if it satisfies the following equation:

$$(A_i x + B_i w)^T P (A_i x + B_i w) \le 1 \tag{23}$$

for the disturbance w and all i. The following Theorem holds by applying a previous study [9]. where $\rho(A_i)$ is the spectral radius of the matrix A_i and represents the absolute value of the maximum eigenvalue.

Theorem 2 In the switched linear system (14) with arbitrary switching signals, consider that $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$ is given for all i with bounded disturbance $w_k \in \overline{\mathcal{W}_c}$. Then, A necessary and sufficient condition for an ellipsoid $\mathcal{E}(P) = \{x \in \mathbb{R}^n \mid x^T P x \le 1\}$ to be a robust invariant set of this system is that there exists an $\alpha_i \in [0, 1 - \rho(A_i)^2]$ satisfying

$$\begin{bmatrix} A_i^T P A_i - (1-\alpha_i) P & {A_i}^T P B_i \\ {B_i}^T P A_i & {B_i}^T P B_i - \frac{\alpha_i}{\overline{w}^2} I \end{bmatrix} \le 0 \tag{24}$$

for all i. ∎

That is, By finding P that satisfies the condition of the Theorem 2, the ellipsoid $\mathcal{E}(P)$ becomes a robust invariant set of systems in (14), assuming an arbitrary switching signal and a disturbance $w_k \in \mathcal{W}_p$.

If Theorem 2 is satisfied, the following relation holds from (5):

$$\mathcal{R}_\infty \subset \mathcal{E}(P). \tag{25}$$

In Theorem 2, there is an infinite number of P that satisfy the inequality condition, and it is essential to find the smallest possible robust invariant ellipsoid $\mathcal{E}(P)$ among them. Based on this, we consider the following inequality conditions:

$$\frac{1}{\gamma} x^T x \le x^T P x. \tag{26}$$

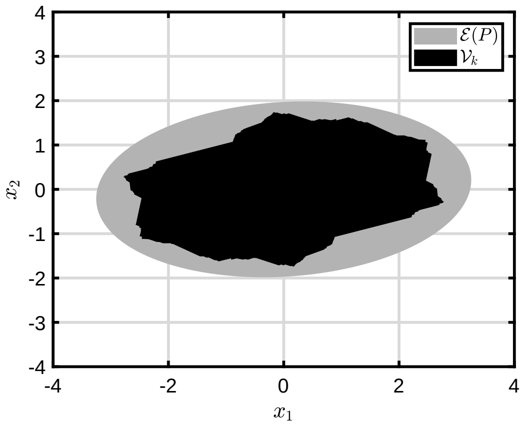


Fig. 2: Comparison of $\mathcal{V}_8$ and $\mathcal{E}(P)$

Here, since the ellipsoid is given by $\{x \in \mathbb{R}^n \mid x^T P x \le 1\}$, if (26) is satisfied for all i, then, the following inequality:

$$\frac{1}{\gamma} x^T x \le 1 \tag{27}$$

is satisfied. In other words, the ellipsoid fits within a sphere (circle) of radius $\sqrt{\gamma}$, which enables us to evaluate the ellipsoid based on its major axis. Applying the Shur's complement to the inequality (26), we can convert it to a matrix inequality such that

$$\begin{bmatrix} P & \mathbf{1} \\ \mathbf{1} & \gamma I \end{bmatrix} \ge 0. \tag{28}$$

The conditional expressions (24) and (28) can be regarded as BMI with each element of P and α_i and γ as variables, or LMI with α_i fixed in small increments. Thus, we can estimate $\mathcal{E}(P)$ based on the major axis of the smallest ellipsoid by finding the smallest γ that satisfies the conditions of (24) and (28) for all i.

In this paper, we take the long axis of an ellipsoid as the evaluation function, but there are various ways to uniquely determine the ellipsoid, such as using the output as the evaluation signal, depending on the purpose of the analysis.

Note that the conservativeness may be higher than in the linear time-invariant case [9], since the common Lyapunov matrix P should be found for all i.

Then, in the switched linear system (14), we give a numerical example of (20) and illustrate $\mathcal{E}(P)$ with a positive definite matrix P common to all subsystems. In order to compare the accuracy with the method in Section 3.2, we set $w^T w \le \overline{w}^2 = 1$ as the disturbance. When analyzing the state reachable set for a disturbance $w \in \mathcal{W}_p$, it is necessary to determine $\mathcal{W}_p \subset \overline{\mathcal{W}_c}$ and $\overline{w}$ as small as possible from the point of view of conservativeness.

To solve the matrix inequality in (24), we compute α_i as a fixed LMI with a total of 33^3 patterns in increments of 0.03 from 1. As a result, the evaluation signal for determining the unique $\mathcal{E}(P)$ is $\gamma = 10.61$, and the calculated $\mathcal{E}(P)$ is shown in Fig. 2. Here we also show the sum set of the polyhedral set $\mathcal{V}_8$ calculated in Section 3.2.

As shown in (25), $\mathcal{E}(P)$ is the outer set of $\mathcal{R}_\infty$. Therefore, the performance of the analysis results is guaranteed when $\mathcal{E}(P)$ is used for the analysis. Then, we discuss the accuracy of the analysis using a robust invariant ellipsoid $\mathcal{E}(P)$.

From Fig. 2, we can see that we approximate $\mathcal{V}_8$, a reasonable estimate of $\mathcal{R}_\infty$, from the outside by using $\mathcal{E}(P)$. However, we can see that there is a large gap between each set. This is due to the fact that $\mathcal{R}_\infty$ is a polyhedron with a massive number of parameters, which is approximated using an ellipsoid with few parameters.

Thus, if there is a large gap between the estimated robust invariant set and $\mathcal{R}_\infty$, the analysis using it will result in conservative analysis results. If the analysis results are highly conservative in control system analysis, it is not only difficult to clarify the effects of disturbances in the actual system but also difficult to use in the design of the control system. Therefore, it is necessary to improve the gap with $\mathcal{R}_\infty$ in the estimation of the robust invariant set to reduce the conservatism of the analysis results.

4. Main Result

4.1 Combining recursive state updating and robust invariant ellipsoid

In this section, in order to guarantee the state reachable set $\mathcal{R}_\infty$ of the switched linear system from the outside, and to improve the conservativeness of the results of control system analysis, we describe a method for estimating robust invariant set using the method of our previous work [20], which combines recursive state updating and robust invariant ellipsoid $\mathcal{E}(P)$. In this paper, we use a robust invariant ellipsoid $\mathcal{E}(P)$ from the viewpoint of computational simplicity, but it should be noted that any form of robust invariant set is acceptable. First, we discuss an idea for combining the two methods.

4.2 Basic idea

In this section, we explain the basic idea of the method in this paper, which uses recursive state updating and robust invariant ellipsoid $\mathcal{E}(P)$ in combination to reduce conservativeness. First, if there exists a robust invariant set $\mathcal{X}$, the state at a given time k exists in that set. Assuming that the value of the disturbance w is zero after that time, the state will converge to the origin. Therefore, the robust invariant set also becomes smaller and smaller. Therefore, by estimating the invariant set $\mathcal{X}_k$ when a k step has elapsed after applying a disturbance $w = 0$ to $\mathcal{X}$, the set is smaller than the original robust invariant set $\mathcal{X}$. In other words, the following relation holds.

$$\{0\} \subset \cdots \subset \mathcal{X}_{k+1} \subset \mathcal{X}_k \subset \mathcal{X}_{k-1} \subset \cdots \subset \mathcal{X}_0 = \mathcal{X} \tag{29}$$

Also, theoretically, $\lim_{k\to\infty} \mathcal{X}_k = \{0\}$ is valid by computing up to $k = \infty$.

We further consider that the state set is a vector space of states and that the state vector in a linear time-invariant system is represented by the sum of zero input vector and zero state vector. Then, we compute the sum of the vector space of states for k steps of disturbance and the vector space of state transitions with k steps of disturbance as zero. This configuration corresponds to considering the direct sum of the set. The following Theorem holds.

Theorem 3 For any positive integer ℓ, $\mathcal{X}_\ell \oplus \mathcal{L}_\ell$ is a robust invariant set and the following relation holds.

$$\mathcal{X}_{\ell+1} \oplus \mathcal{L}_{\ell+1} \subset \mathcal{X}_\ell \oplus \mathcal{L}_\ell \tag{30}$$

Proof 1 First, if $\mathcal{X}_\ell \oplus \mathcal{L}_\ell$ is a robust invariant set, the set of states that the states in the invariant set take after 1 steps is

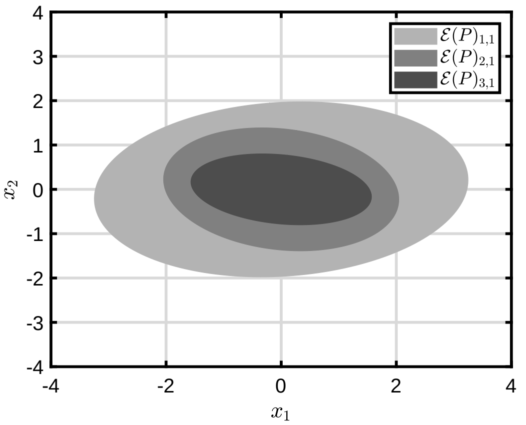


Fig. 3: $\mathcal{E}(P)_{k,1}(k = 1, 2, 3)$

$\mathcal{X}_{\ell+1} \oplus \mathcal{L}_{\ell+1}$. From this, it follows that $\mathcal{X}_{\ell+1} \oplus \mathcal{L}_{\ell+1}$ is a subset of $\mathcal{X}_{\ell} \oplus \mathcal{L}_{\ell}$ and (30) holds. Here, for a disturbance $w \in \mathcal{W}_p$,

$$x \in \mathcal{X}_{\ell} \oplus \mathcal{L}_{\ell} \ \rightarrow Ax + Bw \in \mathcal{X}_{\ell+1} \oplus \mathcal{L}_{\ell+1} \quad (31)$$

holds. Furthermore, since the relation (30) leads to

$$x \in \mathcal{X}_{\ell+1} \oplus \mathcal{L}_{\ell+1} \ \rightarrow Ax + Bw \in \mathcal{X}_{\ell+1} \oplus \mathcal{L}_{\ell+1}, \quad (32)$$

$\mathcal{X}_{\ell+1} \oplus \mathcal{L}_{\ell+1}$ is a robust invariant set. Therefore, we can show by mathematical induction that $\mathcal{X}_{\ell} \oplus \mathcal{L}_{\ell}$ is a robust invariant set for any ℓ. □

Since the direct sum of the set is a subset of the original set from Theorem 3, the following relation holds.

$$\mathcal{X}_k \oplus \mathcal{L}_k \subset \mathcal{X}_{k-1} \oplus \mathcal{L}_{k-1} \subset \cdots \subset \mathcal{X}_0 = \mathcal{X} \quad (33)$$

In the following, we will use this relationship to approximate a robust invariant set for the switched linear system from the outside. We can expect to find a set close to $\mathcal{R}_\infty$ by generating a robust invariant set with a large value of k to the extent possible.

4.3 Analysis of robust invariant set for switched linear systems

In this section, based on the ideas in the previous section, we present a specific analysis procedure for the switched linear system. First, consider applying a zero input to the robust invariant set for the switched linear system (14). Note that since the constraint equations of the switched linear system increase exponentially, the robust invariant set handled here is a robust invariant ellipsoid $\mathcal{E}(P)$, which is relatively easy to calculate from the viewpoint of computational simplicity. When a zero disturbance is applied to the initial set $\mathcal{E}(P)$, the invariant set $\mathcal{E}(P)_{k,j_k}$ is calculated as

$$\begin{aligned} \mathcal{E}(P)_{0,1} =& \mathcal{E}(P) = \{x \in \mathbb{R}^n \mid x^T P x \leq 1\}, \\ \mathcal{E}(P)_{k,N(j_{k-1}-1)+i} =& \{x \in \mathbb{R}^n | x = A_i z, z \in \mathcal{E}(P)_{k-1,j_{k-1}}\}, \\ & i, j_k, k \textbf{ satisfy } (18). \end{aligned} \quad (34)$$

Then $\mathcal{E}(P)_{k,j_k}$ is determined by (34) for any $j_k \in \{1, \cdots, N^k\}$. Fig. 3 shows $\mathcal{E}(P)_{k,j_k}$ for $i = 1$ (i.e. $j_k = 1$ in (34)) for all k, where we will use the numerical example in (20). From Fig. 3, we can confirm that $\mathcal{E}(P)_{k,j_k}$ becomes smaller as k is increased. This corresponds to (29).

Also based on the ideas in the previous section, considering the direct sum of the state set $\mathcal{V}_{k,j_k}$ for zero states and the state set $\mathcal{E}(P)_{k,j_k}$ for zero disturbances for each j_k, we obtain the following state set $\mathcal{P}_{k,j_k}$:

$$\mathcal{P}_{k,j_k} = \mathcal{V}_{k,j_k} \oplus \mathcal{E}(P)_{k,j_k}. \quad (35)$$

In addition, we obtain the robust invariant set $\mathcal{P}_k$ of the switched linear system by finding the sum set of robust invariant set $\mathcal{P}_{k,j_k}$ at each time k for all j_k as follows.

$$\mathcal{P}_k = \bigcup_{j_k=1}^{N^k} \mathcal{P}_{k,j_k} \quad (36)$$

From the Theorem 3, $\mathcal{P}_k$ is a robust invariant set, and $\mathcal{P}_{k+1} \subset \mathcal{P}_k$ is available at any time k. To compute the set $\mathcal{P}_k$, we combine the set $\mathcal{E}(P)_k$, which becomes smaller at every step when a zero disturbance is applied to a robust invariant ellipsoid as the initial set, and the set $\mathcal{V}_k$, which becomes larger at every step when a disturbance is applied to the origin as the initial set. Therefore, from (19) and (29), the following relationship between $\mathcal{P}_k$ and $\mathcal{R}_\infty$ holds.

$$\mathcal{P}_0 = \mathcal{E}(P) \supset \cdots \supset \mathcal{P}_{k-1} \supset \mathcal{P}_k \supset \mathcal{P}_{k+1} \supset \mathcal{R}_\infty \quad (37)$$

Also, theoretically, if we calculate up to $k = \infty$, $\lim_{k\to\infty} \mathcal{E}(P)_{k,j_k} = \{0\}$, $\lim_{k\to\infty} \mathcal{V}_k = \mathcal{R}_\infty$, then $\lim_{k\to\infty} \mathcal{P}_k = \mathcal{R}_\infty$ holds. Therefore, by computing $\mathcal{P}_k$ with a computationally large value of k, we can obtain a robust invariant set with low conservativeness. Furthermore, it should be noted that for any $k \geq 0$ the following relationship holds.

$$\mathcal{V}_k \subset \mathcal{R}_\infty \subset \mathcal{P}_k \quad (38)$$

5. Numeral Examples

In this section, we give numerical examples of (20) for a switched linear system given by (14), and illustrate a robust invariant set using the proposed method. The set of disturbances $\mathcal{W}_p$ is given by $M_W = [1\ -1]^T$, $m_W = [1\ 1]^T$. where the disturbance $w_k \in \overline{\mathcal{W}_c}$ ($\overline{w} = 1$) is used to computed $\mathcal{E}(P)$, $\mathcal{W}_p \subset \mathcal{W}_c$.

The robust invariant set $\mathcal{P}_2$, $\mathcal{P}_6$ and $\mathcal{P}_8$ computed with the proposed method are shown in Fig. 4,5,6 respectively. In addition, we also show a robust invariant ellipsoid $\mathcal{E}(P)$ and a good estimate of the reachable set $\mathcal{V}_8$.

Fig. 4,5,6 shows that proposed robust invariant set $\mathcal{P}_2, \mathcal{P}_6, \mathcal{P}_8$ by the proposed method approximates from the outside the polyhedral set $\mathcal{V}_8$, which seems to be a good estimate of the state reachable set $\mathcal{R}_\infty$, and is a better estimate than the robust invariant ellipsoid $\mathcal{E}(P)$. Also, we can see that $\mathcal{P}_8 \subset \mathcal{P}_6 \subset \mathcal{P}_2$ is used, which corresponds to (37). From these results, it is confirmed that the larger the value of k is taken, the more accurate the robust invariant set is for the calculation of $\mathcal{P}_k$, which approximates the state reachable set $\mathcal{R}_\infty$. In addition, the drawing time of the proposed robust invariant set $\mathcal{P}_8$ is 1162 sec (Not including the computation time of $\mathcal{V}_{8,i}$ and $\mathcal{E}(P)_{8,i}, \forall i$). Note that $\mathcal{P}_k = \mathcal{R}_\infty$ does not hold since it is impossible to compute $k = \infty$ in practice. However, even if we terminate the calculation at k, $\mathcal{P}_k$ is the set that covers $\mathcal{R}_\infty$ from the outside, and the estimation accuracy is higher than the initial set.

In addition, we compare the method in [19]. The method for estimating the state-reachable set using Lyapunov function based inequalities based on multiple Lyapunov strategies was provided in [19]. It is a method to approximate the state-reachable set from the outside by determining the optimal plural ellipsoid using a genetic algorithm and obtaining their product

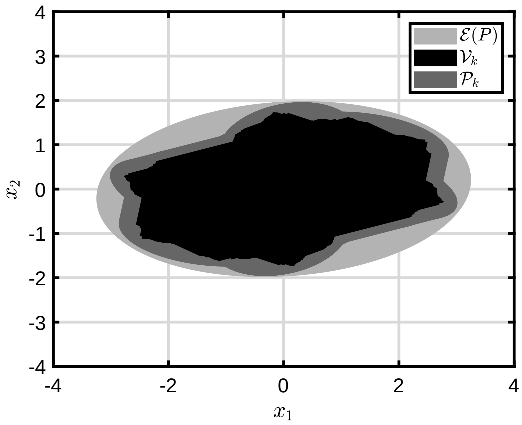


Fig. 4: Comparison of $\mathcal{P}_2$, $\mathcal{V}_8$ and $\mathcal{E}(P)$

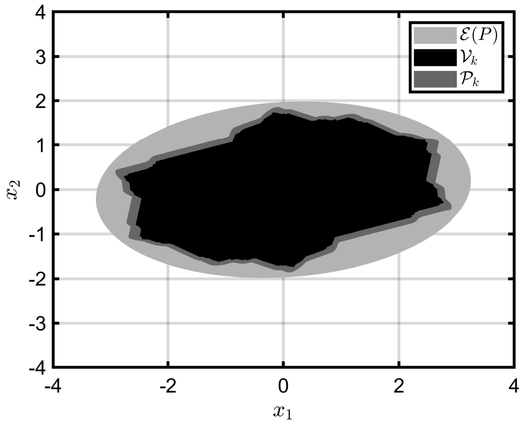


Fig. 5: Comparison of $\mathcal{P}_6$, $\mathcal{V}_8$ and $\mathcal{E}(P)$

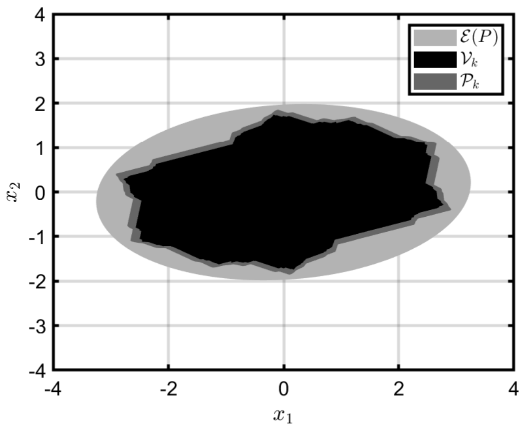


Fig. 6: Comparison of $\mathcal{P}_8$, $\mathcal{V}_8$ and $\mathcal{E}(P)$

set $\mathcal{F}$. It is shown to be less conservative than the robust invariant ellipsoid $\mathcal{E}(P)$, which is constructed by the Lyapunov function common to all subsystems. Here, a comparison between $\mathcal{F}$ and the proposed robust invariant set $\mathcal{P}_8$ is shown in Fig. 7. From Fig. 7, we can confirm that the robust invariant set obtained by the method of this paper is smaller than the robust invariant set. As described above, the robust invariant set obtained by the proposed method enables us to estimate the state of the system more accurately than the conventional method [19], i.e., with less conservativeness, while guaranteeing how disturbances affect the state of the system.

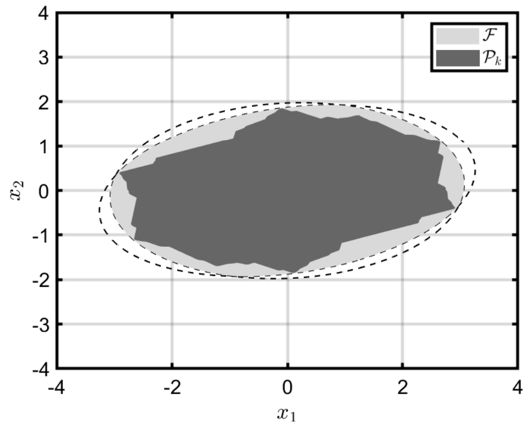


Fig. 7: Comparison of $\mathcal{P}_8$ and $\mathcal{F}$ in [19]

6. Conclusion

In this paper, we propose a method for estimating robust invariant set for switched linear systems by combining recursive state updating and robust invariant ellipsoid. The direct sum of the two sets is calculated to obtain a smaller robust invariant set. Through simulations, we confirmed that the robust invariant set obtained by the proposed method approaches $\mathcal{R}_\infty$ when the number of steps is increased, and it is possible to provide a smaller estimated invariant set.

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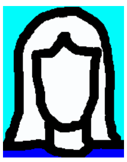

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