Riemannian Median and Its Applications for Orientation Distribution Function Computing

Jian Cheng, Aurobrata Ghosh, Tianzi Jiang, Rachid Deriche

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J. Cheng[1,2], A. Ghosh[1], T. Jiang[2], and R. Deriche[1]
[1]INRIA Sophia Antipolis, Sophia Antipolis, Valbonne, France, [2]Institute of Automation, Chinese Academy of Sciences, Beijing, Beijing, China, People's Republic of

Introduction. The geometric median is a classic robust estimator of centrality for data in Euclidean spaces, and it has been generalized in analytical manifold in [1]. Recently, an intrinsic Riemannian framework for Orientation Distribution Function (ODF) was proposed for the calculation in ODF field [2]. In this work, we prove the unique existence of the Riemannian median in ODF space. Then we explore its two potential applications, median filtering and atlas estimation.

Theory. If we denote $\{x_i, i = 1,\ldots,n\}$ in manifold M as given samples, and $\{w_i, i = 1,\ldots,n\}$ as given weights, $\sum_{i=1}^{n} w_i = 1$, then the weighted Riemannian mean is defined as $\mu = \arg\min_{x\in M} \sum_{i=1}^{n} d(x, x_i)^2$ and the weighted Riemannian median is defined as $m = \arg\min_{x\in M} \sum_{i=1}^{n} d(x, x_i)$, where $d(x, x_i)$ is the Riemannian distance. In [1], the authors proved that weighted Riemannian median exists and is unique if (a) the sectional curvatures of M are non-positive, or if (b) the sectional curvatures of M are bounded above by $\Delta > 0$ and $diam(U) < \pi / (2\sqrt{\Delta})$, where U is the convex set which contains $\{x_i\}$ and $diam(U)$ is the diameter of U. In [2], the authors demonstrated that the space of ODFs is a closed convex subset of a high dimensional semi-sphere, where the weighted Riemannian uniquely exists. And the space is contained by a convex cone with 90^o. Thus the sectional curvature of the ODF space is 1 as well as the sphere and $diam(U) < \pi / 2$, which means the condition (b) is satisfied. So the Riemannian median of ODFs uniquely exists and it could be found via the gradient descent method in [1]. In our experiments, this method will converge in 5 iteration steps.

Application: Median Filtering. Median filtering is a traditional filtering technique in image processing field. It is often used to remove outlier signal with large noise. Based on the theoretical results of unique existence and numerical solution of Riemannian median, we could apply Riemannian median filtering on an ODF field. The value in each voxel in the output ODF field is the Riemannian median of ODFs in a local kernel window in the original ODF field. Here we present an experiment on synthetic data. In Fig. 1, the original synthetic ODF field is shown in the left, which was generated using the multi-tensor model [4] and ODF is defined as the marginal probability of diffusion PDF in [3]. We use a tensor profile with eigenvalues [300, 300, 1700] x 10^{-6} mm^2/s (Fractional Anisotropy = 0.8) and the Gaussian noise is added in the tangent space. The b-value is 3000 s/mm^2. Gaussian noise with large deviation (std=0.1) was added in Riemannian space, which is shown in the center. And the median filtered ODF field with Gaussian kernel (3x3, std=0.3) is in the right. It could been seen that the Riemannian median filtering for ODFs is a edge-preserving smoothing, as well as the median filtering in Euclidean space.

Application: Atlas Estimation. Compared with the mean, the median is a much more robust estimator, whose breakdown point is larger [1]. The measure of robustness of an estimator is the breakdown point, which is the fraction of the data that can be completely corrupted without affecting the boundedness of the estimator. This property makes the median more appropriate than the mean in atlas estimation. Here we demonstrate the robustness of the mean and the median calculated from the synthetic data with different percentage of outliers. From a ground truth ODF, we generated 10 random ODFs as the images under the exponential map of Gaussian random tangent vectors. Then we replace some of these ODFs in a given percentage with some outliers. In Fig. 2, we compare the robustness between Euclidean mean, Riemannian mean and Riemannian median. We could conclude that Euclidean mean and Riemannian mean estimators seem to have the same behavior in robustness and Riemannian median is more robust than both Euclidean mean and Riemannian mean. We also show a constructed atlas from a real human dataset of 5 subjects. We use high angular resolution diffusion-weighted data (50 diffusion encoding gradients with a b-value of 1126 s/mm^2, twice-refocused spin-echo EPI sequence, TE = 100 ms, 1.5×1.5×1.5 mm^3 voxel resolution, three repetitions, corrected for subject motion). Since so far there is no common registration method for ODF data and it is also not the focus in this work, we just use a naïve way to registrate the ODF data. The GFA images [4] were calculated from DWIs in each subject and were affinely coregistrated together. The affine transformation was used to register all DWIs and re-orientate the gradient directions for each subject through the finite strain (FS) method in [5]. After that, ODF images could be estimated from registrated DWIs and reorientated gradient directions. The ODF in every voxel in the atlas was estimated as the median of the ODFs in the same voxel in ODF images of 5 subjects. ODF data of one subject and the estimated atlas are shown in Fig. 3. In order to estimate the robustness, we also add some noise in one subject and estimated atlas from the noisy data, which is shown in Fig.3. The atlas from noisy data is much similar with the one from the real data, which demonstrate the robustness of Riemannian median.

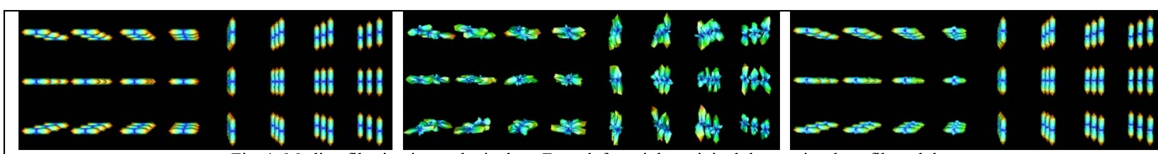

Fig. 1. Median filtering in synthetic data. From left to right: original data, noise data, filtered data.

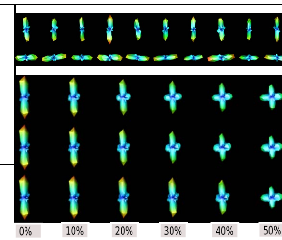

Fig. 2. Robustness with different percentage of outliers in different metrics. From top to bottom: regular ODFs, outliers, Euclidean mean, Riemannian mean, Riemannian median with different percentage of outliers: 0-50%

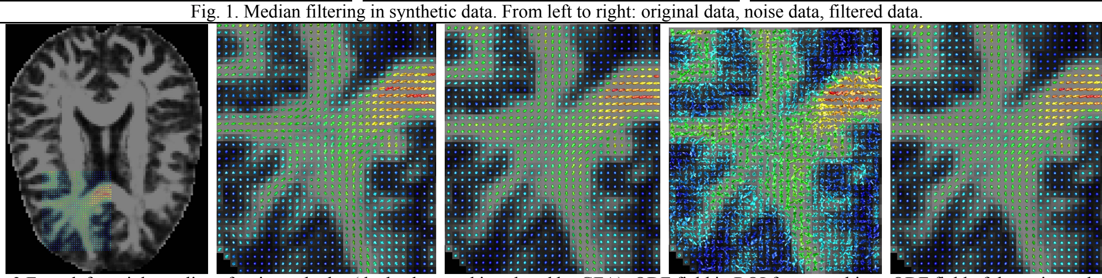

Fig. 3 From left to right: a slice of estimated atlas (the background is colored by GFA); ODF field in ROI for one subject; ODF field of the estimated atlas, ODF field in ROI with Gaussian noise in Riemannian space (std=0.1) for one subject; ODF field of estimated atlas from the data with noise.

Conclusion. Riemannian framework has been successfully used for tensor calculation in [6] and recently was generalized to ODF computation in [2]. However, in tensor space, there is no proof for the existence of median. Here we found that in ODF space the median uniquely exists. And we also proposed two potential applications, median filtering and atlas estimation. Our experiments showed that Riemannian median is much more robust than Riemannian mean and Euclidean mean and it is much useful for edge-preserving filtering and atlas estimation.

References [1] P. T. Fletcher, et al, NeuroImage 45 (2009) S143–S152, [2] J. Cheng et al, MICCAI 2009, [3] I. Aganj et al, ISBI 2009, [4] D.S. Tuch, MRM 52:1358–1372, 2004, [5] D. C. Alexander, et al, TMI vol 20, 2001. [6] X. Pennec, et al, IJCV 2006.