Lipschitz-Killing curvatures of the Excursion Sets of Skew Student's t Random fields

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***Abstract*—In many real applications related with Geostatistics, medical imaging and material science, the real observations have asymmetric, and heavy-tailed multivariate distributions. These observations are spatially correlated and they could be modelled by the skew random fields. However, certain statistical analysis problems require giving analytical expectations of some integral geometric characteristics of these random fields, such as Lipschitz-Killing curvatures, specifically Euler-Poincaré characteristic. This paper considers a class of skew random fields, namely skew student's t random fields. The goal is to give the analytical expressions of the Lipschitz-Killing curvatures of the skew student's t excursion sets on a compact subset S of $\mathbb{R}^2$. The motivation comes from the need to model the roughness of some engineering surfaces, involved in the total hip replacement application, which is characterized by the Euler-Poincaré characteristic function. The analytical and estimated Euler-Poincaré characteristics are fitted in order to test the skew student's t random field with the surface roughness topography.**

***Keywords*-Skew student-*t* random field; Excursion sets; Lipschitz-Killing curvatures; Skewness; Surface roughness topography**

I. Introduction

In a wide range of real applications, there is a great need to model observations that have asymmetric behaviour and heavy-tailed multivariate or univariate distributions. One of the most interesting models, that are developed for this aim, are the skew-normal and skew student's t families [1]–[7]. In many statistical problems, the random observations are spatially correlated, and the best solution to represent them is by the random field theory [8]–[10]. In certain applications the distribution of these correlated samples exhibit both skewness and heavy-tailed behaviour. These kind of samples can not be modelled by Gaussian random fields or skew Gaussian random fields [4], [6], [11]. Skew student-t random fields arise among the flexible class of stochastic models that could be used for such real problems. They are characterized by the skewness parameter and the degree of freedom, which controls the heavy-tail shape of the probability density function. The motivation to study this class of skew random fields is due to the need to model the roughness of certain engineering surfaces during wear, specifically those who are used for the total hip replacement. The surface topography is represented on a subset $S \subset \mathbb{R}^2$ as a 3D height maps. One of the interesting issues to study the roughness of the surface is the evaluation of the number of peak/valleys and their size distribution over certain height level which is related with the maxima of the random field. Many researches have been investigated for approximating the maxima of the random fields in applications related to modelling the brain anomalies and studying its functionality using the integral geometry of the excursion sets [12]–[16].

The integral geometry provides interesting characteristic functions that could be used to evaluate the morphology and the size of these excursion sets, namely Minkowski functionals or their extensions Lipschitz-Killing curvatures [13], [16]. They measure the intrinsic volumes, (i.e., $k-$th dimensional size), of these excursion sets such as the volume, half-surface area, mean curvature length, in three-dimensional case, and the area, half-boundary length in two-dimensional case. Furthermore, they measure the connectivity of the components inside these sets, namely Euler-Poincaré characteristic. It was proved in [9], [16] that the Euler-Poincaré characteristic function is a good approximation of the the maxima of the random field at the high thresholds, which make this characteristic function be considered as a useful technique for the statistical analysis-based problems, surface roughness characterization, and for model testing problems. This paper considers a class of skew random fields, called skew student's t random field with ν degrees of freedom, section IV. The interest of this work is to present these random fields on a subset $S \subset \mathbb{R}^2$ and to give the analytical expressions of the Lipschitz-Killing Curvatures of its excursion sets, specifically Euler-Poincaré characteristic, section V. A simulation is reported to validate the estimated characteristic functions computed from the simulated random field with the analytical ones for many realizations, in section VI. The skew student's t random field has been tested on the real 3D surfaces roughness topography by fitting the analytical Euler-Poincaré characteristic function of the skew student's

t excursion sets with the estimated one computed from the real surface, section VII.

II. Preliminaries

A real-valued random field, denoted $Y = Y(x), (x \in S)$, will be defined on a compact subset $S \subset \mathbb{R}^2$, with a mean μ_Y and variance σ_Y^2. Y will be supposed stationary, but not necessarily isotropic. The probability density function of Y will be denoted by p_Y.

III. The skew student's t distribution

This section will consider one of the most interesting skew student-t distribution's family defined in [7] in order to introduce the skew student-t random field.

A. Univariate skew student-t distribution

Consider a skew student-t random variable Y with ν degrees of freedom, skewness index α, mean value μ_Y and variance σ_Y^2, then, the probability density function of Y is expressed as follows:

$$p_Y(h;\alpha,\nu) = 2t_\nu(h;\mu_Y,\sigma_Y)T\left(\alpha s\sqrt{\frac{\nu+1}{\nu+s^2}};\nu+1\right) \tag{1}$$

where $s = (h-\mu_Y)/\sigma_Y$, t_ν is the student's t distribution, and T is the student-t cumulative distribution function with $\nu+1$ degrees of freedom. The parameter α controls the skewness, $\alpha > 0$ ($\alpha < 0$) the distribution has positive (negative) skewness. When $\alpha = 0$ the distribution is symmetric and it turns back to the known student's t distribution.

The stochastic representation of the skew student-t random variable Y that has the probability density function defined in 1 is given in [7] by:

$$Y = \mu_Y + \sigma_Y V^{-1/2} Z \tag{2}$$

where V is the chi-squared random variable, and $\boldsymbol{Z}$ is the known skew-normal random variable [1], [2], [4]–[6] which is defined as follows:

$$Z = \delta\,|U| + \sqrt{1-\delta^2}G \tag{3}$$

where $U, G \sim N(0,1)$ are independent normal random variables, and $\delta = \alpha/\sqrt{(1+\alpha^2)}$.

B. Multivariate skew student-t distribution

The $p-$variate skew student's t distribution with ν degrees of freedom has been defined in [1], [3], [7] for a random vector $\boldsymbol{Y} = (Y_1, ..., Y_p)^t$ as follows:

$$p_{\boldsymbol{Y}}(\boldsymbol{h};\boldsymbol{\mu_Y},\boldsymbol{\alpha},\Omega,\nu) =$$
$$2t_p(\boldsymbol{h};\boldsymbol{\mu_Y},\Omega,\nu)T\left(\boldsymbol{\alpha}^t\Omega^{-1/2}(\boldsymbol{h}-\boldsymbol{\mu_Y})\sqrt{\frac{\nu+p}{\nu+\boldsymbol{d_h}}};\nu+p\right)$$

where $\boldsymbol{d_h} = (\boldsymbol{h}-\boldsymbol{\mu})^t\Omega^{-1}(\boldsymbol{h}-\boldsymbol{\mu})$. $t_p(.;\boldsymbol{\mu_Y},\Omega,\nu)$ is the density of the $p-$variate student's t distribution with the $p-$dimensional mean vector $\boldsymbol{\mu_Y}$, $(p\times p)$ covariance matrix Ω, ν degrees of freedom, and α refereed to the $p-$dimensional skewness vector. $T(.;\nu+p)$ is the scalar cumulative distribution function of the standard student's t distribution with $\nu+p$ degrees of freedom.

The stochastic representation of the skew student's t random vector, $\boldsymbol{Y}$, is defined as:

$$\boldsymbol{Y} = \boldsymbol{\mu_Y} + V^{-1/2}(\delta\,|z| + \sqrt{1-\delta^2}\boldsymbol{G}) \tag{4}$$

where $\boldsymbol{G} \sim Normal_p(0,\Omega)$ is the multivariate Gaussian random vector with $p\times p$ covariance matrix Ω, $V \sim \chi^2_\nu/\nu$ independent of $\boldsymbol{G}$, and $z \sim Normal(0,1)$ is a standard normal random variable independent of both $\boldsymbol{G}$ and V.

IV. Skew student's t random field

The skew student's t random field introduced, in this paper, is based on the definitions given in [1]–[3], [7]. We will focus on the geometric properties of this random field using the integral geometry framework.

Let us first briefly review the student's t random field defined in [14]:

Definition IV.1. *Let $Z(x), G_1(x), ..., G_\nu(x)$, $x \in \mathbb{R}^d$ be independent, identically distributed, homogenous, real-valued Gaussian random fields with zero mean, unit variance, and with the $(d\times d)$ matrix, denoted Λ, such that $\Lambda = Var(\partial Z/\partial x) = Var(\partial G_i/\partial x)$, $i = 1, ..., \nu$. Then, the student's t random field with ν degrees of freedom, denoted T^ν, is defined, at any fixed point x, as:*

$$T^\nu(x) = \frac{\sqrt{\nu}Z(x)}{\sqrt{\sum_{k=1}^{\nu} G_k^2(x)}} \tag{5}$$

Definition IV.2 (Skew student's t random field)**.** *Let U be a stationary student's t random fields defined as above. Let U_0 be defined as follows:*

$$U_0 = \frac{\sqrt{\nu}z}{\sqrt{\sum_{k=1}^{\nu} G_k^2(x)}} \tag{6}$$

where $z \sim Normal(0,1)$ is a standard normal random variable independent of G_i, $i = 1, ..., \nu$. Let δ be a real value such that $|\delta| < 1$. Then, the random field, Y, defined as:

$$Y(x) = \delta\,|U_0(x)| + \sqrt{1-\delta^2}U(x) \tag{7}$$

is called a skew student's t random field with ν degrees of freedom and skewness index δ.

The marginal distribution of Y at any fixed point x is the standard skew student's t probability density function p_Y with $\mu_Y = 0$ and $\sigma_Y^2 = 1$.

V. Lipschitz-Killing curvatures of the skew student's t excursion sets

A. Theory

The direct way to study the geometry of the skew student's t excursion sets is by Minkowski functionals, for the isotropic random fields, or their extensions Lipschitz-Killing curvatures, for the non-isotropic random fields [13]. An excursion set, denoted E_h, of a random field Y, on a subset $S \subset \mathbb{R}^d, d \geq 1$, is the result of thresholding the random field at h. It is defined, [9], [17], as:

$$E_h \equiv E_h(Y, S) = \{x \in S,\ Y(x) \geq h\} \tag{8}$$

The explicit formulae of the $j-$th dimensional Lipschitz-Killing curvatures of the excursion set for a non-isotropic and twice differentiable random field, defined as a function in terms of (i.i.d.) stationary Gaussian random fields is expressed as follows, [13]:

$$\mathbb{E}[\mathcal{L}_j(E_h(Y, S))] = \sum_{k=0}^{d-j} \mathcal{L}_k(S)\rho_k(h) \tag{9}$$

where $\mathcal{L}_k(S)$ are the Lipschitz-Killing curvatures (LKC) of S. They measure the $k-$th dimensional volume of S in the Riemannian metric defined by the variogram. This means, replacing the local Euclidean distance between any two points of S by the $(d \times d)$ matrix Λ, of the second spectral moments of the Gaussian component, G. It's elements are defined as:

$$\lambda_{kl} = \mathbb{E}\left[\frac{\partial G}{\partial x_l}\frac{\partial G}{\partial x_k}\right] \tag{10}$$

where $k, l = 1, ..., d$, and $Y = f(G)$ is a function in terms of G.

Then, the $d-$th dimensional LKC $\mathcal{L}_d(S)$ of the subset S is defined as:

$$\mathcal{L}_d(S) = \int_S \det(\Lambda)^{1/2} \tag{11}$$

and the $(d-1)-$th dimensional LKC of S is:

$$\mathcal{L}_{d-1}(S) = \frac{1}{2}\int_{\partial S} \det(\Lambda_x)^{1/2} dx \tag{12}$$

where Λ_x is the local version of Λ, [16].

The functions $\rho_k(h)$, in the equation (9), are the Euler characteristic (EC) densities of the excursion set above the threshold h. They depends on the type of the random field, and not on the geometry of S.

B. Notation

In this paper, the skew student's t random field is considered stationary on the 2-dimensional space, where the subset S is defined as a rectangle $[0, a] \times [0, b]$ in $\mathbb{R}^2$. Then, the LKCs, of S become:

$$\begin{aligned} \mathcal{L}_0(S) &= 1, \\ \mathcal{L}_1(S) &= \frac{1}{2}(a\lambda_{11}^{1/2} + b\lambda_{22}^{1/2}), \\ \mathcal{L}_2(S) &= ab \times \det(\Lambda)^{1/2} \end{aligned} \tag{13}$$

where $\mathcal{L}_0(S)$ is the Euler-Poincaré characteristic of S, $\mathcal{L}_1(S)$ is half the boundary length of S, and $\mathcal{L}_2(S)$ is the two dimensional area of S.

In the following, we will concern on giving the analytical formulae of the $k-$th dimensional EC densities for the two-dimensional stationary skew student's t random field defined in IV.2.

C. Expectations

The expected $j-$th dimensional Lipschitz-Killing curvatures, $\mathbb{E}[\mathcal{L}_j(E_h(Y, S))]$, of the skew student's t excursion sets, E_h, on $\mathbb{R}^2$ measure the area, the half-boundary length, and the Euler-Poincaré characteristic of the excursion set E_h above the threshold h. They will be denoted by $A(E_h)$, $C(E_h)$, and $\chi(E_h)$ respectively.

The $k-$th dimensional EC density functions of the skew student's t random field, $Y(x)$, could be calculated using Morse theory [18]. They are expressed for any twice-differentiable and isotropic random field, [13], [14], [16], as follows:

$$\begin{aligned} \rho_k(h) &= \mathbb{E}\left(1_{(Y \geq h)}\det(-\ddot{Y}_{|k})|\dot{Y}_{|k} = 0\right)\mathbb{P}(\dot{Y}_{|k} = 0) \\ &= \mathbb{E}\left(\dot{Y}_k^+\det(-\ddot{Y}_{|k-1})|\dot{Y}_{|k-1} = 0, Y = h\right)p_{|k-1}(0, h) \end{aligned} \tag{14}$$

where $\dot{Y}_{|k-1}$ and $\ddot{Y}_{|k-1}$ are the first and second order partial derivatives of the first $(k-1)-$th elements of $Y(x)$ at any point $x \in S$, (i.e., the $(k-1) \times (k-1)$ sub matrix of $\dot{Y}$ and $\ddot{Y}$, respectively). $p_{|k-1}(0, h)$ is the joint probability density function of $\dot{Y}_{|k-1}$ at zero and Y, whereas $\dot{Y}_j^+ = max(0, \dot{Y}_j)$

Proposition V.1. *The k-th dimensional EC density functions, $\rho_k(.)$, $j = 0, 1, 2$, of a stationary skew student-t random field, Y, with ν degrees of freedom, and skewness index $\alpha \in \mathbb{R}$, on $\mathbb{R}^2$ for a given level h are expressed as:*

$$(i) \quad \rho_0(h) = 2\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\int_h^\infty \left(1 + \frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}} \times T\left(\alpha y\sqrt{\frac{\nu+1}{y^2+\nu}}; \nu+1\right)dy$$

$$(ii) \quad \rho_1(h) = \frac{2}{2\pi}\sqrt{1-\delta^2}\left(1 + \frac{h^2}{\nu(1-\delta^2)}\right)\left(1 + \frac{h^2}{\nu}\right)^{-\frac{\nu+1}{2}} T\left(\alpha h\sqrt{\frac{\nu+1}{\nu+h^2}}; \nu+1\right)\Bigg\{1 + \frac{4\alpha}{\sqrt{\pi}(\nu-1)}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \times \left(1 + \frac{h^2}{\nu(1-\delta^2)}\right)^{-1}\left\{\frac{1}{\nu-1}\left[\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\right]^2 - \frac{h}{\sqrt{\nu}\sqrt{1-\delta^2}}\right\} + \frac{2\pi\alpha^2}{(\nu-2)\sqrt{1-\delta^2}}\left(1 + \frac{h^2}{\nu(1-\delta^2)}\right)^{-1}\Bigg\}$$

$$(iii) \quad \rho_2(h) = \frac{2}{(2\pi)^{3/2}}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu/2}\Gamma\left(\frac{\nu}{2}\right)}h\left(1 + \frac{h^2}{\nu(1-\delta^2)}\right) \times$$

$$\left(1+\frac{h^2}{\nu}\right)^{-\frac{\nu+1}{2}} T\left(\alpha h\sqrt{\frac{\nu+1}{h^2+\nu}};\nu+1\right)\Bigg\{1-\frac{2\delta\sqrt{\nu}}{h(\nu-1)}\times \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu}{2}\right)}\left[\frac{3h^2+\nu(1-\delta^2)}{h^2+\nu(1-\delta^2)}\right]+\frac{\nu(\nu-1)\delta^2}{h(\nu-2)^2} \left[\frac{\frac{3h(\nu-1)}{\nu-2}+\nu(1-\delta^2)}{h^2+\nu(1-\delta^2)}\right]-\frac{\nu\delta\alpha^2}{h(\nu-2)(\nu-4)}\times \left(1+\frac{h^2}{\nu(1-\delta^2)}\right)^{-1}\Bigg\} \tag{15}$$

Proof: The proof of the proposition is based on the theorems and the lemmas reported in [14], [17], [19].
Let consider the random field Y as defined in equation (7), Y satisfies the regularity conditions, (i.e., Y is twice differentiable inside S and on the boundaries of S), with the restriction to the condition that Y has the degree of freedom $\nu > 4$.
For simplicity, Y will be assumed centred, $\mu_Y = 0$, with unit variance, $\sigma_Y^2 = 1$, and skewness index $\delta > 0$. However, in the case that Y has a negative skewness, $\delta < 0$, one can transform Y to have a positive skewness dealing with $-Y$. Using the conditional expectations and the formulae in equation (9), the EC densities can be obtained, conditioning on U_0, as follows:

$$\begin{aligned}\rho_k(h) &= \mathbb{E}\left[\rho_k^{T^\nu}\left(\frac{h-\delta u}{\sqrt{1-\delta^2}}\right)||U_0|=u\right]\\ &= 2\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi\nu}\Gamma(\frac{\nu}{2})}\int_0^\infty \rho_k^{T^\nu}\left(\frac{h-\delta u}{\sqrt{1-\delta^2}}\right)\left(1+\frac{u^2}{\nu}\right)^{-\frac{\nu+1}{2}} du\end{aligned} \tag{16}$$

where $\rho_k^{T^\nu}(h;u)$ is the $k-$th EC density of the student's t random field, T^ν, of ν degrees of freedom at $(h-\delta u)/\sqrt{1-\delta^2}$, (see [14]).
Doing the integration over $|U_0|$, we get the formulae in equation (15), for $k=1$ and $k=2$.
For $k=0$, $\rho_0(h)$ is simply equal to $\mathbb{P}[Y \geq h]$.

■

The interesting result of the Libchetz-Killing curvatures is the expected value of Euler-Poincaré characteristic of the skew student's t excursion sets, $\mathbb{E}[\chi(E_h(Y,S))]$, which is approximately equal to the maxima of Y at very high thresholds $(h \to \infty)$, which is expressed as:

$$\mathbb{E}[\chi(E_h(Y,S))] \cong 2ab\frac{\det(\Lambda)\Gamma\left(\frac{\nu+1}{2}\right)}{(2\pi)^{3/2}(\sqrt{\nu/2})\Gamma\left(\frac{\nu}{2}\right)}h\times \left(1+\frac{h^2}{\nu(1-\delta^2)}\right)\left(1+\frac{h^2}{\nu}\right)^{-\frac{\nu+1}{2}} T\left(\alpha h\sqrt{\frac{\nu+1}{h^2+\nu}};\nu+1\right) \tag{17}$$

VI. Simulation results

n this section, simulation examples of the skew student's t random fields are illustrated for positive (resp. negative) values of the skewness index δ. An illustration of the analytical expressions of the $j-$th dimensional Lipschitz-Killing curvatures are given comparing with the numerical ones that are computed from the simulations for validation. For this aim, 50 realizations of the skew student's t random field have been generated for two different examples. The simulations are investigated on a lattice of 500×500 points in both x and y directions within the unit square $[0,1]\times[0,1]$. Figures 1(a) and 2(a) illustrate two examples of stationary skew student's t random fields with 5 degrees of freedom and with skewness indexes $\delta = -0.9$ and $\delta = 0.4$, respectively. Their analytical and numerical Lipschitz-Killing curvatures are represented in figures 1(b) and 2(b), respectively.

VII. Application

The stochastic model has been tested on a real 3D microstructured rough surface of a UHMWPE (Ultra High Molecular Weight Polyethylene) component, [20]. The surface has been measured by a non-contact white light interferometry, (Bruker nanoscope Wyko® NT 9100), on a lattice of 480×640 points with a spatial resolution equal to $1.8\mu m$ in both x and y directions, see figure 3(a). There are two main reasons for using the expected Euler-Poincaré characteristic, in such applications: firstly, at high threshold, it counts the number of connected components which approximate the number of the local maxima and minima of Y such that [21], [9]:

$$\begin{aligned}&\mathbb{P}[Y_{max} > h] \cong \mathbb{E}[\chi(E_h(Y,S))]\\ &\mathbb{P}[Y_{min} < h] \cong \mathbb{E}[\chi(E_{-h}(-Y,S))] = -\mathbb{E}[\chi(E_h(Y,S))]\end{aligned} \tag{18}$$

when $(h \to \infty)$.
Thus it can describe its roughness. Secondly, its explicit formulae enable estimating the random field's parameters, or estimating, for a specific test of significance, the threshold h the can detect if some materials of the surface are lost.
The results presented in this paper concerns on fitting the surface roughness stochastic model with the real 3D heights map using the Euler-Poincaré characteristic function. Figure 3(b) shows the analytical and the numerical Euler-Poincaré characteristic of the skew student's t excursion sets and the real surface upcrossings, respectively, for 10 degrees of freedom and skewness index $\delta = 0.7$.
The results show the ability to use this stochastic model for studying the morphology and the size distribution of the excursion sets at certain significant thresholds in order to describe the functionality of the surface and to determine its performance before and maybe after involving it in the hip replacement surgery.

VIII. Conclusion

A special class of skew student's t random fields with ν degrees of freedom has been introduced, and studied geometrically. We derived, in this paper, the analytical formulae of the Euler characteristic densities of the skew student's t excursion sets in order to expect their Lipschitz-Killing Curvatures on a rectangular subset $S \subset \mathbb{R}^d$, $(d = 2)$, and they can be extended to higher dimensions $d > 2$.
Simulation results have been performed to validate the analytical formulae. Then, an application of modelling the roughness topography of a worn engineering surface has been investigated. The results are promising and show the ability of using the skew student's t random field for describing the surface roughness changes between the different spatial samples and/or during a specific temporal process, which is the aim of our future work.

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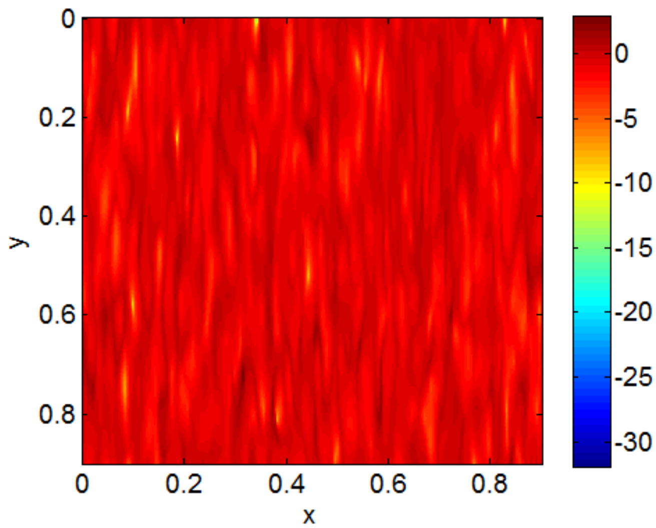
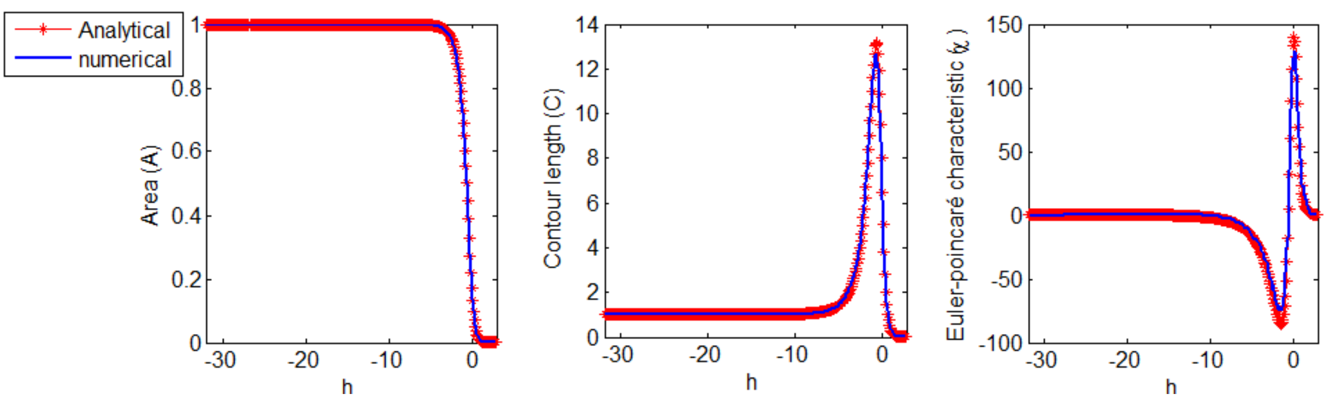
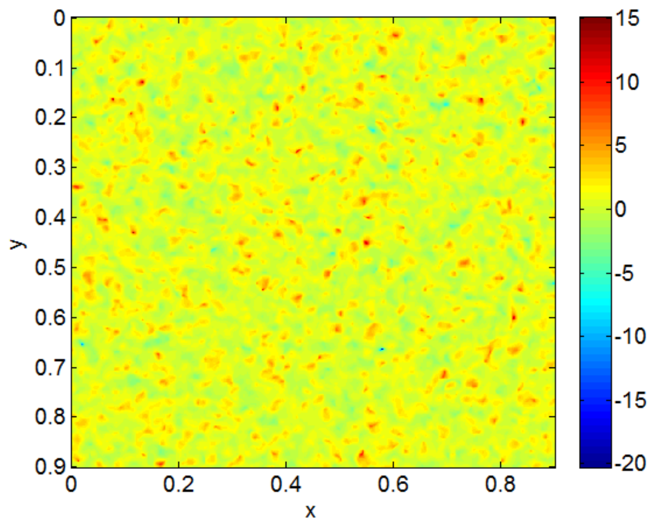


(a)

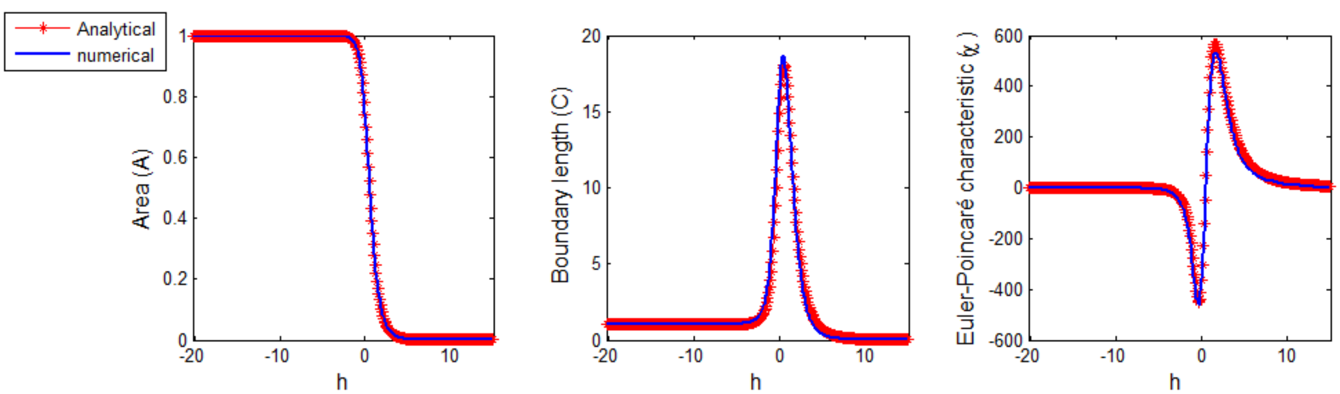


(b)

Figure 1. First simulation example. (a) Anisotropic skew student's t random field with 5 degrees of freedom and skewness index $\delta = -0.9$ realized on a lattice of 500×500 points in $[0, 1]^2$. (b) The numerical and the analytical Lipschitz-Killing Curvatures, A, C and χ, respectively.

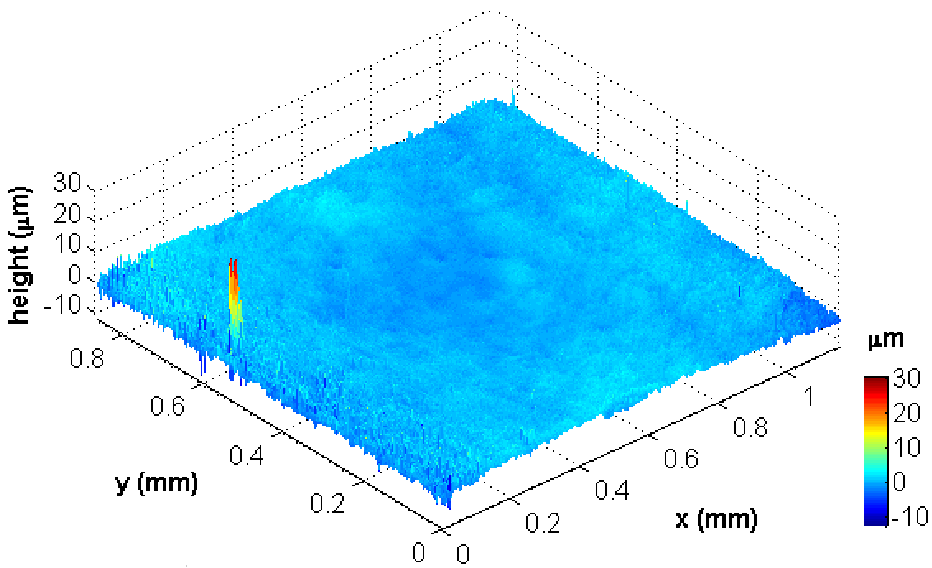


(a)

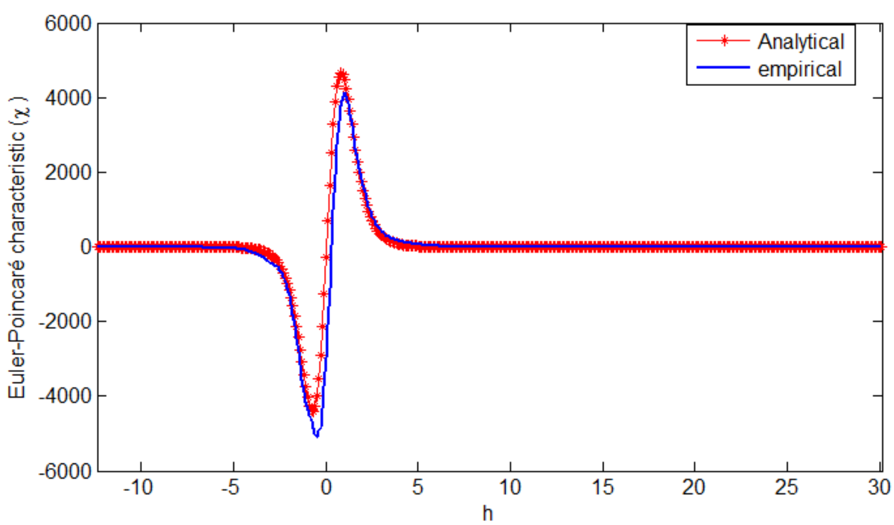


(b)

Figure 2. Second simulation example. (a) Isotropic skew student's t random field with 5 degrees of freedom and skewness index $\delta = 0.4$ realized on a lattice of 500×500 points in the unit square $[0, 1]^2$. (b) The numerical and the analytical Lipschitz-Killing Curvatures, A, C and χ, respectively.



(a)



(b)

Figure 3. An application example. (a) A real 3D surface roughness topography digitized on a lattice of 480 points with a spatial sampling steps equal to $1.8\mu m$ in x and y directions. (b) Fitting the empirical and the analytical Euler-Poincaré characteristic functions of the real surface upcrossings and the skew student's t excursion sets for 10 degrees of freedom and skewness index $\delta = 0.7$.