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Про задачу оцінювання стаціонарних процесів неперервного часу за спостереженнями у спеціальних множинах точок

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On estimation problem for continuous time stationary processes from observations in special sets of points

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Досліджується задача оптимального оцінювання лінійних функціоналів від невідомих значень стаціонарного стохастичного процесу за спостереженнями процесу у спеціальних множинах точок. Знайдені формули для обчислення значення середньокваратичної похибки та спектральної характеристики оптимальної лінійної оцінки функціоналів за умови спектральної визначеності, коли спектральна щільність процесу точно відома. У випадку, коли спектральна щільність процесу точно невідома, а задаються лише деякі класи допустимих спектральних щільностей, застосовується мінімаксно-робастний метод. Знайдені формули для визначення найменш сприятливих спектральних щільностей та мінімаксних спектральних характеристик для оптимального лінійного оцінювання функціоналів для конкретних класів спектральних щільностей.
Ключові слова: стаціонарний стохастичний процес, мінімаксно-робастна оцінка, найменш сприятлива спектральна щільність, мінімаксно-робастні спектральні характеристики.

The problem of the mean-square optimal estimation of the linear functionals which depend on the unknown values of a stochastic stationary process from observations of the process with missings is considered. Formulas for calculating the mean-square error and the spectral characteristic of the optimal linear estimate of the functionals are derived under the condition of spectral certainty, where the spectral density of the process is exactly known. The minimax (robust) method of estimation is applied in the case where the spectral density of the process is not known exactly while some sets of admissible spectral densities are given. Formulas that determine the least favourable spectral densities and the minimax spectral characteristics are derived for some special sets of admissible densities.
Key Words: stationary stochastic process, minimax-robust estimate, least favorable spectral density, minimax-robust spectral characteristics.

Introduction

Investigation of the properties of stationary stochastic processes plays an important role both in the theory of stochastic processes and application it to the practice. A great number of scientific papers deals with the problem of estimation of unknown values of a stationary process. The formulation and effective methods of solution of the problems of interpolation, extrapolation and filtering of stationary sequences and processes belong to A. N. Kolmogorov [16]. Further analysis can be found in the works by Yu. A. Rozanov [37] and E. J. Hannan [11]. A significant contribution to the theory of forecasting was made by H. Wold [41, 42], T. Nakazi [33]. Constructive methods of solution of estimation problems for stationary stochastic sequences and processes were developed by N. Wiener [40] and A. M. Yaglom [43].

The basic assumption of most of the methods of estimation of the unobserved values of stochastic processes is that the spectral densities of the considered stochastic processes are exactly known. However, in practice, these methods are not applicable since the complete information on the spectral densities is impossible in most cases. In order to solve the problem parametric or nonparametric estimates of the unknown spectral densities are found. Then, one of traditional estimation methods is applied, provided that the selected densities are the true ones. This procedure can result in significant increasing of the value of error as K. S. Vastola and H. V. Poor [39] have demonstrated with the help of some examples. To avoid this effect one can search the estimates which are optimal for all densities from a certain class of admissible spectral densities. These estimates are called minimax since they minimize the maximum value of the error. The paper by Ulf Grenander [10] should be marked as the first one where this approach to extrapolation problem for stationary processes was proposed.

Several models of spectral uncertainty and minimax-robust methods of data processing can be found in the survey paper by S. A. Kassam and H. V. Poor [15]. J. Franke [5], J. Franke and H. V. Poor [6] investigated the minimax extrapolation and filtering problems for stationary sequences with the help of convex optimization methods. This approach makes it possible to find equations that determine the least favorable spectral densities for different classes of densities.

A great amount of papers by M. Moklyachuk [22] – [25] is dedicated to the investigation of problems of the optimal linear estimation of functionals which depend on unknown values of stationary sequences and processes. In papers by M. Moklyachuk and A. Masyutka a minimax technique of the estimation for vector-valued stationary stochastic processes is proposed [21], [26]–[28]. Methods of solution of problems of interpolation, extrapolation and filtering problems for periodically correlated stochastic processes were developed by M. Moklyachuk and I. Dubovetska [4, 8]. Estimation problems for functionals which depend on unknown values of stochastic processes with stationary increments were investigated by M. Luz and M. Moklyachuk [17]–[20]. The problem of interpolation of a stationary sequence with missing values was investigated by M. Moklyachuk and M. Sidei [29] – [32].

Prediction of stationary processes with missing observations was investigated in papers by P. Bondon [1, 2], Y. Kasahara, M. Pourahmadi and A. Inoue [14, 34], R. Cheng, A. G. Miamee, M. Pourahmadi [3]. The problem of interpolation of stationary sequences was considered in the paper of H. Salehi [38].

In this paper we deal with the problem of the mean-square optimal linear estimation of the functionals

$$A_1\xi=\int_{-\infty}^{-M}a(t)\xi(t)dt+\int_0^T a(t)\xi(t)dt,$$

$$A_2\xi=\int_0^T a(t)\xi(t)dt+\int_{T+N}^{\infty}a(t)\xi(t)dt,$$

$$A_3\xi=\int_{-\infty}^{-M}a(t)\xi(t)dt+\int_0^T a(t)\xi(t)dt+\int_{T+N}^{\infty}a(t)\xi(t)dt,$$

$$A_4\xi=\int_{-M-M_1}^{-M}a(t)\xi(t)dt+\int_0^T a(t)\xi(t)dt,$$

$$A_5\xi=\int_0^T a(t)\xi(t)dt+\int_{T+N}^{T+N+N_1}a(t)\xi(t)dt,$$

$$A_6\xi=\int_{-M-M_1}^{-M}a(t)\xi(t)dt+ +\int_0^T a(t)\xi(t)dt+\int_{T+N}^{T+N+N_1}a(t)\xi(t)dt,$$

which depend on the unknown values of a stochastic stationary process $\xi(t)$ from observations of the process $\xi(t)+\eta(t)$ at points $t\in\mathbb{R}\backslash S_k$, respectively, where

$$S_1=[-\infty;-M]\cup[0;T],\ S_2=[0;T]\cup[T+N;\infty],$$
$$S_3=S_1\cup S_2,\ S_4=[-M-M_1;-M]\cup[0;T],$$
$$S_5=[0;T]\cup[T+N;T+N+N_1],\ S_6=S_4\cup S_5.$$

The case of spectral certainty as well as the case of spectral uncertainty are considered. Formulas for calculating the spectral characteristic and the mean square error of the optimal linear estimate of the functionals are derived under the condition that spectral densities of the processes are exactly known. In the case of spectral uncertainty, where the spectral densities are not exactly known but a set of admissible spectral densities is given, the minimax method is applied. Formulas for determination the least favorable spectral densities and the minimax-robust spectral characteristics of the optimal estimates of the functionals are proposed for some specific classes of admissible spectral densities.

1 Classical interpolation problem for stationary processes

Let $\xi(t), t \in \mathbb{R}$, and $\eta(t), t \in \mathbb{R}$, be uncorrelated stationary stochastic processes with zero first moments $E\xi(t) = 0$, $E\eta(t) = 0$. Correlation functions

$$R_\xi(t) = E\xi(t+s)\overline{\xi(s)},\ R_\eta(t) = E\eta(t+s)\overline{\eta(s)}$$

of stationary processes $\xi(t), t \in \mathbb{R}$ and $\eta(t), t \in \mathbb{R}$, respectively, admit the spectral decomposition [7]

$$R_\xi(t) = \int\limits_{-\infty}^{\infty} e^{it\lambda} F(d\lambda), \quad R_\eta(t) = \int\limits_{-\infty}^{\infty} e^{it\lambda} G(d\lambda),$$

where $F(d\lambda)$ and $G(d\lambda)$ are spectral measures of processes $\xi(t)$ and $\eta(t)$, respectively. Consider stationary stochastic processes $\xi(t), t \in \mathbb{R}$ and $\eta(t), t \in \mathbb{R}$ with absolutely continuous spectral measures $F(d\lambda)$, $G(d\lambda)$ and correlation functions of the form

$$R_\xi(t) = \int\limits_{-\infty}^{\infty} e^{it\lambda} f(\lambda) d\lambda, \quad R_\eta(t) = \int\limits_{-\infty}^{\infty} e^{it\lambda} g(\lambda) d\lambda,$$

where $f(\lambda)$ and $g(\lambda)$ are spectral densities of processes $\xi(t)$ and $\eta(t)$, respectively, and the minimality condition holds true

$$\int_{-\infty}^{\infty} \frac{|\gamma_k(\lambda)|^2}{f(\lambda)+g(\lambda)} d\lambda < \infty, \qquad (1)$$

where

$$\gamma_1(\lambda) = \int\limits_{-\infty}^{-M} \alpha_1(t) e^{it\lambda} dt + \int\limits_{0}^{T} \alpha_1(t) e^{it\lambda} dt,$$

$$\gamma_2(\lambda) = \int\limits_{0}^{T} \alpha_2(t) e^{it\lambda} dt + \int\limits_{T+N}^{\infty} \alpha_2(t) e^{it\lambda} dt,$$

$$\gamma_3(\lambda) = \int\limits_{-\infty}^{-M} \alpha_3(t) e^{it\lambda} dt + \int\limits_{0}^{T} \alpha_3(t) e^{it\lambda} dt + \int\limits_{T+N}^{\infty} \alpha_3(t) e^{it\lambda} dt,$$

$$\gamma_4(\lambda) = \int\limits_{-M-M_1}^{-M} \alpha_4(t) e^{it\lambda} dt + \int\limits_{0}^{T} \alpha_4(t) e^{it\lambda} dt,$$

$$\gamma_5(\lambda) = \int\limits_{0}^{T} \alpha_5(t) e^{it\lambda} dt + \int\limits_{T+N}^{T+N+N_1} \alpha_5(t) e^{it\lambda} dt,$$

$$\gamma_6(\lambda) = \int\limits_{-M-M_1}^{-M} \alpha_6(t) e^{it\lambda} dt + \int\limits_{0}^{T} \alpha_6(t) e^{it\lambda} dt + \int\limits_{T+N}^{T+N+N_1} \alpha_6(t) e^{it\lambda} dt$$

are nontrivial functions of the exponential type. Under this condition the error-free estimation is impossible (see, for example, [37]).

Processes $\xi(t)$ and $\eta(t)$ admit the spectral decomposition [13]

$$\xi(t) = \int\limits_{-\infty}^{\infty} e^{it\lambda} Z_\xi(d\lambda), \qquad \eta(t) = \int\limits_{-\infty}^{\infty} e^{it\lambda} Z_\eta(d\lambda), \qquad (2)$$

where $Z_\xi(d\lambda)$ and $Z_\eta(d\lambda)$ are the orthogonal stochastic measures that correspond to spectral functions $F(d\lambda)$ and $G(d\lambda)$. The following properties hold true

$$EZ_\xi(\Delta_1)\overline{Z_\xi(\Delta_2)} = F(\Delta_1 \cap \Delta_2) = \int_{\Delta_1 \cap \Delta_2} f(\lambda) d\lambda,$$

$$EZ_\eta(\Delta_1)\overline{Z_\eta(\Delta_2)} = G(\Delta_1 \cap \Delta_2) = \int_{\Delta_1 \cap \Delta_2} g(\lambda) d\lambda.$$

Consider the problem of the mean-square optimal linear estimation of the functionals

$$A_1\xi = \int\limits_{-\infty}^{-M} a(t)\xi(t) dt + \int\limits_{0}^{T} a(t)\xi(t) dt,$$

$$A_2\xi = \int\limits_{0}^{T} a(t)\xi(t) dt + \int\limits_{T+N}^{\infty} a(t)\xi(t) dt,$$

$$A_3\xi = \int_{-\infty}^{-M} a(t)\xi(t)dt + \int_{0}^{T} a(t)\xi(t)dt + \int_{T+N}^{\infty} a(t)\xi(t)dt,$$

$$A_4\xi = \int_{-M-M_1}^{-M} a(t)\xi(t)dt + \int_{0}^{T} a(t)\xi(t)dt,$$

$$A_5\xi = \int_{0}^{T} a(t)\xi(t)dt + \int_{T+N}^{T+N+N_1} a(t)\xi(t)dt,$$

$$A_6\xi = \int_{-M-M_1}^{-M} a(t)\xi(t)dt + \int_{0}^{T} a(t)\xi(t)dt + \int_{T+N}^{T+N+N_1} a(t)\xi(t)dt,$$

which depend on the unknown values of a stochastic stationary process $\xi(t)$ from observations of the process $\xi(t)+\eta(t)$ at points $t \in \mathbb{R}\backslash S_k$, respectively, where

$$S_1 = [-\infty; -M] \cup [0; T],\ S_2 = [0; T] \cup [T+N; \infty],$$

$$S_3 = S_1 \cup S_2,\ S_4 = [-M-M_1; -M] \cup [0; T],$$

$$S_5 = [0; T] \cup [T+N; T+N+N_1],\ S_6 = S_4 \cup S_5.$$

Making use of spectral decomposition (2) of stationary process $\xi(t)$ we can write the functionals $A_k\xi$ in the following form $A_k\xi = \int_{-\infty}^{\infty} A_k(\lambda) Z_\xi(d\lambda)$ where

$$A_1(\lambda) = \int_{-\infty}^{-M} a(t)e^{it\lambda}dt + \int_{0}^{T} a(t)e^{it\lambda}dt,$$

$$A_2(\lambda) = \int_{0}^{T} a(t)e^{it\lambda}dt + \int_{T+N}^{\infty} a(t)e^{it\lambda}dt,$$

$$A_3(\lambda) = \int_{-\infty}^{-M} a(t)e^{it\lambda}dt + \int_{0}^{T} a(t)e^{it\lambda}dt + \int_{T+N}^{\infty} a(t)e^{it\lambda}dt,$$

$$A_4(\lambda) = \int_{-M-M_1}^{-M} a(t)e^{it\lambda}dt + \int_{0}^{T} a(t)e^{it\lambda}dt,$$

$$A_5(\lambda) = \int_{0}^{T} a(t)e^{it\lambda}dt + \int_{T+N}^{T+N+N_1} a(t)e^{it\lambda}dt,$$

$$A_6(\lambda) = \int_{-M-M_1}^{-M} a(t)e^{it\lambda}dt + \int_{0}^{T} a(t)e^{it\lambda}dt + \int_{T+N}^{T+N+N_1} a(t)e^{it\lambda}dt.$$

Denote by $\hat{A}_k\xi$ the optimal linear estimate of the functional $A_k\xi$ from observations of the process $\xi(t)+\eta(t)$ at points $t \in \mathbb{R}\backslash S_k$ and by $\Delta_k(F,G) = E\left|A_k\xi - \hat{A}_k\xi\right|^2$ the mean-square error of the estimate $\hat{A}_k\xi$. Since spectral densities of stationary processes $\xi(t)$ and $\eta(t)$ are known, we can use the method of orthogonal projections in Hilbert spaces [16] to find the estimate $\hat{A}_k\xi$.

Consider values $\xi(t)$ and $\eta(t)$ as elements of the Hilbert space $H = L_2(\Omega, \mathcal{F}, P)$ generated by random variables ξ with 0 mathematical expectations, $E\xi = 0$, finite variations, $E|\xi|^2 < \infty$, and inner product $(\xi, \eta) = E\xi\overline{\eta}$. Denote by $H^k(\xi+\eta)$ the closed linear subspace generated by elements $\{\xi(t)+\eta(t) : t \in \mathbb{R}\backslash S_k\}$ in the Hilbert space $H = L_2(\Omega, \mathcal{F}, P)$. Denote by $L_2(f+g)$ the Hilbert space of functions $a(\lambda)$ such that

$$\int_{-\infty}^{\infty} |a(\lambda)|^2 (f(\lambda)+g(\lambda))d\lambda < \infty.$$

Denote by $L_2^k(f+g)$ the subspace of $L_2(f+g)$ generated by functions $\{e^{it\lambda}, t \in \mathbb{R}\backslash S_k\}$.

The mean-square optimal linear estimate $\hat{A}_k\xi$ of the functional $A_k\xi$ can be represented in the form

$$\hat{A}_k\xi = \int_{-\infty}^{\infty} h_k(\lambda)(Z_\xi(d\lambda) + Z_\eta(d\lambda)),$$

where $h_k(\lambda) \in L_2^k(f+g)$ is the spectral characteristic of the estimate.

The mean-square error $\Delta(h_k; f, g)$ of the estimate $\hat{A}_k\xi$ is given by the formula

$$\Delta(h_k; f, g) = E\left|A_k\xi - \hat{A}_k\xi\right|^2 =$$

$$= \int_{-\infty}^{\infty} |A_k(\lambda) - h_k(\lambda)|^2 f(\lambda)d\lambda + \int_{-\infty}^{\infty} |h_k(\lambda)|^2 g(\lambda)d\lambda.$$

According to the Hilbert space projection method proposed by A. N. Kolmogorov [16], the optimal estimation of the functional $A_k\xi$ is a projection of the element $A_k\xi$ of the space H on the space $H^k(\xi+\eta)$. It can be found from the following conditions:

$$1)\hat{A}_k\xi \in H^k(\xi+\eta),$$
$$2)A_k\xi - \hat{A}_k\xi \perp H^k(\xi+\eta).$$

It follows from the second condition that the spectral characteristic $h_k(\lambda)$ of the optimal linear estimate $\hat{A}_k\xi$ for any $t \in \mathbb{R}\backslash S_k$ satisfies equations

$$\int_{-\infty}^{\infty} (A_k(\lambda)f(\lambda) - h_k(\lambda)(f(\lambda)+g(\lambda))e^{-it\lambda}d\lambda = 0. \tag{3}$$

Define the function $C_k(\lambda) = A_k(\lambda))f(\lambda) - h_k(\lambda)(f(\lambda)+g(\lambda))$ and its Fourier transformation

$$c_k(t) = \int_{-\infty}^{\infty} C_k(\lambda)e^{-it\lambda}d\lambda, \quad t \in \mathbb{R}.$$

It follows from the condition (3) that the function $c_k(t)$ is nonzero only on the set S_k. Hence,

$$C_1(\lambda) = \int_{-\infty}^{-M} c_1(t)e^{it\lambda}dt + \int_{0}^{T} c_1(t)e^{it\lambda}dt,$$

$$C_2(\lambda) = \int_{0}^{T} c_2(t)e^{it\lambda}dt + \int_{T+N}^{\infty} c_2(t)e^{it\lambda}dt,$$

$$C_3(\lambda) = \int_{-\infty}^{-M} c_3(t)e^{it\lambda}dt + \\ + \int_{0}^{T} c_3(t)e^{it\lambda}dt + \int_{T+N}^{\infty} c_3(t)e^{it\lambda}dt,$$

$$C_4(\lambda) = \int_{-M-M_1}^{-M} c_4(t)e^{it\lambda}dt + \int_{0}^{T} c_4(t)e^{it\lambda}dt,$$

$$C_5(\lambda) = \int_{0}^{T} c_5(t)e^{it\lambda}dt + \int_{T+N}^{T+N+N_1} c_5(t)e^{it\lambda}dt,$$

$$C_6(\lambda) = \int_{-M-M_1}^{-M} c_6(t)e^{it\lambda}dt + \\ + \int_{0}^{T} c_6(t)e^{it\lambda}dt + \int_{T+N}^{T+N+N_1} c_6(t)e^{it\lambda}dt$$

and the spectral characteristic of the estimate $\hat{A}_k\xi$ is of the form

$$h_k(\lambda) = A_k(\lambda)f(\lambda)(f(\lambda)+g(\lambda))^{-1} - \\ - C_k(\lambda)(f(\lambda)+g(\lambda))^{-1}. \tag{4}$$

It follows from the first condition, $\hat{A}_k\xi \in H^k(\xi+\eta)$, which determine the optimal linear estimate of the functional $A_k\xi$, that for some function $v_k(t) \in L_2^k(f+g)$

$$h_k(\lambda) = \int_{\mathbb{R}\backslash S_k} v_k(t)e^{it\lambda}d\lambda,$$

therefore for any $t \in S_k$, the following relation holds true

$$\int_{-\infty}^{\infty} \Big[A_k(\lambda)f(\lambda)(f(\lambda)+g(\lambda))^{-1} - \\ - C_k(\lambda)(f(\lambda)+g(\lambda))^{-1}\Big]e^{-it\lambda}d\lambda = 0. \tag{5}$$

The last relation can be represented in terms of linear operators in the space $L_2^k(f+g)$. Let us define operators

$$(\mathbf{B}_k\mathbf{a})(t) = \int_{S_k}\int_{-\infty}^{\infty} a(u)(f(\lambda)+g(\lambda))^{-1}e^{i\lambda(u-t)}d\lambda du,$$

$$(\mathbf{R}_k\mathbf{a})(t) = \\ \int_{S_k}\int_{-\infty}^{\infty} a(u)f(\lambda)(f(\lambda)+g(\lambda))^{-1}e^{i\lambda(u-t)}d\lambda du,$$

$$(\mathbf{Q}_k\mathbf{a})(t) = \\ = \int_{S_k}\int_{-\infty}^{\infty} a(u)f(\lambda)(f(\lambda)+g(\lambda))^{-1}g(\lambda)e^{i\lambda(u-t)}d\lambda du, \\ a(t) \in L_2^k(f+g), \quad t \in S_k.$$

The relation (5) can be rewritten in the form

$$(\mathbf{R}_k\mathbf{a})(t) = (\mathbf{B}_k\mathbf{c_k})(t), \quad t \in S_k. \tag{6}$$

Consider the operator $\mathbf{B}_k$ is invertible. Then the function $c_k(t)$ can be calculated by the formula

$$c_k(t) = (\mathbf{B}_k^{-1}\mathbf{R}_k\mathbf{a})(t), \quad t \in S_k.$$

Consequently, the spectral characteristic $h_k(\lambda)$ of the estimate $\hat{A}_k\xi$ is calculated by the formula

$$h_k(\lambda) = h_k(f, g) = A_k(\lambda)f(\lambda)(f(\lambda) + g(\lambda))^{-1} - C_k(\lambda)(f(\lambda) + g(\lambda))^{-1}, \quad (7)$$

$$C_k(\lambda) = \int_{S_k} (\mathbf{B}_k^{-1}\mathbf{R}_k\mathbf{a})(t)e^{it\lambda}dt.$$

The mean-square error of the estimate $\hat{A}_k\xi$ can be calculated by the formula

$$\Delta_k(f, g) = \int_{-\infty}^{\infty} |A_k(\lambda)g(\lambda) + C_k(\lambda)|^2((f(\lambda) + g(\lambda))^2)^{-1}f(\lambda)d\lambda + \int_{-\infty}^{\infty} |A_k(\lambda)f(\lambda) - C_k(\lambda)|^2((f(\lambda) + g(\lambda))^2)^{-1}g(\lambda)d\lambda$$
$$= \langle(\mathbf{R}_k\mathbf{a})(t), (\mathbf{B}_k^{-1}\mathbf{R}_k\mathbf{a})(t)\rangle + \langle(\mathbf{Q}_k\mathbf{a})(t), a(t)\rangle, \quad (8)$$

where $\langle a(t), b(t)\rangle$ is the inner product in the space $L_2^k(f + g)$.

Let us summarize results and present them in the form of a theorem.

Theorem 1.1. *Let $\xi(t)$ and $\eta(t)$ be uncorrelated stationary processes with spectral densities $f(\lambda)$ and $g(\lambda)$ which satisfy the minimality condition (1). The spectral characteristics $h_k(\lambda)$ and the mean-square error $\Delta_k(F, G)$ of the optimal linear estimate of the functional $A_k\xi$ which depends on the unknown values of the process $\xi(t)$ based on observations of the process $\xi(t) + \eta(t)$, $t \in \mathbb{R}\backslash S_k$ can be calculated by formulas (7), (8).*

Consider the case when the stationary process $\xi(t)$ is observed without noise. Then the spectral characteristic of the estimate $\hat{A}_k\xi$ is of the form

$$h_k(\lambda) = A_k(\lambda) - C_k(\lambda)(f(\lambda))^{-1},$$
$$C_k(\lambda) = \int_{S_k} c_k(t)e^{it\lambda}dt, \quad (9)$$

the relation(6) is of the form

$$a(t) = (\mathbf{B}_k\mathbf{c_k})(t), \quad t \in S_k. \quad (10)$$

If the operator $\mathbf{B}_k$ is invertible then the unknown function $c_k(t)$ can be calculated by the formula

$$c_k(t) = (\mathbf{B}_k^{-1}\mathbf{a})(t), \quad t \in S_k,$$

and the spectral characteristic of the estimate $\hat{A}_k\xi$ is of the form

$$h_k(\lambda) = h_k(f) = A_k(\lambda) - C_k(\lambda)(f(\lambda))^{-1},$$
$$C_k(\lambda) = \int_{S_k} (\mathbf{B}_k^{-1}\mathbf{a})(t)e^{it\lambda}dt. \quad (11)$$

The mean-square error of the estimate can be calculated by the formula

$$\Delta_k(f) = < \mathbf{B}_k^{-1}\mathbf{a}, a(t) > . \quad (12)$$

The following theorem holds true.

Theorem 1.2. *Let $\xi(t)$ be a stationary stochastic process with the spectral density $f(\lambda)$, which satisfies the minimality condition*

$$\int_{-\infty}^{\infty} \frac{|\gamma_k(\lambda)|^2}{f(\lambda)}d\lambda < \infty, \quad (13)$$

for some nonzero function of the exponential type. The spectral characteristic $h_k(\lambda)$ and the mean-square error $\Delta_k(f)$ of the optimal linear estimate of the functional $A_k\xi$ which depends on the unknown values of the process $\xi(t)$ based on observations of the process $\xi(t)$ at time points $t \in \mathbb{R}\backslash S_k$ where

$$S_1 = [-\infty; -M] \cup [0; T], \ S_2 = [0; T] \cup [T + N; \infty],$$

$$S_3 = S_1 \cup S_2, \ S_4 = [-M - M_1; -M] \cup [0; T],$$

$$S_5 = [0; T] \cup [T + N; T + N + N_1], \ S_6 = S_4 \cup S_5$$

can be calculated by formulas (11),(12).

2 Minimax method of interpolation

Derived Theorem 1.1 and Theorem 1.2 can be used only in the case of spectral certainty, when spectral densities of processes are exactly known. However, in practice, this case is not common, we do not have the exact values of spectral densities. If there is given a class of admissible spectral densities where spectral densities of the processes belong to, minimax method can be used to estimate the functionals. This method allows us to find estimates that minimize the maximum values of the mean-square errors of the estimates for all spectral densities from the given class of admissible spectral densities.

Definition 2.1. For a given class of spectral densities $D = D_f \times D_g$ the spectral densities $f_k^0(\lambda) \in D_f$, $g_k^0(\lambda) \in D_g$ are called least favorable in the class D for the optimal linear interpolation of the functional $A_k\xi$ if the following relation holds true

$$\Delta\left(f_k^0, g_k^0\right) = \Delta\left(h_k\left(f_k^0, g_k^0\right); f_k^0, g_k^0\right) = \\ = \max_{(f,g)\in D_f\times D_g} \Delta\left(h_k\left(f,g\right); f, g\right).$$

Definition 2.2. For a given class of spectral densities $D = D_f \times D_g$ the spectral characteristic $h_k^0(\lambda)$ of the optimal linear interpolation of the functional $A_k\xi$ is called minimax-robust if there are satisfied conditions

$$h_k^0(\lambda) \in H_D^k = \bigcap_{(f,g)\in D_f\times D_g} L_2^k(f+g),$$

$$\min_{h\in H_D^k} \max_{(f,g)\in D} \Delta\left(h; f, g\right) = \max_{(f,g)\in D} \Delta\left(h_k^0; f, g\right).$$

From the introduced definitions and formulas derived in previous section we can obtain the following statements.

Lemma 1. *Spectral densities $f_k^0(\lambda) \in D_f$, $g_k^0(\lambda) \in D_g$ satisfying the minimality condition (1) are the least favorable in the class $D = D_f \times D_g$ for the optimal linear interpolation of the functional $A_k\xi$ if the Fourier coefficients of the functions*

$$(f_k^0(\lambda) + g_k^0(\lambda))^{-1},\ f_k^0(\lambda)(f_k^0(\lambda) + g_k^0(\lambda))^{-1},$$
$$f_k^0(\lambda)(f_k^0(\lambda) + g_k^0(\lambda))^{-1} g_k^0(\lambda)$$

determine the operators $\mathbf{B}_k^0, \mathbf{R}_k^0, \mathbf{Q}_k^0$, which determine a solution to the constrain optimization problem

$$\max_{(f,g)\in D_f\times D_g} \langle(\mathbf{R}_k\mathbf{a})(t), (\mathbf{B}_k^{-1}\mathbf{R}_k\mathbf{a})(t)\rangle + \\ + \langle(\mathbf{Q}_k\mathbf{a})(t), a(t)\rangle = \\ = \langle(\mathbf{R}_k^0\mathbf{a})(t), ((\mathbf{B}_k^0)^{-1}\mathbf{R}_k^0\mathbf{a})(t)\rangle + \langle(\mathbf{Q}_k^0\mathbf{a})(t), a(t)\rangle. \tag{14}$$

The minimax spectral characteristic $h_k^0 = h_k(f_k^0, g_k^0)$ is determined by the formula (7) if $h(f_k^0, g_k^0) \in H_D^k$.

Corollary 2.1. Let the spectral density $f_k^0(\lambda) \in D_f$ satisfy the minimality condition (13). The spectral density $f_k^0(\lambda) \in D_f$ is the least favorable in the class D_f for the optimal linear interpolation of the functional $A_k\xi$ from the observation of the process $\xi(t)$ at time points $t \in \mathbb{R}\backslash S_k$ if the Fourier coefficients of the function $(f_k^0(\lambda))^{-1}$ determine the operator $\mathbf{B}_k^0$ which determines a solution to the constrain optimization problem

$$\max_{f\in D_f} \langle(\mathbf{B}_k^{-1}\mathbf{a})(t), a(t)\rangle = \langle((\mathbf{B}_k^0)^{-1}\mathbf{a})(t), a(t)\rangle. \tag{15}$$

The minimax spectral characteristic $h_k^0 = h_k(f_k^0)$ is determined by the formula (11) if $h_k(f_k^0) \in H_{D_f}^k$.

The least favorable spectral densities $f_k^0(\lambda)$, $g_k^0(\lambda)$ and the minimax spectral characteristic $h_k^0 = h_k(f_k^0, g_k^0)$ form a saddle point of the function $\Delta\left(h; f, g\right)$ on the set $H_D^k \times D$. The saddle point inequalities

$$\Delta\left(h_k^0; f, g\right) \leqslant \Delta\left(h_k^0; f_k^0, g_k^0\right) \leqslant \Delta\left(h; f_k^0, g_k^0\right),$$

$$\forall h \in H_D^k, \forall f \in D_f, \forall g \in D_g,$$

hold true if $h_k^0 = h_k(f_k^0, g_k^0)$ and $h_k(f_k^0, g_k^0) \in H_D^k$, where (f_k^0, g_k^0) is a solution to the constrained optimization problem

$$\sup_{(f,g)\in D_f\times D_g} \Delta\left(h_k(f_k^0, g_k^0); f, g\right) = \\ = \Delta\left(h_k(f_k^0, g_k^0); f_k^0, g_k^0\right), \tag{16}$$

$$\Delta\left(h_k\left(f_k^0, g_k^0\right); f, g\right) =$$

$$\int_{-\infty}^{\infty} |A_k(\lambda)g_k^0(\lambda) + C_k^0(\lambda)|^2((f_k^0(\lambda) + g_k^0(\lambda))^2)^{-1} f(\lambda)d\lambda +$$

$$\int_{-\infty}^{\infty} |A_k(\lambda)f_k^0(\lambda) - C_k^0(\lambda)|^2((f_k^0(\lambda) + g_k^0(\lambda))^2)^{-1} g(\lambda)d\lambda,$$

$$C_k^0(\lambda) = \int_{S_k} ((\mathbf{B}_k^0)^{-1}\mathbf{R}_k^0\mathbf{a})(t)e^{it\lambda}dt.$$

The constrained optimization problem (16) is equivalent to the unconstrained optimization problem [35]:

$$\Delta_D(f,g) = \\ -\Delta(h_k(f_k^0, g_k^0); f, g) + \delta((f,g)\,|D_f \times D_g) \to \inf, \tag{17}$$

where $\delta((f,g)|D_f \times D_g)$ is the indicator function of the set $D = D_f \times D_g$. Solution of the problem (17) is characterized by the condition $0 \in \partial\Delta_D(f_k^0, g_k^0)$, where $\partial\Delta_D(f_k^0, g_k^0)$ is the subdifferential of the convex functional $\Delta_D(f,g)$ at point (f_k^0, g_k^0) [36].

The form of the functional $\Delta(h_k(f_k^0, g_k^0); f, g)$ admits finding the derivatives and differentials of the functional in the space $L_1 \times L_1$. Therefore the complexity of the optimization problem (17) is determined by the complexity of calculating the subdifferential of the indicator functions $\delta((f, g)|D_f \times D_g)$ of the sets $D_f \times D_g$ [12].

Lemma 2. *Let (f_k^0, g_k^0) be a solution to the optimization problem (17). The spectral densities $f_k^0(\lambda)$, $g_k^0(\lambda)$ are the least favorable in the class $D = D_f \times D_g$ and the spectral characteristic $h_k^0 = h_k(f_k^0, g_k^0)$ is the minimax of the optimal linear estimate of the functional $A_k\xi$ if $h_k(f_k^0, g_k^0) \in H_D^k$.*

3 Least favorable spectral densities in the class $D = D_0 \times D_\varepsilon$

Consider the problem of the mean-square optimal interpolation of the functional $A_k\xi$ in the case when spectral densities of processes belong to the class of admissible spectral densities $D = D_0 \times D_\varepsilon$,

$$D_0 = \left\{ f(\lambda) \left| \int_{-\infty}^{\infty} f(\lambda) d\lambda = p \right. \right\},$$

$$D_\varepsilon = \left\{ g(\lambda) \middle| g(\lambda) = (1-\varepsilon) g_1(\lambda) + \varepsilon w(\lambda), \int_{-\infty}^{\infty} g(\lambda) d\lambda = q \right\},$$

where $g_1(\lambda)$ is known and fixed spectral density, $w(\lambda)$ is unknown spectral density.

From the condition $0 \in \partial\Delta_D(f_k^0, g_k^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities

$$|A_k(\lambda) g_k^0(\lambda) + C_k^0(\lambda)| = \alpha^2 (f_k^0(\lambda) + g_k^0(\lambda)), \quad (18)$$

$$|A_k(\lambda) f_k^0(\lambda) - C_k^0(\lambda)| = (\beta^2 + \gamma(\lambda))(f_k^0(\lambda) + g_k^0(\lambda)), \quad (19)$$

where $\gamma(\lambda) \leqslant 0$ and $\gamma(\lambda) = 0$ if $g_k^0(\lambda) > (1-\varepsilon) g_1(\lambda)$, and α^2, β^2 are unknown Lagrange multipliers.

Thus, the following statements hold true.

Theorem 3.1. *Let the minimality condition (1) hold true. The least favorable spectral densities $f_k^0(\lambda)$, $g_k^0(\lambda)$ in the class $D_0 \times D_\varepsilon$ for the optimal linear interpolation of the functional $A_k\xi$ are determined by relations (18), (19), constrained optimization problem (14) and restrictions on densities from the class $D_0 \times D_\varepsilon$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A_k\xi$ is determined by the formula (7).*

Theorem 3.2. *Let the spectral densities $f_k^0(\lambda) \in D_0^f$, $g_k^0(\lambda) \in D_0^g$ where*

$$D_0^f = \left\{ f(\lambda) \left| \int_{-\infty}^{\infty} f(\lambda) d\lambda = p \right. \right\},$$

$$D_0^g = \left\{ g(\lambda) \left| \int_{-\infty}^{\infty} g(\lambda) d\lambda = q \right. \right\},$$

and the minimality condition (1) hold true. The least favorable spectral densities $f_k^0(\lambda)$, $g_k^0(\lambda)$ in the class $D_0^f \times D_0^g$ for the optimal linear interpolation of the functional $A_k\xi$ are determined by relations

$$|A_k(\lambda) g_k^0(\lambda) + C_k^0(\lambda)| = \alpha^2 (f_k^0(\lambda) + g_k^0(\lambda)), \quad (20)$$

$$|A_k(\lambda) f_k^0(\lambda) - C_k^0(\lambda)| = \beta^2 (f_k^0(\lambda) + g_k^0(\lambda)), \quad (21)$$

constrained optimization problem (14) and restrictions on densities from the class $D_0^f \times D_0^g$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A_k\xi$ is determined by the formula (7).

Corollary 3.1. Let the minimality condition (13) hold true. The least favorable spectral densities $f_k^0(\lambda)$ in the class D_0^f for the optimal linear interpolation of the functional $A_k\xi$, which depends on the unknown values of the process $\xi(t)$ based on observations of the process $\xi(t)$ at time points $t \in \mathbb{R} \backslash S_k$, are determined by the following equation

$$|C_k^0(\lambda)| = \alpha^2 (f_k^0(\lambda)), \quad (22)$$

constrained optimization problem (15) and restrictions on densities from the class D_0^f. The minimax spectral characteristic of the optimal estimate of the functional $A_k\xi$ is determined by the formula (11).

Corollary 3.2. Let the minimality condition (13) hold true. The least favorable spectral densities $f_k^0(\lambda)$ in the class

$$D_\varepsilon^f = \left\{ f(\lambda) \middle| f(\lambda) = (1-\varepsilon) f_1(\lambda) + \varepsilon w(\lambda), \int_{-\infty}^{\infty} f(\lambda) d\lambda = p \right\}$$

for the optimal linear interpolation of the functional $A_k\xi$, which depends on the unknown values of the process $\xi(t)$ based on observations of the process $\xi(t)$ at time points $t \in \mathbb{R}\backslash S_k$, are determined by the following equation

$$|C_k^0(\lambda)| = (\alpha^2 + \gamma(\lambda))(f_k^0(\lambda)), \qquad (23)$$

where $\gamma(\lambda) \leqslant 0$ and $\gamma(\lambda) = 0$ if $f_k^0(\lambda) > (1-\varepsilon)f_1(\lambda)$, constrained optimization problem (15) and restrictions on densities from the class D_ε^f. The minimax spectral characteristic of the optimal estimate of the functional $A_k\xi$ is determined by the formula (11).

4 Least favorable spectral densities in the class $D = D_v^u \times D_{2\delta}$

Consider the problem of mean square optimal interpolation of the functional $A_k\xi$ in the case when spectral densities of the processes belong to the class of admissible spectral densities $D = D_v^u \times D_{2\delta}$,

$$D_v^u = \left\{ f(\lambda) \middle| v(\lambda) \leqslant f(\lambda) \leqslant u(\lambda), \int_{-\infty}^{\infty} f(\lambda) d\lambda = p \right\},$$

$$D_{2\delta} = \left\{ g(\lambda) \middle| \int_{-\infty}^{\infty} |g(\lambda) - g_1(\lambda)|^2 \, d\lambda \leqslant \delta \right\},$$

where spectral densities $v(\lambda), u(\lambda), g_1(\lambda)$ are known and fixed. The class D_v^u describes the "strip" model of stochastic processes, $D_{2\delta}$ describes "δ-district"in the space L_2 of the given bounded spectral density $g_1(\lambda)$.

From the condition $0 \in \partial\Delta_D(f_k^0, g_k^0)$ we find the following equations which determine the least favourable spectral densities for these given sets of admissible spectral densities

$$|A_k(\lambda)g_k^0(\lambda) + C_k^0(\lambda)| = (\alpha^2 + \gamma_1(\lambda) + \gamma_2(\lambda))(f_k^0(\lambda) + g_k^0(\lambda)), \qquad (24)$$

$$|A_k(\lambda)f_k^0(\lambda) - C_k^0(\lambda)|^2 = \beta^2(g_k^0(\lambda) - g_1(\lambda))(f_k^0(\lambda) + g_k^0(\lambda))^2, \qquad (25)$$

$$\int_{-\infty}^{\infty} \left|g_k^0(\lambda) - g_1(\lambda)\right|^2 d\lambda = \delta, \qquad (26)$$

where $\gamma_1(\lambda) \leqslant 0$ and $\gamma_1(\lambda) = 0$ if $f_k^0(\lambda) > v(\lambda)$, $\gamma_2(\lambda) \geqslant 0$ and $\gamma_2(\lambda) = 0$ if $f_k^0(\lambda) < u(\lambda)$, and α^2, β^2 are unknown Lagrange multipliers.

The following theorem and corollaries hold true.

Theorem 4.1. *Let the minimality condition (1) hold true. The least favorable spectral densities $f_k^0(\lambda)$, $g_k^0(\lambda)$ in the class $D = D_v^u \times D_{2\delta}$ for the optimal linear interpolation of the functional $A_k\xi$ are determined by relations (24) – (26), constrained optimization problem (14) and restrictions on densities from the corresponding classes $D = D_v^u \times D_{2\delta}$. The minimax-robust spectral characteristic of the optimal estimate of the functional $A_k\xi$ is determined by the formula (7).*

Corollary 4.1. Let the minimality condition (13) hold true. The least favorable spectral densities $f_k^0(\lambda)$ in the classes D_v^u for the optimal linear estimation of the functional $A_k\xi$, which depends on the unknown values of the process $\xi(t)$ based on observations of the process $\xi(t)$ at time points $t \in \mathbb{R}\backslash S_k$, are determined by the following equation

$$|C_k^0(\lambda)| = (\alpha^2 + \gamma_1(\lambda) + \gamma_2(\lambda))f_k^0(\lambda), \qquad (27)$$

constrained optimization problem (15) and restrictions on densities from the class D_v^u. The minimax spectral characteristic of the optimal estimate of the functional $A_k\xi$ is determined by the formula (11).

Corollary 4.2. Let the minimality condition (13) hold true. The least favorable spectral densities $f_k^0(\lambda)$ in the class

$$D_{2\delta} = \left\{ f(\lambda) \middle| \int_{-\infty}^{\infty} |f(\lambda) - f_1(\lambda)|^2 \, d\lambda \leqslant \delta \right\}$$

for the optimal linear estimation of the functional $A_k\xi$, which depends on the unknown values of the process $\xi(t)$ based on observations of the process $\xi(t)$ at time points $t \in \mathbb{R}\backslash S_k$, are determined by the following equations

$$|C_k^0(\lambda)|^2 = \beta^2(f_k^0(\lambda) - f_1(\lambda))(f_k^0(\lambda))^2, \qquad (28)$$

constrained optimization problem (15) and the following restrictions on densities from the class $D_{2\delta}$

$$\int_{-\infty}^{\infty} \left|f_k^0(\lambda) - f_1(\lambda)\right|^2 d\lambda = \delta. \qquad (29)$$

The minimax spectral characteristic of the optimal estimate of the functional $A_k\xi$ is determined by the formula (11).

5 Conclusions

In the article we propose the methods of the mean-square optimal linear interpolation of the functionals which depend on the unknown values of the process based on observed data of the process with noise and missing values. Under condition of spectral certainty when the spectral densities of the stationary processes are known we derive formulas for calculating the spectral characteristics and the mean-square errors of the estimates of the functionals. Analogous results are derived for the case of observations of the process without noise. In the case of spectral uncertainty when certain sets of admissible densities are given we derive the relations which the least spectral densities satisfy.

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