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Non-linear bending and stress analysis of a transversely loaded isotropic plates material using 3-D plate theory

F. C. Onyeka [a] *, T. E. Okeke [b], O. Ikhazuagbe [c]

[a] *Department of Civil Engineering, Edo State University, Uzairue, Edo State, Nigeria*
[b] *Department of Civil Engineering, University of Nigeria, Nsukka, Enugu State, Nigeria*
[c] *Department of Civil Engineering, Auchi Polytechnic, Auchi, Edo State, Nigeria*

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ABSTRACT

This paper presents the bending stress analysis of anisotropic plate material under transverse loading using a three-dimensional (3-D) plate theory. The static elastic theory was used to formulate the total energy expression of the plate thereafter, transformed into a compatibility equation through general variation to get the slope and deflection relationship. The solution of equations of the equilibrium gave rise to the exact polynomial deflection function while the coefficient of deflection and shear deformation of the plate was gotten from the governing equation through the direct variation method. These solutions were used to obtain the characteristic expression for analyzing the displacement and stresses of the rectangular plate. This formula was used for the solution of the bending problem of the rectangular plate that is clamped at the first-two edge and the other edges simply supported (CCSS). The result of the deflection and stresses decrease as the span-thickness ratio increases. More so, the aspect ratio effect of the shear stress of isotropic plates is investigated and discussed after a comparative analysis between the present work and previous studies. The result shows that the present study differs from that refined plate theory (RPT) of assumed deflection by 5.5% whereas exact 2-D RPT by 5.3%. This shows the efficacy of the exact 3-D plate theory for flexural characteristics of CCSS isotropic rectangular thick plate.

1. Introduction

Applications of plate structures are found in ship and airplane structures, bridges, etc. Plate's application in engineering is enormous and different theories which use linear strain-displacement expressions have been developed [1-3]. Studies have shown that results obtained using linear strain-displacement may appear inconsistent for nonlinear stress and bending analyzes [4-6]. Hence, with the increasing use of plates, the need for an improved theory is easily discovered. Thus, the need to develop a non-linear displacement model [7-9] that considers the transverse shear stress effect at a given boundary condition to achieve consistency is urgent.

Isotropic plates are widely used in structures under heavy transverse loading which generates large stresses [10-12]. The actual stress induced in such a plate is analyzed using one of the shear deformation theories [13-15] of different shape functions [16-18]. A Series of theories have been developed and applied to analyze the bending behavior of plates. Results were obtained using refined theories with Fourier series over-predicted stresses in the plate when subjected to a combination of boundary conditions [19-21]. The advantage of polynomial shear deformation functions over exponential, hyperbolic and trigonometric shear deformation functions is that they are easier to apply, which helps to reduce error in the analysis [22-24].

In [16], the authors used a combination of the Hamilton principle and Navier with hyperbolic shear deformation theory for the bending and free vibration analysis of isotropic orthotropic plate. Authors in [18] applied Trigonometric shear deformation theory (TSDT) which involves the shear deformation effect, but its results are quite difficult to apply in the analysis. Their work did not consider the isotropic plate material in the analysis.

The stability study can be carried out using either equilibrium, numerical, energy methods, or a combination of any [25-27]. Numerical methods such

* Corresponding author
E-mail address: onyeka.festus@edouniversity.edu.ng
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as finite difference methods, boundary element methods, and truncated double Fourier series, often yield approximate solutions to the plate problem. To obtain exact solutions requires so much time and lots of work. The energy method differs from the numerical and the equilibrium method in that it adds all the strain energy and potential energy or external work on the continuum to be equal to the total potential energy [28-30]. Their work did not analyze the stresses that might be induced due to applied load on the structure.

In [31], the authors used a polynomial shear deformation theory for the analysis of a rectangular thick plate that is clamped and simply support at the second and the fourth edge (CSFS). They developed the calculation formula for the critical load of the CSFS thick rectangular plate. They neither determine the displacement and stresses nor check the effect of shear stress in responses to the applied load which can lead to failure of the structures.

In [32], the authors applied 2-D theories to investigate the stability of elastic thick plates. Their study did not consider the stresses in the direction of the thickness axis. The outcome of their analysis was not a closed-form solution as the shape function used was assumed. The authors failed to cover plates with CCSS boundary conditions. The authors in [33] employed the displacement potential function method and used an assumed shape function to obtain the solution of buckling of thick plates that are simply supported. The authors applied the method of variable separation and satisfied the support conditions of the plate in order to establish the governing differential equations. Both authors in [28] and [29] did not use the displacement function that stems from the compatibility equation, and their study did not perform a bending analysis of CCSS rectangular plates.

The distinguishing feature of this current study from other works is that previous studies used an assumed deflection function to produce an approximate solution instead of an exact functional from the equilibrium equation, thereby making their solution unrealistic for the plate analysis of any type of load and thickness configuration. This unrealistic result produced could be attributed to the inability of the previous scholars to achieve an exact displacement function produced from the equilibrium equation formulated from the first principle of elasticity [34].

Authors in [35] applied an exact shear deformation theory for the analysis of a simply supported rectangular thick plate under a uniformly distributed load. They have obtained the expression for the displacement, and buckling load that may occur due to the compressive forces for a rectangular plate from the equilibrium equation but neither analyze for the stresses which produce the real exact solution for bending analysis nor solve for the present boundary condition. As well, the author in [36] considered all the stress elements to produce an exact solution for bidirectional composites and sandwich plate analysis but did not achieve the deflection function from the equilibrium equation.

Apart from the work of authors in [36-38], no work can be seen in the literature that adopted the 3-D plate theory to evaluate the bending and stress analysis of rectangular thick plates. Meanwhile, both authors did not study the plate at the first-two edge and the other edges simply (CCSS) or use the polynomial shape function which is easily applied to complex boundary conditions such as CCSS, so this research work is needed.

In this study, nonlinear strain-displacement polynomial shape functions are employed for the analysis of a rectangular plate and suggest a more reliable plate theory that gives exact solutions for plates with varying thicknesses and still satisfies the boundary condition. This theory, which is based on the 3-D theory of elasticity and includes all of the transverse stress components is presented and applied to the bending analysis of CCSS rectangular plate analysis under uniformly distributed load. Furthermore, the effects of the aspect ratio of the transverse shear stress of isotropic plates are investigated and discussed.

2. Methodology

2.1. Kinematics/Constitutive Relationship

By considering a rectangular plate subjected to uniformly distributed load as shown in Fig. 1 the displacement-strain relationship is established applying a consideration of non-linear deformation of the plate section as presented.

The constitutive equations for five stress components are:

$$u = \frac{zdw}{dy} + F.\theta_{sX} \tag{1}$$

$$v = \frac{zdw}{dy} + F.\theta_{sy} \tag{2}$$

Where, F is the profile of the plate.

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{3}$$

$$\varepsilon_y = \frac{dv}{\partial y} \tag{4}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \tag{5}$$

$$\sigma_x = E\,\frac{\left[\left(-\frac{z\partial^2 w}{dx^2}+\frac{F\partial\theta_{Sx}}{\partial x}\right)-\mu\left(\frac{z\partial^2 w}{dy^2}+\frac{F\partial\theta_{Sy}}{\partial y}\right)\right]}{(1-\mu^2)} \tag{6}$$

$$\sigma_y = E\,\frac{\left[\left(-\frac{z\partial^2 w}{dy^2}+\frac{F\partial\theta_{Sx}}{\partial x}\right)-\mu\left(\frac{z\partial^2 w}{\partial x^2}+\frac{F\partial\theta_{Sy}}{\partial y}\right)\right]}{(1-\mu^2)} \tag{7}$$

$$\tau_{xy} = 2G\left[-\frac{z\partial^2 w}{\partial x\partial y} + F\left(\frac{\partial\theta_{Sx}}{dy}+\frac{\partial\theta_{Sy}}{dx}\right)\right] \tag{8}$$

$$\tau_{xz} = 2G\left[\frac{z\partial^2 w}{\partial x\partial z} + F\left(\frac{d\theta_{Sx}}{\partial z}+\frac{\partial\theta_{Sz}}{\partial x}\right)\right] \tag{9}$$

$$\tau_{yz} = 2G\left[\frac{z\partial^2 w}{\partial y\partial z} + F\left(\frac{\partial\theta_{Sy}}{\partial z}+\frac{\partial\theta_{Sz}}{\partial y}\right)\right] \tag{10}$$

$$G = \frac{E(1-\mu)}{2(1-\mu^2)} \tag{11}$$

Let the non-dimensional coordinates be $R = x/a$, $Q = y/b$ and $S = z/t$ corresponding to x, y and z-axes respectively. Substituting appropriately, the six stresses in the plate becomes:

$$\sigma_x = \frac{Ets}{(1+\mu)(1-2\mu)a}\left[(1-\mu)\,.\frac{\partial \theta_x}{\partial R} + \frac{\mu}{\beta}.\frac{\partial \theta_y}{\partial Q} + \frac{\mu a}{st^2}.\frac{\partial w}{\partial S}\right] \quad (12)$$

$$\sigma_y = \frac{Ets}{(1+\mu)(1-2\mu)a}\left[\mu.\frac{\partial \theta_x}{\partial R} + \frac{(1-\mu)}{\beta}.\frac{\partial \theta_y}{\partial Q} + \frac{\mu a}{st^2}.\frac{\partial w}{\partial S}\right] \quad (13)$$

$$\sigma_z = \frac{Ets}{(1+\mu)(1-2\mu)a}\left[\mu.\frac{\partial \theta_x}{\partial R} + \frac{\mu}{\beta}.\frac{\partial \theta_y}{\partial Q} + \frac{(1-\mu)a}{st^2}.\frac{\partial w}{\partial S}\right] \quad (14)$$

$$\tau_{xy} = \frac{E(1-2\mu)ts}{2(1+\mu)(1-2\mu)a}.\left[\frac{1}{\beta}\frac{\partial \theta_x}{\partial Q} + \frac{\partial \theta_y}{\partial R}\right] \quad (15)$$

$$\tau_{xz} = \frac{E(1-2\mu)ts}{2(1+\mu)(1-2\mu)a}.\left[\frac{a}{ts}\,\theta_x + \frac{1}{ts}\frac{\partial w}{\partial R}\right] \quad (16)$$

$$\tau_{yz} = \frac{E(1-2\mu)ts}{2(1+\mu)(1-2\mu)a}.\left[\frac{a}{ts}\,\theta_y + \frac{1}{\beta ts}\frac{\partial w}{\partial Q}\right] \quad (17)$$

Given that: β = a/t.

2.2. Potential Energy Expressions

Total energy expression being the algebraic summation of strain energy (M) and external work (V) is expressed mathematically as [39-41]:

$$\prod = \mathrm{M} - \mathrm{V} \quad (18)$$

The strain energy equation which is the dot product of stresses and strain [42-44] as obtained in equations (3) – (17) as follows:

$$M = \frac{abt}{2}\int_0^1\int_0^1\int_{-0.5}^{0.5}\big(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\big)\,dR\,dQ\,dS \quad (19)$$

While the potential energy for the plate with a uniformly distributed load is given as:

$$V = abq\int_0^1\int_0^1 w\,dR\,dQ \quad (20)$$

Where, the symbol w denotes the deflection function of the plate, while; q, a, and b denote the uniformly distributed load, length, and breadth of the plate, respectively.

Thus, putting equations (19) and (20) into equation (18) after substituting equation (17) into (19) gives:

$$\Pi = \frac{D^*ab}{2a^2}\int_0^1\Bigg[(1-\mu)\left(\frac{\partial\theta_{sx}}{\partial R}\right)^2 + \frac{1}{\beta}\frac{\partial\theta_{sx}}{\partial R}.\frac{\partial\theta_{sy}}{\partial Q} + \frac{(1-\mu)}{\beta^2}\left(\frac{\partial\theta_{sy}}{\partial Q}\right)^2 + \frac{(1-2\mu)}{2\beta^2}\left(\frac{\partial\theta_{sx}}{\partial Q}\right)^2 + \frac{(1-2\mu)}{2}\left(\frac{\partial\theta_{sy}}{\partial R}\right)^2 + \frac{6(1-2\mu)}{t^2}\left(a^2\theta_{sx}{}^2 + a^2\theta_{sy}{}^2 + \left(\frac{\partial w}{\partial R}\right)^2 + \frac{1}{\beta^2}\left(\frac{\partial w}{\partial Q}\right)^2 + 2a.\,\theta_{sx}\frac{\partial w}{\partial R} + \frac{2a.\theta_{sy}}{\beta}\frac{\partial w}{\partial Q}\right) + \frac{(1-\mu)a^2}{t^4}\left(\frac{\partial w}{\partial S}\right)^2 - \frac{2qa^4w}{D^*}\Bigg]\,dR\,dQ \quad (21)$$

2.3. Governing Energy Equation

The solution of the general governing equation is obtained in line with the work of the authors in [35] to get the exact deflection equation and slope at the x and y-axes of the plate as presented in equations (22), (23), and (24) respectively.

$$w = \left(a_0 + a_1R + \frac{a_2R^2}{2} + \frac{a_3R^3}{6} + \frac{qa^4}{D}\left(\frac{n_1}{w_3}\right).\frac{R^4}{24}\right).\left(b_0 + b_1Q + \frac{b_2Q^2}{2} + \frac{b_3Q^3}{6} + \frac{qa^4}{D}\left(\frac{n_1}{w_3}\right).\frac{Q^4}{24}\right) \quad (22)$$

$$\theta_x = \left(a_4 + a_5R + \frac{a_6R^2}{2} + \frac{qa^3}{D}\left(\frac{n_4}{g_2\theta_3}\right).\frac{R^3}{6}\right).\left(b_7 + b_8Q + \frac{b_9Q^2}{2} + \frac{b_{10}Q^3}{6} + \frac{b_{11}Q^4}{24}\right) \quad (23)$$

$$\theta_y = \left(a_7 + a_8R + \frac{a_9R^2}{2} + \frac{a_{10}R^3}{6} + \frac{a_{11}R^4}{24}\right).\left(b_4 + b_5Q + \frac{b_6Q^2}{2} + \frac{qa^3}{D}\left(\frac{\propto^3 n_5}{g_2\theta_1}\right).\frac{Q^3}{6}\right) \quad (24)$$

Let:

$$w = C_1.h \quad (25)$$

$$\theta_x = \left[\frac{dh}{dR}\right]\left[\frac{C_2}{a}\right] \quad (26)$$

$$\theta_y = \left[\frac{dh}{dQ}\right]\left[\frac{C_3}{a\beta}\right] \quad (27)$$

Where; h, C_1, C_2, and C_3 are the plate shape function, coefficient of deflection, coefficient of shear deformation along the x-axis, and coefficient of shear deformation along the y-axis respectively.

The solution of the governing equation is achieved by differentiation equation (21) with respect to C_1, C_2 , and C_3 to get:

$$\frac{\partial \Pi}{\partial C_1} = \frac{\partial \Pi}{\partial C_2} = \frac{\partial \Pi}{\partial C_3} = 0 \quad (28)$$

The solution of equation (28) gives:

$$\Pi = \frac{D^*ab}{2a^4}\Bigg[(1-\mu)C_2{}^2k_x + \frac{1}{\beta^2}\left[C_2.C_3 + \frac{(1-2\mu)C_2{}^2}{2} + \frac{(1-2\mu)C_3{}^2}{2}\right]k_{xy} + \frac{(1-\mu)C_3{}^2}{\beta^4}k_y + 6(1-2\mu)\left(\frac{a}{t}\right)^2\Big(\big[C_2{}^2 + C_1{}^2 + 2C_1C_2\big].k_z + \frac{1}{\beta^2}.\big[C_3{}^2 + 2C_1C_3\big].k_{2z}\Big) - \frac{2qa^4k_hC_1}{D^*}\Bigg] \quad (29)$$

Where:

$$k_x = \int_0^1\int_0^1\left(\frac{\partial^2 h}{\partial R^2}\right)^2 dRdQ \quad (30)$$

$$k_{xy} = \int_0^1\int_0^1\left(\frac{\partial^2 h}{\partial R\partial Q}\right)^2 dRdQ \quad (31)$$

$$k_h = \int_0^1\int_0^1 h\,.dRdQ \quad (32)$$

$$k_y = \int_0^1\int_0^1\left(\frac{\partial^2 h}{\partial Q^2}\right)^2 dRdQ \quad (33)$$

$$k_z = \int_0^1\int_0^1\left(\frac{\partial h}{\partial R}\right)^2 dRdQ \quad (34)$$

$$k_{2z} = \int_0^1\int_0^1\left(\frac{\partial h}{\partial Q}\right)^2 dRdQ \quad (35)$$

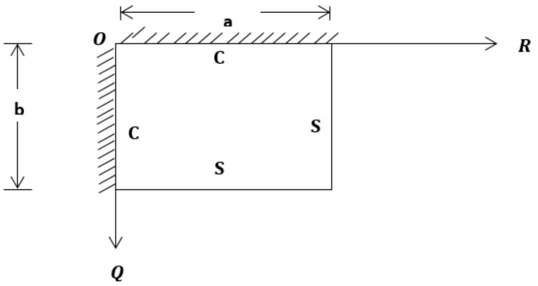

Fig. 1 CCSS rectangular plate.

Minimizing equation (21) with respect to C_2 and simplifying the outcome gives:

$$\left[(1-\mu)k_x + \frac{1}{2\beta^2}(1-2\mu)k_{xy} + 6(1-2\mu)\left(\frac{a}{t}\right)^2 k_z\right] C_2 + \left[\frac{1}{2\beta^2}k_{xy}\right] C_3 = \left[-6(1-2\mu)\left(\frac{a}{t}\right)^2 k_z\right] C_1 \quad (36)$$

Minimizing equation (21) with respect to C_3 and simplifying the outcome gives*:*

$$\left[\frac{1}{2\beta^2}k_{xy}\right] C_2 + \left[\frac{(1-\mu)}{\beta^4}k_y + \frac{1}{2\beta^2}(1-2\mu)k_{xy} + \frac{6}{\beta^2}(1 -2\mu)\left(\frac{a}{t}\right)^2 k_{2z}\right] C_3 = \left[-\frac{6}{\beta^2}(1-2\mu)\left(\frac{a}{t}\right)^2 k_Q\right] C_1 \quad (37)$$

Equations (38) and (39) are the solutions of equations (36) and (37) respectively.

$$C_2 = AC_1 \quad (38)$$

$$C_3 = BC_1 \quad (39)$$

Minimizing equation (33) with respect to C_1 and simplifying the outcome gives:

$$6(1-2\mu)\left(\frac{a}{t}\right)^2 \left([C_1 + C_1 A].k_z + \frac{1}{\beta^2}.[C_1 + C_1 B].k_{2z}\right) - \frac{qa^4 k_h}{D^*} = 0 \quad (40)$$

Factorizing equation (40) and simplifying further gives:

$$6(1-2\mu)\left(\frac{a}{t}\right)^2 C_1 \left([1 + A].k_z + \frac{1}{\beta^2}.[1 + B].k_{2z}\right) = \frac{qa^4 k_h}{D^*} \quad (41)$$

Thus:

$$C_1 = \frac{qa^4}{D^*}\left(\frac{k_h}{T}\right) \quad (42)$$

Let:

$$K = 6(1-2\mu)\left(\frac{a}{t}\right)^2 * \left([1 + G_2].k_z + \frac{1}{\beta^2}.[1 + G_3].k_{2z}\right) \quad (43)$$

$$A = \frac{(s_{12}s_{23} - s_{13}s_{22})}{(s_{12}s_{12} - s_{11}s_{22})} \quad (44)$$

$$B = \frac{(s_{12}s_{13} - s_{11}s_{23})}{(s_{12}s_{12} - s_{11}s_{22})} \quad (45)$$

Where,

$$s_{11} = (1-\mu)k_x + \frac{1}{2\beta^2}(1-2\mu)k_{xy} + 6(1-2\mu)\left(\frac{a}{t}\right)^2 k_z \quad (46)$$

$$s_{22} = \frac{(1-\mu)}{\beta^4}k_y + \frac{1}{2\beta^2}(1-2\mu)k_{xy} + \frac{6}{\beta^2}(1-2\mu)\left(\frac{a}{t}\right)^2 k_{2z} \quad (47)$$

$$s_{23} = s_{32} = -\frac{6}{\beta^2}(1-2\mu)\left(\frac{a}{t}\right)^2 k_{2z} \quad (48)$$

$$s_{13} = -6(1-2\mu)\left(\frac{a}{t}\right)^2 k_z \quad (49)$$

$$s_{12} = s_{21} = \frac{1}{2\beta^2}k_{xy} \quad (50)$$

$$D^* = \frac{Et^3}{12(1+\mu)(1-2\mu)} \quad (51)$$

D^*, E, t denotes the modulus of rigidity of 3-D plate, modulus of elasticity and thickness of the plate respectively.

2.4. Numerical Analysis

The isotropic rectangular plate subjected to CCSS plate boundary condition shown in Fig. 1 at varying span to depth and aspect ratios of the plate is analyzed using the established polynomial deflection function in the previous section.

The boundary conditions of the plate in Fig. 1 are as follows, At:

$$R = Q = 0; \text{Deflection } (w) = 0 \quad (52)$$

$$R = Q = 0; \text{Slope}\left(\frac{dw}{dR} = \frac{dw}{dQ}\right) = 0 \quad (53)$$

$$R = Q = 1; \text{Deflection } (w) = 0 \quad (54)$$

$$R = Q = 1; \text{Bending moment}\left(\frac{d^2w}{dR^2} \,\&\, \frac{d^2w}{dQ^2}\right) = 0 \quad (55)$$

The deflection function that satisfies the boundary conditions for one edge free and the other three edges clamped rectangular plate boundary conditions are determined as follows:

Substituting equations (52) – (55) into the derivatives of *w* and solving gave constants:

$R = 0, w = 0$

$Q = 0, w = 0$

Solving, gave the following constants:

$$a_0 = 0, b_0 = 0 \quad (56)$$

For $R = 0$, $\frac{dw}{dR} = 0$ and $Q = 0$, $\frac{dw}{dQ} = 0$, thus

$$a_1 = 0; b_1 = 0 \quad (57)$$

For $R = 1$, $\frac{d^2w}{dR^2} = 0$ and $Q = 1$, $\frac{d^2w}{dQ^2} = 0$, thus

$$a_2 = \frac{F_{a4}}{8}; b_2 = \frac{F_{b4}}{8} \quad (58)$$

For $R = 1$, $w = 0$ and $Q = 1$, $w = 0$,

$$a_3 = \frac{-5F_{a4}}{8}; b_3 = \frac{-5F_{b4}}{8} \quad (59)$$

Substituting these constants back into equations (56) – (59) back into equation (22) and solving gives:

$$w = \frac{F_{a4}}{48}(3R^2 - 5R^3 + 2R^4).\frac{F_{b4}}{48}(3Q^2 - 5Q^3 + 2Q^4) \quad (60)$$

Let the amplitude,

$$C_1 = \frac{(F_{a4} \cdot F_{b4})}{2304} \quad (61)$$

and

$$h = (1.5R^2 - 2.5R^3 + R^4).(1.5Q^2 - 2.5Q^3 + Q^4) \quad (62)$$

Using equations (30) – (35), the stiffness coefficients of the CCSS rectangular plate are established as presented in Fig. 2.

2.5. Exact Displacement and Stress Expression

The expression for the in-plane displacement (u and v), deflection (w) and stress of 3-D plate were derived by substituting the values of C_1, C_2, and C_3 in equations (38), (39) and (42) into equations (12) – (17), simplify appropriately, gives:

$$u = \mathrm{F}.\frac{C_2}{a}.\frac{\partial h}{\partial Q} \quad (63)$$

$$w = C_1(1.5R^2 - 2.5R^3 + R^4).(1.5Q^2 - 2.5Q^3 + Q^4) \quad (64)$$

$$\sigma_x = \frac{\mathrm{Ets}}{(1+\mu)(1-2\mu)\mathrm{a}}\left[C_2\frac{\partial^2 h}{\partial R^2} + \mu\,\frac{C_3}{\beta^2}\frac{d^2 h}{dQ^2} + \frac{(1-\mu)\mathrm{a}}{st^2}.\frac{\partial \mathrm{w}}{\partial \mathrm{S}}\right] \quad (65)$$

$$\sigma_z = \frac{\mathrm{Ets}}{(1+\mu)(1-2\mu)\mathrm{a}}\left[\frac{\mu}{C_2}.\frac{\partial^2 h}{\partial R^2} + \mu\,\frac{C_3}{\beta}\frac{\partial^2 h}{\partial Q^2} + \frac{(1-\mu)\mathrm{a}}{st^2}.\frac{\partial \mathrm{w}}{\partial \mathrm{S}}\right] \quad (66)$$

$$\tau_{xy} = \frac{\mathrm{Ets}}{2(1+\mu)(1-2\mu)}.\frac{1}{\beta a^3}\left[C_2 + C_3\right]\frac{\partial^2 h}{\partial R\,\partial Q} \quad (67)$$

$$\tau_{yz} = \frac{\mathrm{Ets}}{2(1+\mu)(1-2\mu)}.\frac{1}{\beta a^2}\left[C_1 + \frac{C_3}{t}.\frac{\partial F}{\partial S}\right]\frac{\partial h}{\partial Q} \quad (68)$$

3. Results and Discussion

Fig. 2 shows the values of stiffness coefficient, k_1, k_2, k_3, k_4, k_5, and k_h for various supports while Table 1 contains the result of the non-dimensional value of displacements and stresses in a rectangular thick plate with length to breadth ratio of 1.5 at different span-thickness aspect ratio. These numerical values were obtained from the equations (30) – (35). The table contains the numerical representation of the result of the non-dimensional displacements (u, v, and w) and the stress characteristics of a CCSS rectangular plate using the established exact polynomial displacement function. The numerical and graphical comparison was made to show the disparities between the present study and the literature under review to show the effect of aspect ratio on the 3-D bending and stress analysis of rectangular plates at varying thicknesses. The span to thickness ratio considered ranges between 4, 5, 10, 15, 20, 50, 100, and CPT, which is seen to span from the thick plate, moderately thick plate, and thin plate [37]. The present work obtained a non-dimensional result of the stresses of the plate by expressing the displacement function of the plate in the form of a polynomial to analyze the effect of the aspect ratio of bending characteristics of the plate.

It is observed (Tables 1 – 2) that the value of the normal stress along the x, y, and z axes a shear stress perpendicular to the x-y, x-z, and y-z planes decreases as the span-thickness ratio increases as seen in Tables 1 and 2. It is also observed in the tables that the non-dimensional displacement (u, v, and w) characteristics decrease with increases in the value of the span-thickness ratio. The value of deflection is discovered to vary less as the span to thickness increases. It is also observed in the tables that the non-dimensional displacement (u, v, and w) characteristics decrease with increases in the value of the span-thickness ratio. The value of deflection is discovered to vary less as the span to thickness increases, this is equal to the value of the CPT at a span to thickness ratio of 15 and above.

A critical look at Table 1 shows that the displacement (u, v, and w) and stress characteristics increase as the value of the aspect ratio of the plate increases. Also, it was deduced that the normal stress and shear stress characteristics also decrease as the span-thickness ratio increases. This implies that the failure in a plate structure is bound to occur as more stresses are induced within the plate element which leads to the bending of the elementary section of the plate material.

Fig. 2 Stiffness coefficient for the CCSS plate boundary condition.

Table 1. The result of displacements and stresses of a CCSS square plates.

β = a/t	w	u	v	σ_x	σ_y	σ_z	τ_{xy}	τ_{xz}
4	0.003151	0.001113	0.001113	0.247282	0.247282	0.240569	0.008995	0.004226
5	0.002681	0.001008	0.001008	0.227704	0.227704	0.224145	0.008116	0.002579
10	0.002076	0.000874	0.000874	0.202575	0.202575	0.202639	0.006989	0.000465
15	0.001967	0.000851	0.000851	0.198038	0.198038	0.198699	0.006785	0.000083
20	0.001929	0.000841	0.000841	0.196459	0.196459	0.197323	0.006714	-0.000051
50	0.001889	0.000832	0.000832	0.194758	0.194758	0.195839	0.006416	-0.000110
100	0.001883	0.000831	0.000831	0.194516	0.194516	0.195627	0.006627	-0.000213
CPT	0.001883	0.000831	0.000831	0.194516	0.194516	0.195627	0.006627	-0.000213

Table 2. Displacement and stresses of a CCSS plate aspect ratio of 1.5.

β = a/t	w	u	v	σ_x	σ_y	σ_z	τ_{xy}	τ_{xz}
4	0.005392	0.002076	0.001370	0.251789	0.249589	0.245189	0.010905	0.006215
5	0.004747	0.001940	0.001260	0.235365	0.233165	0.228765	0.010063	0.003861
10	0.003911	0.001765	0.001114	0.213859	0.211659	0.207259	0.008963	0.000789
15	0.003759	0.001734	0.001087	0.209919	0.207719	0.203319	0.008762	0.000227
20	0.003706	0.001723	0.001078	0.208543	0.206343	0.201943	0.008692	0.000031
50	0.003649	0.001711	0.001068	0.207059	0.204859	0.200459	0.008616	-0.000110
100	0.003640	0.001709	0.001066	0.206847	0.204647	0.200247	0.008605	-0.000210
CPT	0.003640	0.001709	0.001066	0.206847	0.204647	0.200247	0.008605	-0.000210

Table 3. Comparative analysis of present study for length to breadth ratio ($\propto = b/a$) of 1.5 and past studies showing their percentage difference calculations of deflection (w) at varying span-depth ratio (β = a/t).

β = a/t	Present work (w)	Previous study [37]	Percentage difference (%)	Previous study [36]	Percentage difference (%)
4	0.005392	0.005612	4.08	0.005545	2.84
5	0.004747	0.004967	4.63	0.004940	4.07
10	0.003911	0.004131	5.62	0.004129	5.57
15	0.003759	0.003979	5.85	0.003978	5.83
20	0.003706	0.003926	5.94	0.003926	5.94
509	0.003649	0.003869	6.03	0.003869	6.03
100	0.003640	0.003860	6.04	0.003864	6.15
CPT	0.003640	0.003860	6.04	0.003864	6.15
Average difference (%)			5.5		5.3
Total average difference (%)			5.4		

Table 1 shows that, at span to thickness ratio between 4 and 15, the value of out-of-plane displacement varies between 0.003151 and 0.001967. These values decrease and maintain a constant value of 0.0019 at the span to a thickness between 15 till 100 which is equal to the value of the CPT. Similarly, Table 2 shows that, at span to thickness ratio between 4 and 15, the value of out-of-plane displacement varies between 0.005392 and 0.003759. These values decrease and maintain a constant value of 0.0037 at the span-thickness between 15 till 100 which is equal to the value of the CPT.

It's worth noting that, the plate whose deflection and transverse shear stress varies very much from zero is categorized as a thick plate. Thus, the span-thickness ratio for these categories of rectangular plates is that the thick plate is categorized as the plate with the span to thickness ratio: while the thin plate is categorized as the plate with the span to thickness ratio.

Meanwhile, the present theory of stress prediction shows that the result of the displacement and stress of thin and moderately thick plates using the 3-D theory is the same for the bending analysis of rectangular plates under the CCSS boundary condition. Thus, the results obtained in the Figures reveal that the values of critical buckling load increase as the span-thickness ratio increases. This reveals that as the span or the depth of the plate is altered, it affects the performance in terms of the serviceability of the plate. Thus, caution must be taken when selecting the depth and other dimensions along the x and y co-ordinate of the plate to ensure the safety and accuracy of the analysis.

The comparative analysis performed (as presented in Table 3) showed that the present model using a derived shape function is safer and more credible to use as it considered the six stress elements to yield the exact solution for the analysis of a thick plate that is clamped supported at the first edge and other three edges simply supported (CCSS). Hence, the result of the present analysis, which contains all the stress elements and ensured that the variation of the stresses through the thickness of the plate which induced buckling are uniformly distributed, showed that the present method can be used with confidence for bending analysis of plate. The plate with the largest thickness (a/t of 4) gives a percentage difference of 2.84% and 4.08% of the work of authors in [36] and [37] respectively when compared with the present study. On the other hand, the thinnest plate (a/t of 100 and above) gives a percentage difference of 6.15% and 6.04% of the work of authors in [35] and [36] respectively, when compared with the present study.

Finally, it can be deduced that the overall average percentage difference values of deflection for the present theory and those of authors in [36] and [37] are 5.3% and 5.5% respectively. This means that the 2-D RPT with exact deflection gives a closer result when compared with the exact 3-D plate theory than those with an assumed deflection in the analysis.

4. Conclusion

The 3-D bending and stress analysis of thick rectangular plates using the 3-D elasticity theory has been investigated. From this study, the following conclusion has been drawn:

(a) It is concluded that the present theory of stress prediction shows that the result of the displacement and stress of thin and moderately thick plates using the 3-D theory is the same for the bending analysis of rectangular plates under the CCSS boundary condition.

(b) Plate analysis required a 3-D analogy for a true solution, but the 2-D shear deformation theory gives an unrealistic solution.

(c) The 3-D exact plate model developed in this study is variationally consistent and can be used in the analysis of any category of the plate.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

ORCID

F. C. Onyeka https://orcid.org/0000-0002-2668-9753
T. E. Okeke https://orcid.org/0000-0003-4743-5055

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