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Superlinear damped vibration problems on time scales with nonlocal boundary conditions*

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Abstract. This paper studies a class of superlinear damped vibration equations with nonlocal boundary conditions on time scales by using the calculus of variations. We consider the Cerami condition, while the nonlinear term does not satisfy Ambrosetti–Rabinowitz condition such that the critical point theory could be applied. Then we establish the variational structure in an appropriate Sobolev's space, obtain the existence of infinitely many large energy solutions. Finally, two examples are given to prove our results.

Keywords: damped vibration, nonlocal boundary condition, Cerami condition, variational structure, critical point theory.

1 Introduction

Vibration is a common form of motion in daily life and engineering technology, such as the reciprocating swing of a pendulum, the vibration of a spring, the vibration of a string in a musical instrument, the vibration of the spindle of a machine tool, the electromagnetic oscillation in a circuit, and so on. The study of vibration problem can be reduced to the second-order constant coefficient differential equation under certain conditions. Assuming that the pendulum is oscillating in a viscous medium, considering the air resistance and the constant external force $F(t)$ acting on the pendulum along its direction of motion, the pendulum's action called forced vibration, its damped forced vibration equation is

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + \frac{\mu}{m}\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \frac{g}{l}\varphi = \frac{1}{ml}F(t).$$

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In recent papers, many authors studied the damped forced vibration problems with local conditions, especially, two-point boundary conditions. Nieto and Xiao [16, 26] studied the existence of weak solutions of the following damped Dirichlet problem:

$$\begin{aligned}
&-y''(s) + g(s)y'(s) + \lambda y(s) = h\bigl(s, y(s)\bigr), \quad \text{a.e. } s \in [0,T],\\
&-\Delta y'(s_j) = I_j\bigl(y(s_j)\bigr), \quad j = 1,2,\dots,p,\\
&y(0) = y(T) = 0.
\end{aligned}$$

They used the classical Lax–Milgram theorem to reveal the variational structure of the problem and get the existence and uniqueness of weak solutions as critical points. Bai et al. [4] studied the following damped vibration problem:

$$\begin{aligned}
&-y''(s) + g(s)y'(s) = h\bigl(s, y(s)\bigr), \quad \text{a.e. } s \in [0,T],\\
&-\Delta\bigl((y')^i(s_j)\bigr) = I_{ij}\bigl(y^i(s_j)\bigr), \quad i = 1,2,\dots,N,\ j = 1,2,\dots,p,\\
&y(0) - y(T) = y'(0) - y'(T) = 0.
\end{aligned}$$

They used the variational method and Brezis–Nirenberg critical point theorem to get the existence of nonzero solutions of the problem. Barilla et al. [5] studied the second-order dynamic Sturm–Liouville boundary value problem on time scale

$$\begin{aligned}
&-\bigl(p(s)x^{\Delta}(s)\bigr)^{\Delta} + q(s)y^{\sigma}(s) = h\bigl(s, y^{\sigma}(s)\bigr), \quad s \in [0,T]_{\mathbb{T}},\\
&\alpha_1 y(0) - \alpha_2 y^{\Delta}(0) = 0, \qquad \alpha_3 y\bigl(\sigma^2(T)\bigr) + \alpha_4 y^{\Delta}\bigl(\sigma(T)\bigr) = 0,
\end{aligned}$$

where $p \in \mathcal{C}^1([0,\sigma(T)]_{\mathbb{T}}, (0,\infty))$, $q \in \mathcal{C}([0,T]_{\mathbb{T}}, (0,\infty))$, $h \in \mathcal{C}([0,T]_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$, $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2 \geqslant 0$, $\alpha_1 + \alpha_3, \alpha_3 + \alpha_4 > 0$. They got the existence of solutions by variational methods on time scales. More literatures we can see [2, 19].

However, we find it is better to impose nonlocal conditions in some problems [9, 14, 20, 21] because the measurements required by nonlocal conditions may be more accurate than those given by local conditions. As example, the following second-order differential equation:

$$y''(s) + h\bigl(s, y(s), y'(s)\bigr) = 0$$

with Robin boundary condition $y(0) = 0$, $y'(1) = 0$. Let the nonlocal condition $y(1) = y(\eta)$ replace the local condition $y'(1) = 0$, then the problem can be transfer to a nonlocal problem. Moreover, because in the actual conditions and the process of numerical calculation, the value of $(y(\eta) - y(1))/(\eta - 1)$ is easier to determine than the value of $y'(1) = 0$, the effect of nonlocal problems is better than that of local problems. Therefore, nonlocal problems can be considered as boundary value problems that include continuous equations and one or more discrete multipoint boundary conditions. The origin of nonlocal boundary value problems of differential equations is related to the mathematical models of nonlinear problems in mathematics, physics, and other disciplines. Many problems in elastic stability theory can also be treated as nonlocal boundary value problems. Therefore, the damped forced vibration problems will be more accurate with nonlocal boundary conditions. However, there are few literatures on solving nonlocal boundary value problems by using variational method. So far, we only know what we have done recently [12, 22–24].

In this paper, we study a class of second-order damped vibration equations with three-point boundary conditions on time scales

$$\begin{aligned} &-y^{\Delta\Delta}(s) + g(s)y^{\Delta}\big(\sigma(s)\big) + \lambda y^{\sigma}(s) = h\big(\sigma(s), y^{\sigma}(s)\big), \quad \Delta\text{-a.e. } s \in J, \\ &y(a) = 0, \qquad y(b) = \zeta y(\eta), \end{aligned} \tag{1}$$

where $g(s) \in \mathcal{L}^1(J, \mathbb{R}^+)$, $\lambda \in \mathbb{R}$, $J := [a, \rho(b))_{\mathbb{T}}$ is an arbitrary closed subinterval of $\mathbb{T}$, and $\mathbb{T} \subset \mathbb{R}$ is an arbitrary bounded time scales, $\min \mathbb{T} = a, \max \mathbb{T} = b$. Otherwise, $0 < \zeta \leqslant 1$, $\eta = k_1/k_2 \in \mathbb{Q} \cap (a, b)_{\mathbb{T}}$ with $k_1, k_2 \in \mathbb{N}$, and they are relatively prime.

In fact, when the time scale $\mathbb{T}$ is equal to the real numbers $\mathbb{R}$ or integers $\mathbb{Z}$, it represents the classical theory of differential equations and difference equations. We assume that η is a right-dense point, denote $T_1 = [a, \rho(\eta))_{\mathbb{T}}$, $T_2 = [\eta, \rho(b))_{\mathbb{T}}$, and let the nonlinearity $h(s, y)$ satisfies the following conditions:

- (A) define the function $H(s, y) = \int_0^y h(s, \tau)\,\mathrm{d}\tau$, and assume $H : J \times \mathbb{R} \to \mathbb{R}$ satisfies: $H(s, y)$ is Δ-measurable in s for each $y \in \mathbb{R}$ and continuously differentiable in y for Δ-a.e. $s \in J$, and there exist $a \in \mathcal{C}(J, \mathbb{R}^+)$, $b \in \mathcal{L}^1_{\Delta}(J, \mathbb{R}^+)$ such that $|H(s, y)| \leqslant a(|y|)b(s)$, $|h(s, y)| \leqslant a(|y|)b(s)$ for all $y \in \mathbb{R}$ and Δ-a.e. $s \in J$;
- (A1) $\liminf_{|y|\to\infty} h(s, y)y/|y|^{\mu} = +\infty$ uniformly for Δ-a.e. $s \in J$ and $2 \leqslant \mu < \infty$;
- (A2) $|h(s, y)| \leqslant C(1 + |y|^{p-1})$ for Δ-a.e. $s \in J$ and $y \in \mathbb{R}$, where $\mu < p < \infty$;
- (A3) there exists a constant $\theta > 0$ such that for Δ-a.e. $s \in J$, there is $h(s, y)y/|y|^{\mu}$ increasing for $y \geqslant \theta$ and decreasing for $y \leqslant -\theta$.

In order to find the existence of infinitely many solutions, the following superquadratic conditions proposed by Ambrosetti and Rabinowitz [3] are needed:

(AR) there exists the constant $\mu > 2$ such that

$$0 < \mu H(s, y) \leqslant y h(s, y) \quad \text{for all } y \in \mathbb{R} \setminus \{0\} \text{ and } s \in \Omega,$$

where h is the nonlinear term, $H(s, y) = \int_0^y h(s, \tau)\,\mathrm{d}\tau$, and $\Omega \subset \mathbb{R}$.

The role of Ambrosetti–Rabinowitz (AR) condition ensure the compactness of Palais–Smale (PS) condition. However, condition (AR) eliminates many nonlinearities and has certain limitations. In fact, some examples show that the nonlinearity $h(s, y)$ could not satisfy the (AR) condition such as $h(s, y) = 2y^2 \log(1 + |y|)$. Thus it is meaningful to study problem (1) without (AR) condition.

The interesting points of this paper are the followings: (i) we establish the variational structure of the superlinear damped vibration problem with nonlocal boundary conditions; (ii) we consider the Cerami condition such that although the Palais–Smale sequence is unbounded, the critical point theory could be applied; (iii) the existence of infinitely many large energy solutions for the problem is obtained without (AR) condition by using the mountain pass theorem and fountain theorem.

2 Preliminaries

In order to solve problem (1), the following definitions and lemmas are needed.

Definition 1. (See [1, Def. 2.3].) Let $\mathcal{A} \subset \mathbb{T}$ and $X \subset \mathbb{T}$. $\mathcal{A}$ is called $\triangle$-null set if $\mu_\triangle(\mathcal{A}) = 0$. Say that property $\mathcal{P}$ holds for $\triangle$-almost all ($\triangle$-a.a.) $s \in X$ if there is a $\triangle$-null set $\mathcal{A} \subset X$ such that $\mathcal{P}$ holds for all $s \in X \setminus \mathcal{A}$. We say that property $\mathcal{P}$ holds $\triangle$-almost everywhere ($\triangle$-a.e.) on X.

Definition 2. (See [7, Def. 1.1].) Suppose $\mathbb{T}$ be a time scale. If $s \in \mathbb{T}$, define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(s) := \inf\{t \in \mathbb{T}\colon\ t > s\}$$

and define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(s) = \sup\{t \in \mathbb{T}\colon\ t < s\},$$

where $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$ (here $\emptyset$ denotes the empty set). If $\sigma(s) > s$, $\rho(s) < s$ hold, then the point $s \in \mathbb{T}$ is called right-scattered, left-scattered, respectively. Points are called isolated when they are right-scattered and left-scattered at the same time. If $s < \sup \mathbb{T}$ and $\sigma(s) = s$, then s is called right-dense. If $s > \inf \mathbb{T}$ and $\rho(s) = s$, then s is called left-dense. Points are called dense when they are right-dense and left-dense at the same time.

Definition 3. A function $g : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous on the right-dense points of $\mathbb{T}$ and there exists the (finite) left limit on the left-dense points of $\mathbb{T}$.

Then we give the following notations throughout this paper:

- $\mathcal{C}_{rd}(\mathbb{T}) = \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R}) = \{g : \mathbb{T} \to \mathbb{R}\colon\ g$ is rd-continuous$\}$,
- $\mathcal{C}^1_{rd}(\mathbb{T}) = \mathcal{C}^1_{rd}(\mathbb{T}, \mathbb{R}) = \{g : \mathbb{T} \to \mathbb{R}\colon\ g$ is differentiable on $\mathbb{T}$ and $g^\triangle \in \mathcal{C}_{rd}(\mathbb{T})\}$,
- $\mathcal{C}^1_{0,\mathrm{rd}}(\mathrm{T}_1) = \{g \in \mathcal{C}^1_{rd}(\mathrm{T}_1, \mathbb{R})\colon\ g(a) = g(\eta) = 0\}$,
- $\mathcal{C}^1_{0,\mathrm{rd}}(\mathrm{T}_2) = \{g \in \mathcal{C}^1_{rd}(\mathrm{T}_2, \mathbb{R})\colon\ g(\eta) = g(b) = 0\}$.

We denote by $\mathcal{W}^{1,q}_\triangle(J, \mathbb{R}) = \{y \in \mathcal{L}^q_\triangle(J^0, \mathbb{R}),\ y^\triangle \in \mathcal{L}^q_\triangle(J^0, \mathbb{R})\}$ the Sobolev's space on J with Lebesgue $\triangle$-measure $\mu_\triangle$, where $1 \leqslant q < \infty$. Then denote

$$\mathcal{H}^{1,q}_\triangle(J, \mathbb{R}) = \bigl\{y \in \mathcal{W}^{1,q}_\triangle(J, \mathbb{R})\colon\ y(a) = 0,\ y(b) = \zeta y(\eta)\bigr\} \subset \mathcal{W}^{1,q}_\triangle(J, \mathbb{R}).$$

Lemma 1. (*See* [1].) *For* $q \in \mathbb{R}$, $q \geqslant 1$, *define*

$$\mathcal{V}^{1,q}_\triangle(J, \mathbb{R}) = \bigl\{y \in \mathcal{AC}(J, \mathbb{R})\colon\ y^\triangle \in \mathcal{L}^q_\triangle(J^0, \mathbb{R}),\ y(a) = 0,\ y(b) = \zeta y(\eta)\bigr\},$$

where $y \in \mathcal{AC}(\mathrm{J}, \mathbb{R})$*, that is,* y *is a absolutely continuous function. If we integrate* $y \in \mathcal{AC}(J, \mathbb{R})$ *by parts, we can get* $\mathcal{V}^{1,q}_\triangle(J, \mathbb{R}) \subset \mathcal{H}^{1,q}_\triangle(J, \mathbb{R})$*. In fact, sets* $\mathcal{V}^{1,q}_\triangle(J, \mathbb{R})$ *and* $\mathcal{H}^{1,q}_\triangle(J, \mathbb{R})$ *are equivalent for a class of functions.*

Definition 4. (See [1, Def. 3.1].) Let $q \in \mathbb{R}$, $q \geqslant 1$, and $y : T_i \to \mathbb{R}$, $i = 1, 2$. We say that $y \in \mathcal{H}^{1,q}_{\Delta}(T_i, \mathbb{R})$ if and only if $y \in \mathcal{L}^q_{\Delta}(T_i^0, \mathbb{R})$, and there exists $g : T_i \to \mathbb{R}$ such that $g \in \mathcal{L}^q_{\Delta}(T_i^0, \mathbb{R})$ and

$$\int\limits_{T_i^0} \big(y(s), \psi^{\Delta}\big)\Delta s = -\int\limits_{T_i^0} \big(g(s), \psi^{\sigma}(s)\big)\Delta s \quad \forall \psi \in \mathcal{C}^1_{0,\mathrm{rd}}(T_i),\ i = 1, 2, \tag{2}$$

where $J = T_1 \cup T_2$.

Lemma 2. *Let $g \in \mathcal{L}^1_{\Delta}(T_i^0, \mathbb{R})$, $i = 1, 2$, then a necessary and sufficient condition for*

$$\int\limits_{T_i^0} \big(g(s), \psi^{\Delta}(s)\big)\Delta s = 0 \quad \forall \psi \in \mathcal{C}^1_{0,\mathrm{rd}}(T_i),\ i = 1, 2,$$

is that there exists a constant $c_i \in \mathbb{R}$ such that $h \equiv c_i$ Δ-a.e. on T_i^0, $i = 1, 2$.

Lemma 3. (*See* [1, Lemma 3.3].) *Assume $g \in \mathcal{L}^1_{\Delta}$ such that for any $\phi \in \mathcal{C}^{\infty}_0(J^0)$, there is*

$$\int\limits_{J^0} (g \cdot \phi)(t)\Delta t = 0.$$

Then $g(t) \equiv 0$ for Δ-a.e. J^0.

Lemma 4. (*See* [1, *Thm.* 3.4].) *Assume $y \in \mathcal{H}^{1,q}_{\Delta}(J, \mathbb{R})$ with $q \in \mathbb{R}$ and $q \geqslant 1$, and* (2) *holds for $z \in \mathcal{L}^q_{\Delta}(J^0)$. Then there exists a unique function $\omega \in \mathcal{V}^{1,q}_{\Delta}(J, \mathbb{R})$ such that the equations*

$$\omega = y, \quad \omega^{\Delta} = z \qquad \Delta\text{-a.e. on } J^0$$

are satisfied.

Lemma 5. (*See* [14, *Lemma* 2.5].) *Let $\eta = k_1/k_2 \in \mathbb{Q} \cap (a, b)$ and $0 < \zeta \leqslant 1$ with $k_1, k_2 \in \mathbb{N}$, k_1, k_2 are relatively prime, and let*

$$k_2^* := \min\big\{\hat{k_2} \in \mathbb{N} \bigm| \chi(\xi + 2\hat{k}_2\pi) = \chi(\xi)\ \forall a \in \mathbb{R}\big\},$$

where $\chi(\xi) = \sin(\xi) - \zeta\sin(\eta\xi)$ and $\mathcal{K} = \{s\colon\ \chi(s) = 0,\ s \in (0, 2k_2^*\pi]\}$. Suppose that the sequence of positive solutions of $\chi(\xi) = 0$ is

$$\xi_1 < \xi_2 < \cdots < \xi_n < \cdots.$$

Then

(i) *The sequence of positive eigenvalues of system*

$$\begin{aligned} &-y''(s) = \lambda y(s), \quad s \in [a, b],\\ &y(a) = 0, \qquad y(b) = \zeta y(\eta), \end{aligned} \tag{3}$$

is exactly given by $\lambda_n = \xi_n^2$, $n = 1, 2, \dots$.

(ii) *For each* $n \in \mathbb{R}$, $\Phi_n(s) = \sin(\sqrt{\lambda_n} s)$ *is the eigenfunction of* λ_n.
(iii) *For each* $n = dl + j$ *with* $d \in \mathbb{N}$, $l \in \mathcal{K}$ *and* $j \in \{1, 2, \dots, l\}$, *there is*

$$\sqrt{\lambda_{ld+j}} = 2dk_2^* \pi + \sqrt{\lambda_j}.$$

Lemma 6. (*See* [28, *Thm.* 2.6].) *Suppose* $q \in \mathbb{R}$ *and* $q \geqslant 1$, *then* $\mathcal{H}_\Delta^{1,q}(J, \mathbb{R})$ *is a Banach space with the norm*

$$\|y\|_{\mathcal{H}_\Delta^{1,q}} = \left(\int_{J^0} \left|y(s)\right|^q \Delta s + \int_{J^0} \left|y^\Delta(s)\right|^q \Delta s \right)^{1/q} \quad \forall y \in \mathcal{H}_\Delta^{1,q}(J, \mathbb{R}).$$

Lemma 7. (*See* [1, *Prop*. 2.7].) *If* $q \in \mathbb{R}$ *and* $q \geqslant 1$, *assume* $q' \in \bar{\mathbb{R}}$ *satisfies* $1/q + 1/q' = 1$. *Then for* $g_1 \in \mathcal{L}_\Delta^q(J^0)$ *and* $g_2 \in \mathcal{L}_\Delta^{q'}(J^0)$, $g_1 \cdot g_2 \in \mathcal{L}_\Delta^1(J^0)$ *and the Hölder's inequality hold*:

$$\|g_1 \cdot g_2\|_{\mathcal{L}_\Delta^1} \leqslant \|g_1\|_{\mathcal{L}_\Delta^q} \|g_2\|_{\mathcal{L}_\Delta^{q'}}.$$

Moreover, when $q = 2$, *the inequality is called Cauchy–Schwarz's inequality.*

For $q \in \mathbb{R}$, $q \geqslant 1$, we set the space

$$\mathcal{L}_\Delta^q(J^0, \mathbb{R}) = \left\{ y : J^0 \to \mathbb{R}\colon \int_{J^0} \left|g(s)\right|^q \Delta s < +\infty \right\}$$

with the norm $\|g\|_{\mathcal{L}_\Delta^q} = (\int_{J^0} |g(s)|^q \Delta s)^{1/q}$.

Definition 5. Assume $g : \mathbb{T} \to \mathbb{R}$ is a regulated function. We define the Cauchy integral by

$$\int_r^\tau g(s)\, \Delta s = G(\tau) - G(r) \quad \forall r, \tau \in \mathbb{T}.$$

Lemma 8. (*See* [8, *Thm.* 4.1].) *If and only if* g *is* Δ*-differentiable* Δ*-a.e. on* J *and* $g^\Delta \in \mathcal{L}_\Delta^1(J^0, \mathbb{R})$ *such that*

$$g(s) = g(a) + \int_{[a,s) \cap \mathbb{T}} g^\Delta(s)\, \Delta s \quad \forall s \in J,$$

then $g : J \to \mathbb{R}$ *is the absolutely continuous function on* J.

Definition 6. Let X be a real Banach space, $\Psi \in \mathcal{C}^1(X, \mathbb{R})$. If any sequence $\{y_j\} \subset X$ such that

$$\Psi(y_j) \text{ being bounded and } \lim_{j \to \infty} \Psi'(y_j) \to 0$$

contains a convergent subsequence, then the functional Ψ is called satisfying the Palais–Smale (PS) condition.

Lemma 9. (*See* [15, *Thm.* 1.1].) *If Ψ is weakly lower semicontinuous* (*w.l.s.c.*) *on a reflexive Banach space X and has a bounded minimizing sequence, then Ψ has a minimum on X.*

Lemma 10. (*See* [17].) *Let X be a real Banach space, and let $\Psi \in \mathcal{C}^1(X,\mathbb{R})$ satisfies* (PS) *condition and the following conditions*:

(P1) *There are constants $\rho, \alpha > 0$ such that $\Psi|_{\partial B_\rho} \geqslant \alpha$.*
(P2) *There is an $r \in X \setminus \partial B_\rho$ such that $\Psi(r) \leqslant 0$.*

Then there exists a critical value $c \geqslant \alpha$, which can be characterized as

$$c = \Psi(y^*) = \inf_{h\in\Gamma} \max_{s\in[0,1]} \Psi\bigl(y(s)\bigr),$$

where

$$\Gamma = \bigl\{y \bigm| y \in \mathcal{C}\bigl([0,1], X\bigr),\ y(0) = 0,\ y(1) = r\bigr\}.$$

Lemma 11 [Fountain theorem]. (*See* [6].)

(i) *The compact group $\mathcal{G}$ acts isometrically on the Banach space $X = \overline{\oplus_{j\in\mathbb{N}} X_j}$ and is invariant*; *there exists a finite dimensional space V such that, for every $j \in \mathbb{N}$, $X_j \simeq V$*; *and the action of $\mathcal{G}$ on $\mathcal{V}$ is admissible.*

Under assumption (i), *let $\Psi \in \mathcal{C}^1(X,\mathbb{R})$ be an invariant functional and satisfies* (PS) *condition. Let for every $k \in \mathbb{N}$, there exits $\rho_k > r_k > 0$ such that*

(ii) $a_k := \max_{y\in Y_k:\ \|y\|=\rho_k} \Psi(y) \leqslant 0$ *for* $Y_k = \oplus_{j=0}^{k} X_j$;
(iii) $b_k := \inf_{y\in Z_k:\ \|y\|=r_k} \Psi(y) \to \infty$, $k \to \infty$, *for* $Z_k = \overline{\oplus_{j=k}^{\infty}} X_j$.

Then Ψ has an unbounded sequence of critical values.

3 Variational formulation of problem (1)

For this superlinear damped three-point boundary problem (1), the variational structure due to the presence of the damped term $g(s)y^\Delta(\sigma(s))$ is not apparent. However, we will be able to transform it into a variational formulation.

We assume that $\lambda > -\lambda_1 m/M$, where λ_1 is the first eigenvalue of system (3) that is the least positive parameter λ_n for system (3), and

$$m = \min_{s\in J} \mathrm{e}^{G(s)} \quad \text{and} \quad M = \max_{s\in J} \mathrm{e}^{G(s)}, \tag{4}$$

$m, M > 0$, and $G(s) = -\int_a^s g(\tau)\Delta\tau$, $s \in J$, where $g(s) \in \mathcal{L}^1_\Delta(J,\mathbb{R})$ and $G'(s) = g(s)$, thus $G(s)$ is absolutely continuous. Multiplying the equation in (1) by $\mathrm{e}^{G(s)}$, we transfer the superlinear damped system (1) into the following equivalent form:

$$\begin{aligned} &-\bigl(\mathrm{e}^{G(s)} y^\Delta(s)\bigr)^\Delta + \lambda \mathrm{e}^{G(s)} y^\sigma(s) = \mathrm{e}^{G(s)} h\bigl(\sigma(s), y^\sigma(s)\bigr), \quad \Delta\text{-a.e. } s \in J,\\ &y(a) = 0, \qquad y(b) = \zeta y(\eta). \end{aligned}$$

In this paper, consider the space $\mathcal{H}_\Delta^1 = \mathcal{H}_\Delta^{1,2}(J,\mathbb{R})$ with the inner product

$$\langle y,\omega\rangle = \int\limits_J e^{G(s)}\big(y^\Delta(s),\omega^\Delta(s)\big)\Delta s + \int\limits_J \lambda \mathrm{e}^{G(s)}\big(y^\sigma(s),\omega^\sigma(s)\big)\Delta s$$

and the induced norm

$$\|y\| = \biggl(\int\limits_J \mathrm{e}^{G(s)}\big|y^\Delta(s)\big|^2\Delta s + \int\limits_J \lambda \mathrm{e}^{G(s)}\big|y^\sigma(s)\big|^2\Delta s\biggr)^{1/2}.$$

Define the functional in $\mathcal{H}_\Delta^1$:

$$\begin{aligned}\Psi(y) = {}& \frac{1}{2}\int\limits_J \mathrm{e}^{G(s)}\big|y^\Delta(s)\big|^2\Delta s + \frac{1}{2}\int\limits_J \lambda \mathrm{e}^{G(s)}\big|y^\sigma(s)\big|^2\Delta s\\ &- \int\limits_J \mathrm{e}^{G(s)} H\big(\sigma(s),y^\sigma(s)\big)\Delta s,\end{aligned} \tag{5}$$

and let

$$A(y,\omega) = \int\limits_J \mathrm{e}^{G(s)} y^\Delta(s)\omega^\Delta(s)\Delta s + \lambda\int\limits_J \mathrm{e}^{G(s)} y^\sigma(s)\omega^\sigma(s)\,\Delta s, \quad y,\omega\in\mathcal{H}_\Delta^1.$$

Then the derivative is

$$\big\langle\Psi'(y),\omega\big\rangle = A(y,\omega) - \int\limits_J \mathrm{e}^{G(s)} h\big(\sigma(s),y^\sigma(s)\big)\omega^\sigma(s)\,\Delta s. \tag{6}$$

Lemma 12. *If assumption* (A3) *are satisfied, then for any $s\in J$, $\mathcal{I}(s,y)$ is increasing for $y\geqslant\theta$ and decreasing for $y\leqslant-\theta$, where $\mathcal{I}(s,y) = h(s,y)y - \mu H(s,y)$. In particular, there exists $c_1>0$ such that*

$$\mathcal{I}(s,\omega)\leqslant\mathcal{I}(s,y)+c_1 \tag{7}$$

for $s\in J$ and $0\leqslant\omega\leqslant y$ or $y\leqslant\omega\leqslant 0$.

Proof. Assume $\theta\leqslant\omega\leqslant y$, there is

$$\begin{aligned}&\mathcal{I}(s,y)-\mathcal{I}(s,y)\\ &\quad=\mu\biggl[\frac{1}{\mu}\big(h(s,y)y-h(s,\omega)\omega\big)-\big(H(s,y)-H(s,\omega)\big)\biggr]\\ &\quad=\mu\biggl[\int\limits_\theta^y\frac{h(s,y)y}{y^\mu}\tau^{\mu-1}\Delta\tau-\int\limits_\theta^\omega\frac{h(s,\omega)\omega}{\omega^\mu}\tau^{\mu-1}\Delta\tau-\int\limits_\omega^y\frac{h(s,\tau)}{\tau^{\mu-1}}\tau^{\mu-1}\Delta\tau\\ &\qquad+\frac{h(s,y)}{\mu y^\mu}\theta^\mu-\frac{h(s,\omega)}{\mu\omega^\mu}\theta^\mu\biggr]\end{aligned}$$

$$= \mu\left[\int_{\omega}^{y}\left(\frac{h(s,y)y}{y^{\mu}} - \frac{h(s,\tau)\tau}{\tau^{\mu}}\right)\tau^{\mu-1}\Delta\tau + \int_{\theta}^{\omega}\left(\frac{h(s,y)y}{y^{\mu}} - \frac{h(s,\omega)\omega}{\omega^{\mu}}\right)\tau^{\mu-1}\Delta\tau\right]$$
$$+ \theta^{\mu}\left(\frac{h(s,y)}{y^{\mu}} - \frac{h(s,\omega)}{\omega^{\mu}}\right)$$
$$\geqslant 0.$$

The case $y \leqslant \omega \leqslant -\theta$ is similar. Finally, by assumption (A) we see that

$$c_1 = 1 + \sum_{(s,y)\in J\times[-\theta,\theta]} \mathcal{I}(s,y) - \inf_{(s,y)\in J\times[-\theta,\theta]} \mathcal{I}(s,y)$$

is finite. The proof is completed. □

Remark 1. Condition (A1) when $\mu = 2$, is originally from the famous (AR) condition. This condition ensures the compactness of (PS) condition. However, some examples show that the nonlinearity $h(s,y)$ could not satisfy (AR) condition such as $h(s,y) = 2y^2\log(1+|y|)$. But it satisfies conditions (A1)–(A3), and we say that Ψ satisfies the Cerami condition if any sequence $\{y_n\} \in \mathcal{H}^1_\Delta$ such that

$$\Psi(y_n) \to c, \quad \text{and} \quad \bigl(1 + \|y_n\|\bigr)\|\Psi'(y_n)\| \to 0$$

has a convergent subsequence.

Actually, the deformation theorem under the Cerami condition has been given in [13].

Lemma 13. (*See* [13, *Prop.* 2.0].) *Let Ψ be a functional of class $\mathcal{C}^1(X,\mathbb{R})$ defined on a real Banach space X. Denote $\Psi^c = \{y \in X\colon \Psi(y) \leqslant c\}$, $\Psi_c = \{y \in X\colon \Psi(y) \geqslant c\}$, $\mathcal{B}_\alpha = \{y \in X\colon \|y\| \leqslant \alpha\}$, $T_c = \{y \in X\colon \Psi(y) = c,\ \Psi'(y) = 0\}$, $P_r = \{y \in X\colon \|y - T_c\| < r\}$, $0 < r \leqslant 2$. Let $\varepsilon, \delta_1 > 0$, $c \in \mathbb{R}$ be such that $(1 + \|y\|)\|\Psi'(y)\| \geqslant \delta_1$, $y \in \Psi^{-1}([c - 2\varepsilon], c + 2\varepsilon) \setminus P_{r/8}$. Then there exits $\eta \in \mathcal{C}([0,1] \times X, X)$ such that*

(i) $\eta(0,y) = y$ *for all* $y \in X$;
(ii) $\Psi(\eta(\cdot,y))$ *is nonincreasing for all* $y \in X$;
(iii) $\Psi(\eta(s,y)) < c$ *for all* $s \in (0,1], y \in \Psi^c \setminus P_r$;
(iv) $\eta(1, (\Psi_{c-\varepsilon}^{c+\varepsilon} \setminus P_r) \cap \mathcal{B}_\alpha) \subset \Psi^{c-\varepsilon}$, $\alpha > 0$;
(v) $\|\eta(s,y) - y\| \leqslant r/2$ *for all* $s \in [0,1], y \in X$.

Here $\|y - T_c\|$ denote the distance form y to T_c.

Remark 2. Since the deformation theorem is still valid under the Cerami condition, we see that the mountain pass theorem and the fountain theorem are true under the Cerami condition, more details can be found in [11, 18].

Lemma 14. *There exists a constant $\delta > 0$ satisfying the inequality*

$$\|y\|_\infty \leqslant \delta\|y\| \quad \forall y \in \mathcal{H}^1_\Delta,$$

where $\|y\|_\infty = \max_{s\in J}|y(s)|$, *and the norm* $\|\cdot\|$ *is equivalent to* $\|\cdot\|_1$, *where*

$$\|y\|_1 = \left(\int_J e^{G(s)}\left|y^\Delta(s)\right|^2\Delta s\right)^{1/2}.$$

Proof. First, define the norm

$$\|y\|_2 = \left(\int_J \left|y^\Delta(s)\right|^2\Delta s + \int_J \left|y^\sigma(s)\right|^2\Delta s\right)^{1/2}.$$

By (4) there is $m \leqslant e^{G(s)} \leqslant M$. Then by Poincaré's inequality and $\lambda > -(\lambda_1 m)/M$ we can get that the norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent, and

$$\left|y(s)\right|^2 \leqslant s \int_{[a,s)_\mathbb{T}} \left|y^\Delta(\tau)\right|^2\Delta\tau.$$

After integrating the above formula, we get

$$\int_J \left|y(s)\right|^2\Delta s \leqslant \frac{b^2-a^2}{2}\int_J \left|y^\Delta(\tau)\right|^2\Delta\tau.$$

Thus the norm $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$. Then the norm $\|\cdot\|$ is equivalent to $\|\cdot\|_1$.

Besides, there is constant $c>0$ such that

$$\begin{aligned}\left|y(s)\right| &\leqslant \int_J |y^\Delta(\tau)|\Delta\tau \leqslant \left(\int_J \frac{1}{e^{G(\tau)}}\Delta\tau\right)^{1/2}\left(\int_J e^{G(\tau)}\left|y^\Delta(\tau)\right|^2\Delta\tau\right)^{1/2}\\ &\leqslant \sqrt{\frac{|b-a|}{m}}\|y\|_1 \leqslant \sqrt{\frac{c|b-a|}{m}}\|y\|.\end{aligned}$$

We can assume that $\delta = \sqrt{(c|b-a|)/m} > 0$, then the proof is completed. □

Lemma 15. *Let condition* (A) *holds, and let* $\lambda > -\lambda_1 m/M$. *Then the functional* $\Psi : \mathcal{H}^1_\Delta \to \mathbb{R}$ *defined by* (5) *is continuously differentiable, and the derivative defined by* (6) *can be obtained.*

Proof. For every point $y \in \mathcal{H}^1_\Delta$, it suffices to prove that Ψ has a directional derivative $\Psi'(y) \in (\mathcal{H}^{1,2}_\Delta(J,\mathbb{R}))^*$ given by (6) and that the mapping $\Psi' : \mathcal{H}^{1,2}_\Delta(J,\mathbb{R}) \to (\mathcal{H}^{1,2}_\Delta(J,\mathbb{R}))^*$ is continuous.

(i) It follows from condition (A) that Ψ is everywhere finite on $\mathcal{H}^{1,2}_\Delta(J,\mathbb{R})$. Then, for y and ω fixed in $\mathcal{H}^1_\Delta$, $s \in J$, $\xi \in [-1,1]$, let

$$\begin{aligned}L\big(\sigma(s), y^\sigma(s), y^\Delta(s)\big) = {}&\frac{1}{2}e^{G(s)}\left|y^\Delta(s)\right|^2 + \frac{1}{2}\lambda e^{G(s)}\left|y^\sigma(s)\right|^2\\ &- e^{G(s)}H\big(\sigma(s), y^\sigma(s)\big),\end{aligned}$$

$$\mathcal{Q}(\xi, s) = L\big(\sigma(s), y^\sigma(s) + \xi\omega^\sigma(s),\, y^\Delta(s) + \xi\omega^\Delta(s)\big),$$

and

$$\psi(\xi) = \int_J \mathcal{Q}(\xi, s)\,\Delta s = \Psi(y + \xi\omega).$$

Then by condition (A) there is

$$\begin{aligned}
\big|D_\xi\mathcal{Q}(\xi, s)\big| &= \big|\big(D_y L\big(\sigma(s),\, y^\sigma(s) + \xi\omega^\sigma(s),\, y^\Delta(s) + \xi\omega^\Delta(s)\big),\, \omega^\sigma(s)\big) \\
&\quad + \big(D_\omega L(\sigma(s),\, y^\sigma(s) + \xi\omega^\sigma(s),\, y^\Delta(s) + \xi\omega^\Delta(s)\big),\, \omega^\Delta(s)\big)\big| \\
&= \big|\big(-e^{G(s)} h\big(\sigma(s),\, y^\sigma(s) + \xi\omega^\sigma(s)\big),\, \omega^\sigma(s)\big) \\
&\quad + \big(\lambda e^{G(s)}\big|y^\sigma(s) + \xi\omega^\sigma(s)\big|,\, \omega^\sigma(s)\big) \\
&\quad + e^{G(s)}\big(\big|y^\Delta(s) + \xi\omega^\Delta(s)\big|,\, \omega^\Delta(s)\big)\big| \\
&\leqslant M a\big(\big|y^\sigma(s) + \xi\omega^\sigma(s)\big|\big) b^\sigma(s)\big|\omega^\sigma(s)\big| + \lambda M\big|y^\sigma(s) + \xi\omega^\sigma(s)\big|\big|\omega^\sigma(s)\big| \\
&\quad + M\big|y^\Delta(s) + \xi\omega^\Delta(s)\big|\big|\omega^\Delta(s)\big| \\
&\leqslant M\bar{a} b^\sigma(s)\big|\omega^\sigma(s)\big| + \lambda M\big(\big|y^\sigma(s)\big| + \big|\omega^\sigma(s)\big|\big)\big|\omega^\sigma(s)\big| \\
&\qquad + M\big|y^\Delta(s) + \omega^\Delta(s)\big|\big|\omega^\Delta(s)\big| \\
&\triangleq d(s),
\end{aligned}$$

where $\bar{a} = \max_{(\lambda,s)\in[-1,1]\times J} a(|y^\sigma(s) + \lambda\omega^\sigma(s)|)$, $b \in \mathcal{L}^1_\Delta(J, \mathbb{R}^+)$, $(|y^\Delta| + |\omega^\Delta|) \in \mathcal{L}^2_\Delta(J, \mathbb{R})$, and $(|y^\sigma(s)| + |\omega^\sigma(s)|) \in \mathcal{L}^2_\Delta(J, \mathbb{R})$. Then we have $|D_\xi\mathcal{Q}(\xi, s)| \leqslant d(s) \in \mathcal{L}^1_\Delta(J, \mathbb{R})$. Thus applying the Leibniz formula, we have

$$\begin{aligned}
\psi'(0) &= \int_J D_\xi\mathcal{Q}(0, s)\,\Delta s \\
&= \int_J \big[\big(D_u L\big(\sigma(s), y^\sigma(s), y^\Delta(s)\big), \omega^\sigma(s)\big) + \big(D_v L\big(\sigma(s), y^\sigma(s), y^\Delta(s)\big), \omega^\Delta(s)\big)\big]\Delta s \\
&= \int_J e^{G(s)}\big[\big(-h\big(\sigma(s), y^\sigma(s)\big), \omega^\sigma(s)\big) + \lambda\big(y^\sigma(s), \omega^\sigma(s)\big) + \big(y^\Delta(s), \omega^\Delta(s)\big)\big]\Delta s.
\end{aligned}$$

Moreover, let $a(|y^\sigma(s)|)b^\sigma(s) + |y^\sigma(s)| \triangleq \psi_1(s)$ and $|y^\Delta(s)| \triangleq \psi_2(s)$, thus $\psi_1 \in \mathcal{L}^1_\Delta(J, \mathbb{R}^+)$, $\psi_2 \in \mathcal{L}^2_\Delta(J, \mathbb{R}^+)$, then by Lemma 14

$$\begin{aligned}
&\int_J \big[\big(D_u L\big(\sigma(s), y^\sigma(s), y^\Delta(s)\big), \omega^\sigma(s)\big) + \big(D_v L\big(\sigma(s), y^\sigma(s), y^\Delta(s)\big), \omega^\Delta(s)\big)\big]\Delta s \\
&\quad = \int_J e^{G(s)}\big[\big(-h\big(\sigma(s), y^\sigma(s)\big),\, \omega^\sigma(s)\big) + \lambda\big(y^\sigma(s), \omega^\sigma(s)\big) + \big(y^\Delta(s), \omega^\Delta(s)\big)\big]\Delta s \\
&\leqslant M\int_J \big(a\big(\big|y^\sigma(s)\big|\big)b^\sigma(s) + \lambda y^\sigma(s),\, \omega^\sigma(s)\big)\Delta s + M\int_J \big(y^\Delta(s), \omega^\Delta(s)\big)\Delta s \\
&\quad \leqslant c_1\|\omega\|_\infty + c_2\|\omega^\Delta\| \leqslant c_3\|\omega\|,
\end{aligned}$$

where c_1, c_2, c_3 are positive constants, and Ψ has a directional derivative at y and $\Psi'(y) \in (\mathcal{H}^1_\triangle)^*$ given by (6).

(ii) By a theorem of Krasnosel'skii, conditions (A) and (A3) imply the mapping from $\mathcal{H}^{1,2}_\triangle(J,\mathbb{R})$ into $\mathcal{L}^1_\triangle(J,\mathbb{R}) \times \mathcal{L}^2_\triangle(J,\mathbb{R})$ defined by

$$y \to \big(D_y L\big(\cdot, y^\sigma, y^\triangle\big), D_\omega L\big(\cdot, y^\sigma, y^\triangle\big)\big)$$

is continuous. So Ψ' is continuous from $\mathcal{H}^{1,2}_\triangle(J,\mathbb{R})$ into $(\mathcal{H}^{1,2}_\triangle(J,\mathbb{R}))^*$, and the proof is completed. □

Lemma 16. *If $y \in \mathcal{H}^{1,2}_\triangle(J,\mathbb{R})$ is a critical point of the functional Ψ, then $y = y(s)$ is a solution of problem* (1).

Proof. By Lemma 15 the functional Ψ is continuously differentiable, and the assumption that y is a critical point of Ψ means that $\langle \Psi'(y), \omega\rangle = 0$ for all $\omega \in \mathcal{H}^1_\triangle$. Obviously, there is $\mathcal{C}^1_{0,\mathrm{rd}}(T_i,\mathbb{R}) \subset \mathcal{H}^{1,2}_\triangle(J,\mathbb{R})$, $i = 1, 2$, then we have

$$\big\langle \Psi'(y), z\big\rangle = 0 \quad \forall z \in \mathcal{C}^1_{0,\mathrm{rd}}(T_i,\mathbb{R}),\ i = 1, 2.$$

Then by Definition 4 there is

$$\begin{aligned} 0 &= \int\limits_{T_i} \mathrm{e}^{G(s)} y^\triangle(s) z^\triangle(s)\,\Delta s + \lambda \int\limits_{T_i} \mathrm{e}^{G(s)} y^\sigma(s) z^\sigma(s)\,\Delta s \\ &\quad - \int\limits_{T_i} \mathrm{e}^{G(s)} \big(h(\sigma(s), y^\sigma(s)), z^\sigma(s)\big)\Delta s \\ &= -\int\limits_{T_i} \big(\mathrm{e}^{G(s)} y^\triangle(s)\big)^\triangle z^\sigma(s)\Delta s + \lambda \int\limits_{T_i} \mathrm{e}^{G(s)} y^\sigma(s) z^\sigma(s)\,\Delta s \\ &\quad - \int\limits_{T_i} \mathrm{e}^{G(s)} \big(h\big(\sigma(s), y^\sigma(s)\big), z^\sigma(s)\big)\Delta s. \end{aligned}$$

Since $z \in \mathcal{C}^1_{0,\mathrm{rd}}(T_i,\mathbb{R})$ is arbitrary and by Lemma 3 we can deduce

$$-\big(\mathrm{e}^{G(s)} y^\triangle(s)\big)^\triangle + \lambda \mathrm{e}^{G(s)} y^\sigma(s) = \mathrm{e}^{G(s)} h\big(\sigma(s), y^\sigma(s)\big), \quad \triangle\text{-a.e. } s \in T_i,\ i = 1, 2,$$

then it holds that

$$-\big(\mathrm{e}^{G(s)} y^\triangle(s)\big)^\triangle + \lambda \mathrm{e}^{G(s)} y^\sigma(s) = \mathrm{e}^{G(s)} h\big(\sigma(s), y^\sigma(s)\big), \quad \triangle\text{-a.e. } s \in J,$$

that is,

$$-y^{\triangle\triangle}(s) + g(s) y^\triangle\big(\sigma(s)\big) + \lambda y^\sigma(s) = h\big(\sigma(s), y^\sigma(s)\big), \quad \triangle\text{-a.e. } s \in J.$$

Thus $y = y(s)$ is a solution of problem (1). □

4 Main results

Theorem 1. *Assume condition* (A) *holds, and* $\lambda > -\lambda_1 m/M$, *then problem* (1) *has at least one solution.*

Proof. By the functional Ψ defined in (5) let

$$\Psi_1(y) = \frac{1}{2}\int_J \mathrm{e}^{G(s)}\left|y^\Delta(s)\right|^2\Delta s + \frac{1}{2}\int_J \lambda \mathrm{e}^{G(s)}\left|y^\sigma(s)\right|^2\Delta s.$$

Since Ψ_1 is continuous and convex, Ψ_1 is weakly lower semicontinuous, and

$$\Psi_2(y) = \int_J \mathrm{e}^{G(s)} H\big(\sigma(s), y^\sigma(s)\big)\Delta s.$$

By condition (A), Ψ_2 is a weakly continuous functional. Thus, Ψ is weakly lower semicontinuous.

For any $y \in \mathcal{H}^1_\Delta$, there exists a enough large constant $c > 0$ such that

$$\begin{aligned}\Psi(y) &= \frac{1}{2}\int_J \mathrm{e}^{G(s)}\left|y^\Delta(s)\right|^2\Delta s + \frac{1}{2}\int_J \lambda \mathrm{e}^{G(s)}\left|y^\sigma(s)\right|^2\Delta s - \int_J \mathrm{e}^{G(s)} H\big(\sigma(s), y^\sigma(s)\big)\Delta s \\ &\geqslant \frac{1}{2}\|y\|^2 - M\int_J a\big(|y|\big)b(s)\,\Delta s \geqslant \frac{1}{2}\|y\|^2 - Mc.\end{aligned}$$

This implies that $\lim_{\|y\|\to\infty}\Psi(y) = +\infty$, that is, Ψ is coercive. Hence Ψ has minimum by Lemma 9, which is also the critical point of Ψ. Therefore, problem (1) has at last one solution. $\square$

Theorem 2. *Assume conditions* (A), (A1), (A2), *and* (A3) *hold, and let* $\lambda > -\lambda_1 m/M$. *Then the functional* Ψ *satisfies the Cerami condition.*

Proof. First, suppose a subsequence $\{y_i\}$ of sequence $\{y_n\}$ such that $y_i \rightharpoonup y$ in $\mathcal{H}^1_\Delta$, then $y_i \to y$ in $\mathcal{C}(J,\mathbb{R})$. Thus when $i \to \infty$, there are

$$\big\langle \Psi'(y_i) - \Psi'(y),\, y_i - y\big\rangle \to 0,$$

$$\int_J \big[h(s,y_i) - h(s,y)\big](y_i - y)\,\Delta s \to 0.$$

Thus

$$\begin{aligned}&\big\langle \Psi'(y_i) - \Psi'(y),\, y_i - y\big\rangle \\ &\quad= \int_J \mathrm{e}^{G(s)}\big(\left|y_i^\Delta(s) - y^\Delta(s)\right|^2 + \lambda\left|y_i^\sigma(s) - y^\sigma(s)\right|^2\big)\Delta s \\ &\qquad- \int_J \mathrm{e}^{G(s)}\big[h\big(\sigma(s),\, y_i^\sigma(s)\big) - h\big(\sigma(s), y^\sigma(s)\big)\big]\big(y_i^\sigma(s) - y^\sigma(s)\big)\Delta s\end{aligned}$$

$$\geqslant m\int_J \left(\left|y_i^{\Delta}(s)-y^{\Delta}(s)\right|^2+\lambda\left|y_i^{\sigma}(s)-y^{\sigma}(s)\right|^2\right)\Delta s$$
$$-M\int_J \left[h\left(\sigma(s), y_i^{\sigma}(s)\right)-h\left(\sigma(s),y^{\sigma}(s)\right)\right]\left(y_i^{\sigma}(s)-y^{\sigma}(s)\right)\Delta s.$$

By $\Psi'(y_i)\to 0$ in $\mathcal{H}_{\Delta}^1$ we have $\|y_i-y\|\to 0$ as $i\to\infty$. That is, sequence $\{y_n\}$ has a convergent subsequence.

Second, we show that the sequence $\{y_n\}\subset\mathcal{H}_{\Delta}^1$ is bounded. If $\{y_n\}$ is unbounded, then for some $c\in\mathbb{R}$, we have

$$\Psi(y_n)\to c,\quad \|y_n\|\to\infty,\quad \left\|\Psi'(y_n)\right\|\cdot\|y_n\|\to 0\quad\text{as } n\to\infty,$$

then

$$\lim_{n\to\infty}\int_J\left(\frac{1}{2}h(s,y_n)y_n-H(s,y_n)\right)\Delta s=\lim_{n\to\infty}\left\{\Psi(y_n)-\frac{1}{2}\left\langle\Psi'(y_n),y_n\right\rangle\right\}$$
$$=c. \tag{8}$$

We consider $v_n:=y_n/\|y_n\|$, then up to the subsequence $\{v_n\}$ in $\mathcal{H}_{\Delta}^1$, we get

$$v_n\rightharpoonup v\quad\text{in } \mathcal{H}_{\Delta}^1,\qquad v_n\to v\quad\text{in } \mathcal{L}^{\varsigma}(J)\text{ for } 2\leqslant\varsigma<\infty,$$
$$v_n(s)\to v(s),\quad \Delta\text{-a.e. } s\in J.$$

Case I. If $v=0$, define a sequence $\{t_n\}\subset[0,1]$ such that

$$\Psi(t_ny_n):=\max_{t\in[0,1]}\Psi(ty_n).$$

For any $m>0$, define $\bar{v}_n:=(4k)^{1/2}v_n$. There is $\bar{v}_n\to 0$ in $\mathcal{L}^2(J)$, and by condition (A2) there exists $D>0$ such that $|H(s,y)|\leqslant D(|y|+|y|^p)$. We see $H(\cdot,\bar{v}_n)\to 0$ in $\mathcal{L}^1(J)$. Thus $\lim_{n\to\infty}\int_J H(s,\bar{v}_n)\,\Delta s=0$. So for n large enough,

$$\Psi(t_ny_n)\geqslant\Psi(\bar{v}_n)=2k-\int_J H(s,\bar{v}_n)\,\Delta s\geqslant k,$$

which implies that $\lim_{n\to\infty}\Psi(t_ny_n)=+\infty$. Then by $\Psi(0)=0$ and $\Psi(y_n)\to c$ get $t_n\in(0,1)$. Thus, if n big enough, then

$$\int_J \mathrm{e}^{G(s)}\left(\left|(t_ny_n)^{\Delta}(s)\right|^2+\lambda\left|t_ny_n^{\sigma}(s)\right|^2\right)\Delta s-\int_J \mathrm{e}^{G(s)}h\left(\sigma(s),t_ny_n^{\sigma}(s)\right)\left(t_ny_n^{\sigma}(s)\right)\Delta s$$
$$=\left\langle\Psi'(t_ny_n),t_ny_n\right\rangle=t_n\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_n}\Psi(ty_n)=0,$$

and, obviously, there is

$$\begin{aligned}
&\int_J e^{G(s)}\left(\frac{1}{2}h\bigl(\sigma(s), t_n y_n^\sigma(s)\bigr)\bigl(t_n y_n^\sigma(s)\bigr) - H\bigl(\sigma(s), t_n y_n^\sigma(s)\bigr)\right)\Delta s \\
&\quad = \frac{1}{2}\int_J e^{G(s)}\bigl(\bigl|(t_n y_n)^\Delta(s)\bigr|^2 + \lambda\bigl|t_n y_n^\sigma(s)\bigr|^2\bigr)\Delta s \\
&\qquad - \int_J e^{G(s)} H\bigl(\sigma(s), t_n y_n^\sigma(s)\bigr)\bigl(t_n y_n^\sigma(s)\bigr)\Delta s \\
&= \Psi(t_n y_n) \to +\infty \quad \text{as } n \to +\infty.
\end{aligned}$$

By (7) we have

$$\mathcal{I}(s, y_n) \geqslant \mathcal{I}(s, t_n y_n) - c_1,$$

then

$$\begin{aligned}
&\int_J \left(\frac{1}{2}h\bigl(s, y_n^\sigma(s)\bigr)y_n^\sigma(s) - H\bigl(s, y_n^\sigma(s)\bigr)\right)\Delta s \\
&\quad \geqslant \int_J \left(\frac{1}{2}h\bigl(\sigma(s), t_n y_n^\sigma(s)\bigr)\bigl(t_n y_n^\sigma(s)\bigr) - H\bigl(\sigma(s), t_n y_n^\sigma(s)\bigr)\right)\Delta s - \frac{c_1}{2}|b-a| \\
&\quad = \Psi(t_n y_n) - \frac{c_1}{2}|b-a| \to \infty \quad \text{as } n \to +\infty,
\end{aligned}$$

which contradicts with (8).

Case II. If $v \neq 0$, we have that

$$\begin{aligned}
&\int_J e^{G(s)}\bigl(|y_n^\Delta(s)|^2 + \lambda|y_n^\sigma(s)|^2\bigr)\Delta s - \int_J e^{G(s)} h\bigl(\sigma(s), y_n^\sigma(s)\bigr)y_n^\sigma(s)\,\Delta s \\
&\quad = \bigl\langle \Psi'(y_n), y_n \bigr\rangle, = o(1)
\end{aligned}$$

then there is

$$\begin{aligned}
1 - o(1) &= \int_J \frac{h(\sigma(s), y_n^\sigma(s))y_n^\sigma(s)}{\|y_n\|^2}\,\Delta s \\
&= \left(\int_{v=0} + \int_{v\neq 0}\right)\frac{h(\sigma(s), y_n^\sigma(s))y_n^\sigma(s)}{|y_n|^2}|v_n|^2\Delta s.
\end{aligned}$$

For $t \in \Theta := \{t \in J\colon\ v(t) \neq 0\}$, we have $|y_n(t)| \to \infty$. Then by condition (A1) there is

$$\int_{v\neq 0} \frac{h(\sigma(s), y_n^\sigma(s))y_n^\sigma(s)}{|y_n|^2}|v_n|^2\Delta s \to +\infty \quad \text{as } n \to \infty,$$

and there exists $\vartheta > -\infty$ such that

$$\frac{h(\sigma(s), y_n^\sigma(s))y_n^\sigma(s)}{|y_n|^2} \geqslant \vartheta \quad \text{for } \triangle\text{-a.e. } s \in J.$$

Then we have

$$\begin{aligned} 1 - o(1) &= \left(\int\limits_{v=0} + \int\limits_{v\neq 0}\right) \frac{h(\sigma(s), y_n^\sigma(s))y_n^\sigma(s)}{|y_n|^2} |v_n|^2 \Delta s \\ &\geqslant \int\limits_{v\neq 0} \frac{h(\sigma(s), y_n^\sigma(s))y_n^\sigma(s)}{|y_n|^2} |v_n|^2 \Delta s + \vartheta \int\limits_{v=0} |v_n|^2 \Delta s, \end{aligned}$$

which is a contradiction.

Therefore $\|y_n\|$ is bounded, and Ψ satisfies Cerami condition. □

Remark 3. The technology we used in Theorem 2 to eliminate the case $v \neq 0$, is derived from Jeanjean [10]. We proved that although there may be unbounded (PS) sequence, every Cerami sequence of the functional Ψ is bounded. To prove the boundedness of Cerami sequence, we refer to Zou [29].

Theorem 3. *Assume $\lambda > -\lambda_1 m/M$, conditions* (A), (A1)–(A3)*, and*

(A′) *$h(s, y) : J \times \mathbb{R} \to \mathbb{R}$ satisfies $h(s, y) = o(|y|)$ as $y \to 0$ uniformly for $\triangle$-a.e. $s \in J$*

hold. Then problem (1) *has a nontrivial solution.*

Proof. By Theorem 2, Ψ satisfies Cerami condition. Given $\varepsilon_1 = 1/(4M\delta^2(b-a)) > 0$, where $M, \delta > 0$ and $b > a$, then by condition (A′) there exists $r > 0$ such that $H(s, y) \leqslant \varepsilon_1 |y|^2$ for all $|y| \leqslant r$, and by Lemma 14 there is

$$\begin{aligned} \Psi(y) &= \frac{1}{2} \int\limits_J \mathrm{e}^{G(s)} \left|y^\triangle(s)\right|^2 \Delta s + \frac{1}{2} \int\limits_J \lambda \mathrm{e}^{G(s)} \left|y^\sigma(s)\right|^2 \Delta s \\ &\quad - \int\limits_J \mathrm{e}^{G(s)} H\bigl(\sigma(s), y^\sigma(s)\bigr) \Delta s \\ &\geqslant \frac{1}{2}\|y\|^2 - M \int\limits_J \varepsilon_1 |y|^2 \Delta s \geqslant \frac{1}{2}\|y\|^2 - M \cdot \varepsilon_1 (b-a) \|y\|_\infty^2 \\ &\geqslant \frac{1}{2}\|y\|^2 - M \cdot \varepsilon_1 \cdot \delta^2 (b-a) \|y\|^2 = \left(\frac{1}{2} - M\varepsilon_1 \delta^2 (b-a)\right) \|y\|^2 \\ &= \frac{1}{4M\delta^2(b-a)} r^2. \end{aligned}$$

Taking into account Lemma 10 and setting $\rho = r$, $\alpha = 1/(4M\delta^2(b-a))r^2 > 0$, one has that (P1) holds.

On the other hand, by condition (A1), for any $\varepsilon_2 > 0$, there exists $C_{\varepsilon_2} > 0$ such that $H(s,y) \geqslant C_{\varepsilon_2}(|y|^\mu + 1)$ for $|y| \geqslant r$. Because in finite-dimensional normed spaces, all norms are equivalent, and $2 \leqslant \mu < \infty$. Let C_{ε_2} satisfies

$$\frac{1}{2}\|y\|^2 \leqslant mC_{\varepsilon_2}\|y\|^\mu_{\mathcal{L}^\mu}.$$

Then

$$\begin{aligned}\Psi(y) &= \frac{1}{2}\int\limits_J \mathrm{e}^{G(s)}\left|y^\Delta(s)\right|^2\Delta s + \frac{1}{2}\int\limits_J \lambda\mathrm{e}^{G(s)}\left|y^\sigma(s)\right|^2\Delta s \\ &\quad - \int\limits_J \mathrm{e}^{G(s)}H\bigl(\sigma(s), y^\sigma(s)\bigr)\Delta s \\ &\leqslant \frac{1}{2}\|y\|^2 - mC_{\varepsilon_2}\int\limits_J \bigl(|y|^\mu + 1\bigr)\Delta s \\ &= \frac{1}{2}\|y\|^2 - mC_{\varepsilon_2}\cdot\|y\|^\mu_{\mathcal{L}^\mu} - mC_{\varepsilon_2}|b-a|\end{aligned}$$

by $\mu \geqslant 2$, and $\Psi(y) \to -\infty$ as $\|y\| \to \infty$, hence (P2) holds. It is clear that $\Psi(0) = 0$, then by Lemma 10 the proof is completed. □

Theorem 4. *Assume $\lambda > -\lambda_1 m/M$, conditions* (A), (A1)–(A3)*, and the following condition are satisfied:*

(A″) $h(s,-y) = -h(s,y)$ *for Δ-a.e. $s \in J$ and $y \in \mathbb{R}$.*

Then problem (1) *has an unbounded sequence $\{y_k\}$ as solutions satisfying*

$$\begin{aligned}&\frac{1}{2}\int\limits_J \mathrm{e}^{G(s)}\left|y_k^\Delta(s)\right|^2\Delta s + \frac{1}{2}\int\limits_J \lambda\mathrm{e}^{G(s)}\left|y_k^\sigma(s)\right|^2\Delta s \\ &\quad - \int\limits_J \mathrm{e}^{G(s)}H\bigl(\sigma(s), y_k^\sigma(s)\bigr)\Delta s \to \infty \quad \textit{as } k \to \infty.\end{aligned}$$

Proof. We know that $\mathcal{X} = \mathcal{H}^1_\Delta$ is a reflexive and separable space, then by [27] there exits $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$ and $\{\Psi_n\}_{n\in\mathbb{N}} \subset \mathcal{X}^*$ such that

$$\langle\Psi_n, \tau_m\rangle = \begin{cases}1, & n = m;\\ 0, & n \neq m,\end{cases} \quad \text{and} \quad \overline{\operatorname{span}}\{\tau_n,\, n \in \mathbb{N}\} = \mathcal{X}.$$

We choose to define $X_j := \mathbb{R}\tau_j$ as $j \in \mathbb{N}$, and let

$$\mathcal{X} = \overline{\oplus_{j\in\mathbb{N}}X_j}, \quad Y_k = \oplus_{j=0}^k X_j, \quad \text{and} \quad Z_k = \overline{\oplus_{j=k}^\infty}X_j.$$

By assumption (A″) there is $\Psi(-y) = \Psi(y)$, $y \in \mathcal{H}^1_\Delta$, and Ψ satisfies the Cerami condition by Theorem 2. On the one hand, by condition (A2), for any $(s,y) \in J \times \mathbb{R}$, there exists $D>0$ such that $|H(s,y)|\leqslant D(|y|+|y|^p)$. Let $\beta_k:=\sup_{y\in Z_k:\ \|y\|=1}\|y\|_{\mathcal{L}^p}$, then $\beta_k\to 0$ as $k\to\infty$ (cf. [25]). Therefore, for $y\in Z_k$,

$$\begin{aligned}\Psi(y)&=\frac{1}{2}\|y\|^2-\int\limits_J \mathrm{e}^{G(s)}H\bigl(\sigma(s),y^\sigma(s)\bigr)\Delta s\\&\geqslant\frac{1}{2}\|y\|^2-M\int\limits_J D\bigl(|y|+|y|^p\bigr)\Delta s\\&\geqslant\frac{1}{2}\|y\|^2-DM\|y\|_\infty-DM\|y\|^p_{\mathcal{L}^p}\\&\geqslant\frac{1}{2}\|y\|^2-DM\delta\|y\|-D\beta_k^pM\|y\|^p,\end{aligned}$$

then for $y\in Z_k$ with $\|y\|=r_k:=(\beta_k)^{-1}$,

$$\Psi(y)\geqslant\frac{(\beta_k)^{-2}}{2}-DM\delta(\beta_k)^{-1}-DM:=\bar{b}_k,$$

which implies $b_k:=\inf_{y\in Z_k:\ \|y\|=r_k}\Psi(y)\geqslant\bar{b}_k\to\infty$ as $k\to\infty$.

By condition (A1), for any $\varepsilon_2>0$, there exists $D_{\varepsilon_2}>0$ such that $H(s,y)\geqslant D_{\varepsilon_2}(|y|^\mu+1)$ and $\|y\|^2/2\leqslant D_{\varepsilon_2}\|y\|^2_{L^2}$ as $y\in Y_k$, then there is

$$\begin{aligned}\Psi(y)&=\frac{1}{2}\|y\|^2-\int\limits_J \mathrm{e}^{G(s)}H\bigl(\sigma(s),y^\sigma(s)\bigr)\Delta s\\&\leqslant\frac{1}{2}\|y\|^2-\int\limits_J \mathrm{e}^{G(s)}D_{\varepsilon_2}\bigl(|y|^\mu+1\bigr)\Delta s\\&\leqslant\frac{1}{2}\|y\|^2-mD_{\varepsilon_2}\|y\|^\mu_{\mathcal{L}^\mu}-mD_{\varepsilon_2}\end{aligned}$$

by $\mu\geqslant 2$, and all norms on the finite-dimensional space Y_k are equivalent. Thus for $\rho_k>0$ large enough, there is $a_k:=\max_{y\in Y_k:\ \|y\|=\rho_k}\Psi(y)\leqslant 0$.

Thus by Lemma 11 there is the unbounded sequence $\{y_k\}$ such that $\Psi(y_k)\to+\infty$ as $k\to\infty$. The proof is completed. □

5 Examples

Example 1. Let $\mathbb{T}=\{2/n,\ n\in\mathbb{N}\}$ and $a=0$, $b=1$, $g(s)=0$, then consider the following problem:

$$\begin{aligned}&-y^{\Delta\Delta}(s)+\lambda y(s)=h(s,y(s)),\quad \Delta\text{-a.e. } s\in J,\\&y(0)=0,\qquad y(1)=y\left(\frac{1}{2}\right),\end{aligned}$$

where $h(s,y)=|y|^3(5|y|+\mathrm{e}^{\sin y}-1)$ and $\lambda>-(2\pi/3)^2/\sqrt{\mathrm{e}}\approx-2.66$. Then all assumptions in Theorem 3 are fulfilled.

Example 2. Let $\mathbb{T} = \{1/n,\ n \in \mathbb{N}\}$ and $a = 0$, $b = 2$, $g(s) = s^2$, there is the problem

$$-y^{\Delta\Delta}(s) + s^2 y^{\Delta}(s) + \lambda y(s) = h(s, y(s)), \quad \Delta\text{-a.e. } s \in J,$$
$$y(0) = 0, \qquad y(2) = \frac{1}{2} y\left(\frac{4}{3}\right),$$

where $h(s, y) = 2|y|^2 y \log(1 + s^2|y|)$ and $\lambda > -\lambda_1 \mathrm{e}^{-3/8}$. Then all assumptions in Theorem 4 are fulfilled, and we can obtain the primitive function of h, but it is almost impossible to check (AR) condition.

Remark 4. The eigenvalue of Example 1 had been investigated in [14]. We get $\lambda_1 = (2\pi/3)^2$ as in Example 3.1 [14], and for more precise values of λ_1 in Example 2, we refer to Theorem 6 in [9].

6 Conclusion

In this paper, the key point in our proof is that although Ψ may possess unbounded Palais–Smale sequences, under appropriate assumptions, the functional Ψ satisfies the Cerami condition. Then by using the mountain pass theorem and fountain theorem, we get the existence of infinitely many large energy solutions.

We give the variational structure of the nonlocal damped vibration problem, existence and multiplicity of solutions for the superlinear equations are obtained without (AR) condition. In addition, our results generalize previous results of others to nonlocal cases.

References

1. R.P. Agarwal, V. Otero-Espinar, K. Perera, D.R. Vivero, Basic properties of Sobolev's spaces on time scales, *Adv. Difference Equ.*, **2006**:38121, 2006, `https://doi.org/10.1155/ADE/2006/38121`.
2. R.P. Agarwal, V. Otero-Espinar, K. Perera, D.R. Vivero, Multiple positive solutions of singular Dirichlet problems on time scales via variational methods, *Nonlinear Anal., Theory Methods Appl., Ser. A*, **67**(2):368–381, 2007, `https://doi.org/10.1016/j.na.2006.05.014`.
3. A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14**:349–381, 1973, `https://doi.org/10.1016/0022-1236(73)90051-7`.
4. L. Bai, B.X. Dai, Existence of nonzero solutions for a class of damped vibration problems with impulsive effects, *Appl. Math., Praha*, **59**(2):145–165, 2014, `https://doi.org/10.1007/S10492-014-0046-6`.
5. D. Barilla, M. Bohner, S. Heidarkhani, S. Moradi, Existence results for dynamic Sturm–Liouville boundary value problems via variational methods, *Appl. Math. Comput.*, **409**:125614, 2021, `https://doi.org/10.1016/j.amc.2020.125614`.
6. T. Bartsch, Infinitely many solutions of a symmetric Dirichlet problem, *Nonlinear Anal., Theory Methods Appl.*, **20**(10):1205–1216, 1993, `https://doi.org/10.1016/0362-546X(93)90151-H`.

7. M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Basel, Boston, 2001, https://doi.org/10.1007/978-1-4612-0201-1.
8. A. Cabada, D.R. Vivero, Criterions for absolute continuity on time scales, *J. Difference Equ. Appl.*, **11**(11):1013–1028, 2005, https://doi.org/10.1080/10236190500272830.
9. E. Şen, A. Štikonas, Asymptotic distribution of eigenvalues and eigenfunctions of a nonlocal boundary value problem, *Math. Model. Anal.*, **26**(2):253–266, 2021, https://doi.org/10.3846/mma.2021.13056.
10. L. Jeanjean, On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer-type problem on $\mathbb{R}^N$, *Proc. R. Soc. Edinb., Sect. A, Math.*, **129**(4):787–809, 1999, https://doi.org/10.1017/S0308210500013147.
11. R. Kajikiya, A critical point theorem related to the symmetric mountain pass lemma and its applications to elliptic equations, *J. Funct. Anal.*, **225**(2):352–370, 2005, https://doi.org/10.1016/j.jfa.2005.04.005.
12. W. Lian, Z.B. Bai, Z.J. Du, Existence of solution of a three-point boundary value problem via variational approach, *Appl. Math. Lett.*, **104**:106283, 2020, https://doi.org/10.1016/j.aml.2020.106283.
13. S. Luan, A. Miao, Periodic solutions of nonautonomous second order Hamiltonian systems, *Acta. Math. Sin., Engl. Ser.*, **21**(4):685–690, 2005, https://doi.org/10.1007/s10114-005-0532-6.
14. R. Ma, D. O'Regan, Nodal solutions for second-order m-point boundary value problems with nonlinearities across several eigenvalues, *Nonlinear Anal., Theory Methods Appl.*, **64**(7):1562–1577, 2006, https://doi.org/10.1016/j.na.2005.07.007.
15. J. Mawhin, M. Willem, *Critical Point Theory and Hamiltonian Systems*, Appl. Math. Sci., Vol. 74, Springer, New York, 1989, https://doi.org/10.1007/978-1-4757-2061-7.
16. J.J. Nieto, Variational formulation of a damped Dirichlet impulsive problem, *Appl. Math. Lett.*, **23**(8):940–942, 2010, https://doi.org/10.1016/j.aml.2010.04.015.
17. P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, Volume 65 of *CBMS Reg. Conf. Ser. Math., Vol. 65*, AMS, Providence, RI, 1986, https://doi.org/10.1090/cbms/065.
18. M. Schechter, A variation of the mountain pass lemma and applications, *J. Lond. Math. Soc., Ser. II*, **44**(3):491–502, 1991, https://doi.org/10.1112/jlms/s2-44.3.491.
19. M. Timoumi, On ground-state homoclinic orbits of a class of superquadratic damped vibration systems, *Mediterr. J. Math.*, **15**(2):53, 2018, https://doi.org/10.1007/s00009-018-1097-9.
20. A. Štikonas, The Sturm–Liouville problem with a nonlocal boundary condition, *Lith. Math. J.*, **47**(3):336–351, 2007, https://doi.org/10.1007/s10986-007-0023-9.
21. A. Štikonas, A survey on stationary problems, Green's functions and spectrum of Sturm–Liouville problem with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **19**(3):301–334, 2014, https://doi.org/10.15388/NA.2014.3.1.

22. Y. Wei, Z. Bai, Multiple solutions for some nonlinear impulsive differential equations with three-point boundary conditions via variational approach, *J. Appl. Anal. Comput.*, **11**(6):3031–3043, 2021, https://doi.org/10.11948/20210113.

23. Y. Wei, S. Shang, Z. Bai, Applications of variational methods to some three-point boundary value problems with instantaneous and noninstantaneous impulses, *Nonlinear Anal. Model. Control*, **27**(3):466–478, 2022, https://doi.org/10.15388/namc.2022.27.26253.

24. Y. Wei, S. Shang, Z. Bai, Solutions for a class of Hamiltonian systems on time scales with non-local boundary conditions, *Appl. Math. Mech., Engl. Ed.*, **43**(4), 2022, https://doi.org/10.1007/s10483-022-2832-9.

25. M. Willem, *Minimax Theorem*, Boston, Birkhäuser, 1996, https://doi.org/10.1007/978-1-4612-4146-1.

26. J. Xiao, J.J. Nieto, Variational approach to some damped dirichlet nonlinear impulsive differential equations, *J. Franklin Inst.*, **348**(2):369–377, 2011, https://doi.org/10.1016/j.jfranklin.2010.12.003.

27. J.F. Zhao, *Structure Theory for Banach Space*, Wuhan Univ. Press, Wuhan, 1991 (in Chinese).

28. J.W. Zhou, Y. Li, Sobolev's spaces on time scales and its applications to a class of second order Hamiltonian systems on time scales, *Nonlinear Anal., Theory Methods Appl.*, **73**(5):1375–1388, 2010, https://doi.org/10.1016/j.na.2010.04.070.

29. W. Zou, Variant fountain theorems and their applications, *Manuscr. Math.*, **104**(3):343–358, 2001, https://doi.org/10.1007/s002290170032.