Nonlinear Analysis: Modelling and Control, Online First, 1–25
https://doi.org/10.15388/namc.2022.27.27587

Vilnius University Press

Controllability of nonlinear higher-order fractional damped stochastic systems involving multiple delays*

Ganesan Arthi[a,1] **Kalaiselvan Suganya**[a], **Juan J. Nieto**[b,2]

[a]Department of Mathematics,
PSGR Krishnammal College for Women,
Coimbatore-641004, India
arthimath@gmail.com

[b]Instituto de Matemáticas,
Universidade de Santiago de Compostela,
Santiago de Compostela, Spain

Received: October 31, 2021 / **Revised:** April 15, 2022 / **Published online:** May 24, 2022

Abstract. This paper is concerned with the controllability problem for higher-order fractional damped stochastic systems with multiple delays, which involves fractional Caputo derivatives of any different orders. In the process of proof, we have proposed the controllability of considered linear system by establishing a controllability Grammian matrix and employing a control function. Sufficient conditions for the considered nonlinear system concerned to be controllable have been derived by constructing a proper control function and utilizing the Banach fixed point theorem with Burkholder–Davis–Gundy's inequality. Finally, two examples are provided to emphasize the applicability of the derived results.

Keywords: controllability, fractional damped systems, stochastic systems, multiple delays, Mittag-Leffler function.

1 Introduction

Fractional calculus is a dynamic mathematical argument and suitable for analyzing various problems in evolving applied mathematical research in dealing with many real-world applications. For more than a decade, many researchers have paid attention on fractional differential equations. The study of fractional differential equation consists of key approaches to examine differential equations including fractional derivatives of unknown function. The fractional derivatives appear as attractive and powerful modeling

*This research was supported by grant Nos. PID2020-113275GB-I00 and ED431C 2019/02.

[1]Corresponding author.

[2]The author has been partially supported by the Agencia Estatal de Investigacion (AEI) of Spain, co-financed by the European Fund for Regional Development (FEDER), project PID2020-113275GB-I00; and by Xunta de Galicia under grant ED431C 2019/02.

tools in variety of areas such as bioengineering, electrical networks, signal processing, viscoelastic materials and many other physical phenomena [17, 18, 21]. In recent years, higher-order fractional systems have been widely and efficiently studied because its potential to model real time problems with more accuracy and its occurrence in control problems [16, 19]. Damping is a force within or beyond an oscillatory system that has the effect of restricting reducing or preventing its oscillations. The fractional oscillator can be modeled by establishing the fractional time derivative in standard harmonic oscillator, which explains the physical occurrence based on the fractional time evolution notion. Specifically, in the field of mechanics, fractional damping may occur towards the modeling of mechanical systems with viscoelastic components. It should be pointed out that the viscoelastic behavior of complex materials in many real time practices has been well characterized by fractional-order components; see [2, 8, 24, 29, 30].

Controllability is an essential aspect of control theory, and it acts a significant role in many control problems. The study of controllability is to verify the presence of a control function that drives the control system from its initial state to a final state in a specific time. More works for the controllability problems have been discussed in recent research; see [9, 20, 25, 27] and references therein. In recent years, the controllability of fractional damped systems has attracted much attention to researchers [11, 14, 28]. On the other hand, the stochastic analysis has gained significance and attractiveness based on its applications in wide-ranging areas of applied mathematics and engineering [4]. The research discussing the uniqueness, existence and stability of several stochastic differential equations gain more interests; see [1, 6, 12, 15] and the references therein. Recently, there has been a very important progress in the study of controllability of stochastic differential systems [13, 22, 26].

Stochastic process or noise is inevitable to model the time evolution of dynamical systems, which are related to random influences. Consequently, it is of intense importance to include the stochastic effects into the analysis of fractional-order systems. Many works have been done concerning the stochastic differential equations involving fractional derivatives in the recent years for their importance in applied sciences. Sun et al. [23] examined the controllability problem for neutral stochastic fractional integro–differential systems involving infinite delay. Guendouzi et al. [10] obtained the controllability concepts for the fractional stochastic dynamical systems involving multiple delays by means of Banach fixed point theorem. Recently, Cui and Yan [3] explored the controllability result for neutral stochastic evolution systems involving fractional Brownian motion. In [7], the authors obtained the controllability problem for fractional stochastic evolution systems involving nonlocal conditions and noncompact semigroups by means of fixed point theory. However, up to now, the controllability concept of higher-order fractional stochastic systems with damping properties and multiple delays has not been considered in the literature. Thus, this topic is an interesting one and essential to analyze it. The analysis includes the contributions, which are stated as follows.

- Most of the earlier investigations on fractional systems have been discussed with single delay. Consequently, it is essential to pay consideration to the analysis of fractional damped stochastic systems with multiple delays.

- Compared with several previous analyses, controllability of higher-order fractional stochastic system with damping effects and multiple delays is firstly presented for designing more general fractional-order model.
- The linear system of higher-order fractional damped stochastic dynamical system involving multiple delays is considered to investigate the controllability concept by utilizing Grammian matrix, it can be expressed in terms of Mittag-Leffler function.
- Further, Burkholder–Davis–Gundy's inequality and fixed point theorem are utilized to derive the sufficient conditions for the nonlinear higher-order fractional damped system involving multiple delays.

Finally, to explain the efficiency and applicability of controllability criteria clearly, we provide two examples. A brief viewpoint on how the obtained results can be extended will be presented in the conclusion section.

2 Preliminaries

Assume the complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ involving filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ generated by the Wiener m-dimensional process with probability measure $\mathbf{P}$ on Ω. Let $\mu - 1 < \rho_1 \leqslant \mu$, $\lambda - 1 < \rho_2 \leqslant \lambda$ and $\lambda \leqslant \mu - 1$, the symbol D represents differential operator. $\mathbb{R}^m$ denotes the m-dimensional Euclidean space $R_+ = [0, \infty)$. The state variable $x(t)$ denoted in the Hilbert space $L^2_{\mathcal{F}_t}(\mathcal{J} \times \Omega, \mathbb{R}^n)$ is equipped with $\|x\|^2_{L^2} = \sup_{t\in\mathcal{J}} \mathbf{E}\|x(t)\|^2$, where $\mathbf{E}(\cdot)$ symbolizes the expectation w.r.t measure $\mathbf{P}$. The continuous map $I = I([0,T]; L^2_{\mathcal{F}_t})$ is defined from $[0,T]$ into $L^2_{\mathcal{F}_t}(\mathcal{J} \times \Omega, \mathbb{R}^n)$ satisfying $\sup_{t\in\mathcal{J}} \mathbf{E}\|x(t)\|^2 \leqslant \infty$. Now we recall several important basic concepts.

Definition 1. Fractional derivative with Caputo sense of order ρ_1 ($0 \leqslant m_0 \leqslant \rho_1 < m_0 + 1$) for a function $h : R^+ \to R$ is stated as

$$ {}^C_0D^{\rho_1}_t h(t) = \frac{1}{\Gamma(m_0 - \rho_1 + 1)} \int\limits_0^t \frac{h^{(m_0+1)}(\theta)}{(t-\theta)^{\rho_1 - m_0}}\, \mathrm{d}\theta. $$

The Laplace Transform (LT) of fractional derivative with Caputo sense is

$$ \mathcal{L}\bigl\{{}^C_0D^{\rho_1}_t h(t)\bigr\}(\xi) = \xi^{\rho_1} H(\xi) - \sum_{k=0}^{m_0-1} h^{(k)}(t)\xi^{\rho_1-1-k}. $$

Definition 2. The Mittag-Leffler function $E_{\rho_1}(z)$ involving $\rho_1 > 0$ is stated as

$$ E_{\rho_1}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\rho_1 j + 1)}, \quad \rho_1 > 0,\ z \in \mathbb{C}. $$

The Mittag-Leffler function $E_{\rho_1,\rho_2}(z)$ involving $\rho_1, \rho_2 > 0$ is stated as

$$ E_{\rho_1,\rho_2}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\rho_1 j + \rho_2)}, \quad \rho_1 > 0,\ z \in \mathbb{C}. $$

The LT of $E_{\rho_1,\rho_2}(z)$ is

$$\mathcal{L}\{t^{\rho_2-1}E_{\rho_1,\rho_2}(\pm at^{\rho_1})\}(\xi) = \frac{\xi^{\rho_1-\rho_2}}{\xi^{\rho_1} \mp a}.$$

For $\rho_2 = 1$, we have

$$\mathcal{L}\{E_{\rho_1}(\pm at^{\rho_1})\}(\xi) = \frac{\xi^{\rho_1-1}}{\xi^{\rho_1} \mp a}.$$

Lemma 1 [Burkholder–Davis–Gundy's inequality]. (*See* [4, 5].) *For any* $r \geqslant 1$ *and for arbitrary* $\mathcal{L}_2^0$*-valued predictable process* $\Psi(t)$, $t \in [0, \mathcal{T}]$, *one has*

$$\mathbf{E}\Bigg(\sup_{0\leqslant t\leqslant \mathcal{T}}\bigg|\int_0^t \Psi(\xi)\,\mathrm{d}w(\xi)\bigg|^{2r}\Bigg) \leqslant C_r \mathbf{E}\Bigg(\int_0^t \big\|\Psi(\xi)\big\|^2_{\mathcal{L}_2^0}\,\mathrm{d}\xi\Bigg)^r,$$

where $C_r = (r(2r-1))^r(2r/(2r-1))^{2r^2}$.

Let the Cauchy fractional problem

$$\begin{aligned} &{}_0^C D_t^{\rho_1} y(t) - \mathcal{A}\,{}_0^C D_t^{\rho_2} y(t) = h(t), \quad t \geqslant 0, \\ &y(0) = y_0, \qquad y'(0) = y_1, \ \ldots, \ y^{\mu-1}(0) = y_{\mu-1} \end{aligned} \tag{1}$$

with $\mu - 1 < \rho_1 \leqslant \mu$, $\lambda - 1 < \rho_2 \leqslant \lambda$ and $\lambda \leqslant \mu - 1$.

Here $h : \mathcal{J} \to \mathbb{R}^n$ is a continuous function, and $\mathcal{A}$ is a $n \times n$ matrix. Applying LT to (1), we get

$$\begin{aligned} &\xi^{\rho_1}Y(\xi) - \xi^{\rho_1-1}y(0) - \xi^{\rho_1-2}y'(0) - \cdots - \xi^{\rho_1-\mu}y^{\mu-1}(0) \\ &\quad - \mathcal{A}\xi^{\rho_2}Y(\xi) + \mathcal{A}\xi^{\rho_2-1}y(0) + \mathcal{A}\xi^{\rho_2-2}y'(0) + \cdots + \mathcal{A}\xi^{\rho_2-\lambda}y^{\lambda-1}(0) \\ &= H(\xi). \end{aligned}$$

Applying inverse LT to the above equation, then utilizing LT of Mittag-Leffler function and convolution operator, we obtain

$$\begin{aligned} y(t) = {} & \sum_{r=0}^{\mu-1} y^r(0)t^r E_{\rho_1-\rho_2,1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big) \\ & - \sum_{r=0}^{\lambda-1} y^r(0)\mathcal{A}t^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big) \\ & + \int_0^t (t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)h(\xi)\,\mathrm{d}\xi. \end{aligned}$$

3 Controllability result for linear system

Consider the linear damped fractional stochastic system involving multiple delays of the form

$$
\begin{aligned}
&{}_0^C D_t^{\rho_1} y(t) - \mathcal{A}{}_0^C D_t^{\rho_2} y(t) \\
&\quad = \mathcal{B}u(t) + \sum_{i=0}^{P} \mathcal{C}_i u(t-\tau_i) + \sigma(t)\frac{\mathrm{d}w(t)}{\mathrm{d}t}, \quad t \in [0,\mathcal{T}] = \mathcal{J}, \qquad (2)
\end{aligned}
$$

$$
y(0) = y_0, \qquad y'(0) = y_1, \ \ldots, \ y^{\mu-1}(0) = y_{\mu-1}, \qquad (3)
$$

$$
u(t) = \varphi(t), \quad t \in [-\tau_P, 0), \qquad (4)
$$

where $\mu-1 < \rho_1 \leqslant \mu$, $\lambda-1 < \rho_2 \leqslant \lambda$ and $\lambda \leqslant \mu-1$, $y \in \mathbb{R}^n$ represents a state variable, $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B}, \mathcal{C}_i \in \mathbb{R}^{n\times m}$, $i = 0,1,\ldots,P$, are constant matrices, $u(t) \in \mathbb{R}^m$ denotes a control input, $0 = \tau_0 < \tau_1 < \cdots < \tau_i < \cdots < \tau_{P-1} < \tau_P$ are constant delays, and φ represents the initial control function. $w(t)$ represents m-dimensional Wiener process involving $\mathcal{F}_t$ generated by $w(\xi)$, $0 \leqslant \xi \leqslant t$, and $\sigma : \mathcal{J} \to \mathbb{R}^{n\times m}$ is a continuous function.

The solution of fractional system (2)–(4) takes the form

$$
\begin{aligned}
y(t) &= \sum_{r=0}^{\mu-1} y^r(0) t^r E_{\rho_1-\rho_2,1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big) \\
&\quad - \sum_{r=0}^{\lambda-1} y^r(0)\mathcal{A}t^{\rho_1-\rho_2+r} E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big) \\
&\quad + \int_0^t (t-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\mathcal{B}u(\xi)\,\mathrm{d}\xi \\
&\quad + \int_0^t (t-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\left[\sum_{i=0}^{P} \mathcal{C}_i u(\xi-\tau_i)\right]\mathrm{d}\xi \\
&\quad + \int_0^t (t-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\left(\int_0^{\eta} \sigma(\vartheta)\,\mathrm{d}w(\vartheta)\right)\mathrm{d}\xi.
\end{aligned}
$$

For $\tau_k < t < \tau_{k+1}$, $k = 0,1,\ldots,P-1$,

$$
\begin{aligned}
y(t) &= \sum_{r=0}^{\mu-1} y^r(0) t^r E_{\rho_1-\rho_2,1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big) \\
&\quad - \sum_{r=0}^{\lambda-1} y^r(0)\mathcal{A}t^{\rho_1-\rho_2+r} E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big) \\
&\quad + \int_0^t (t-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\mathcal{B}u(\xi)\,\mathrm{d}\xi
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=0}^{k}\int\limits_{-\tau_i}^{t-\tau_i}(t-\xi-\tau_i)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_i)^{\rho_1-\rho_2}\big)\mathcal{C}_i u(\xi)\,\mathrm{d}\xi\\
&+\int\limits_0^t(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\Bigg(\int\limits_0^\eta\sigma(\vartheta)\,\mathrm{d}w(\vartheta)\Bigg)\,\mathrm{d}\xi\\
=&\sum_{r=0}^{\mu-1}y^r(0)t^rE_{\rho_1-\rho_2,1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big)\\
&-\sum_{r=0}^{\lambda-1}y^r(0)\mathcal{A}t^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big)\\
&+\int\limits_0^t(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\mathcal{B}u(\xi)\,\mathrm{d}\xi\\
&+\sum_{i=0}^{k-1}\int\limits_{t-\tau_{i+1}}^{t-\tau_i}\Bigg(\sum_{j=0}^{i}(t-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\Bigg)u(\xi)\,\mathrm{d}\xi\\
&+\int\limits_0^{t-\tau_k}\Bigg(\sum_{j=0}^{k}(t-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\Bigg)u(\xi)\,\mathrm{d}\xi\\
&+\int\limits_{-\tau_P}^{0}\Bigg(\sum_{j=0}^{k}(t-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\Bigg)\varphi(\xi)\,\mathrm{d}\xi\\
&+\int\limits_0^t(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\Bigg(\int\limits_0^\eta\sigma(\vartheta)\,\mathrm{d}w(\vartheta)\Bigg)\,\mathrm{d}\xi.
\end{aligned}
$$

For $t>\tau_P$,

$$
\begin{aligned}
y(t)=&\sum_{r=0}^{\mu-1}y^r(0)t^rE_{\rho_1-\rho_2,1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big)\\
&-\sum_{r=0}^{\lambda-1}y^r(0)\mathcal{A}t^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big)\\
&+\int\limits_0^t(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\mathcal{B}u(\xi)\,\mathrm{d}\xi\\
&+\sum_{i=0}^{P-1}\int\limits_{t-\tau_{i+1}}^{t-\tau_i}\Bigg(\sum_{j=0}^{i}(t-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\Bigg)u(\xi)\,\mathrm{d}\xi
\end{aligned}
$$

$$+\int_0^{t-\tau_P}\left(\sum_{j=0}^{P}(t-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\right)u(\xi)\,\mathrm{d}\xi$$
$$+\int_{-\tau_P}^{0}\left(\sum_{j=0}^{k}(t-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\right)\varphi(\xi)\,\mathrm{d}\xi$$
$$+\int_0^{t}(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\left(\int_0^{\eta}\sigma(\vartheta)\,\mathrm{d}w(\vartheta)\right)\mathrm{d}\xi.$$

Controllability Grammian matrix W is as follows:

$$W=\int_0^{\mathcal{T}}\big[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2})\mathcal{B}\big]$$
$$\times\big[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\mathcal{B}\big]^*\,\mathrm{d}\xi$$
$$+\sum_{i=0}^{P-1}\int_{\mathcal{T}-\tau_{i+1}}^{\mathcal{T}-\tau_i}\left[\sum_{j=0}^{i}(\mathcal{T}-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\right]$$
$$\times\left[\sum_{j=0}^{i}(\mathcal{T}-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\right]^*\mathrm{d}\xi$$
$$+\int_0^{\mathcal{T}-\tau_P}\left[\sum_{j=0}^{P}(\mathcal{T}-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\right]$$
$$\times\left[\sum_{j=0}^{P}(\mathcal{T}-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2})\mathcal{C}_j\right]^*\mathrm{d}\xi.$$

Definition 3. System (2)–(4) is known as controllable on $[0,\mathcal{T}]$ if there exists a control $u(t)$ for every $y_0, y_1, \dots, y_{\mu-1}, y_{\mathcal{T}} \in \mathbb{R}^n$. Then the solution $y(t)$ of system (2)–(4) satisfies $y(0)=y_0$, $y'(0)=y_1, \dots, y^{\mu-1}(0)=y_{\mu-1}$, $y(\mathcal{T})=y_{\mathcal{T}}$.

Theorem 1. *The linear fractional system* (2)–(4) *is controllable on $\mathcal{J}$ if and only if the $n\times n$ Grammian matrix*

$$W=\int_0^{\mathcal{T}}\big[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\mathcal{B}\big]$$
$$\times\big[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\mathcal{B}\big]^*\,\mathrm{d}\xi$$

$$+\sum_{i=0}^{P-1}\int\limits_{\mathcal{T}-\tau_{i+1}}^{\mathcal{T}-\tau_i}\left[\sum_{j=0}^{i}(\mathcal{T}-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\right]$$
$$\times\left[\sum_{j=0}^{i}(\mathcal{T}-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\right]^*\mathrm{d}\xi$$
$$+\int\limits_{0}^{\mathcal{T}-\tau_P}\left[\sum_{j=0}^{P}(\mathcal{T}-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\right]$$
$$\times\left[\sum_{j=0}^{P}(\mathcal{T}-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\right]^*\mathrm{d}\xi$$

is nonsingular.

Proof. Assume that W is nonsingular. For every $y_0, y_1, \dots, y_{\mu-1}$ and $y_{\mathcal{T}}$, we can take the following input function $u(t)$:

$$u(t)=\begin{cases}\mathbb{D}_1^*(\mathcal{T},t)W^{-1}(\hat{k}), & t\in[0,\mathcal{T}],\\ \mathbb{D}_2^*(\mathcal{T},t)W^{-1}(\hat{k}), & t\in[\mathcal{T}-\tau_{i+1},\mathcal{T}-\tau_i],\\ \mathbb{D}_3^*(\mathcal{T},t)W^{-1}(\hat{k}), & t\in[0,\mathcal{T}-\tau_P],\end{cases}$$

where

$$\mathbb{D}_1(\mathcal{T},t)=(\mathcal{T}-t)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-t)^{\rho_1-\rho_2}\big)\mathcal{B}$$

for $t\in[0,\mathcal{T}]$,

$$\mathbb{D}_2(\mathcal{T},t)=\left[\sum_{j=0}^{i}(\mathcal{T}-t-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-t-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\right],$$

for $t\in[\mathcal{T}-\tau_{i+1},\,\mathcal{T}-\tau_i]$,

$$\mathbb{D}_3(\mathcal{T},t)=\left[\sum_{j=0}^{P}(\mathcal{T}-t-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-t-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\right],$$

for $t\in[0,\,\mathcal{T}-\tau_P]$,

$$\hat{k}=\frac{1}{3}\Bigg[y_{\mathcal{T}}-\sum_{r=0}^{\mu-1}y_r\mathcal{T}^rE_{\rho_1-\rho_2,1+r}\big(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2}\big)$$
$$-\sum_{r=0}^{\lambda-1}y_r\mathcal{A}\mathcal{T}^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}\big(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2}\big)$$

$$+ \int_{-\tau_P}^{0} \left(\sum_{j=0}^{k} (\mathcal{T} - \xi - \tau_j)^{\rho_1 - 1} E_{\rho_1 - \rho_2,\, \rho_1}\bigl(\mathcal{A}(\mathcal{T} - \xi - \tau_j)^{\rho_1 - \rho_2}\bigr)\mathcal{C}_j \right) \varphi(\xi)\, \mathrm{d}\xi$$
$$+ \int_{0}^{\mathcal{T}} (\mathcal{T} - \xi)^{\rho_1 - 1} E_{\rho_1 - \rho_2,\, \rho_1}\bigl(\mathcal{A}(\mathcal{T} - \xi)^{\rho_1 - \rho_2}\bigr) \left(\int_0^{\eta} \sigma(\vartheta)\, \mathrm{d}w(\vartheta) \right) \mathrm{d}\xi \Biggr].$$

At $t = \mathcal{T}$, the solution of system (2)–(4) can be written in the following form:

$$\begin{aligned}
y(\mathcal{T}) ={}& \sum_{r=0}^{\mu-1} y^r(0) \mathcal{T}^r E_{\rho_1-\rho_2, 1+r}\bigl(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2}\bigr) \\
&- \sum_{r=0}^{\lambda-1} y^r(0) \mathcal{A}\mathcal{T}^{\rho_1-\rho_2+r} E_{\rho_1-\rho_2, \rho_1-\rho_2+1+r}\bigl(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2}\bigr) \\
&+ \int_0^{\mathcal{T}} \mathbb{D}_1(\mathcal{T}, t)\mathbb{D}_1^*(\mathcal{T}, t) W^{-1}(\hat{k})\, \mathrm{d}\xi + \sum_{i=0}^{P-1} \int_{\mathcal{T}-\tau_{i+1}}^{\mathcal{T}-\tau_i} \mathbb{D}_2(\mathcal{T}, t)\mathbb{D}_2^*(\mathcal{T}, t) W^{-1}(\hat{k})\, \mathrm{d}\xi \\
&+ \int_0^{\mathcal{T}-\tau_P} \mathbb{D}_3(\mathcal{T}, t)\mathbb{D}_3^*(\mathcal{T}, t) W^{-1}(\hat{k})\, \mathrm{d}\xi \\
&+ \int_{-\tau_P}^{0} \left(\sum_{j=0}^{k} (\mathcal{T} - \xi - \tau_j)^{\rho_1 - 1} E_{\rho_1 - \rho_2,\, \rho_1}(\mathcal{A}(\mathcal{T} - \xi - \tau_j)^{\rho_1 - \rho_2})\mathcal{C}_j \right) \varphi(\xi)\, \mathrm{d}\xi \\
&+ \int_{0}^{\mathcal{T}} (\mathcal{T} - \xi)^{\rho_1 - 1} E_{\rho_1 - \rho_2,\, \rho_1}\bigl(\mathcal{A}(\mathcal{T} - \xi)^{\rho_1 - \rho_2}\bigr) \left(\int_0^{\eta} \sigma(\vartheta)\, \mathrm{d}w(\vartheta) \right) \mathrm{d}\xi = y_{\mathcal{T}}.
\end{aligned}$$

Therefore, system (2)–(4) is controllable on $[0, \mathcal{T}]$.

On the other hand, assume that system (2)–(4) is controllable, but the matrix W is singular. Then there exists a vector $z \neq 0$ such that

$$\begin{aligned}
z^* W z ={}& z^* \int_0^{\mathcal{T}} \bigl[(\mathcal{T} - \xi)^{\rho_1 - 1} E_{\rho_1 - \rho_2,\, \rho_1}(\mathcal{A}(\mathcal{T} - \xi)^{\rho_1 - \rho_2})\mathcal{B}\bigr] \\
&\quad \times \bigl[(\mathcal{T} - \xi)^{\rho_1 - 1} E_{\rho_1 - \rho_2,\, \rho_1}\bigl(\mathcal{A}(\mathcal{T} - \xi)^{\rho_1 - \rho_2}\bigr)\mathcal{B}\bigr]^* z\, \mathrm{d}\xi \\
&+ z^* \sum_{i=0}^{P-1} \int_{\mathcal{T}-\tau_{i+1}}^{\mathcal{T}-\tau_i} \left[\sum_{j=0}^{i} (\mathcal{T}-\xi-\tau_j)^{\rho_1 - 1} E_{\rho_1 - \rho_2,\, \rho_1}\bigl(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1 - \rho_2}\bigr)\mathcal{C}_j \right] \\
&\quad \times \left[\sum_{j=0}^{i} (\mathcal{T}-\xi-\tau_j)^{\rho_1 - 1} E_{\rho_1 - \rho_2,\, \rho_1}\bigl(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1 - \rho_2}\bigr)\mathcal{C}_j \right]^* z\, \mathrm{d}\xi
\end{aligned}$$

$$+ z^* \int_0^{\mathcal{T}-\tau_P} \left[\sum_{j=0}^{P} (\mathcal{T}-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \right] \times \left[\sum_{j=0}^{P} (\mathcal{T}-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \right]^* z \,\mathrm{d}\xi$$

$$= 0.$$

Hence

$$z^*(\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\mathcal{B} = 0,$$

$$z^* \sum_{i=0}^{P-1} \int_{\mathcal{T}-\tau_{i+1}}^{\mathcal{T}-\tau_i} \left[\sum_{j=0}^{i} (\mathcal{T}-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \right] = 0$$

and

$$z^* \int_0^{\mathcal{T}-\tau_P} \left[\sum_{j=0}^{P} (\mathcal{T}-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \right] = 0$$

for $\mathcal{T} \in \mathcal{J}$.

Since system (2)–(4) is controllable, it can be driven from the initial points $y_0 = y_1 = \cdots = y_{\mu-1} = 0$ to the final point $y_{\mathcal{T}} = z$. So there exists a control $u(t)$ that drives the initial state to $y_{\mathcal{T}} = z$ at $t = \mathcal{T}$,

$$\begin{aligned}
y_{\mathcal{T}} &= z \\
&= \int_0^{\mathcal{T}} (\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big) B u(\xi)\,\mathrm{d}\xi \\
&\quad + \sum_{i=0}^{P-1} \int_{\mathcal{T}-\tau_{i+1}}^{\mathcal{T}-\tau_i} \left(\sum_{j=0}^{i} (\mathcal{T}-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \right) u(\xi)\,\mathrm{d}\xi \\
&\quad + \int_0^{\mathcal{T}-\tau_P} \left(\sum_{j=0}^{P} (\mathcal{T}-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \right) u(\xi)\,\mathrm{d}\xi \\
&\quad + \int_{-\tau_P}^{0} \left(\sum_{j=0}^{k} (\mathcal{T}-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \right) \varphi(\xi)\,\mathrm{d}\xi \\
&\quad + \int_0^{\mathcal{T}} (\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big) \left(\int_0^{\eta} \sigma(\vartheta)\,\mathrm{d}w(\vartheta) \right) \mathrm{d}\xi.
\end{aligned}$$

Thus

$$\begin{aligned}
z^*z = &\int_0^{\mathcal{T}} z^*(\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\mathcal{B}u(\xi)\,\mathrm{d}\xi \\
&+ \sum_{i=0}^{P-1} \int_{\mathcal{T}-\tau_{i+1}}^{\mathcal{T}-\tau_i} z^* \Bigg(\sum_{j=0}^{i} (\mathcal{T}-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \Bigg) \\
&\qquad \times u(\xi)\,\mathrm{d}\xi \\
&+ \int_0^{\mathcal{T}-\tau_P} z^* \Bigg(\sum_{j=0}^{P} (\mathcal{T}-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \Bigg) u(\xi)\,\mathrm{d}\xi \\
&+ \int_{-\tau_P}^{0} z^* \Bigg(\sum_{j=0}^{k} (\mathcal{T}-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \Bigg) \varphi(\xi)\,\mathrm{d}\xi \\
&+ \int_0^{\mathcal{T}} z^*(\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big) \Bigg(\int_0^{\eta} \sigma(\vartheta)\,\mathrm{d}w(\vartheta) \Bigg)\,\mathrm{d}\xi.
\end{aligned}$$

Then, taking into account that

$$\begin{aligned}
&\int_0^{\mathcal{T}} z^*(\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\mathcal{B}u(\xi)\,\mathrm{d}\xi \\
&\quad + \sum_{i=0}^{P-1} \int_{\mathcal{T}-\tau_{i+1}}^{\mathcal{T}-\tau_i} z^* \Bigg(\sum_{j=0}^{i} (\mathcal{T}-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \Bigg) \\
&\qquad \times u(\xi)\,\mathrm{d}\xi \\
&\quad + \int_0^{\mathcal{T}-\tau_P} z^* \Bigg(\sum_{j=0}^{P} (\mathcal{T}-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \Bigg) u(\xi)\,\mathrm{d}\xi
\end{aligned}$$

and

$$\begin{aligned}
&+ \int_{-\tau_P}^{0} z^* \Bigg(\sum_{j=0}^{k} (\mathcal{T}-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \Bigg) \varphi(\xi)\,\mathrm{d}\xi \\
&+ \int_0^{\mathcal{T}} z^*(\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big) \Bigg(\int_0^{\eta} \sigma(\vartheta)\,\mathrm{d}w(\vartheta) \Bigg)\,\mathrm{d}\xi
\end{aligned}$$

tend to zero, it follows that $z^*z = 0$. This implies the contradiction to $z \neq 0$. Hence the matrix W is nonsingular. □

4 Controllability result for nonlinear system

In this section, we analyze the controllability criteria of nonlinear fractional damped stochastic dynamical system (5)–(7) based on contraction mapping principle. Consider the nonlinear damped fractional stochastic system involving multiple delays of the form

$$
\begin{aligned}
&{}_0^C D_t^{\rho_1} y(t) - \mathcal{A}\,{}_0^C D_t^{\rho_2} y(t) \\
&\quad = \mathcal{B}u(t) + \sum_{i=0}^{P} \mathcal{C}_i u(t-\tau_i) + h\big(t, y(t)\big) + \sigma\big(t, y(t)\big)\frac{\mathrm{d}w(t)}{\mathrm{d}t}, \quad t \in [0, \mathcal{T}], \qquad (5)\\
&y(0) = y_0, \qquad y'(0) = y_1,\ \dots,\ y^{\mu-1}(0) = y_{\mu-1}, \qquad (6)\\
&u(t) = \varphi(t), \quad t \in [-\tau_P, 0), \qquad (7)
\end{aligned}
$$

where $\mu - 1 < \rho_1 \leqslant \mu$, $\lambda - 1 < \rho_2 \leqslant \lambda$ and $\lambda \leqslant \mu - 1$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}_i$ and $w(t)$ are defined as in previous section, $0 = \tau_0 < \tau_1 < \cdots < \tau_i < \cdots < \tau_{P-1} < \tau_P$ are constant delays, $y \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $h : \mathcal{J} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathcal{J} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$. Then the solution of system (5)–(7) is defined as

$$
\begin{aligned}
y(t) = {} & \sum_{r=0}^{\mu-1} y^r(0) t^r E_{\rho_1-\rho_2, 1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big) \\
& - \sum_{r=0}^{\lambda-1} y^r(0)\mathcal{A}t^{\rho_1-\rho_2+r} E_{\rho_1-\rho_2,\, \rho_1-\rho_2+1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big) \\
& + \int_{-\tau_P}^{0} \left(\sum_{j=0}^{k} (t-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\, \rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \right) \varphi(\xi)\,\mathrm{d}\xi \\
& + \int_0^t (t-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\, \rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big) h\big(\xi, y(\xi)\big)\,\mathrm{d}\xi \\
& + \int_0^t (t-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\, \rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big) \left(\int_0^{\eta} \sigma\big(\vartheta, y(\vartheta)\big)\,\mathrm{d}w(\vartheta) \right) \mathrm{d}\xi \\
& + \int_0^t (t-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\, \rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big) \mathcal{B}u(\xi)\,\mathrm{d}\xi \\
& + \sum_{i=0}^{P-1} \int_{t-\tau_{i+1}}^{t-\tau_i} \left(\sum_{j=0}^{i} (t-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\, \rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \right) u(\xi)\,\mathrm{d}\xi \\
& + \int_0^{t-\tau_P} \left(\sum_{j=0}^{P} (t-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\, \rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \right) u(\xi)\,\mathrm{d}\xi, \qquad (8)
\end{aligned}
$$

and

$$u(t) = \begin{cases} \mathbb{D}_1^*(\mathcal{T}, t) W^{-1}\gamma, & t \in [0, \mathcal{T}], \\ \mathbb{D}_2^*(\mathcal{T}, t) W^{-1}\gamma, & t \in [\mathcal{T} - \tau_{i+1}, \mathcal{T} - \tau_i], \\ \mathbb{D}_3^*(\mathcal{T}, t) W^{-1}\gamma, & t \in [0, \mathcal{T} - \tau_P], \end{cases} \tag{9}$$

$$\mathbb{D}_1(\mathcal{T}, t) = (\mathcal{T} - t)^{\rho_1 - 1} E_{\rho_1 - \rho_2,\, \rho_1}\big(\mathcal{A}(\mathcal{T} - t)^{\rho_1 - \rho_2}\big)\mathcal{B}$$

for $t \in [0, \mathcal{T}]$,

$$\mathbb{D}_2(\mathcal{T}, t) = \left[\sum_{j=0}^{i} (\mathcal{T} - t - \tau_j)^{\rho_1 - 1} E_{\rho_1 - \rho_2,\, \rho_1}\big(\mathcal{A}(\mathcal{T} - t - \tau_j)^{\rho_1 - \rho_2}\big)\mathcal{C}_j\right],$$

for $t \in [\mathcal{T} - \tau_{i+1},\, \mathcal{T} - \tau_i]$,

$$\mathbb{D}_3(\mathcal{T}, t) = \left[\sum_{j=0}^{P} (\mathcal{T} - t - \tau_j)^{\rho_1 - 1} E_{\rho_1 - \rho_2,\, \rho_1}\big(\mathcal{A}(\mathcal{T} - t - \tau_j)^{\rho_1 - \rho_2}\big)\mathcal{C}_j\right]$$

for $t \in [0,\, \mathcal{T} - \tau_P]$,

$$\begin{aligned} \gamma = \frac{1}{3}\Bigg[& y_{\mathcal{T}} - \sum_{r=0}^{\mu - 1} y_r \mathcal{T}^r E_{\rho_1 - \rho_2, 1+r}\big(\mathcal{A}\mathcal{T}^{\rho_1 - \rho_2}\big) \\ & - \sum_{r=0}^{\lambda - 1} y_r \mathcal{A}\mathcal{T}^{\rho_1 - \rho_2 + r} E_{\rho_1 - \rho_2, \rho_1 - \rho_2 + 1 + r}\big(\mathcal{A}\mathcal{T}^{\rho_1 - \rho_2}\big) \\ & + \int_{-\tau_P}^{0} \left(\sum_{j=0}^{k} (\mathcal{T} - \xi - \tau_j)^{\rho_1 - 1} E_{\rho_1 - \rho_2,\, \rho_1}\big(\mathcal{A}(\mathcal{T} - \xi - \tau_j)^{\rho_1 - \rho_2}\big)\mathcal{C}_j\right)\varphi(\xi)\,\mathrm{d}\xi \\ & + \int_{0}^{\mathcal{T}} (\mathcal{T} - \xi)^{\rho_1 - 1} E_{\rho_1 - \rho_2,\, \rho_1}\big(\mathcal{A}(\mathcal{T} - \xi)^{\rho_1 - \rho_2}\big) h\big(\xi, y(\xi)\big)\,\mathrm{d}\xi \\ & + \int_{0}^{\mathcal{T}} (\mathcal{T} - \xi)^{\rho_1 - 1} E_{\rho_1 - \rho_2,\, \rho_1}\big(\mathcal{A}(\mathcal{T} - \xi)^{\rho_1 - \rho_2}\big) \left(\int_{0}^{\eta} \sigma\big(\vartheta, y(\vartheta)\big)\,\mathrm{d}w(\vartheta)\right)\mathrm{d}\xi \Bigg]. \end{aligned}$$

We impose the following assumptions.

(H1) The linear damped fractional stochastic system involving multiple delays (2)–(4) is controllable on $[0, \mathcal{T}]$.

(H2) There exist the constants $\tilde{N}, \tilde{L} > 0$ such that the continuous functions h and σ satisfy the following:

$$\big\|h(t, y)\big\|^2 \leqslant \tilde{N}\big(1 + \|y\|^2\big), \qquad \big\|\sigma(t, y)\big\|^2 \leqslant \tilde{L}\big(1 + \|y\|^2\big).$$

(H3) For every $t \geqslant 0$ and $x, y \in \mathbb{R}^n$, there exist constants $N, L > 0$ such that the functions h and σ satisfy the following Lipschitz form:

$$\big\|h(t,x) - h(t,y)\big\|^2 \leqslant N\|x-y\|^2, \qquad \big\|\sigma(t,x) - \sigma(t,y)\big\|^2 \leqslant L\|x-y\|^2.$$

For transience, we present the following representations:

$$\begin{aligned}
a_1 &= \big\|t^r E_{\rho_1-\rho_2,1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big)\big\|^2,\\
a_2 &= \big\|\mathcal{A}t^{\rho_1-\rho_2+r} E_{\rho_1-\rho_2,\,\rho_1-\rho_2+1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big)\big\|^2,\\
a_3 &= \big\|E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\big\|^2,\\
v &= \int_{-\tau_P}^{0}\sum_{j=0}^{k}\mathbf{E}\big\|(t-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\big\|^2\,\mathrm{d}\xi,\\
\omega &= \Bigg[\mathbf{E}\|y_{\mathcal{T}}\|^2 + a_1\sum_{r=0}^{\mu-1}\mathbf{E}\|y_r\|^2 + a_2\sum_{r=0}^{\lambda-1}\mathbf{E}\|y_r\|^2 + uv + 4a_3L_\sigma\tilde{L}\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\\
&\quad\times\int_0^{\mathcal{T}}\Bigg(\int_0^{\eta}\big(1+\mathbf{E}\big\|y(\vartheta)\big\|^2\big)\,\mathrm{d}\vartheta\Bigg)\,\mathrm{d}\xi + a_3\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\tilde{N}\mathcal{T}\int_0^{\mathcal{T}}\big(1+\mathbf{E}\big\|y(\xi)\big\|^2\big)\,\mathrm{d}\xi\Bigg],\\
u &= \big\|\varphi(\xi)\big\|^2, \qquad l = \big\|W^{-1}\big\|,\\
M &= \big\|\mathbb{D}_1(\mathcal{T},t)\big\|, \qquad \tilde{M} = \big\|\mathbb{D}_2(\mathcal{T},t)\big\|, \qquad \hat{M} = \big\|\mathbb{D}_3(\mathcal{T},t)\big\|.
\end{aligned} \tag{10}$$

Theorem 2. *Assume that* (H1)–(H3) *hold, then the nonlinear fractional system* (5)–(7) *is controllable on* $\mathcal{J}$.

Proof. Define an operator $\xi : I \to I$ as follows:

$$\begin{aligned}
(\xi y)(t) &= \sum_{r=0}^{\mu-1} y^r(0)t^r E_{\rho_1-\rho_2,1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big)\\
&\quad - \sum_{r=0}^{\lambda-1} y^r(0)\mathcal{A}t^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big)\\
&\quad + \int_{-\tau_P}^{0}\Bigg(\sum_{j=0}^{k}(t-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\Bigg)\varphi(\xi)\,\mathrm{d}\xi\\
&\quad + \int_0^t (t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)h\big(\xi,y(\xi)\big)\,\mathrm{d}\xi\\
&\quad + \int_0^t (t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\Bigg(\int_0^{\eta}\sigma\big(\vartheta,y(\vartheta)\big)\,\mathrm{d}w(\vartheta)\Bigg)\,\mathrm{d}\xi
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t (t-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\mathcal{B}u(\xi)\,\mathrm{d}\xi \\
&+ \sum_{i=0}^{P-1} \int_{t-\tau_{i+1}}^{t-\tau_i} \left(\sum_{j=0}^{i} (t-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \right) \\
&\qquad \times u(\xi)\,\mathrm{d}\xi \\
&+ \int_0^{t-\tau_P} \left(\sum_{j=0}^{P} (t-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \right) \\
&\qquad \times u(\xi)\,\mathrm{d}\xi,
\end{aligned} \tag{11}
$$

where the control function $u(t)$ is defined as in (9).

By Theorem 1, the control $u(t)$ (9) transfers $y(t)$ (8) from the initial state y_0 to the final state $y_{\mathcal{T}}$, provided that the operators ξ has a fixed point in I. So, if the operator ξ has a fixed point, then system (5)–(7) is controllable. As mentioned before, to prove the controllability of system (5)–(7), it is enough to show that ξ has a fixed point in I. To do this, we can employ the contraction mapping principle. In the following, we will divide the proof into two steps.

Based on contraction mapping principle, we shall prove that ξ maps I into itself. By Eq. (11), we have

$$
\begin{aligned}
&\sup_{0\leqslant t\leqslant \mathcal{T}} \mathbf{E}\big\|(\xi y)(t)\big\|^2 \\
&\quad = 8 \sup_{0\leqslant t\leqslant \mathcal{T}} \mathbf{E}\left\| \sum_{r=0}^{\mu-1} y^r(0) t^r E_{\rho_1-\rho_2,1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big) \right\|^2 \\
&\qquad + 8 \sup_{0\leqslant t\leqslant \mathcal{T}} \mathbf{E}\left\| \sum_{r=0}^{\lambda-1} y^r(0)\mathcal{A}t^{\rho_1-\rho_2+r} E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big) \right\|^2 \\
&\qquad + 8 \sup_{0\leqslant t\leqslant \mathcal{T}} \mathbf{E}\left\| \int_{-\tau_P}^{0} \left(\sum_{j=0}^{k} (t-\xi-\tau_j)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j \right) \varphi(\xi)\,\mathrm{d}\xi \right\|^2 \\
&\qquad + 8 \sup_{0\leqslant t\leqslant \mathcal{T}} \mathbf{E}\left\| \int_0^t (t-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big) h\big(\xi, y(\xi)\big)\,\mathrm{d}\xi \right\|^2 \\
&\qquad + 8 \sup_{0\leqslant t\leqslant \mathcal{T}} \mathbf{E}\left\| \int_0^t (t-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big) \left(\int_0^{\eta} \sigma\big(\vartheta, y(\vartheta)\big)\,\mathrm{d}w(\vartheta) \right) \mathrm{d}\xi \right\|^2 \\
&\qquad + 8 \sup_{0\leqslant t\leqslant \mathcal{T}} \mathbf{E}\left\| \int_0^t (t-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\mathcal{B}u(\xi)\,\mathrm{d}\xi \right\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+8\sup_{0\leqslant t\leqslant\mathcal{T}}\mathbf{E}\Bigg\|\sum_{i=0}^{P-1}\int\limits_{t-\tau_{i+1}}^{t-\tau_i}\Bigg(\sum_{j=0}^{i}(t-\xi-\tau_j)^{\rho_1-1}\\
&\qquad\times E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\Bigg)u(\xi)\,\mathrm{d}\xi\Bigg\|^2\\
&+8\sup_{0\leqslant t\leqslant\mathcal{T}}\mathbf{E}\Bigg\|\int\limits_{0}^{t-\tau_P}\Bigg(\sum_{j=0}^{P}(t-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\Bigg)u(\xi)\,\mathrm{d}\xi\Bigg\|^2\\
&=\sum_{b=1}^{8}\mathcal{R}_b.
\end{aligned}
$$

Using Hölder inequality, Burkholder–Davis–Gundy's inequality (here $C_1 = 4$) and (10), we have the following estimates:

$$
\begin{aligned}
\mathcal{R}_1&\leqslant 8\sum_{r=0}^{\mu-1}\mathbf{E}\big\|y_r t^r E_{\rho_1-\rho_2,1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big)\big\|^2\leqslant 8a_1\sum_{r=0}^{\mu-1}\mathbf{E}\|y_r\|^2,\\
\mathcal{R}_2&\leqslant 8\sum_{r=0}^{\lambda-1}\mathbf{E}\big\|y_r\mathcal{A}t^{\rho_1-\rho_2+r}E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}\big(\mathcal{A}t^{\rho_1-\rho_2}\big)\big\|^2\leqslant 8a_2\sum_{r=0}^{\lambda-1}\mathbf{E}\|y_r\|^2,\\
\mathcal{R}_3&\leqslant 8\sup_{0\leqslant t\leqslant\mathcal{T}}\mathbf{E}\Bigg\|\int\limits_{-\tau_P}^{0}\Bigg(\sum_{j=0}^{k}(t-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\Bigg)\varphi(\xi)\,\mathrm{d}\xi\Bigg\|^2\\
&\leqslant 8uv,\\
\mathcal{R}_4&\leqslant 8\sup_{0\leqslant t\leqslant\mathcal{T}}\mathbf{E}\Bigg\|\int\limits_{0}^{t}(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)h\big(\xi,y(\xi)\big)\,\mathrm{d}\xi\Bigg\|^2\\
&\leqslant 8a_3\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\tilde{N}\mathcal{T}\int\limits_{0}^{\mathcal{T}}\big(1+\mathbf{E}\big\|y(\xi)\big\|^2\big)\,\mathrm{d}\xi,\\
\mathcal{R}_5&\leqslant 8\sup_{0\leqslant t\leqslant\mathcal{T}}\mathbf{E}\Bigg\|\int\limits_{0}^{t}(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\Bigg(\int\limits_{0}^{\eta}\sigma\big(\vartheta,y(\vartheta)\big)\,\mathrm{d}w(\vartheta)\Bigg)\mathrm{d}\xi\Bigg\|^2\\
&\leqslant 32a_3\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}L_\sigma\tilde{L}\int\limits_{0}^{\mathcal{T}}\Bigg(\int\limits_{0}^{\eta}\big(1+\mathbf{E}\big\|y(\vartheta)\big\|^2\big)\,\mathrm{d}\vartheta\Bigg)\,\mathrm{d}\xi,\\
\mathcal{R}_6&\leqslant 8\sup_{0\leqslant t\leqslant\mathcal{T}}\mathbf{E}\Bigg\|\int\limits_{0}^{t}(t-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\mathcal{B}u(\xi)\,\mathrm{d}\xi\Bigg\|^2\\
&\leqslant\frac{8}{3}\mathbf{E}\Bigg\|\int\limits_{0}^{\mathcal{T}}\mathbb{D}_1(\mathcal{T},t)\mathbb{D}_1^*(\mathcal{T},t)W^{-1}\Bigg[y_{\mathcal{T}}-\sum_{r=0}^{\mu-1}y_r\mathcal{T}^rE_{\rho_1-\rho_2,1+r}\big(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2}\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{r=0}^{\lambda-1} y_r \mathcal{A}\mathcal{T}^{\rho_1-\rho_2+r} E_{\rho_1-\rho_2,\rho_1-\rho_2+1+r}\big(\mathcal{A}\mathcal{T}^{\rho_1-\rho_2}\big)\\
&-\int_{-\tau_P}^{0}\left(\sum_{j=0}^{k}(\mathcal{T}-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\right)\varphi(\xi)\,\mathrm{d}\xi\\
&-\int_{0}^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)h\big(\xi,y(\xi)\big)\,\mathrm{d}\xi\\
&-\int_{0}^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\left(\int_{0}^{\eta}\sigma\big(\vartheta,y(\vartheta)\big)\,\mathrm{d}w(\vartheta)\right)\mathrm{d}\xi\Bigg]\,\mathrm{d}\xi\Bigg\|^2\\
\leqslant{}& \frac{8}{3}M^2l^2\mathcal{T}\Bigg[\mathbf{E}\|y_{\mathcal{T}}\|^2 + a_1\sum_{r=0}^{\mu-1}\mathbf{E}\|y_r\|^2 + a_2\sum_{r=0}^{\lambda-1}\mathbf{E}\|y_r\|^2 + uv\\
&+4a_3L_\sigma\tilde{L}\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\int_{0}^{\mathcal{T}}\left(\int_{0}^{\eta}\big(1+\mathbf{E}\big\|y(\vartheta)\big\|^2\big)\,\mathrm{d}\vartheta\right)\mathrm{d}\xi\\
&+a_3\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\tilde{N}\mathcal{T}\int_{0}^{\mathcal{T}}\big(1+\mathbf{E}\big\|y(\xi)\big\|^2\big)\,\mathrm{d}\xi\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_7 \leqslant{}& \frac{8}{3}\tilde{M}^2l^2\big((\mathcal{T}-\tau_i)-(\mathcal{T}-\tau_{i+1})\big)\Bigg[\mathbf{E}\|y_{\mathcal{T}}\|^2 + a_1\sum_{r=0}^{\mu-1}\mathbf{E}\|y_r\|^2\\
&+a_2\sum_{r=0}^{\lambda-1}\mathbf{E}\|y_r\|^2 + uv + 4a_3L_\sigma\tilde{L}\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\int_{0}^{\mathcal{T}}\left(\int_{0}^{\eta}\big(1+\mathbf{E}\big\|y(\vartheta)\big\|^2\big)\,\mathrm{d}\vartheta\right)\mathrm{d}\xi\\
&+a_3\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\tilde{N}\mathcal{T}\int_{0}^{\mathcal{T}}\big(1+\mathbf{E}\big\|y(\xi)\big\|^2\big)\,\mathrm{d}\xi\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_8 \leqslant{}& \frac{8}{3}\hat{M}^2l^2(\mathcal{T}-\tau_P)\Bigg[\mathbf{E}\|y_{\mathcal{T}}\|^2 + a_1\sum_{r=0}^{\mu-1}\mathbf{E}\|y_r\|^2\\
&+a_2\sum_{r=0}^{\lambda-1}\mathbf{E}\|y_r\|^2 + uv + 4a_3L_\sigma\tilde{L}\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\int_{0}^{\mathcal{T}}\left(\int_{0}^{\eta}\big(1+\mathbf{E}\big\|y(\vartheta)\big\|^2\big)\,\mathrm{d}\vartheta\right)\mathrm{d}\xi\\
&+a_3\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\tilde{N}\mathcal{T}\int_{0}^{\mathcal{T}}\big(1+\mathbf{E}\big\|y(\xi)\big\|^2\big)\,\mathrm{d}\xi\Bigg].
\end{aligned}
$$

Then

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant \mathcal{T}} \mathbf{E}\big\|(\xi y)(t)\big\|^2 \\
&\quad\leqslant 8a_1\sum_{r=0}^{\mu-1}\mathbf{E}\|y_r\|^2 + 8a_2\sum_{r=0}^{\lambda-1}\mathbf{E}\|y_r\|^2 + uv + 8a_3\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}\tilde{N}\mathcal{T}\int_0^{\mathcal{T}}\big(1+\mathbf{E}\big\|y(\xi)\big\|^2\big)\,\mathrm{d}\xi \\
&\qquad + 32a_3\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-1}L_\sigma\tilde{L}\int_0^{\mathcal{T}}\left(\int_0^{\eta}\big(1+\mathbf{E}\big\|y(\vartheta)\big\|^2\big)\,\mathrm{d}\vartheta\right)\mathrm{d}\xi + \frac{8}{3}M^2l^2\mathcal{T}\omega \\
&\qquad + \frac{8}{3}\tilde{M}^2l^2\big((\mathcal{T}-\tau_i)-(\mathcal{T}-\tau_{i+1})\big)\omega + \frac{8}{3}\hat{M}^2l^2(\mathcal{T}-\tau_P)\omega \\
&\quad\leqslant C(1+\mathcal{T})\left[\int_0^{\mathcal{T}}\big(1+\mathbf{E}\big\|y(\xi)\big\|^2\big)\,\mathrm{d}\xi\right] \leqslant C\Big(1+\mathcal{T}\sup_{0\leqslant t\leqslant \mathcal{T}}\mathbf{E}\big\|y(\xi)\big\|^2\Big), \quad t\in[0,\mathcal{T}].
\end{aligned}$$

Here C is a constant, which gives that ξ maps I into itself.

Next, for any $x, y \in I$, we shall prove that ξ is a contraction mapping on I,

$$\begin{aligned}
&\mathbf{E}\big\|(\xi x)(t)-(\xi y)(t)\big\|^2 \\
&\quad\leqslant 8\sup_{0\leqslant t\leqslant \mathcal{T}}\mathbf{E}\Bigg\|\mathbb{D}_1(\mathcal{T},t)\mathbb{D}_1^{\mathcal{T}}(\mathcal{T},t)W^{-1} \\
&\qquad\times\Bigg[\int_0^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\left(\int_0^{\eta}\big[\sigma\big(\vartheta,x(\vartheta)\big)-\sigma\big(\vartheta,y(\vartheta)\big)\big]\,\mathrm{d}w(\vartheta)\right)\mathrm{d}\xi \\
&\qquad+\int_0^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\big(h\big(\xi,x(\xi)\big)-h\big(\xi,y(\xi)\big)\big)\,\mathrm{d}\xi\Bigg] \\
&\qquad+\mathbb{D}_2(\mathcal{T},t)\mathbb{D}_2^{\mathcal{T}}(\mathcal{T},t)W^{-1} \\
&\qquad\times\Bigg[\int_0^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\left(\int_0^{\eta}\big[\sigma\big(\vartheta,x(\vartheta)\big)-\sigma\big(\vartheta,y(\vartheta)\big)\big]\,\mathrm{d}w(\vartheta)\right)\mathrm{d}\xi \\
&\qquad+\int_0^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\big(h\big(\xi,x(\xi)\big)-h\big(\xi,y(\xi)\big)\big)\,\mathrm{d}\xi\Bigg] \\
&\qquad+\mathbb{D}_3(\mathcal{T},t)\mathbb{D}_3^{\mathcal{T}}(\mathcal{T},t)W^{-1} \\
&\qquad\times\Bigg[\int_0^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\left(\int_0^{\eta}\big[\sigma\big(\vartheta,x(\vartheta)\big)-\sigma\big(\vartheta,y(\vartheta)\big)\big]\,\mathrm{d}w(\vartheta)\right)\mathrm{d}\xi \\
&\qquad+\int_0^{\mathcal{T}}(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\big(h\big(\xi,x(\xi)\big)-h\big(\xi,y(\xi)\big)\big)\,\mathrm{d}\xi\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t (t-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\Bigg(\int_0^\eta \big[\sigma\big(\vartheta, x(\vartheta)\big) - \sigma\big(\vartheta, y(\vartheta)\big)\big]\,\mathrm{d}w(\vartheta)\Bigg)\,\mathrm{d}\xi \\
&+ \int_0^{\mathcal{T}} (\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(t-\xi)^{\rho_1-\rho_2}\big)\big(h\big(\xi, x(\xi)\big) - h\big(\xi, y(\xi)\big)\big)\,\mathrm{d}\xi\Bigg\|^2 \\
&\leqslant \big(8M^2l^2 + 8\tilde{M}^2l^2 + 8\hat{M}^2l^2 + 8\big) \\
&\quad\times \mathbf{E}\Bigg\|\int_0^{\mathcal{T}} (\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\Bigg(\int_0^\eta \big[\sigma\big(\vartheta, x(\vartheta)\big) - \sigma\big(\vartheta, y(\vartheta)\big)\big]\,\mathrm{d}w(\vartheta)\Bigg)\mathrm{d}\xi\Bigg\|^2 \\
&\quad+ \big(8M^2l^2 + 8\tilde{M}^2l^2 + 8\hat{M}^2l^2 + 8\big) \\
&\quad\times \mathbf{E}\Bigg\|\int_0^{\mathcal{T}} (\mathcal{T}-\xi)^{\rho_1-1} E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi)^{\rho_1-\rho_2}\big)\big(h\big(\xi, x(\xi)\big) - h\big(\xi, y(\xi)\big)\big)\,\mathrm{d}\xi\Bigg\|^2 \\
&\leqslant 32a_3\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-2}L_\sigma\tilde{L}\big(M^2l^2 + \tilde{M}^2l^2 + \hat{M}^2l^2 + 1\big)\int_0^{\mathcal{T}}\Bigg(\int_0^\eta \big(\mathbf{E}\big\|x(\vartheta) - y(\vartheta)\big\|^2\big)\,\mathrm{d}\vartheta\Bigg)\,\mathrm{d}\xi \\
&\quad+ 8a_3\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-2}\tilde{N}\mathcal{T}\big(M^2l^2 + \tilde{M}^2l^2 + \hat{M}^2l^2 + 1\big)\int_0^{\mathcal{T}}\big(\mathbf{E}\big\|x(\xi) - y(\xi)\big\|^2\big)\,\mathrm{d}\xi \\
&\leqslant 8a_3\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-2}\big(M^2l^2 + \tilde{M}^2l^2 + \hat{M}^2l^2 + 1\big)\big(4L_\sigma\tilde{L} + \tilde{N}\mathcal{T}\big)\sup_{0\leqslant t\leqslant\mathcal{T}}\mathbf{E}\big\|x(t) - y(t)\big\|^2\,\mathrm{d}\xi.
\end{aligned}
$$

Hence, if

$$
8a_3\frac{\mathcal{T}^{2\rho_1-1}}{2\rho_1-2}\big(M^2l^2 + \tilde{M}^2l^2 + \hat{M}^2l^2 + 1\big)\big(4L_\sigma\tilde{L} + \tilde{N}\mathcal{T}\big) \leqslant 1,
$$

then ξ is a contraction mapping on I. Now the Banach contraction fixed point theorem guarantees that ξ has a unique fixed point. Therefore, the solution of system (5)–(7) is $y(t)$ that given by (8), and we can see that $y(\mathcal{T}) = y_{\mathcal{T}}$. Moreover, the control $u(t)$ drives the state of system (5)–(7) from y_0 to final state $y_{\mathcal{T}}$ on $[0, \mathcal{T}]$. Consequently, system (5)–(7) is controllable on $[0, \mathcal{T}]$. □

5 Examples

Example 1. Consider the following linear damped fractional stochastic system involving multiple delays:

$$
\begin{aligned}
&{}_0^C D_t^{\rho_1} y(t) - \mathcal{A}\,{}_0^C D_t^{\rho_2} y(t) \\
&\qquad= \mathcal{B}u(t) + \sum_{i=0}^{P} \mathcal{C}_i u(t-\tau_i) + \sigma(t)\frac{\mathrm{d}w(t)}{\mathrm{d}t}, \quad t \in [0, \mathcal{T}],
\end{aligned}
\tag{12}
$$

where $\mu - 1 < \rho_1 \leqslant \mu$, $\lambda - 1 < \rho_2 \leqslant \lambda$, $\mu, \lambda \in N$, $y \in \mathbb{R}^2$, $t \in \mathcal{J}$ and

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{C}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathcal{C}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$
$$\sigma(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}.$$

By Theorem 1, the Grammian matrix W defined as

$$\begin{aligned} W = {} & \int_0^{\mathcal{T}} \bigl[(\mathcal{T} - \xi)^{\rho_1 - 1} E_{\rho_1 - \rho_2, \rho_1}\bigl(\mathcal{A}(\mathcal{T} - \xi)^{\rho_1 - \rho_2}\bigr)\mathcal{B}\bigr] \\ & \times \bigl[(\mathcal{T} - \xi)^{\rho_1 - 1} E_{\rho_1 - \rho_2, \rho_1}\bigl(\mathcal{A}(\mathcal{T} - \xi)^{\rho_1 - \rho_2}\bigr)\mathcal{B}\bigr]^* \,\mathrm{d}\xi \\ & + \sum_{i=0} \int_{\mathcal{T} - \tau_1}^{\mathcal{T} - \tau_0} \bigl[(\mathcal{T} - \xi - \tau_0)^{\rho_1 - 1} E_{\rho_1 - \rho_2, \rho_1}\bigl(\mathcal{A}(\mathcal{T} - \xi - \tau_0)^{\rho_1 - \rho_2}\bigr)\mathcal{C}_0\bigr] \\ & \times \bigl[(\mathcal{T} - \xi - \tau_0)^{\rho_1 - 1} E_{\rho_1 - \rho_2, \rho_1}\bigl(\mathcal{A}(\mathcal{T} - \xi - \tau_0)^{\rho_1 - \rho_2}\bigr)\mathcal{C}_0\bigr]^* \,\mathrm{d}\xi \\ & + \int_0^{\mathcal{T} - \tau_1} \Biggl[\sum_{j=0}^{1} (\mathcal{T} - \xi - \tau_j)^{\rho_1 - 1} E_{\rho_1 - \rho_2, \rho_1}\bigl(\mathcal{A}(\mathcal{T} - \xi - \tau_j)^{\rho_1 - \rho_2}\bigr)\mathcal{C}_j\Biggr] \\ & \times \Biggl[\sum_{j=0}^{1} (\mathcal{T} - \xi - \tau_j)^{\rho_1 - 1} E_{\rho_1 - \rho_2, \rho_1}\bigl(\mathcal{A}(\mathcal{T} - \xi - \tau_j)^{\rho_1 - \rho_2}\bigr)\mathcal{C}_j\Biggr]^* \,\mathrm{d}\xi. \end{aligned}$$

The Mittag-Leffler function is given by

$$E_{\rho_1 - \rho_2, \rho_1}\bigl(At^{\rho_1 - \rho_2}\bigr) = \sum_{k=0}^{\infty} \frac{A^k t^{k(\rho_1 - \rho_2)}}{\Gamma((\rho_1 - \rho_2)k + \rho_1)},$$

$$\begin{aligned} & E_{\rho_1 - \rho_2, \rho_1}(A(t - \xi)^{\rho_1 - \rho_2}) \\ & \quad = \frac{1}{\Gamma(\rho_1)} I + \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix} \frac{(t - \xi)^{\rho_1 - \rho_2}}{\Gamma(2\rho_1 - \rho_2)} + \begin{pmatrix} -1 & 4 & 2 \\ -1 & 3 & 1 \\ -1 & 2 & 0 \end{pmatrix} \frac{(t - \xi)^{2(\rho_1 - \rho_2)}}{\Gamma(3\rho_1 - 2\rho_2)} \\ & \qquad + \begin{pmatrix} -3 & 8 & 2 \\ -2 & 5 & 1 \\ -1 & 2 & 0 \end{pmatrix} \frac{(t - \xi)^{3(\rho_1 - \rho_2)}}{\Gamma(4\rho_1 - 3\rho_2)} + \cdots, \end{aligned}$$

$$(\mathcal{T} - \xi)^{\rho_1 - 1} E_{\rho_1 - \rho_2, \rho_1}\bigl(A(\mathcal{T} - \xi)^{\rho_1 - \rho_2}\bigr) = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix},$$

where

$$a_1 = \frac{(\mathcal{T}-\xi)^{\rho_1-1}}{\Gamma(\rho_1)} + \frac{(\mathcal{T}-\xi)^{2\rho_1-\rho_2-1}}{\Gamma(2\rho_1-\rho_2)} + \frac{(\mathcal{T}-\xi)^{3\rho_1-2\rho_2-1}}{\Gamma(3\rho_1-2\rho_2)} + \cdots,$$

$$a_2 = \frac{4(\mathcal{T}-\xi)^{3\rho_1-2\rho_2-1}}{\Gamma(3\rho_1-2\rho_2)} + \frac{8(\mathcal{T}-\xi)^{4\rho_1-3\rho_2-1}}{\Gamma(4\rho_1-3\rho_2)} + \cdots,$$

$$a_3 = \frac{2(\mathcal{T}-\xi)^{2\rho_1-\rho_2-1}}{\Gamma(2\rho_1-\rho_2)} + \frac{2(\mathcal{T}-\xi)^{3\rho_1-2\rho_2-1}}{\Gamma(3\rho_1-2\rho_2)} + \frac{2(\mathcal{T}-\xi)^{4\rho_1-3\rho_2-1}}{\Gamma(4\rho_1-3\rho_2)} + \cdots,$$

$$a_4 = \frac{(-1)(\mathcal{T}-\xi)^{3\rho_1-2\rho_2-1}}{\Gamma(3\rho_1-2\rho_2)} + \frac{(-2)(\mathcal{T}-\xi)^{4\rho_1-3\rho_2-1}}{\Gamma(4\rho_1-3\rho_2)} + \cdots,$$

$$a_5 = \frac{(\mathcal{T}-\xi)^{\rho_1-1}}{\Gamma(\rho_1)} + \frac{(\mathcal{T}-\xi)^{2\rho_1-\rho_2-1}}{\Gamma(2\rho_1-\rho_2)} + \frac{3(\mathcal{T}-\xi)^{3\rho_1-2\rho_2-1}}{\Gamma(3\rho_1-2\rho_2)} + \cdots,$$

$$a_6 = \frac{(\mathcal{T}-\xi)^{2\rho_1-\rho_2-1}}{\Gamma(2\rho_1-\rho_2)} + \frac{(\mathcal{T}-\xi)^{3\rho_1-2\rho_2-1}}{\Gamma(3\rho_1-2\rho_2)} + \frac{(\mathcal{T}-\xi)^{4\rho_1-3\rho_2-1}}{\Gamma(4\rho_1-3\rho_2)} + \cdots,$$

$$a_7 = \frac{(-1)(\mathcal{T}-\xi)^{2\rho_1-\rho_2-1}}{\Gamma(2\rho_1-\rho_2)} + \frac{(-1)(\mathcal{T}-\xi)^{3\rho_1-2\rho_2-1}}{\Gamma(3\rho_1-2\rho_2)} + \frac{(-1)(\mathcal{T}-\xi)^{4\rho_1-3\rho_2-1}}{\Gamma(4\rho_1-3\rho_2)} + \cdots,$$

$$a_8 = \frac{2(\mathcal{T}-\xi)^{2\rho_1-\rho_2-1}}{\Gamma(2\rho_1-\rho_2)} + \frac{2(\mathcal{T}-\xi)^{3\rho_1-2\rho_2-1}}{\Gamma(3\rho_1-2\rho_2)} + \frac{2(\mathcal{T}-\xi)^{4\rho_1-3\rho_2-1}}{\Gamma(4\rho_1-3\rho_2)} + \cdots,$$

$$a_9 = \frac{(\mathcal{T}-\xi)^{\rho_1-1}}{\Gamma(\rho_1)},$$

$$(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\bigl(A(\mathcal{T}-\xi)^{\rho_1-\rho_2}\bigr)\mathcal{B} = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_2 & a_3 \\ a_5 & a_6 \\ a_8 & a_9 \end{pmatrix},$$

$$\begin{aligned}
&\bigl[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\bigl(A(\mathcal{T}-\xi)^{\rho_1-\rho_2}\bigr)\mathcal{B}\bigr] \\
&\qquad\times \bigl[(\mathcal{T}-\xi)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\bigl(A(\mathcal{T}-\xi)^{\rho_1-\rho_2}\bigr)\mathcal{B}\bigr]^* \\
&\quad= \begin{pmatrix} a_2 & a_3 \\ a_5 & a_6 \\ a_8 & a_9 \end{pmatrix}\begin{pmatrix} a_2 & a_5 & a_8 \\ a_3 & a_6 & a_9 \end{pmatrix} = \begin{pmatrix} a_2^2+a_3^2 & a_2a_5+a_3a_6 & a_2a_8+a_3a_9 \\ a_2a_5+a_3a_6 & a_5^2+a_6^2 & a_5a_8+a_6a_9 \\ a_2a_8+a_3a_9 & a_5a_8+a_6a_9 & a_8^2+a_9^2 \end{pmatrix} \\
&\quad= a^*.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\bigl[(\mathcal{T}-\xi-\tau_0)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\bigl(\mathcal{A}(\mathcal{T}-\xi-\tau_0)^{\rho_1-\rho_2}\bigr)\mathcal{C}_0\bigr] \\
&\quad= \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b_2 \\ b_5 \\ b_8 \end{pmatrix},
\end{aligned}$$

where

$$b_1 = \frac{(\mathcal{T}-\xi-\tau_0)^{\rho_1-1}}{\Gamma(\rho_1)} + \frac{(\mathcal{T}-\xi-\tau_0)^{2\rho_1-\rho_2-1}}{\Gamma(2\rho_1-\rho_2)} + \frac{(\mathcal{T}-\xi-\tau_0)^{3\rho_1-2\rho_2-1}}{\Gamma(3\rho_1-2\rho_2)} + \cdots,$$

$$
\begin{aligned}
b_2 &= \frac{4(\mathcal{T}-\xi-\tau_0)^{3\rho_1-2\rho_2-1}}{\Gamma(3\rho_1-2\rho_2)} + \frac{8(\mathcal{T}-\xi-\tau_0)^{4\rho_1-3\rho_2-1}}{\Gamma(4\rho_1-3\rho_2)} + \cdots, \\
b_3 &= \frac{2(\mathcal{T}-\xi-\tau_0)^{2\rho_1-\rho_2-1}}{\Gamma(2\rho_1-\rho_2)} + \frac{2(\mathcal{T}-\xi-\tau_0)^{3\rho_1-2\rho_2-1}}{\Gamma(3\rho_1-2\rho_2)} + \frac{2(\mathcal{T}-\xi-\tau_0)^{4\rho_1-3\rho_2-1}}{\Gamma(4\rho_1-3\rho_2)} + \cdots, \\
&\ldots, \\
b_9 &= \frac{(\mathcal{T}-\xi-\tau_0)^{\rho_1-1}}{\Gamma(\rho_1)}.
\end{aligned}
$$

$$
\begin{aligned}
&\big[(\mathcal{T}-\xi-\tau_1)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_1)^{\rho_1-\rho_2}\big)\mathcal{C}_1\big] \\
&\quad = \begin{pmatrix} d_1 & d_2 & d_3 \\ d_4 & d_5 & d_6 \\ d_7 & d_8 & d_9 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} d_3 \\ d_6 \\ d_9 \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{(\mathcal{T}-\xi-\tau_1)^{\rho_1-1}}{\Gamma(\rho_1)} + \frac{(\mathcal{T}-\xi-\tau_1)^{2\rho_1-\rho_2-1}}{\Gamma(2\rho_1-\rho_2)} + \frac{(\mathcal{T}-\xi-\tau_1)^{3\rho_1-2\rho_2-1}}{\Gamma(3\rho_1-2\rho_2)} + \cdots, \\
d_2 &= \frac{4(\mathcal{T}-\xi-\tau_1)^{3\rho_1-2\rho_2-1}}{\Gamma(3\rho_1-2\rho_2)} + \frac{8(\mathcal{T}-\xi-\tau_1)^{4\rho_1-3\rho_2-1}}{\Gamma(4\rho_1-3\rho_2)} + \cdots, \\
d_3 &= \frac{2(\mathcal{T}-\xi-\tau_1)^{2\rho_1-\rho_2-1}}{\Gamma(2\rho_1-\rho_2)} + \frac{2(\mathcal{T}-\xi-\tau_1)^{3\rho_1-2\rho_2-1}}{\Gamma(3\rho_1-2\rho_2)} \\
&\quad + \frac{2(\mathcal{T}-\xi-\tau_1)^{4\rho_1-3\rho_2-1}}{\Gamma(4\rho_1-3\rho_2)} + \cdots, \\
&\ldots, \\
d_9 &= \frac{(\mathcal{T}-\xi-\tau_1)^{\rho_1-1}}{\Gamma(\rho_1)}.
\end{aligned}
$$

$$
\begin{aligned}
&\big[(\mathcal{T}-\xi-\tau_0)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_0)^{\rho_1-\rho_2}\big)\mathcal{C}_0\big] \\
&\quad \times \big[(\mathcal{T}-\xi-\tau_0)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_0)^{\rho_1-\rho_2}\big)\mathcal{C}_0\big]^* \\
&\quad = \begin{pmatrix} b_2^2 & b_2b_5 & b_2b_8 \\ b_2b_5 & b_5^2 & b_5b_8 \\ b_2b_8 & b_5b_8 & b_8^2 \end{pmatrix} = b^*,
\end{aligned}
$$

$$
\begin{aligned}
&\left[\sum_{j=0}^{1}(\mathcal{T}-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\right] \\
&\quad \times \left[\sum_{j=0}^{1}(\mathcal{T}-\xi-\tau_j)^{\rho_1-1}E_{\rho_1-\rho_2,\,\rho_1}\big(\mathcal{A}(\mathcal{T}-\xi-\tau_j)^{\rho_1-\rho_2}\big)\mathcal{C}_j\right]^* \\
&\quad = \begin{pmatrix} b_2^2+d_3^2 & b_2b_5+d_3d_6 & b_2b_8+d_3d_9 \\ b_2b_5+d_3d_6 & b_5^2+d_6^2 & b_5b_8+d_6d_9 \\ b_2b_8+d_3d_9 & b_5b_8+d_6d_9 & b_8^2+d_9^2 \end{pmatrix} = d^*.
\end{aligned}
$$

Thus, the matrix W is obtained by

$$W = \int_0^{\mathcal{T}} a^* \,\mathrm{d}\xi + \int_{\mathcal{T}-\tau_1}^{\mathcal{T}-\tau_0} b^* \,\mathrm{d}\xi + \int_0^{\mathcal{T}-\tau_1} d^* \,\mathrm{d}\xi > 0, \quad \mathcal{T} > 0.$$

From the above, we have shown that W is a nonsingular matrix. Thus, we can ensure that system (12) is controllable on $[0, \mathcal{T}]$.

Example 2. Consider the following nonlinear damped fractional stochastic system involving multiple delays:

$$\begin{aligned} &{}_0^C D_t^{\rho_1} y(t) - \mathcal{A}\,{}_0^C D_t^{\rho_2} y(t) \\ &\quad = \mathcal{B}u(t) + \sum_{i=0}^{P} \mathcal{C}_i u(t-\tau_i) + h\big(t, y(t)\big) + \sigma\big(t, y(t)\big)\frac{\mathrm{d}w(t)}{\mathrm{d}t}, \quad t \in [0, \mathcal{T}], \end{aligned} \tag{13}$$

where $\mu - 1 < \rho_1 \leqslant \mu$, $\lambda - 1 < \rho_2 \leqslant \lambda$, $\mu, \lambda \in \mathbb{N}$, $y \in \mathbb{R}^2$, $t \in \mathcal{J}$ and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}_0$, $\mathcal{C}_1$ and $y(t)$ are as above,

$$\sigma\big(t, y(t)\big) = \begin{pmatrix} \ln(\cosh(y_1)) \\ \frac{\tan^{-1}(y_2)}{10t} \\ \sin(y_3) \end{pmatrix}, \qquad h\big(t, y(t)\big) = \begin{pmatrix} \sin y_1(t) \\ \cos y_2(t) \\ y_3(t) \end{pmatrix}.$$

Since the corresponding linear fractional system is controllable by Example 1, $\sigma(t, y(t))$ and $f(t, y(t))$ satisfy the assumptions of Theorem 2. Based on that, we conclude that system (13) is controllable on $[0, \mathcal{T}]$.

6 Conclusion

In this paper, we have analyzed the higher-order fractional damped stochastic system involving multiple delays in both linear and nonlinear cases. Based on controllability Grammian matrix, controllability results for the considered linear damped fractional stochastic system have been attained under suitable assumptions. Some sufficient conditions, which ensure the controllability of nonlinear damped fractional stochastic system containing multiple delays in control have been derived by applying the fixed point technique. Examples were provided to verify the established criteria. The proposed technique could be implemented to other type of fractional-order dynamical systems. An interesting extension would be to study the controllability concept for the fractional damped nonlinear equation involving time-varying delay or fractional damped stochastic system with various delay effects. This area will be the future focus of our research.

References

1. A. Anguraj, K. Ravikumar, J.J. Nieto, On stability of stochastic differential equations with random impulses driven by Poisson jumps, *Stochastics*, **93**(5):682–696, 2021, `https://doi.org/10.1080/17442508.2020.1783264`.

2. R.L. Bagley, P.J. Torvik, Fractional calculus in the transient analysis of viscoelastically damped structures, *AIAA Journal*, **23**(6):918–925, 1985, https://doi.org/10.2514/3.9007.
3. J. Cui, L. Yan, Controllability of neutral stochastic evolution equations driven by fractional Brownian motion, *Acta Math. Sci., Ser. B, Engl. Ed.*, **37**(1):108–118, 2017, https://doi.org/10.1016/S0252-9602(16)30119-9.
4. G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge Univ. Press, Cambridge, 1992, https://doi.org/10.1017/CBO9780511666223.
5. X. Dai, F. Yang, Complete controllability of impulsive stochastic integrodifferential systems in Hilbert space, *Abstr. Appl. Anal.*, **2013**:783098, 2013, https://doi.org/10.1155/2013/783098.
6. X.L. Ding, C.L. Daniel, J.J. Nieto, A new generalized Gronwall inequality with a double singularity and its applications to fractional stochastic differential equations, *Stochastic Anal. Appl.*, **37**(6):1042–1056, 2019, https://doi.org/10.1080/07362994.2019.1640612.
7. Y. Ding, Y. Li, Controllability of fractional stochastic evolution equations with nonlocal conditions and noncompactsemigroups, *Open Math.*, **18**(1):616–631, 2020, https://doi.org/10.1515/math-2020-0034.
8. J. Du, D. Cui, Y. Sun, J.Xu, Approximate controllability for a kind of fractional neutral differential equations with damping, *Math. Probl. Eng.*, **2020**, 2020, https://doi.org/10.1155/2020/7592818.
9. S.B. Fernandez, J.J. Nieto, Basic control theory for linear fractional differential equations with constant coefficients, *Front. Phys.*, **8**, 2020, https://doi.org/10.3389/fphy.2020.00377.
10. T. Guendouzi, I. Hamada, Relative controllability of fractional stochastic dynamical systems with multiple delays in control, *Malaya J. Mat.*, **1**(1):86–97, 2013.
11. B.B. He, H.C. Zhou, C.H. Kou, The controllability of fractional damped dynamical systems with control delay, *Commun. Nonlinear Sci. Numer. Simul.*, **32**:190–198, 2016, https://doi.org/10.1016/j.cnsns.2015.08.011.
12. L. Hu, Y. Ren, Existence results for impulsive neutral stochastic functional integro–differential equations with infinite delays, *Acta Appl. Math.*, **111**:303–317, 2010, https://doi.org/10.1007/s10440-009-9546-x.
13. D.D. Huan, On the controllability of nonlocal second-order impulsive neutral stochastic integro–differential equations with infinite delay, *Asian J. Control*, **17**(4):1233–1242, 2015, https://doi.org/10.1002/asjc.972.
14. X. Li, Z. Liu, J. Li, C. Tisdell, Existence and controllability for nonlinear fractional control systems with damping in hilbert spaces, *Acta Math. Sci.*, **39**(1):229–242, 2019, https://doi.org/10.1007/s10473-019-0118-5.
15. A. Lin, L. Hu, Existence results for impulsive neutral stochastic functional integro–differential inclusions with nonlocal initial conditions, *Comput. Math. Appl.*, **59**(1):64–73, 2010, https://doi.org/10.1016/j.camwa.2009.09.004.
16. R.L. Magin, *Fractional Calculus in Bioengineering*, Begell House, Redding, 2006.

17. F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*, World Scientific, Singapore, 2010.

18. K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, 1993.

19. C.A. Monje, Y. Chen, B.M. Vinagre, D. Xue, V. Feliu-Batlle, *Fractional-order Systems and Controls: Fundamentals and Applications*, Springer, London, 2010.

20. M. Nawaz, J.Wei, S.Jiale, The controllability of fractional differential system with state and control delay, *Adv. Differ. Equ.*, **30**:1–11, 2020, `https://doi.org/10.1186/s13662-019-2479-4`.

21. J. Sabatier, O.P. Agarwal, J.A. Tenreiro Machado, *Advances in Fractional Calculus* , Springer, Netherlands, 2007.

22. L. Shen, J. Sun, Relative controllability of stochastic nonlinear systems with delay in control, *Nonlinear Anal., Real World Appl.*, **13**:2880–2887, 2012, `https://doi.org/10.1016/j.nonrwa.2012.04.017`.

23. X. Sun, L. Yan, J. Cui, Controllability of fractional neutral stochastic integro–differential systems with infinite delay, *Math. Probl. Eng.*, **2013**, 2013, `https://doi.org/10.1155/2013/316906`.

24. A. Tofighi, The intrinsic damping of the fractional oscillator, *Physica A*, **329**(1):29–34, 2003, `https://doi.org/10.1016/S0378-4371(03)00598-3`.

25. J. Wang, A.G. Ibrahim, M. Feckan, Y. Zhou, Controllability of fractional non-instantaneous impulsive differential inclusions without compactness, *IMA J. Math. Control Inf.*, **36**(2):443–460, 2017, `https://doi.org/10.1093/imamci/dnx055`.

26. Y. Wang, X. Zhou, Exact controllability of stochastic differential equations with memory, *Syst. Control Lett.*, **142**, 2020, `https://doi.org/j.sysconle.2020.104732`.

27. J. Wei, The controllability of fractional control systems with control delay, *Comput. Math. Appl.*, **64**(10):3153–3159, 2012, `https://doi.org/10.1016/j.camwa.2012.02.065`.

28. Z. Xu, F. Xie, The controllability of nonlinear fractional damped dynamical systems with control delay, *J. Shanghai Norm. Univ., Nat. Sci.*, **46**(3):375–383, 2017, `https://doi.org/10.3969/J.ISSN.100-5137.2017.03.005`.

29. J. Yang, A. Liu, T. Liu, Forced oscillation of nonlinear fractional differential equations with damping term, *Adv. Differ. Equ.*, **2015**:1, 2015, `https://doi.org/10.1186/s13662-014-0331-4`.

30. C. Yin, F. Liu, V. Anh, Numerical simulation of the nonlinear fractional dynamical systems with fractional damping for the extensible and inextensible pendulum, *J. Algorithms Comput. Technol.*, **1**(4):427–447, 2007, `https://doi.org/10.1260/174830107783133888`.