Nonlinear Analysis: Modelling and Control, Online First, 1–16
https://doi.org/10.15388/namc.2022.27.27538

Infinitely many sign-changing solutions for an elliptic equation involving double critical Hardy–Sobolev–Maz'ya terms*

Lixia Wang, Pingping Zhao, Dong Zhang

School of Sciences, Tianjin Chengjian University,
Tianjin 300384, China
wanglixia0311@126.com

Received: September 26, 2021 / **Revised:** January 3, 2022 / **Published online:** May 15, 2022

Abstract. In this paper, we consider the existence of infinitely many sign-changing solutions for an elliptic equation involving double critical Hardy–Sobolev–Maz'ya terms. By using a compactness result obtained in [C.H. Wang, J. Yang, Infinitely many solutions for an elliptic problem with double Hardy–Sobolev–Maz'ya terms, *Discrete Contin. Dyn. Syst.*, 36(3):1603–1628, 2016], we prove the existence of these solutions by a combination of invariant sets method and Ljusternik–Schnirelman-type minimax method.

Keywords: Hardy–Sobolev–Maz'ya exponents, invariant sets, sign-changing solutions, minimax method.

1 Introduction and main results

Let $N \geqslant 3$, $\mu \geqslant 0$, $0 \leqslant t < s < 2$, $2^*(t) = 2(N-t)/(N-2)$ and $2^*(s) = 2(N-s)/(N-2)$ are the critical Hardy–Sobolev–Maz'ya exponents, Ω is an open bounded domain in $\mathbb{R}^N$. We study the following equation:

$$\begin{aligned} &-\Delta u = \mu \frac{|u|^{2^*(t)-2}u}{|y|^t} + \frac{|u|^{2^*(s)-2}u}{|y|^s} + a(x)u \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $a(x)$ is a positive function, $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $2 \leqslant k < N$. It is well known that solutions of (1) are critical points of the corresponding functional $J : H_0^1(\Omega) \to \mathbb{R}$

*This research was supported by Scientific Research Program of Tianjin Education Commission (grant No. 2020KJ045).

given by

$$J(u) = \frac{1}{2}\int\limits_{\Omega} \left(|\nabla u|^2 - a(x)u^2\right) \mathrm{d}x - \frac{\mu}{2^*(t)}\int\limits_{\Omega} \frac{|u|^{2^*(t)}}{|y|^t}\,\mathrm{d}x - \frac{1}{2^*(s)}\int\limits_{\Omega} \frac{|u|^{2^*(s)}}{|y|^s}\,\mathrm{d}x. \quad (2)$$

By using the following Hardy–Sobolev–Maz'ya inequality (Lemma 1), we know that J is well defined and C^1 functional on $H_0^1(\Omega)$ for any open subset of $\mathbb{R}^N$.

Since (1) involves the double critical Hardy–Sobolev–Maz'ya exponents, we can use the pioneering idea of Brézis and Nirenberg [5], or the concentration compactness principle of Lions [16, 17], or the global compactness of Struwe [23] to show that (2) has a critical point, then get a positive solution to (1).

When $s = \mu = 0$, $a(x) = \lambda$ and $k = N$, (1) is related to the well-known Brézis–Nirenberg problem [5]

$$\begin{aligned} &-\Delta u = \lambda u + |u|^{2^*-2}u \quad \text{in } \Omega,\\ &u = 0 \quad \text{on} \partial\Omega, \end{aligned} \qquad (3)$$

where $2^* = 2N/(N-2)$ is the critical Sobolev exponent. Since the pioneering work of [5], there are some important results on this problem. See, e.g., [6, 8, 9, 11, 25]. Here we would like to point out [10]. In this paper, Devillanova and Solimini proved that when $N \geqslant 7$, (3) has infinitely many solutions for each $\lambda > 0$. Let us now briefly recall the main results concerning the sign-changing solutions of (3) obtained before. If $N \geqslant 4$ and Ω is a ball, then for any $\lambda > 0$, (3) has infinitely many nodal solutions, which are built by using particular symmetries of the domain Ω (see [12]). In [22], Solimini proved that if Ω is a ball and $N \geqslant 7$, for each $\lambda > 0$, (3) has infinitely many sign-changing radial solutions. When Ω is a ball and $4 \leqslant N \leqslant 6$, there is a $\lambda^* > 0$ such that (3) has no radial solutions, which change sign if $\lambda \in (0, \lambda^*)$ (see [2]). In [12, 22], the symmetry of the ball plays an essential role, hence their methods are invalid for general domains.

When $t = 2$, $a(x) = \lambda$, $k = N$, (1) is becoming Hardy–Sobolev–Maz'ya equation

$$\begin{aligned} &-\Delta u - \frac{\mu u}{|y|^2} = \lambda u + \frac{|u|^{2^*(s)-2}u}{|y|^s} \quad \text{in } \Omega,\\ &u = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

By using the idea of [10], the authors of [26] obtained infinitely many solutions for Hardy–Sobolev–Maz'ya equation. Ganguly [13] and Wang [29] used different methods to get infinitely many sign-changing solutions. For the existence of infinitely many solutions or infinitely many sign-changing solutions for the related equations, see [14, 24, 30, 32] and the references therein. Very recently, Wang and Yang [27] proved the existence of infinitely many sign-changing solutions for (1).

Theorem 1. *Suppose that $a((0, z^*)) > 0$ and Ω is a bounded domain. If $(0, z^*) \in \partial\Omega$, $(x - (0, z^*)) \cdot \nu \leqslant 0$ in a neighborhood of $(0, z^*)$, where ν is the outward normal of $\partial\Omega$. If $N > 6 + t$ when $\mu > 0$ and if $N > 6 + s$ when $\mu = 0$, then* (1) *has infinitely many sign-changing solutions.*

Wang and Yang also considered the following nonexistence theorem.

Theorem 2. (*See* [27].) *Suppose that* $N \geqslant 3$*,* $a(x) \in C^1(\bar{\Omega})$ *and* $(a(x)+(x/2)\cdot\nabla a) \leqslant 0$ *for every* $x \in \Omega$*. Then* (1) *does not have nontrivial solution in a domain, which is star shaped domain with respect to the origin.*

Remark 1. Let λ_1 be the first eigenvalue of

$$\begin{aligned} &-\Delta u = \lambda a(x)u \quad \text{in } \Omega \\ &u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4}$$

Since $a(x) \in C^1(\bar{\Omega})$ and is strictly positive, system (4) has infinitely many eigenvalues $\{\lambda_1, \lambda_2, \dots\}$ such that $0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leqslant \lambda_3(\Omega) \leqslant \cdots \leqslant \lambda_m(\Omega) \leqslant \cdots$. It is characterized by the following variational principle:

$$\lambda_1(\Omega) = \inf_{u\in H_0^1(\Omega),\, u\neq 0} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega a(x)u^2}. \tag{5}$$

Let e_m be the orthonormal eigenfunction corresponding to λ_m and $e_m > 0$. Denote

$$H_m := \operatorname{span}\{e_1, e_2, \dots, e_m\}.$$

Then $H_m \subset H_{m+1}$ and $H_0^1(\Omega) = \overline{\cup_{m=1}^{\infty} H_m}$. It is easy to know that if $\lambda_1 \leqslant 1$, equation (1) has infinitely many sign-changing solutions. Indeed, by multiplying the first eigenfunction e_1 and integrating both sides, then we can check that if $\lambda_1 \leqslant 1$, any nontrivial solution of (1) has to change sign. Therefore, by the result of [28], to prove Theorem 1 it suffices to consider the case of $\lambda_1 > 1$.

Remark 2. When $s = 0$, $t = 2$ and $k = N$, Cao and Peng [6] considered the following system:

$$\begin{aligned} &-\Delta u = |u|^{2^*-2}u + \mu\frac{u}{|x|^2} + \lambda u \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{6}$$

They obtained a pair of sign-changing solutions to (6). In [8,32], the authors get infinitely many sign-changing solutions for (6). They only considered the case $0 \in \Omega$. In another case, $0 \in \partial\Omega$, the mean curvature of $\partial\Omega$ at 0 plays an important role in the existence of mountain pass solutions, see [3,6,14]. As it is pointed in [4,31], there are some differences between the case $t = 2$ and $t \in (0, 2)$. When $t = 2$, solutions of (6) have a singularity at 0, and the authors of [8,32] impose the condition $\mu \in [0, (N-2)^2/4 - 4)$. If $t \in (0, 2)$, no such condition is needed. So the estimates for the case $t \in (0, 2)$ and the case $t = 2$ are very different. Therefore we have generalize the results in [32] to the case $0 \in \partial\Omega$.

Remark 3. In order to prove the results, Wang and Yang [27] first used an abstract theorem, which is introduced by Schechter and Zou [21]. Then by combining with the uniform bounded theorem due to [28], the authors of [27] obtained infinitely many sign-changing solutions. The methods introduced in [4, 8, 13, 21, 31] sometimes are limited because, by general minimax procedure to get the Morse indices of sign-changing critical points, sometimes are not clear. Another limited condition is that the corresponding functional is also needed to be C^2.

Before giving our main results, we give some notations first. We will always denote $0 < t < 2$. Let $E = H_0^1(\Omega)$ be endowed with the standard scalar and norm

$$(u, v) = \int_\Omega \nabla u \nabla v \, \mathrm{d}x; \qquad \|u\| = \left(\int_\Omega |\nabla u|^2 \, \mathrm{d}x \right)^{1/2}.$$

The norm on $L^s = L^s(\Omega)$ with $1 \leqslant s < \infty$ is given by $|u|_s = (\int_\Omega |u|^s \, \mathrm{d}x)^{1/s}$. $L_t^q(\Omega)$ $(1 \leqslant q < \infty, 0 \leqslant t < 2)$ with the norm $|u|_{q,t,\Omega} = (\int_\Omega |u|^q/|x|^t \, \mathrm{d}x)^{1/q}$, where $\mathrm{d}x$ denote the Lebesgue measure in $\mathbb{R}^N$. Denote $B_r = \{u \in E\colon \|u\| \leqslant r\}$ and $B_r^c := E \setminus B_r$.

We will use the usual Ljusternik–Schnirelman-type minimax method and invariant set method to prove Theorem 1. Our method is much simpler than the proof of [27]. In fact, our approach also works for the Brézis–Nirenberg problem involving subcritical perturbation term $f(x, u)$, which is not C^1. However, the techniques developed by Wang and Yang [27] or Schechter and Zou [21] cannot be applied directly. Let us outline the proof of Theorem 1 and explain the difficulties we will encounter.

In general, by using the combination of invariant sets method and minimax method to obtain infinitely many nodal critical points, we need the energy functional satisfies the Palais–Smale condition in all energy level. This fact prevents us from using the variational methods directly to prove the existence of infinitely many sign-changing solutions for (1) because $J(u)$ does not satisfy the Palais–Smale condition for large energy level due to the double critical Hardy–Sobolev–Maz'ya exponents $2^*(t)$ and $2^*(s)$.

In order to overcome the difficulty, we will adopt the idea in [10, 24] and [4, 31]. We first study the following perturbed problem:

$$\begin{aligned} &-\Delta u = \frac{\mu |u|^{2^*(t)-2-\varepsilon} u}{|y|^t} + \frac{|u|^{2^*(s)-2-\varepsilon} u}{|y|^s} + a(x) u \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{7}$$

where $\varepsilon > 0$ is a small constant. The corresponding energy functional is

$$\begin{aligned} J_\varepsilon(u) = {}& \frac{1}{2} \int_\Omega \left(|\nabla u|^2 - a(x) u^2 \right) - \frac{\mu}{2^*(t) - \varepsilon} \int_\Omega \frac{|u|^{2^*(t)-\varepsilon}}{|y|^t} \\ & - \frac{1}{2^*(s) - \varepsilon} \int_\Omega \frac{|u|^{2^*(s)-\varepsilon}}{|y|^s}. \end{aligned} \tag{8}$$

By the following lemmas, we will know $J_\varepsilon(u)$ is a C^1 function on $H_0^1(\Omega)$ and satisfies the Palais–Smale condition. It follows from [1, 20] that $J_\varepsilon(u)$ has infinitely many critical points. More precisely, there are positive numbers $c_{\varepsilon,l}$, $l = 2, 3, \ldots$, with $c_{\varepsilon,l} \to +\infty$ as $l \to \infty$. Moreover, a critical point $u_{\varepsilon,l}$ for $J_\varepsilon(u)$ satisfies $J_\varepsilon(u_{\varepsilon,l}) = c_{\varepsilon,l}$.

Next, we will show that for any fixed $l \geqslant 2$, $\|u_{\varepsilon,l}\|$ are uniformly bounded with respect to ε, then we can apply the following compactness result Proposition 1 (see [28, Thm. 1.3]), which essentially follows from the uniform bounded theorem due to Devillanova and Solimini [10], to show that $u_{\varepsilon,l}$ converges strongly to u_l in E as $\varepsilon \to 0$. Therefore it is easy to prove that u_l is a solution of (1) with $J(u_l) = c_l := \lim_{\varepsilon\to 0} c_{\varepsilon,l}$.

Proposition 1. (*See* [28].) *Suppose that* $a((0, z^*)) > 0$ *and* Ω *satisfies the conditions in Theorem* 1. *If* $N \geqslant 6 + t$ *when* $\mu > 0$ *and* $N > 6 + s$ *when* $\mu = 0$, *then for any sequence* u_n, *which is a solution of* (7) *with* $\varepsilon = \varepsilon_n \to 0$ *satisfying* $\|u_n\| \leqslant C$ *for some constant independent of* n, u_n *has a sequence, which converges strongly in* $H_0^1(\Omega)$ *as* $n \to \infty$.

In the end, we will distinguish two cases to prove that $J(u)$ has infinitely many sign-changing critical points.

Case I. There are $2 \leqslant l_1 < \cdots < l_i < \cdots$ satisfying $c_{l_1} < \cdots < c_{l_i} < \cdots$.

Case II. There is a positive integer L such that $c_l = c$ for all $l \geqslant L$.

The central task in this procedure is to deal with case II. In fact, we can prove that the usual Krasnoselskii genus of $K_c \setminus W$ (W is denoted in Section 2) is at least two, where $K_c := \{u \in E\colon\ J(u) = c,\ J'(u) = 0\}$. Then our result is obtained.

Throughout this paper, the letters $C, C_1, C_2, \dots$ will be used to denote various positive constants, which may vary from line to line and are not essential to the problem. The closure and the boundary of set G are denoted by $\bar{G}$ and ∂G, respectively. We denote "$\rightharpoonup$" weak convergence and by "$\rightarrow$" strong convergence. Also if we take a subsequence of a sequence $\{u_n\}$, we shall denote it again $\{u_n\}$.

The paper is organized as follows. In Section 2, we introduce some notations and Hardy–Sobolev–Maz'ya inequality. In Section 3, we give an auxiliary operator A_ε and construct the invariant sets. We give the proof of Theorem 1 in Section 4.

2 Preliminaries

Now we give some integrals inequalities, for details we refer to [19].

Lemma 1 [Hardy–Sobolev–Maz'ya inequality]. *Let* $N \geqslant 3$, $0 \leqslant t < 2$, *then there exist a positive constant* $S = S(\Omega, s)$ *such that*

$$\left(\int\limits_{\Omega} \frac{|u|^{2^*(s)}}{|y|^s}\,\mathrm{d}x\right)^{2/(2^*(s))} \leqslant S^{-1} \int\limits_{\Omega} |\nabla u|^2\,\mathrm{d}x \tag{9}$$

for all $u \in H_0^1(\Omega)$.

Lemma 2. (*See* [13].) *If* Ω *is a bounded subset of* $\mathbb{R}^N$, $0 \leqslant t < 2$, $N \geqslant 3$, *then*

$$L_t^p(\Omega) \subset L_t^q(\Omega)$$

with the inclusion being continuous whenever $1 \leqslant q \leqslant p < \infty$.

Remark 4. If $f \in L^p_t(\Omega)$ for $1 \leqslant p < \infty$, then $f \in L^p(\Omega)$ with

$$|f|_p \leqslant C|f|_{p,t,\Omega}.$$

For each ε and $u \in E$, we define

$$\|u\|_* = \mu\biggl(\int\limits_{\Omega} \frac{|u|^{2^*(t)-\varepsilon}}{|y|^t}\,\mathrm{d}x\biggr)^{1/(2^*(t)-\varepsilon)} + \biggl(\int\limits_{\Omega} \frac{|u|^{2^*(s)-\varepsilon}}{|y|^s}\,\mathrm{d}x\biggr)^{1/(2^*(s)-\varepsilon)}.$$

Lemma 3. (*See* [13].) *Let $1 \leqslant q < 2^*(t)$, $0 \leqslant t < 2$ and $N \geqslant 3$, then the embedding $H^1_0(\Omega) \hookrightarrow L^q_t(\Omega)$ is compact.*

By Lemmas 2, 3 and Hardy–Sobolev–Maz'ya inequality, we know that the singular term $\int_\Omega |u|^{2^*(t)-\varepsilon}/|y|^t$ and $\int_\Omega |u|^{2^*(s)-\varepsilon}/|y|^s$ are finite and $\|u\|_* \leqslant C\|u\|$, where C is independent of ε. Therefore J_ε is a C^1 function on $H^1_0(\Omega)$. By Lemma 3, J_ε satisfies the Palais–Smale condition. In order to prove Theorem 1, it is enough to obtain sign-changing critical points for the functional J_ε.

Fix $\xi \in (2, 2^*(s))$. In the following, we will always assume that $\varepsilon \in (0, 2^*(s)-\xi)$. In order to construct the minimax values for the perturbed functional J_ε, the following two technique lemmas are needed.

Lemma 4. *Assume $m \geqslant 1$. Then there exists $R = R(H_m) > 0$ such that for all $\varepsilon \in (0,\, 2^*(s)-\xi)$,*

$$\sup_{B^c_R \cap H_m} J_\varepsilon(u) < 0,$$

where $B^c_R := E \setminus B_R$.

Proof. Since H_m is finite dimensional, by Lemma 2, we know that $\|\cdot\|_*$ is defined as the norm on $H^1_0(\Omega)$. There is a constant $C > 0$ such that $\|u\|^{2^*-\varepsilon} \leqslant C\|u\|_*^{2^*-\varepsilon}$ for all $u \in H_m$. Therefore

$$\begin{aligned} J_\varepsilon(u) &\leqslant \frac{1}{2}\|u\|^2 - \frac{\mu}{2^*(t)-\varepsilon}\int\limits_{\Omega} \frac{|u|^{2^*(t)-\varepsilon}}{|y|^t} - \frac{1}{2^*(s)-\varepsilon}\int\limits_{\Omega} \frac{|u|^{2^*(s)-\varepsilon}}{|y|^s} \\ &\leqslant \frac{1}{2}\|u\|^2 - C\|u\|^{2^*(s)-\varepsilon}. \end{aligned}$$

Since $2^*(s)-\varepsilon > 2$ and $\lambda_1 > 1$, we have that $\lim_{\|u\|\to\infty,\, u\in H_m} J_\varepsilon(u) = -\infty$. The proof is complete. □

Lemma 5. *For any $\varepsilon \in (0, 2^*(s)-\xi)$, $\lambda_1 > 1$, there exists $\rho = \rho(\varepsilon)$, $\alpha = \alpha(\varepsilon) > 0$ such that*

$$\inf_{\partial B_\rho} J_\varepsilon(u) \geqslant \alpha.$$

Proof.

$$\begin{aligned} J_\varepsilon(u) &= \frac{1}{2}\int\limits_\Omega \left(|\nabla u|^2 - a(x)u^2\right) - \frac{\mu}{2^*(t)-\varepsilon}\int\limits_\Omega \frac{|u|^{2^*(t)-\varepsilon}}{|y|^t} - \frac{1}{2^*(s)-\varepsilon}\int\limits_\Omega \frac{|u|^{2^*(s)-\varepsilon}}{|y|^s} \\ &\geqslant \frac{1}{2}(\lambda_1 - 1)\int\limits_\Omega a(x)u^2 - C_1|u|_{2^*(t)-\varepsilon}^{2^*(t)-\varepsilon} - C_2|u|_{2^*(s)-\varepsilon}^{2^*(s)-\varepsilon}. \end{aligned}$$

Since $2^* - \varepsilon > 2^*(t) - \varepsilon > 2^*(s) - \varepsilon > \xi > 2$ and $\lambda_1 > 1$, there exists $\rho = \rho(\varepsilon)$, $\alpha = \alpha(\varepsilon) > 0$ such that $\inf_{\partial B_\rho} J_\varepsilon(u) \geqslant \alpha$. The proof is complete. □

Lemma 5 implies that 0 is a strict local minimum critical point. Then we can construct invariant sets containing all the positive and negative solutions of (1) for the gradient flow of J_ε. Therefore nodal solutions can be found outside of these sets.

3 Auxiliary operator and invariant subsets of descending flow

For any $\varepsilon \in (0,\, 2^*(s) - \xi)$, let $A_\varepsilon : E \to E$ be given by

$$A_\varepsilon(u) := (-\Delta)^{-1}\left(\mu\frac{|u|^{2^*(t)-2-\varepsilon}u}{|y|^t} + \frac{|u|^{2^*(s)-2-\varepsilon}u}{|y|^s} + a(x)u\right)$$

for $u \in E$. Then the gradient of J_ε has the form

$$\nabla J_\varepsilon(u) = u - A_\varepsilon(u).$$

Note that the set of fixed points of A_ε is the same as the set of critical points of J_ε, which is $K := \{u \in E\colon\ \nabla J_\varepsilon(u) = 0\}$. It is easy to check that ∇J_ε is locally Lipschitz continuous.

We consider the negative gradient flow ϕ_ε of J_ε defined by

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\phi_\varepsilon(t,u) = -\nabla J_\varepsilon(\phi_\varepsilon(t,u)) \quad \text{for } t \geqslant 0, \\ &\phi_\varepsilon(0,u) = u. \end{aligned}$$

Here and in the sequel, for $u \in E$, denote $u^\pm(x) := \max\{\pm u(x), 0\}$, the convex cones

$$+P = \{u \in E\colon\ u \geqslant 0\}, \quad -P = \{u \in E\colon\ -u \geqslant 0\}.$$

For $\theta > 0$, we define

$$(\pm P)_\theta := \bigl\{u \in E\colon\ \operatorname{dist}(u, \pm P) < \theta\bigr\}.$$

In the following, we will show that there exists $\theta_0 > 0$ such that $(\pm P)_\theta$ is an invariant set under the descending flow for all $0 < \theta \leqslant \theta_0$. Note that $E \setminus W$ contains only sign-changing functions, where

$$W := \overline{(+P)_\theta} \cup \overline{(-P)_\theta},$$

since $E \setminus W$ contains only sign-changing functions. By a version of the symmetric mountain pass theorem, which provides the minimax critical values on $E \setminus W$, we can prove that (6) has infinitely many sign-changing solutions.

For any $N \subset E$ and $\delta > 0$, N_δ denotes the open δ-neighborhood of N, i.e.,

$$N_\delta := \{u \in E\colon \ \mathrm{dist}(u, N) < \delta\},$$

whose closure and boundary are denoted by $\overline{N_\delta}$ and ∂N_δ. By the following result, we can know that a neighborhood of $\pm P$ is an invariant set. We can use similar way as Lemma 2 in [9] and Lemma 3.1 in [3] to get the following lemma.

Lemma 6. *There exists $\theta_0 > 0$ such that for any $\theta \in (0, \theta_0]$, there holds*

$$A_\varepsilon\bigl(\partial(\pm P)_\theta\bigr) \subset (\pm P)_\theta,$$

and

$$\phi_\varepsilon(t, u) \in (\pm P)_\theta \quad \text{for all } t > 0 \text{ and } u \in \overline{(\pm P)_\theta}.$$

Moreover, every nontrivial solutions $u \in (+P)_\theta$ and $u \in (-P)_\theta$ of (5) *are positive and negative, respectively.*

By using the combination of invariant sets method and minimax method, we can construct a nodal solution first, then to prove our main result. We need a deformation lemma in the presence of invariant sets.

Definition 1. A subset $W \subset E$ is an invariant set with respect to ϕ_ε if, for any $u \in W$, $\phi_\varepsilon(t, u) \in W$ for all $t \geqslant 0$.

From Lemma 6 we may choose an $\theta > 0$ sufficiently small such that $\overline{(\pm P)_\theta}$ are invariant set. Set $W := \overline{(+P)_\theta} \cup \overline{(-P)_\theta}$. Note that $\phi_\varepsilon(t, \partial W) \subset \mathrm{int}(W)$ and $Q := E \setminus W$ only contains sign-changing functions.

Since J_ε satisfies the Palais–Smale condition, we have the following deformation lemma, which follows from Lemma 5.1 in [18] (also see Lemma 2.4 in [15]).

Define $K^1_{\varepsilon,c} := K_{\varepsilon,c} \cap W$, $K^2_{\varepsilon,c} := K_{\varepsilon,c} \cap Q$, where $K_{\varepsilon,c} := \{u \in E\colon \ J_\varepsilon(u) = c, J'_\varepsilon(u) = 0\}$. Let $\rho > 0$ be such that $(K^1_{\varepsilon,c})_\rho \subset W$, where $(K^1_{\varepsilon,c})_\rho := \{u \in E\colon \mathrm{dist}(u, K^1_{\varepsilon,c}) < \rho\}$.

We can use the similar method to the proof of Lemma 5.1 [18] and Lemma 2.4 [15] to prove the following lemma.

Lemma 7. *Assume that J_ε satisfies Palais–Smale condition, then there exists an $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$, there exists $\eta \in C([0,1] \times E, E)$ satisfying*:

(i) *$\eta(t, u) = u$ for $t = 0$ or $u \notin J_\varepsilon^{-1}([c - \delta_0,\ c + \delta_0]) \setminus (K^2_{\varepsilon,c})_\rho$.*
(ii) *$\eta(1, J_\varepsilon^{c+\delta} \cup W \setminus (K^2_{\varepsilon,c})_{3\rho}) \subset J_\varepsilon^{c-\delta} \cup W$ and $\eta(1, J_\varepsilon^{c+\delta} \cup W) \subset J_\varepsilon^{c-\delta} \cup W$ if $K^2_{\varepsilon,c} = \emptyset$. Here $J_\varepsilon^d = \{u \in E\colon \ J_\varepsilon(u) \leqslant d\}$ for any $d \in R$.*
(iii) *$\eta(t, \cdot)$ is odd and a homeomorphism of E for $t \in [0,1]$.*
(iv) *$J_\varepsilon(\eta(\cdot, u))$ is nonincreasing.*
(v) *$\eta(t, W) \subset W$ for any $t \in [0,1]$.*

4 The proof of Theorem 1

In the following, we assume that $\lambda_1 > 1$. For any $\varepsilon \in (0, 2^*(s) - \xi)$ small, we define the minimax value $c_{\varepsilon,l}$ for the perturbed functional $J_\varepsilon(u)$ with $l = 2, 3, \dots$. We now define a family of sets for the minimax procedure here. We essentially follow [3], also see [18] and [20]. Define

$$G_m := \big\{h \in C(B_R \cap H_m, E)\colon\ h \text{ is odd and } h = id \text{ on } \partial B_R \cap H_m\big\},$$

where $R > 0$ is given by Lemma 4. Note that $G_m \neq \emptyset$ since $id \in G_m$. Set

$$\Gamma_l := \big\{h(B_R \cap H_m \setminus Y)\colon\ h \in G_m,\ m \geqslant l,\ Y = -Y \text{ is open and } \gamma(Y) \leqslant m - l\big\}$$

for $k \geqslant 2$. From [20] Γ_l possess the following properties:

(i) $\Gamma_l \neq \emptyset$ and $\Gamma_{l+1} \subset \Gamma_l$ for all $l \geqslant 2$.
(ii) If $\phi \in C(E, E)$ is odd and $\phi = id$ on $\partial B_R \cap H_m$, then $\phi(A) \in \Gamma_l$ if $A \in \Gamma_l$ for all $l \geqslant 2$.
(iii) If $A \in \Gamma_l$, $Z = -Z$ is open and $\gamma(Z) \leqslant s < l$ and $l - s \geqslant 2$, then $A \setminus Z \in \Gamma_{l-s}$.

Now, for $l \geqslant 2$, we can define the minimax value $c_{\varepsilon,l}$ by

$$c_{\varepsilon,l} := \inf_{A \in \Gamma_l} \sup_{u \in A \cap Q} J_\varepsilon(u).$$

Lemma 8. *For any $A \in \Gamma_l$ and $l \geqslant 2$, $A \cap Q \neq \emptyset$, then $c_{\varepsilon,l}$ is well defined, and $c_{\varepsilon,l} \geqslant \alpha > 0$, where α is given by Lemma* 5.

Proof. Consider the attracting domain of 0 in E:

$$D := \big\{u \in E\colon\ \phi_\varepsilon(t, u) \to 0,\ \text{as } t \to \infty\big\}.$$

Note that D is open since 0 is a local minimum of J_ε and by the continuous dependence of ODE on initial data. Moreover, ∂D is an invariant set, and $\overline{(+P)_\delta} \cap \overline{(-P)_\delta} \subset D$. In particular, the following holds

$$J_\varepsilon(u) > 0$$

for every $u \in \overline{(+P)_\delta} \cap \overline{(-P)_\delta} \setminus \{0\}$ (see [3, Lemma 3.4]). Now we claim that for any $A \in \Gamma_l$ with $l \geqslant 2$, it holds

$$A \cap Q \cap \partial D \neq \emptyset. \tag{10}$$

If this is true, then we have $A \cap Q \neq \emptyset$ and $c_{\varepsilon,2} \geqslant \alpha > 0$ because $\partial B_\rho \subset D$ and $\sup_{A \cap Q} J_\varepsilon(u) \geqslant \inf_{\partial D} J_\varepsilon(u) \geqslant \inf_{\partial B_\rho} J_\varepsilon(u) \geqslant \alpha > 0$ by Lemma 5.

To prove (10), let

$$A = h(B_R \cap H_m \setminus Y)$$

with $\gamma(Y) \leqslant m - l$ and $l \geqslant 2$. Define

$$O := \big\{u \in B_R \cap H_m\colon\ h(u) \in D\big\}.$$

Then O is a bounded open symmetric set with $0 \in O$ and $\overline{O} \subset B_R \cap H_m$. Thus, it follows from the Borsuk–Ulam theorem that $\gamma(\partial O) = m$ and, by the continuity of h, $h(\partial O) \subset \partial D$. As a consequence,

$$h(\partial O \setminus Y) \subset A \cap \partial D,$$

and therefore

$$\gamma(A \cap \partial D) \geqslant \gamma\bigl(h(\partial O \setminus Y)\bigr) \geqslant \gamma(\partial O \setminus Y) \geqslant \gamma(\partial O) - \gamma(Y) \geqslant l$$

by the "monotone, subadditive and supervariant" property of the genus [23, Prop. 5.4]. Since $(+P)_\delta \cap (-P)_\delta \cap \partial D = \emptyset$,

$$\gamma(W \cap \partial D) \leqslant 1.$$

Thus for $l \geqslant 2$, we conclude that

$$\gamma(A \cap Q \cap \partial D) \geqslant \gamma(A \cap \partial D) - \gamma(W \cap \partial D) \geqslant l - 1 \geqslant 1,$$

which proves (10).

Thus $c_{\varepsilon,l}$ is well defined for all $l \geqslant 2$ and $0 < \alpha \leqslant c_{\varepsilon,2} \leqslant c_{\varepsilon,3} \leqslant \cdots \leqslant c_{\varepsilon,l} \leqslant \cdots$. The proof is complete. □

Lemma 9.

$$K_{\varepsilon,c_{\varepsilon,l}} \cap Q \neq \emptyset. \tag{11}$$

Proof. If not, we assume that

$$K_{\varepsilon,c_{\varepsilon,l}} \cap Q = \emptyset.$$

By Lemma 7, for the functional J_ε, there exist $\delta > 0$ and a map $\eta \in C([0,1] \times E, E)$ such that $\eta(1,\cdot)$ is odd, $\eta(1,u) = u$ for $u \in J_\varepsilon^{c_{\varepsilon,l}-2\delta}$ and

$$\eta\bigl(1,\, J_\varepsilon^{c_{\varepsilon,l}+\delta} \cup W\bigr) \subset J_\varepsilon^{c_{\varepsilon,l}-\delta} \cup W. \tag{12}$$

By the definition of $c_{\varepsilon,l}$, there exists $A \in \Gamma_l$ such that $\sup_{A\cap Q} J_\varepsilon(u) \leqslant c_{\varepsilon,l} + \delta$. Let $B = \eta(1,A)$. It follows from (14) that

$$\sup_{B\cap Q} J_\varepsilon(u) \leqslant c_{\varepsilon,l} - \delta.$$

On the other hand, it is easy to show that $B \in \Gamma_l$ by Lemma 4 and the property (ii) of Γ_l above. As a result, $c_{\varepsilon,l} \leqslant c_{\varepsilon,l} - \delta$. This contradicts with $\delta > 0$. The proof is complete. □

Lemma 9 implies that there exists a sign-changing critical point $u_{\varepsilon,l}$ such that

$$J_\varepsilon\bigl(u_{\varepsilon,l}\bigr) = c_{\varepsilon,l}.$$

As a consequence of Lemma 8, we have that $c_{\varepsilon,l}$ is well defined for all $l \geqslant 2$ and $0 < \alpha \leqslant c_{\varepsilon,2} \leqslant c_{\varepsilon,3} \leqslant \cdots \leqslant c_{\varepsilon,l} \leqslant \cdots$. Now we can show the following lemma.

Lemma 10. *$c_{\varepsilon,l} \to \infty$ as $l \to \infty$.*

Proof. Here we deduce by a negation. Suppose $c_{\varepsilon,l} \to \bar{c}_\varepsilon < \infty$ as $l \to \infty$. Since J_ε satisfies Palais–Smale condition, it follows that $K_{\varepsilon,\bar{c}_\varepsilon} \neq \emptyset$ and is compact. Moreover, we have

$$K^2_{\varepsilon,\bar{c}_\varepsilon} := K_{\varepsilon,\bar{c}_\varepsilon} \cap Q \neq \emptyset.$$

Indeed, assume $\{u_{\varepsilon,l_i}\}_{i\in N}$ is a sequence of sign-changing solutions to (6) with $J_\varepsilon(u_{\varepsilon,l_i}) = c_{\varepsilon,l_i}$, and we have

$$\int\limits_\Omega \left|\nabla u^\pm_{\varepsilon,l_i}\right|^2 - a(x)\left|u^\pm_{\varepsilon,l_i}\right|^2 = \mu \int\limits_\Omega \frac{|u^\pm_{\varepsilon,l_i}|^{2^*(t)-\varepsilon}}{|y|^t} + \int\limits_\Omega \frac{|u^\pm_{\varepsilon,l_i}|^{2^*(s)-\varepsilon}}{|y|^s}.$$

By using the variantional principle of (5), we obtain

$$\left(1 - \frac{1}{\lambda_1}\right)\|u^\pm_{\varepsilon,l_i}\|^2 \leqslant \mu \int\limits_\Omega \frac{|u^\pm_{\varepsilon,l_i}|^{2^*(t)-\varepsilon}}{|y|^t} + \int\limits_\Omega \frac{|u^\pm_{\varepsilon,l_i}|^{2^*(s)-\varepsilon}}{|y|^s}.$$

It follows that, by Sobolev embedding theorem, $\|u^\pm_{\varepsilon,l_i}\| \geqslant c_0 > 0$, where c_0 is a constant independent of i. This implies that the limit $\bar{u}_\varepsilon \in K_{\varepsilon,\bar{c}_\varepsilon}$ of the subsequence of $\{u_{\varepsilon,l_i}\}_{i\in N}$ is still sign-changing.

Assume $\gamma(K^2_{\varepsilon,\bar{c}_\varepsilon}) = \tau$. Since $0 \notin K^2_{\varepsilon,\bar{c}_\varepsilon}$ and $K^2_{\varepsilon,\bar{c}_\varepsilon}$ is compact, by the "continuous" property of the genus [23, Prop. 5.4], there exists an open neighborhood N in E with $K^2_{\varepsilon,\bar{c}_\varepsilon} \subset N$ such that $\gamma(N) = \tau$. Now using Lemma 7 for the functional J_ε, there exist $\delta > 0$ and a map $\eta \in C([0,1]\times E, E)$ such that $\eta(1,\cdot)$ is odd, $\eta(1,u) = u$ for $u \in J_\varepsilon^{\bar{c}_\varepsilon - 2\delta}$ and

$$\eta\bigl(1,\ J_\varepsilon^{\bar{c}_\varepsilon+\delta} \cup W \setminus N\bigr) \subset J_\varepsilon^{\bar{c}_\varepsilon-\delta} \cup W. \tag{13}$$

Since $c_{\varepsilon,l} \to \bar{c}_\varepsilon$ as $l \to \infty$, we can choose l sufficiently large such that $c_{\varepsilon,l} \geqslant \bar{c}_\varepsilon - \delta/2$. Clearly, $c_{\varepsilon,\,l+\tau} \geqslant c_{\varepsilon,l} \geqslant \bar{c}_\varepsilon - \delta/2$. By the definition of $c_{\varepsilon,\,l+\tau}$, we can find a set $A \in \varGamma_{l+\tau}$, that is, $A = h(B_R \cap H_m \setminus Y)$, where $h \in G_m$, $m \geqslant l+\tau$, $\gamma(Y) \leqslant m - (l+\tau)$, such that

$$J_\varepsilon(u) \leqslant c_{\varepsilon,l+\tau} + \frac{1}{4}\delta < \bar{c}_\varepsilon + \delta$$

for any $u \in A \cap Q$, which implies $A \subset J_\varepsilon^{\bar{c}_\varepsilon+\delta} \cup W$. It follows from (13) that

$$\eta(1,\ A \setminus N) \subset J_\varepsilon^{\bar{c}_\varepsilon-\delta} \cup W. \tag{14}$$

Let $Y_1 = Y \cup h^{-1}(N)$. Then Y_1 is symmetric and open, and

$$\gamma(Y_1) \leqslant \gamma(Y) + \gamma\bigl(h^{-1}(N)\bigr) \leqslant m - (l+\tau) + \tau = m - l.$$

Then it is easy to check $\widetilde{A} := \eta(1,\ h(B_R \cap H_m \setminus Y_1)) \in \varGamma_l$ by (ii) and (iii) above. As a result, by (14),

$$c_{\varepsilon,l} \leqslant \sup_{\widetilde{A}\cap Q} J_\varepsilon(u) \leqslant \sup_{\eta(1,\,A\setminus N)\cap Q} J_\varepsilon(u) \leqslant \bar{c}_\varepsilon - \delta.$$

This is a contradiction to $c_{\varepsilon,l} \geqslant \bar{c}_\varepsilon - \delta/2$. The proof is complete. □

Lemma 11. *For any fixed $l \geqslant 2$, $\|u_{\varepsilon,l}\|$ is uniformly bounded with respect to ε, and then $u_{\varepsilon,l}$ converges strongly to u_l in E as $\varepsilon \to 0$.*

Proof. Indeed, by using the same Γ_l above, we can also define the minimax value for the following auxiliary function:

$$J_*(u) = \frac{1}{2}\int\limits_{\Omega} \left(|\nabla u|^2 - a(x)|u|^2\right) - \frac{\mu}{2^*}\int\limits_{\Omega} \frac{|u|^{\xi}}{|y|^t} - \frac{1}{2^*}\int\limits_{\Omega} \frac{|u|^{\xi}}{|y|^s},$$

$$\alpha_l := \inf_{A\in\Gamma_l}\sup_{u\in A} J_*(u), \quad l = 2, 3, \dots .$$

Here we choose $R > 0$ sufficiently large if necessary such that Lemma 4 also holds for J_*. Then by a Z_2 version of the mountain pass theorem [20, Thm. 9.2], for each $l \geqslant 2$, $\alpha_l > 0$ is well defined, and $\alpha_l \to \infty$ as $l \to \infty$ because

$$\begin{aligned} J_\varepsilon(u) &\leqslant \frac{1}{2}\int\limits_{\Omega} \left(|\nabla u|^2 - a(x)|u|^2\right) - \frac{\mu}{2^*}\int\limits_{\Omega} \left(\frac{|u|^{\xi} - 1}{|y|^t}\right) - \frac{1}{2^*}\int\limits_{\Omega} \frac{|u|^{\xi} - 1}{|y|^s} \\ &= J_*(u) + d_0, \end{aligned}$$

where

$$d_0 = \frac{\mu}{2^*}\int\limits_{\Omega} \frac{1}{|y|^t} + \frac{1}{2^*}\int\limits_{\Omega} \frac{1}{|y|^s}.$$

Therefore, for any fixed $l \geqslant 2$, $c_{\varepsilon,l}$ is uniformly bounded for $\varepsilon \in (0,\, 2^*(s) - \xi)$, that is, there is $C = C(\alpha_l, \Omega) > 0$ independent on ε such that $c_{\varepsilon,l} \leqslant C$ uniformly for ε because $u_{\varepsilon,l}$ is a nodal solution of (6) and $J_\varepsilon(u_{\varepsilon,l}) = c_{\varepsilon,l}$. By the definition of λ_1, we can obtain the following:

$$\begin{aligned} C \geqslant c_{\varepsilon,l} &= J_\varepsilon(u_{\varepsilon,l}) = J_\varepsilon(u_{\varepsilon,l}) - \frac{1}{2^*(s) - \varepsilon}\left\langle J'_\varepsilon(u_{\varepsilon,l}), u_{\varepsilon,l}\right\rangle \\ &= \left(\frac{1}{2} - \frac{1}{2^*(s) - \varepsilon}\right)\int\limits_{\Omega} \left(|\nabla u_{\varepsilon,l}|^2 - a(x)u_{\varepsilon,l}^2\right) \\ &\quad + \mu\left(\frac{1}{2^*(s) - \varepsilon} - \frac{1}{2^*(t) - \varepsilon}\right)\int\limits_{\Omega} \frac{|u_{\varepsilon,l}|^{2^* - \varepsilon}}{|y|^t} \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2^*(s) - \varepsilon}\right)\int\limits_{\Omega} \left(|\nabla u_{\varepsilon,l}|^2 - a(x)u_{\varepsilon,l}^2\right) \\ &\geqslant \left(\frac{1}{2} - \frac{1}{\xi}\right)\left(1 - \frac{1}{\lambda_1}\right)\|u_{\varepsilon,l}\|^2 > 0, \end{aligned}$$

where $\lambda_1 > 1$, $\varepsilon \in (0,\, 2^*(s) - \xi)$ and $2 < \xi < 2^*(s)$. Therefore $\|u_{\varepsilon,l}\|$ is uniformly with respect to ε. So we can apply Proposition 1 and obtain a subsequence $\{u_{\varepsilon_l,l}\}_{l\in N}$ such that $u_{\varepsilon_l,l} \to u_l$ strongly in E for some u_l and also $c_{\varepsilon_l,l} \to c_l$. Thus u_l is a solution of (5), and $J_\varepsilon(u_l) = c_l$. Moreover, since $u_{\varepsilon_l,l}$ is sign-changing, similar to Lemma 10, by Sobolev embedding theorem, we can prove that u_l is still sign-changing. The proof is complete. □

Proof. Proof of Theorem 1 Noting that c_l is nondecreasing with respect to l, we have the following two cases:

Case I. There are $2 \leqslant l_1 < \cdots < l_i < \cdots$ satisfying $c_{l_1} < \cdots < c_{l_i} < \cdots$. Obviously, in this case, equation (1) has infinitely many sign solutions such that $J(u_i) = c_{l_i}$.

Case II. There is a positive integer L such that $c_l = c$ for all $l \geqslant L$.

From now on we assume that there exists a $\delta > 0$ such that $J(u)$ has no sign-changing critical point u with

$$J(u) \in [c-\delta,\, c) \cup (c,\, c+\delta].$$

Otherwise, we are done. In this case, we claim that $\gamma(K_c^2) \geqslant 2$, where $K_c := \{u \in E\colon J(u) = c,\ J'(u) = 0\}$ and $K_c^2 = K_c \cap Q$. Then as a consequence, $J(u)$ has infinitely many sign-changing critical points.

Now we adopt a technique in the proof of Theorem 1.1 in [7]. Suppose, on the contrary, that $\gamma(K_c^2) = 1$ (note that $K_c^2 \neq \emptyset$). Moreover, we assume K_c^2 contains only finitely many critical points, otherwise, we are done. Then it follows that K_c^2 is compact. Obviously, $0 \notin K_c^2$. Then there exists a open neighborhood N in E with $K_c^2 \subset N$ such that $\gamma(N) = \gamma(K_c^2)$.

Define

$$V_\varepsilon := \bigl(J_\varepsilon^{c+\delta} \setminus J_\varepsilon^{c-\delta}\bigr) \setminus N.$$

We now claim that if $\varepsilon > 0$ small, J_ε has no sign-changing critical point $u \in V_\varepsilon$. Indeed, arguing indirectly, suppose that there exist $\varepsilon \to 0$ and $u_\varepsilon \in V_\varepsilon$ satisfying $J'_\varepsilon(u_\varepsilon) = 0$ with $u_\varepsilon^\pm \neq 0$ and $u_\varepsilon \notin N$.

Then, by Proposition 1, up to a subsequence, u_n converges strongly to u in E. Therefore $J'(u) = 0$,

$$J(u) \in [c-\delta,\, c+\delta]$$

and $u \notin K_c^2$.

This is a contradiction to our assumption and the fact that u is still sign-changing. The following proof is similar to that of Lemma 9. By using Lemma 7, for the functional J_ε, there exist $\delta > 0$ and a map $\eta \in C([0,1] \times E, E)$ such that $\eta(1,\cdot)$ is odd, $\eta(1,u) = u$ for $u \in J_\varepsilon^{c-2\delta}$ and

$$\eta\bigl(1,\, J_\varepsilon^{c+\delta} \cup W \setminus N\bigr) \subset J_\varepsilon^{c-\delta} \cup W. \tag{15}$$

Now fix $l > L$. Since $c_{\varepsilon,l}, c_{\varepsilon,\,l+1} \to c$ as $\varepsilon \to 0$, we can find an $\varepsilon > 0$ small such that $c_{\varepsilon,l}, c_{\varepsilon,\,l+1} \in (c-\delta/2,\, c+\delta/2)$. By the definition of $c_{\varepsilon,\,l+1}$, we can find a set $A \in \Gamma_{l+1}$, that is,

$$A = h(B_R \cap H_m \setminus Y),$$

where $h \in G_m$, $m \geqslant l+1$, $\gamma(Y) \leqslant m-(l+1)$ such that

$$J_\varepsilon(u) \leqslant c_{\varepsilon,\,l+1} + \frac{1}{4}\delta < c+\delta$$

for any $u \in A \cap Q$, which implies $A \subset J_\varepsilon^{c+\delta} \cup W$. Then by (15), we have

$$\eta(1,\, A \setminus N) \subset J_\varepsilon^{c-\delta} \cup W. \tag{16}$$

Let $\widetilde{Y} = Y \cup h^{-1}(N)$. Then $\widetilde{Y}$ is symmetric and open, and

$$\gamma(\widetilde{Y}) \leqslant \gamma(Y) + \gamma\bigl(h^{-1}(N)\bigr) \leqslant m-(l+1)+1 = m-l.$$

Then it is easy to check $\widehat{A} := \eta(1,\, h(B_R \cap H_m \setminus \widetilde{Y})) \in \varGamma_l$ by (ii) and (iii) above. As a result, by (16),

$$c_{\varepsilon,l} \leqslant \sup_{\widehat{A}\cap Q} J_\varepsilon(u) \leqslant \sup_{\eta(1,\, A\setminus N)\cap Q} J_\varepsilon(u) \leqslant c-\delta.$$

This contradicts to $c_{\varepsilon,l} > c - \delta/2$. Then the proof for case II is finished. The proof is complete. □

Acknowledgment. The first authors like to thank Chern Institute of Mathematics for visiting scholar programme and the authors are grateful to the anonymous referees for their useful suggestions, which improve the contents of this article.

References

1. A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14**(4):349–381, 1973, https://doi.org/10.1016/0022-1236(73)90051-7.
2. F.V. Atkinson, H. Brézis, L.A. Peletier, Nodal solutions of elliptic equations with critical Sobolev exponents, *J. Differ. Equations*, **85**:151–170, 1990, https://doi.org/10.1016/0022-0396(90)90093-5.
3. T. Bartsch, S.J. Peng, Z.T. Zhang, Existence and non-existence of solutions to elliptic equations related to the Caffarelli–Kohn–Nirenberg inequalities, *Calc. Var. Partial Differ. Equ.*, **30**:113–136, 2007, https://doi.org/10.1007/s00526-006-0086-1.
4. M. Bhakta, Infinitely many sign-changing solutions of an elliptic problem involving critical Sobolev and Hardy–Sobolev exponent, *Proc. Indian Acad. Sci., Math. Sci.*, **127**(2):337–347, 2017, https://doi.org/10.1007/s12044-016-0304-5.
5. H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Commun. Pure Appl. Math.*, **36**:437–477, 1983, https://doi.org/10.1002/cpa.3160360405.
6. D.M. Cao, S.J. Peng, A note on the sign-changing solutions to elliptic problems with critical Sobolev and Hardy terms, *J. Differ. Equations*, **193**:424–434, 2003, https://doi.org/10.1016/S0022-0396(03)00118-9.

7. D.M. Cao, S.J. Peng, S.S. Yan, Infinitely many solutions for p-Laplacian equation involving critical Sobolev growth, *J. Funct. Anal.*, **262**:2861–2902, 2012, https://doi.org/10.1016/j.jfa.2012.01.006.
8. Z.J. Chen, W.M. Zou, On an elliptic problem with critical exponent and Hardy potential, *J. Differ. Equations*, **252**:969–987, 2012, https://doi.org/10.1016/j.jde.2011.09.042.
9. M. Clapp, T. Weth, Multiple solutions for the Brezis–Nirenberg problem, *Adv. Differ. Equ.*, **10**(4):463–480, 2005.
10. G. Devillanova, S. Solimini, Concentration estimates and multiple solutions to elliptic problems at critical growth, *Adv. Differ. Equ.*, **7**:1257–1280, 2002.
11. G. Devillanova, S. Solimini, A multiplicity result for elliptic equations at critical growth in low dimension, *Commun. Contemp. Math.*, **15**:171–177, 2003, https://doi.org/10.1142/S0219199703000938.
12. D. Fortunato, E. Jannelli, Infinitely many solutions for some nonlinear elliptic problems in symmetrical domains, *Proc. R. Soc. Edinb., Sect. A, Math.*, **105**:205–213, 1987, https://doi.org/10.1017/S0308210500022046.
13. D. Ganguly, Sign changing solutions of the Hardy–Sobolev–Maz'ya equation, *Adv. Nonlinear Anal.*, **3**(3):187–196, 2014, https://doi.org/10.1515/anona-2014-0005.
14. N. Ghoussoub, X.S. Kang, Hardy-Sobolev critical elliptic equations with boundary singularities, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, **21**(6):767–793, 2004, https://doi.org/10.1016/j.anihpc.2003.07.002.
15. S. Li, Z.-Q. Wang, Ljusternik-Schnirelman theory in partially ordered Hilbert spaces, *Trans. Am. Math. Soc.*, **354**(8):3207–3227, 2002, https://doi.org/10.1090/S0002-9947-02-03031-3.
16. P.L. Lions, The concentration-compactness principle in the calculus of variations: The limit case. I, *Rev. Mat. Iberoam.*, **1**(1):145–201, 1985, https://doi.org/10.4171/RMI/6.
17. P.L. Lions, The concentration-compactness principle in the calculus of variations: The limit case. II, *Rev. Mat. Iberoam.*, **1**(2):45–121, 1985, https://doi.org/10.4171/RMI/12.
18. Z.L. Liu, F.A. van Heerden, Z.-Q. Wang, Nodal type bound states of Schrödinger equations via invariant set and minimax methods, *J. Differ. Equations*, **214**(2):358–390, 2005, https://doi.org/10.1016/j.jde.2004.08.023.
19. V.G. Maz'ja, *Sobolev Spaces*, Springer Series in Soviet Mathematics, Springer, Berlin, Heidelberg, 1985, https://doi.org/10.1007/978-3-662-09922-3.
20. P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Ser. Math., Vol. 65, AMS, Providence, RI, 1986, https://doi.org/10.1090/cbms/065.
21. M. Schechter, W.M. Zou, On the Brézis–Nirenberg problem, *Arch. Ration. Mech. Anal.*, **197**:337–356, 2010, https://doi.org/10.1007/s00205-009-0288-8.
22. S. Solimini, A note on compactness-type properties with respect to Lorentz norms of bounded subsets of a Sovolev space, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, **12**(3):319–337, 1995, https://doi.org/10.1016/S0294-1449(16)30159-7.

23. M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, *Math. Z.*, **187**:511–517, 1984, `https://doi.org/10.1007/BF01174186`.
24. J. Sun, S. Ma, Infinitely many sign-changing solutions for the Brézis–Nirenberg problem, *Commun. Pure Appl. Anal.*, **13**(6):2317–2330, 2014, `https://doi.org/10.3934/cpaa.2014.13.2317`.
25. A. Szulkin, T. Weth, M. Willem, Ground state solutions for a semilinear problem with critical exponent, *Differ. Integral Equ.*, **22**(6):913–926, 2009, `https://doi.org/10.1007/s10623-009-9287-x`.
26. C.H. Wang, J.T. Wang, Infinitely many solutions for Hardy–Sobolev–Maz'ya equation involving critical growth, *Commun. Contemp. Math.*, **14**(6):1250044, 2012, `https://doi.org/10.1142/S0219199712500447`.
27. C.H. Wang, J. Yang, A note on the sign-changing solutions for a double critial Hardy–Sobolev–Maz'ya problem, *Adv. Nonlinear Stud.*, **16**(3):519–528, 2016, `https://doi.org/10.1515/ans-2015-5043`.
28. C.H. Wang, J. Yang, Infinitely many solutions for an elliptic problem with double Hardy–Sobolev–Maz'ya terms, *Discrete Contin. Dyn. Syst.*, **36**(3):1603–1628, 2016, `https://doi.org/10.3934/dcds.2016.36.1603`.
29. L.X. Wang, Infinitely many sign-changing solutions for Hardy–Sobolev–Maz'ya equation involving critical growth, *Rocky Mt. J. Math.*, **49**(4):1371–1390, 2019, `https://doi.org/10.1216/RMJ-2019-49-4-1371`.
30. Y.Z. Wu, Y.S. Huang, Infinitely many sign-changing solutions for p-Laplacian equation involving the critical Sobolev exponent, *Bound. Value Probl.*, **2013**:149, 2013, `https://doi.org/10.1186/1687-2770-2013-149`.
31. S.S. Yan, J.F. Yang, Infinitely many solutions for an elliptic problem involving critical Sobolev and Hardy-Sobolev exponents, *Calc. Var. Partial Differ. Equ.*, **2013**:587–610, 2013, `https://doi.org/10.1007/s00526-012-0563-7`.
32. J. Zhang, S.W. Ma, Infinitely many sign-changing solutions for the Brézis–Nirenberg problem involving Hardy potential, *Acta Math. Sci.*, **36**(B(2)):527–536, 2016, `https://doi.org/10.3969/j.issn.0252-9602.2016.02.018`.