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Positive almost periodicity on SICNNs incorporating mixed delays and D operator*

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Abstract. This article involves a kind of shunting inhibitory cellular neural networks incorporating D operator and mixed delays. First of all, we demonstrate that, under appropriate external input conditions, some positive solutions of the addressed system exist globally. Secondly, with the help of the differential inequality techniques and exploiting Lyapunov functional approach, some criteria are established to evidence the globally exponential stability on the positive almost periodic solutions. Eventually, a numerical case is provided to test and verify the correctness and reliability of the proposed findings.

Keywords: positive almost periodic solution, stability, shunting inhibitory cellular neural networks, D operator, mixed delay.

1 Introduction

In the early 1990s, Bouzerdout and Pinter first established the shunt inhibitory cellular neural networks (SICNNs) system [2], which has attracted extensive attention because

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the lateral inhibition of shunting not only can greatly enhance the edge and contrast, but also be of significant influence in vision. With the increasing improvement of neural networks, the above systems and their variants are widespread used in the fields of pronunciation, robotics, associative memories, psychophysics and optimization [1]. It has been discovered that time delay is inevitable and it may be one of the important reasons leading to the instability and terrible efficiency of the system [7, 10, 17, 28]. Hence, numerous scholars focus on the dynamic researches of cellular neural networks and biomathematical models accompanying bounded time-varying delays, and many interesting findings have been published in [8, 9, 11, 18]. In addition, in the large-scale networks models, since the occurrence of many parallel ways containing a various of axon lengths and sizes, it is meaningful to reveal the dynamical characteristics of neural networks incorporating continuously distributed delays [3, 30]. Because neural cells have complex dynamic characteristics in the real world, in order to further simulate the dynamics of this complex neural reactions, the neural networks system should contain some messages on derivatives of past states [13, 14], which inspired people to study the neutral-type systems. Generally speaking, neutral-type neural network systems can be expressed as non-D operator and D operator, and the cellular neural networks accompanying D operator are more practical than those ones touching non-D operator [23, 24]. Usually, let $ij \in J = \{11, 12, \dots, 1n, \dots, m1, m2, \dots, mn\}$, and mn be the units amount, the neutral-type SICNNs incorporating D operators, continuously distributed delays and time-varying delays are often modelled as the neutral-type functional differential equations

$$\begin{aligned} &\big[x_{ij}(t) - p_{ij}(t)x_{ij}\big(t - r_{ij}(t)\big)\big]' \\ &\quad = -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(t)f\big(x_{kl}\big(t-\tau_{kl}(t)\big)\big)x_{ij}(t) \\ &\qquad - \sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(t)\int_0^{+\infty} K_{ij}(u)g\big(x_{kl}(t-u)\big)\,\mathrm{d}u\, x_{ij}(t) + I_{ij}(t). \end{aligned} \tag{1}$$

Here C_{ij} labels the cell at lattice (i, j), $N_r(i, j) = \{C_{kl}\colon \max(|k - i|, |l - j|) \leqslant r,\ 1 \leqslant k \leqslant m,\ 1 \leqslant l \leqslant n\}$ is designated as the r neighborhood. $N_q(i, j)$ is identically declared. $x_{ij}(t)$ designates the ijth neuron state, a_{ij} is the decay rate, p_{ij}, B_{ij}^{kl} and C_{ij}^{kl} denote the jointing or coupling intensity of postsynaptic action of the cell C_{kl} conveyed to the cell C_{ij}, $r_{ij}(t)$, $\tau_{kl}(t)$ and $K_{ij}(u)$ are transfer delay functions. f and g stand for the activation functions substituting the firing rate or output of the cell C_{kl}, $I_{ij}(t)$ is correlating to the external input, and one can consult [16, 26] for more detailed biological explanations. Furthermore, the initial value conditions of SICNNs (1) are denoted as

$$x_{ij}(s) = \varphi_{ij}(s), \quad s \in (-\infty, 0],\ ij \in J, \tag{2}$$

where $\varphi_{ij}(\cdot) \in C((-\infty, 0], \mathbb{R})$ is bounded.

During recent decades, great efforts have been put into the periodicity and almost periodicity of the population and ecology models [25, 29]. More precisely, many biological and cognitive activities need to be repeated, for example, oscillators [20], which are essential in many electronic circuits, usually generate almost periodic signals. Meanwhile, almost periodicity can better describe the changes in the natural environment and has a significant impact in illustrating the behavior of nonlinear dynamic systems [15, 17, 21, 22, 28]. It is worth pointing out that, in neural networks dynamics involving the field of biomathematics, the relevant state variables are currently treated as light intensity levels, proteins and electric or molecules charge, and they are surely positive restraints [19]. Such biological systems are often handled as positive systems [4]. However, the positive almost periodic stability for neutral-type SICNNs incorporating D operator has not been involved, which needs further research.

Inspired by the above considerations, in this article, we focus on the positive almost periodic stability on SICNNs system. Briefly speaking, the main contributions and highlights of this article can be summarized as below. (i) The positiveness of bounded solutions of SICNNs (1) are demonstrated with the help of some differential inequality methods; (ii) Under certain hypothesises, by exploiting the fixed point theory and Lyapunov functional approach, the positiveness and global exponential stability for the almost periodic solutions of SICNNs (1) are proved for the first time; (iii) Numerical simulations accompanying comparison discussions are supplied to validate the effectiveness of our theoretical findings.

The rest framework of this article is outlined as below. In Section 2, we shall present some definitions and preliminary results. In Section 3, we afford the main theorems and their comprehensive proofs. Section 4 furnishes a numerical example to check the advantage and validity of our results. We terminate this article by a concise conclusion in Section 5.

2 Preliminaries

In what follows, a few definitions, lemmas and presumptions are provided, which are advantageous in the following verification process of the main findings.

Notations. For convenience and simplicity, the n-dimensional real vectors assemble is denoted by $\mathbb{R}^n$ ($\mathbb{R} = \mathbb{R}^1$). For each $x = \{x_{ij}\} \in \mathbb{R}^{mn}$, label $|x| = \{|x_{ij}|\}$, and $\|x\| = \max_{ij\in J} |x_{ij}(t)|$. For a real function ϑ, define

$$\vartheta^+ = \sup_{t\in\mathbb{R}} \bigl|\vartheta(t)\bigr|, \qquad \vartheta^- = \inf_{t\in\mathbb{R}} \bigl|\vartheta(t)\bigr|.$$

Letting the supremum norm $\|z\|_\infty = \sup_{t\in\mathbb{R}} \|z(t)\|$, the bounded and continuous functions collection $BC(\mathbb{R}, \mathbb{R}^n)$ is a Banach space.

Definition 1. (See [5].) Let $g \in C(\mathbb{R}, \mathbb{R}^{mn})$, and the assemble $T(g, \epsilon) = \{\delta\colon \|g(t+\delta) - g(t)\| < \epsilon\ \forall t \in \mathbb{R}\}$ be relative density, that is to say, for each $\epsilon > 0$, one can find a constant $l = l(\epsilon) > 0$ agreeing that every interval of length $l(\epsilon)$ contains a δ such that $\|g(t + \delta) - g(t)\| < \epsilon$ for arbitrary $t \in \mathbb{R}$. Then g is said as an almost periodic function in $\mathbb{R}$.

Denote $AP(\mathbb{J}_1, \mathbb{J}_2)$ as the assemble of the almost periodic functions from $\mathbb{J}_1$ to $\mathbb{J}_2$. For all $ij \in J$, we also presume that $a_{ij}, r_{ij} \in AP(\mathbb{R}, (0, +\infty))$, $a_{ij}^- > 0$, $C_{ij}^{kl}, B_{ij}^{kl} \in AP(\mathbb{R}, \mathbb{R})$, $p_{ij}, \tau_{ij}, I_{ij} \in AP(\mathbb{R}, \mathbb{R}^+)$, $r = \min_{ij \in J} r_{ij}^- > 0$ and $\mathbb{R}^+ = [0, +\infty)$.

The following hypotheses will be adopted later.

(S0) For arbitrary $u, v \in \mathbb{R}$, we can take constants M_f, M_g, μ and γ satisfying

$$\begin{aligned} \bigl|f(u) - f(v)\bigr| &\leqslant \mu|u - v|, \qquad \sup_{x \in \mathbb{R}} \bigl|f(x)\bigr| = M_f < +\infty, \\ \bigl|g(u) - g(v)\bigr| &\leqslant \gamma|u - v|, \qquad \sup_{x \in \mathbb{R}} \bigl|g(x)\bigr| = M_g < +\infty. \end{aligned}$$

(S1) $K_{ij} \in C(\mathbb{R}^+, \mathbb{R})$, and $\int_0^{+\infty} |K_{ij}(t)| \mathrm{e}^{\alpha t} \, \mathrm{d}t < +\infty$ for a positive number α.

(S2) There are positive numbers I, ω, A_{ij} and J_{ij} obeying that

$$\begin{aligned} I &= \max_{ij \in J} \Biggl\{ \sup_{t \in \mathbb{R}} \int_{-\infty}^{t} \mathrm{e}^{-\int_s^t a_{ij}(u) \, \mathrm{d}u} I_{ij(s) \, \mathrm{d}s} \Biggr\} > 0, \\ A_{ij} &= \sup_{t \in R} \frac{1}{a_{ij}(t)} \Biggl[a_{ij}(t) p_{ij}(t) + M_f \sum_{C_{kl} \in N_r(i,j)} \bigl|B_{ij}^{kl}(t)\bigr| \\ &\quad + M_g \sum_{C_{kl} \in N_q(i,j)} \bigl|C_{ij}^{kl}(t)\bigr| \int_0^{+\infty} \bigl|K_{ij}(u)\bigr| \, \mathrm{d}u \Biggr] \leqslant \frac{\omega}{\omega + I} - p_{ij}^+, \\ J_{ij} &= \sup_{t \in R} \frac{1}{a_{ij}(t)} \Biggl[a_{ij}(t) p_{ij}(t) + \sum_{C_{kl} \in N_r(i,j)} \bigl|B_{ij}^{kl}(t)\bigr| \bigl(\mu(\omega + I) + M_f\bigr) \\ &\quad + \sum_{C_{kl} \in N_q(i,j)} \bigl|C_{ij}^{kl}(t)\bigr| \int_0^{+\infty} \bigl|K_{ij}(u)\bigr| \, \mathrm{d}u \bigl(\gamma(\omega + I) + M_g\bigr) \Biggr] < 1 - p_{ij}^+, \end{aligned}$$

and

$$\begin{aligned} &\sup_{t \in \mathbb{R}} \Biggl\{ -a_{ij}(t) + a_{ij}(t) p_{ij}(t) \frac{1}{1 - p_{ij}^+} + \sum_{C_{kl} \in N_r(i,j)} \bigl|B_{ij}^{kl}(t)\bigr| \biggl(M_f \frac{1}{1 - p_{ij}^+} \\ &\quad + \mu(\omega + I) \frac{1}{1 - p_{kl}^+} \biggr) + \sum_{C_{kl} \in N_q(i,j)} \bigl|C_{ij}^{kl}(t)\bigr| \int_0^{+\infty} \bigl|K_{ij}(u)\bigr| \\ &\quad \times \biggl(M_g \frac{1}{1 - p_{ij}^+} + \gamma(\omega + I) \frac{1}{1 - p_{kl}^+} \biggr) \, \mathrm{d}u \Biggr\} < 0, \quad ij \in J. \end{aligned}$$

First, on account of $r = \min_{ij \in J} r_{ij}^- > 0$, employing a discussion similar to that used in Lemma 2.2 of [26], one can reveal the global existence and uniqueness on all solutions of the initial-value problem (1)–(2).

Lemma 1. *Suppose that* (S0)–(S2) *are obeyed. Then every solution for system* (1) *involving initial values* (2) *possesses global existence and uniqueness on* $[0, +\infty)$.

Hereafter, to show the positiveness of bounded solutions on SICNNs (1), we make the following hypotheses:

$$\eta_{ij} = \inf_{t\in\mathbb{R}}\Biggl\{a_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} \bigl|B_{ij}^{kl}(t)\bigr| M_f\biggl(1 + \frac{p_{ij}(t)}{1-p_{ij}^+}\biggr) - \sum_{C_{kl}\in N_q(i,j)} \bigl|C_{ij}^{kl}(t)\bigr| \int_0^{+\infty} \bigl|K_{ij}(u)\bigr| \,\mathrm{d}u M_g\biggl(1 + \frac{p_{ij}(t)}{1-p_{ij}^+}\biggr)\Biggr\} > 0 \tag{3}$$

and

$$\inf_{t\in\mathbb{R}}\Biggl\{I_{ij}(t) - a_{ij}(t)p_{ij}(t)\frac{I_{ij}^+}{\eta_{ij}(1-p_{ij}^+)} - \sum_{C_{kl}\in N_r(i,j)} \bigl|B_{ij}^{kl}(t)\bigr| p_{ij}(t) M_f \frac{I_{ij}^+}{\eta_{ij}(1-p_{ij}^+)} - \sum_{C_{kl}\in N_q(i,j)} \bigl|C_{ij}^{kl}(t)\bigr| \int_0^{+\infty} \bigl|K_{ij}(u)\bigr| \,\mathrm{d}u p_{ij}(t) M_g \frac{I_{ij}^+}{\eta_{ij}(1-p_{ij}^+)}\Biggr\} > 0 \quad \forall ij \in J. \tag{4}$$

Then, for all $t \in \mathbb{R}$, $ij \in J$, one can select a constant $0 < \kappa < (I_{ij}^+ + \kappa)/\eta_{ij}$ agreeing with

$$\begin{aligned}
&-a_{ij}(t)\kappa - a_{ij}(t)p_{ij}(t)\frac{I_{ij}^+ + \kappa}{\eta_{ij}(1-p_{ij}^+)} - \sum_{C_{kl}\in N_r(i,j)} \bigl|B_{ij}^{kl}(t)\bigr| M_f \kappa \\
&\quad - \sum_{C_{kl}\in N_r(i,j)} \bigl|B_{ij}^{kl}(t)\bigr| p_{ij}(t) M_f \frac{I_{ij}^+ + \kappa}{\eta_{ij}(1-p_{ij}^+)} - \sum_{C_{kl}\in N_q(i,j)} \bigl|C_{ij}^{kl}(t)\bigr| \int_0^{+\infty} \bigl|K_{ij}(u)\bigr| \,\mathrm{d}u\, M_g\kappa \\
&\quad - \sum_{C_{kl}\in N_q(i,j)} \bigl|C_{ij}^{kl}(t)\bigr| \int_0^{+\infty} \bigl|K_{ij}(u)\bigr| \,\mathrm{d}u\, p_{ij}(t)\, M_g \frac{I_{ij}^+ + \kappa}{\eta_{ij}(1-p_{ij}^+)} + I_{ij}(t) > 0.
\end{aligned} \tag{5}$$

Adopting the above assumptions of external inputs, some positive solutions of the addressed system can be presented as follows.

Lemma 2. *Let* (S0)–(S2), (3) *and* (4) *be satisfied. In addition, mark* $\tilde{x}(t) = \{\tilde{x}_{ij}(t)\}$ *as the solution of* (1) *with*

$$\tilde{x}_{ij}(\theta) = \tilde{\varphi}_{ij}(\theta), \quad \theta \in (-\infty, 0],\ \tilde{\varphi}_{ij} \in C\bigl((-\infty, 0], (0, +\infty)\bigr), \tag{6}$$

$$\kappa < \tilde{\varphi}_{ij}(\theta) - p_{ij}(\theta)\tilde{\varphi}_{ij}\bigl(\theta - r_{ij}(\theta)\bigr) < \frac{I_{ij}^+ + \kappa}{\eta_{ij}}, \quad \theta \in (-\infty, 0]. \tag{7}$$

Then

$$\kappa < \tilde{x}_{ij}(t) - p_{ij}(t)\tilde{x}_{ij}\bigl(t - r_{ij}(t)\bigr) < \frac{I_{ij}^+ + \kappa}{\eta_{ij}} \quad \forall t \geqslant 0,\ ij \in J. \tag{8}$$

Proof. Striving for a contradiction, suppose that (8) is not true. We shall deal with two scenarios as follows.

Case 1. There arise $\tilde{i}\tilde{j} \in J$ and $T > 0$ obeying that

$$\tilde{x}_{\tilde{i}\tilde{j}}(T) - p_{\tilde{i}\tilde{j}}(T)\tilde{x}_{\tilde{i}\tilde{j}}\bigl(T - r_{\tilde{i}\tilde{j}}(T)\bigr) = \kappa$$

and

$$\kappa < \tilde{x}_{ij}(s) - p_{ij}(s)\tilde{x}_{ij}\bigl(s - r_{ij}(s)\bigr) < \frac{I_{ij}^{+} + \kappa}{\eta_{ij}} \quad \forall s \in (-\infty, T),\ ij \in J. \tag{9}$$

Case 2. There are $\tilde{i}\tilde{j} \in J$ and $T > 0$ agreeing with (9), and

$$\tilde{x}_{\tilde{i}\tilde{j}}(T) - p_{\tilde{i}\tilde{j}}(T)\tilde{x}_{\tilde{i}\tilde{j}}\bigl(T - r_{\tilde{i}\tilde{j}}(T)\bigr) = \frac{I_{\tilde{i}\tilde{j}}^{+} + \kappa}{\eta_{\tilde{i}\tilde{j}}}.$$

If Case 1 holds, one can assert that for arbitrary $ij \in J$,

$$0 < \tilde{x}_{ij}(t) \quad \forall t \in (-\infty, T). \tag{10}$$

On the contrary, suppose that (10) is false. Then there are $i^0 j^0 \in J$ and $T^0 \in (0, T)$ satisfying that for $ij \in J$,

$$\tilde{x}_{i^0 j^0}(T^0) = 0 \quad \text{with} \quad 0 < \tilde{x}_{ij}(t) \quad \forall t \in \bigl(-\infty, T^0\bigr).$$

Hence it follows that

$$\begin{aligned}\kappa &< \tilde{x}_{i^0 j^0}\bigl(T^0\bigr) - p_{i^0 j^0}\bigl(T^0\bigr)\tilde{x}_{i^0 j^0}\bigl(T^0 - r_{i^0 j^0}\bigl(T^0\bigr)\bigr)\\ &= -p_{i^0 j^0}\bigl(T^0\bigr)\tilde{x}_{i^0 j^0}\bigl(T^0 - r_{i^0 j^0}\bigl(T^0\bigr)\bigr) \leqslant 0,\end{aligned}$$

which contradicts the positiveness of κ and proves (10). Consequently,

$$\begin{aligned}\tilde{x}_{ij}(\vartheta) &= \tilde{x}_{ij}(\vartheta) - p_{ij}(\vartheta)\tilde{x}_{ij}\bigl(\vartheta - r_{ij}(\vartheta)\bigr) + p_{ij}(\vartheta)\tilde{x}_{ij}(\vartheta - r_{ij}(\vartheta))\\ &> \kappa + \min_{-\infty \leqslant s \leqslant \vartheta} p_{ij}^{-}\tilde{x}_{ij}(s)\\ &\geqslant \kappa + p_{ij}^{-} \min_{-\infty \leqslant s \leqslant t} \tilde{x}_{ij}(s) \quad \forall \vartheta \in (-\infty, t] \subseteq (-\infty, T)\end{aligned}$$

and

$$\tilde{x}_{ij}(t) \geqslant \min_{-\infty \leqslant s \leqslant t} \tilde{x}_{ij}(s) > \frac{\kappa}{1 - p_{ij}^{-}} \quad \forall t \in (-\infty, T). \tag{11}$$

Meanwhile,

$$\begin{aligned}\tilde{x}_{ij}(\vartheta) &= \tilde{x}_{ij}(\vartheta) - p_{ij}(\vartheta)\tilde{x}_{ij}\bigl(\vartheta - r_{ij}(\vartheta)\bigr) + p_{ij}(\vartheta)\tilde{x}_{ij}\bigl(\vartheta - r_{ij}(\vartheta)\bigr)\\ &< \frac{I_{ij}^{+} + \kappa}{\eta_{ij}} + \max_{-\infty \leqslant s \leqslant \vartheta} p_{ij}^{+}\tilde{x}_{ij}(s)\\ &\leqslant \frac{I_{ij}^{+} + \kappa}{\eta_{ij}} + p_{ij}^{+} \max_{-\infty \leqslant s \leqslant t} \tilde{x}_{ij}(s) \quad \forall \vartheta \in (-\infty, t] \subseteq (-\infty, T)\end{aligned}$$

and

$$\tilde{x}_{ij}(t) \leqslant \max_{-\infty \leqslant s \leqslant t} \tilde{x}_{ij}(s) < \frac{I_{ij}^{+} + \kappa}{\eta_{ij}(1 - p_{ij}^{+})} \quad \forall t \in (-\infty, T). \tag{12}$$

In view of (1), (5), (11), (12) and (S0), we gain

$$\begin{aligned}
0 &\geqslant \big[\tilde{x}_{\tilde{i}\tilde{j}}(t) - p_{\tilde{i}\tilde{j}}(t)\tilde{x}_{\tilde{i}\tilde{j}}\big(t - r_{\tilde{i}\tilde{j}}(t)\big)\big]'\big|_{t=T} \\
&= -a_{\tilde{i}\tilde{j}}(T)\big[\tilde{x}_{\tilde{i}\tilde{j}}(T) - p_{\tilde{i}\tilde{j}}(T)\tilde{x}_{\tilde{i}\tilde{j}}\big(T - r_{\tilde{i}\tilde{j}}(T)\big)\big] - a_{\tilde{i}\tilde{j}}(T)p_{\tilde{i}\tilde{j}}(T)\tilde{x}_{\tilde{i}\tilde{j}}\big(T - r_{\tilde{i}\tilde{j}}(T)\big) \\
&\quad - \sum_{C_{kl} \in N_r(\tilde{i},\tilde{j})} B_{\tilde{i}\tilde{j}}^{kl}(T) f\big(\tilde{x}_{kl}\big(T - \tau_{kl}(T)\big)\big) \\
&\quad \times \big[\tilde{x}_{\tilde{i}\tilde{j}}(T) - p_{\tilde{i}\tilde{j}}(T)\tilde{x}_{\tilde{i}\tilde{j}}\big(T - r_{\tilde{i}\tilde{j}}(T)\big) + p_{\tilde{i}\tilde{j}}(T)\tilde{x}_{\tilde{i}\tilde{j}}\big(T - r_{\tilde{i}\tilde{j}}(T)\big)\big] \\
&\quad - \sum_{C_{kl} \in N_q(\tilde{i},\tilde{j})} C_{\tilde{i}\tilde{j}}^{kl}(T) \int_0^{+\infty} K_{\tilde{i}\tilde{j}}(u) g\big(\tilde{x}_{kl}(T-u)\big)\,\mathrm{d}u \\
&\quad \times \big[\tilde{x}_{\tilde{i}\tilde{j}}(T) - p_{\tilde{i}\tilde{j}}(T)\tilde{x}_{\tilde{i}\tilde{j}}\big(T - r_{\tilde{i}\tilde{j}}(T)\big) + p_{\tilde{i}\tilde{j}}(T)\tilde{x}_{\tilde{i}\tilde{j}}\big(T - r_{\tilde{i}\tilde{j}}(T)\big)\big] + I_{\tilde{i}\tilde{j}}(T) \\
&\geqslant -a_{\tilde{i}\tilde{j}}(T)\kappa - a_{\tilde{i}\tilde{j}}(T)p_{\tilde{i}\tilde{j}}(T)\frac{I_{\tilde{i}\tilde{j}}^{+} + \kappa}{\eta_{\tilde{i}\tilde{j}}(1 - p_{\tilde{i}\tilde{j}}^{+})} \\
&\quad - \sum_{C_{kl} \in N_r(\tilde{i},\tilde{j})} \big|B_{\tilde{i}\tilde{j}}^{kl}(T)\big| M_f \kappa - \sum_{C_{kl} \in N_r(\tilde{i},\tilde{j})} \big|B_{\tilde{i}\tilde{j}}^{kl}(T)\big| p_{\tilde{i}\tilde{j}}(T) M_f \frac{I_{\tilde{i}\tilde{j}}^{+} + \kappa}{\eta_{\tilde{i}\tilde{j}}(1 - p_{\tilde{i}\tilde{j}}^{+})} \\
&\quad - \sum_{C_{kl} \in N_q(\tilde{i},\tilde{j})} \big|C_{\tilde{i}\tilde{j}}^{kl}(T)\big| \int_0^{+\infty} \big|K_{\tilde{i}\tilde{j}}(u)\big|\,\mathrm{d}u\, M_g \kappa \\
&\quad - \sum_{C_{kl} \in N_q(\tilde{i},\tilde{j})} \big|C_{\tilde{i}\tilde{j}}^{kl}(T)\big| \int_0^{+\infty} \big|K_{\tilde{i}\tilde{j}}(u)\big|\,\mathrm{d}u\, p_{\tilde{i}\tilde{j}}(T) M_g \frac{I_{\tilde{i}\tilde{j}}^{+} + \kappa}{\eta_{\tilde{i}\tilde{j}}(1 - p_{\tilde{i}\tilde{j}}^{+})} + I_{\tilde{i}\tilde{j}}(T) \\
&> 0,
\end{aligned}$$

which is absurd.

If Case 2 holds, we can also deduce (11) and (12), which, together with (1), (3) and (S0), results in

$$\begin{aligned}
0 &\leqslant \big[\tilde{x}_{\tilde{i}\tilde{j}}(t) - p_{\tilde{i}\tilde{j}}(t)\tilde{x}_{\tilde{i}\tilde{j}}\big(t - r_{\tilde{i}\tilde{j}}(t)\big)\big]'\big|_{t=T} \\
&= -a_{\tilde{i}\tilde{j}}(T)\big[\tilde{x}_{\tilde{i}\tilde{j}}(T) - p_{\tilde{i}\tilde{j}}(T)\tilde{x}_{\tilde{i}\tilde{j}}\big(T - r_{\tilde{i}\tilde{j}}(T)\big)\big] - a_{\tilde{i}\tilde{j}}(T)p_{\tilde{i}\tilde{j}}(T)\tilde{x}_{\tilde{i}\tilde{j}}\big(T - r_{\tilde{i}\tilde{j}}(T)\big) \\
&\quad - \sum_{C_{kl} \in N_r(\tilde{i},\tilde{j})} B_{\tilde{i}\tilde{j}}^{kl}(T) f\big(\tilde{x}_{kl}\big(T - \tau_{kl}(T)\big)\big) \\
&\quad \times \big[\tilde{x}_{\tilde{i}\tilde{j}}(T) - p_{\tilde{i}\tilde{j}}(T)\tilde{x}_{\tilde{i}\tilde{j}}\big(T - r_{\tilde{i}\tilde{j}}(T)\big) + p_{\tilde{i}\tilde{j}}(T)\tilde{x}_{\tilde{i}\tilde{j}}\big(T - r_{\tilde{i}\tilde{j}}(T)\big)\big]
\end{aligned}$$

$$
\begin{aligned}
&\quad - \sum_{C_{kl}\in N_q(\tilde{i},\tilde{j})} C_{\tilde{i}\tilde{j}}^{kl}(T) \int_0^{+\infty} K_{\tilde{i}\tilde{j}}(u) g\big(\tilde{x}_{kl}(T-u)\big)\,\mathrm{d}u \\
&\quad \times \big[\tilde{x}_{\tilde{i}\tilde{j}}(T) - p_{\tilde{i}\tilde{j}}(T)\tilde{x}_{\tilde{i}\tilde{j}}\big(T - r_{\tilde{i}\tilde{j}}(T)\big) + p_{\tilde{i}\tilde{j}}(T)\tilde{x}_{\tilde{i}\tilde{j}}\big(T - r_{\tilde{i}\tilde{j}}(T)\big)\big] + I_{\tilde{i}\tilde{j}}(T) \\
&\leqslant -a_{\tilde{i}\tilde{j}}(T)\frac{I_{\tilde{i}\tilde{j}}^{+} + \kappa}{\eta_{\tilde{i}\tilde{j}}} + \sum_{C_{kl}\in N_r(\tilde{i},\tilde{j})} \big|B_{\tilde{i}\tilde{j}}^{kl}(T)\big| M_f \frac{I_{\tilde{i}\tilde{j}}^{+} + \kappa}{\eta_{\tilde{i}\tilde{j}}} \\
&\quad + \sum_{C_{kl}\in N_r(\tilde{i},\tilde{j})} \big|B_{\tilde{i}\tilde{j}}^{kl}(T)\big| p_{\tilde{i}\tilde{j}}(T) M_f \frac{I_{\tilde{i}\tilde{j}}^{+} + \kappa}{\eta_{\tilde{i}\tilde{j}}(1 - p_{\tilde{i}\tilde{j}}^{+})} \\
&\quad + \sum_{C_{kl}\in N_q(\tilde{i},\tilde{j})} \big|C_{\tilde{i}\tilde{j}}^{kl}(T)\big| \int_0^{+\infty} \big|K_{\tilde{i}\tilde{j}}(u)\big|\,\mathrm{d}u\, M_g \frac{I_{\tilde{i}\tilde{j}}^{+} + \kappa}{\eta_{\tilde{i}\tilde{j}}} \\
&\quad + \sum_{C_{kl}\in N_q(\tilde{i},\tilde{j})} \big|C_{\tilde{i}\tilde{j}}^{kl}(T)\big| \int_0^{+\infty} \big|K_{\tilde{i}\tilde{j}}(u)\big|\,\mathrm{d}u\, M_g p_{\tilde{i}\tilde{j}}(T) \frac{I_{\tilde{i}\tilde{j}}^{+} + \kappa}{\eta_{\tilde{i}\tilde{j}}(1 - p_{\tilde{i}\tilde{j}}^{+})} + I_{\tilde{i}\tilde{j}}(T) \\
&= \Bigg[-a_{\tilde{i}\tilde{j}}(T) + \sum_{C_{kl}\in N_r(\tilde{i}\tilde{j})} \big|B_{\tilde{i}\tilde{j}}^{kl}(T)\big| M_f \bigg(1 + \frac{p_{\tilde{i}\tilde{j}}(T)}{1 - p_{\tilde{i}\tilde{j}}^{+}}\bigg) \\
&\quad + \sum_{C_{kl}\in N_q(\tilde{i},\tilde{j})} \big|C_{\tilde{i}\tilde{j}}^{kl}(T)\big| \int_0^{+\infty} \big|K_{\tilde{i}\tilde{j}}(u)\big|\,\mathrm{d}u\, M_g \bigg(1 + \frac{p_{\tilde{i}\tilde{j}}(T)}{1 - p_{\tilde{i}\tilde{j}}^{+}}\bigg) \Bigg] \frac{I_{\tilde{i}\tilde{j}}^{+} + \kappa}{\eta_{\tilde{i}\tilde{j}}} + I_{\tilde{i}\tilde{j}}(T) \\
&< 0,
\end{aligned}
$$

which produces a conflict and demonstrates that (8) is correct. This verifies Lemma 2. □

Remark 1. When the hypotheses adopted in Lemma 2 are obeyed, (8), (11) and (12) entail that for any solution of (1) incorporating assumptions (6), (7),

$$
0 < \frac{\kappa}{1 - p_{ij}^{-}} < \tilde{x}_{ij}(t) < \frac{I_{ij}^{+} + \kappa}{\eta_{ij}(1 - p_{ij}^{+})} \quad \forall t \in [0, +\infty). \tag{13}
$$

3 Main result

Proposition 1. (*See* [28, *Prop.* 3.1].) *For* $\theta(t) \in AP(\mathbb{R}, \mathbb{R})$,

$$
\limsup_{t\to+\infty} \theta(t) = \sup_{t\in\mathbb{R}} \theta(t) \quad \textit{and} \quad \liminf_{t\to+\infty} \theta(t) = \inf_{t\in\mathbb{R}} \theta(t).
$$

Theorem 1. *Under the presumptions of* (S0)–(S2), (3) *and* (4), *SICNNs* (1) *possesses just one almost periodic solution* $x^*(t)$, *which is positive and globally exponentially stable. Moreover, for arbitrary solution* $x(t)$ *of* (1) *incorporating the initial values* (2), *it can be* *discovered two constants* λ *and* B_{φ,x^*} *satisfying*

$$\left\|x(t) - x^*(t)\right\| \leqslant B_{\varphi,x^*}\mathrm{e}^{-\lambda t} \tag{14}$$

and

$$0 < \frac{\kappa}{1 - p_{ij}^-} \leqslant x_{ij}^*(t) \leqslant \frac{I_{ij}^+ + \kappa}{\eta_{ij}(1 - p_{ij}^+)} \quad \forall t \in [0, +\infty),\ ij \in J,$$

where κ *can be found in* (7).

Proof. Firstly, we evidence that the possible existing almost periodic solution has eventual positiveness. To do this, assume that the SICNNs (1) possesses a globally exponentially stable almost periodic solution $x^*(t)$ and denote by $v(t) = \{v_{ij}(t)\}$ an arbitrary solution of (1) incorporating assumptions (6), (7). It follows from (13) and (14) that

$$\begin{aligned}\frac{\kappa}{1 - p_{ij}^-} &\leqslant \liminf_{t\to+\infty} v_{ij}(t) = \liminf_{t\to+\infty} x_{ij}^*(t) \leqslant \limsup_{t\to+\infty} x_{ij}^*(t) \\ &= \limsup_{t\to+\infty} v_{ij}(t) \leqslant \frac{I_{ij}^+ + \kappa}{\eta_{ij}(1 - p_{ij}^+)}, \quad ij \in J.\end{aligned}$$

By Proposition 1, one can acquire

$$0 < \frac{\kappa}{1 - p_{ij}^-} \leqslant x_{ij}^*(t) \leqslant \frac{I_{ij}^+ + \kappa}{\eta_{ij}(1 - p_{ij}^+)} \quad \forall t \in [0, +\infty),\ ij \in J,$$

which reveals that $x^*(t)$ is positive on $[0, +\infty)$.

Secondly, we shall verify that SICNNs (1) possesses an almost periodic solution. Denote $H_{ij}(t) = x_{ij}(t) - p_{ij}(t)x_{ij}(t - r_{ij}(t))$, we gain

$$\begin{aligned}H_{ij}'(t) &= \left[x_{ij}(t) - p_{ij}(t)x_{ij}\left(t - r_{ij}(t)\right)\right]' \\ &= -a_{ij}(t)H_{ij}(t) - a_{ij}(t)p_{ij}(t)x_{ij}\left(t - r_{ij}(t)\right) \\ &\quad - \sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(t) f\left(x_{kl}\left(t - \tau_{kl}(t)\right)\right)x_{ij}(t) \\ &\quad - \sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(t) \int_0^{+\infty} K_{ij}(u) g\left(x_{kl}(t-u)\right) \mathrm{d}u\, x_{ij}(t) \\ &\quad + I_{ij}(t).\end{aligned}$$

Given $\varphi \in AP(\mathbb{R}, \mathbb{R}^{mn})$, from $r_{ij}, \tau_{ij} \in AP(\mathbb{R}, \mathbb{R})$ and Lemma 2.2 in [12] we acquire

$$\varphi_{ij}\left(t - r_{ij}(t)\right),\ \varphi_{ij}\left(t - \tau_{ij}(t)\right) \in AP(\mathbb{R}, \mathbb{R}), \quad ij \in J,$$

which, combined with (S0) and the argument process of Lemma 2.3 in [15], indicates that

$$\sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(t) f\left(\varphi_{kl}\left(t - \tau_{kl}(t)\right)\right)\varphi_{ij}(t) \in AP(\mathbb{R}, \mathbb{R})$$

and

$$\sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(t)\int_0^{+\infty} K_{ij}(u)g\big(\varphi_{kl}(t-u)\big)\,\mathrm{d}u\,\varphi_{ij}(t)\in AP(\mathbb{R},\mathbb{R}),\quad ij\in J.$$

Now, we take into consideration the following auxiliary equations:

$$\begin{aligned} H_{ij}'(t) &= -a_{ij}(t)H_{ij}(t)-a_{ij}(t)p_{ij}(t)\varphi_{ij}\big(t-r_{ij}(t)\big)\\ &\quad-\sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(t)f\big(\varphi_{kl}\big(t-\tau_{kl}(t)\big)\big)\varphi_{ij}(t)\\ &\quad-\sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(t)\int_0^{+\infty} K_{ij}(u)g\big(\varphi_{kl}(t-u)\big)\,\mathrm{d}u\,\varphi_{ij}(t)\\ &\quad+I_{ij}(t),\quad ij\in J. \end{aligned}\tag{15}$$

Combining $M[a_{ij}]=\lim_{T\to+\infty}\int_t^{t+T}a_{ij}(s)\,\mathrm{d}s/T>0$ and Theorem 2.3 in [27], we know that (15) possesses a sole almost periodic solution:

$$\begin{aligned} H^\varphi(t) &= \big\{H_{ij}^\varphi(t)\big\}\\ &= \Bigg\{\int_{-\infty}^{t} \mathrm{e}^{-\int_s^t a_{ij}(u)\,\mathrm{d}u}\bigg[-a_{ij}(s)p_{ij}(s)\varphi_{ij}\big(s-r_{ij}(s)\big)\\ &\quad-\sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(s)f\big(\varphi_{kl}\big(s-\tau_{kl}(s)\big)\big)\varphi_{ij}(s)\\ &\quad-\sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(s)\int_0^{+\infty} K_{ij}(u)g\big(\varphi_{kl}(s-u)\big)\,\mathrm{d}u\,\varphi_{ij}(s)+I_{ij}(s)\bigg]\,\mathrm{d}s\Bigg\}. \end{aligned}\tag{16}$$

Manifestly,

$$\big\{p_{ij}(t)\varphi_{ij}\big(t-r_{ij}(t)\big)\big\}+H^\varphi(t)\in AP\big(\mathbb{R},\mathbb{R}^{mn}\big),$$

and $Q(t)=\{Q_{ij}(t)\}=\{\int_{-\infty}^t \mathrm{e}^{-\int_s^t a_{ij}(u)\,\mathrm{d}u}I_{ij}(s)\,\mathrm{d}s\}\in AP(\mathbb{R},\mathbb{R}^{mn})$ is the sole almost periodic solution of

$$H_{ij}'(t)=-a_{ij}(t)H_{ij}(t)+I_{ij}(t),\quad ij\in J.$$

We set

$$\Omega=\big\{\varphi\colon\ \varphi\in AP\big(\mathbb{R},\mathbb{R}^{mn}\big),\ \|\varphi-Q\|_\infty\leqslant\omega\big\}.$$

If $\varphi\in\Omega$, then

$$\|\varphi\|_\infty\leqslant\|\varphi-Q\|_\infty+\|Q\|_\infty\leqslant\omega+I,\tag{17}$$

here $I = \|Q\|_\infty$. In addition, we set a mapping $\Pi : \Omega \to \Omega$:

$$(\Pi\varphi)(t) = \{p_{ij}(t)\varphi_{ij}(t - r_{ij}(t))\} + H^\varphi(t) \quad \forall \varphi \in \Omega.$$

Next, it will be proven that for any $\varphi \in \Omega$, $\Pi\varphi \in \Omega$. Indeed, with the help of (S0), (S2), (16) and (17), one can discover that

$$\begin{aligned}
&\bigl|(\Pi\varphi)(t) - Q(t)\bigr| \\
&\quad = \Biggl\{\Biggl|p_{ij}(t)\varphi_{ij}(t - r_{ij}(t)) + \int_{-\infty}^{t} \mathrm{e}^{-\int_s^t a_{ij}(u)\,\mathrm{d}u}\Biggl[-a_{ij}(s)p_{ij}(s)\varphi_{ij}(s - r_{ij}(s)) \\
&\qquad - \sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(s) f(\varphi_{kl}(s - \tau_{kl}(s)))\varphi_{ij}(s) \\
&\qquad - \sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(s) \int_0^{+\infty} K_{ij}(u) g(\varphi_{kl}(t-u))\,\mathrm{d}u\,\varphi_{ij}(s)\Biggr]\,\mathrm{d}s\Biggr|\Biggr\} \\
&\quad \leqslant \Biggl\{p_{ij}^+\|\varphi\|_\infty + \int_{-\infty}^{t} \mathrm{e}^{-\int_s^t a_{ij}(u)\,\mathrm{d}u}\Biggl[a_{ij}(s)p_{ij}(s) + M_f \sum_{C_{kl}\in N_r(i,j)} \bigl|B_{ij}^{kl}(s)\bigr| \\
&\qquad + M_g \sum_{C_{kl}\in N_q(i,j)} \bigl|C_{ij}^{kl}(s)\bigr| \int_0^{+\infty} \bigl|K_{ij}(u)\bigr|\,\mathrm{d}u\Biggr]\,\mathrm{d}s\,\|\varphi\|_\infty\Biggr\} \\
&\quad \leqslant \Biggl\{p_{ij}^+ + \int_{-\infty}^{t} \mathrm{e}^{-\int_s^t a_{ij}(u)\,\mathrm{d}u}\Biggl[a_{ij}(s)p_{ij}(s) + M_f \sum_{C_{kl}\in N_r(i,j)} \bigl|B_{ij}^{kl}(s)\bigr| \\
&\qquad + M_g \sum_{C_{kl}\in N_q(i,j)} \bigl|C_{ij}^{kl}(s)\bigr| \int_0^{+\infty} \bigl|K_{ij}(u)\bigr|\,\mathrm{d}u\Biggr]\,\mathrm{d}s\Biggr\}(\omega + I) \\
&\quad \leqslant \Biggl\{p_{ij}^+ + \int_{-\infty}^{t} \mathrm{e}^{-\int_s^t a_{ij}(u)\,\mathrm{d}u}\biggl(\frac{\omega}{\omega + I} - p_{ij}^+\biggr) a_{ij}(s)\,\mathrm{d}s\Biggr\}(\omega + I) \leqslant \{\omega\} \quad \forall t \in \mathbb{R},
\end{aligned}$$

which indicates that $\Pi\varphi \in \Omega$.

Moreover, we show that Π is a contractive mapping. In fact, (S0), (S2), (16) and (17) yield

$$\begin{aligned}
&\bigl|(\Pi\varphi)(t) - (\Pi\psi)(t)\bigr| \\
&\quad = \Biggl\{\Biggl|p_{ij}(t)\bigl(\varphi_{ij}(t - r_{ij}(t)) - \psi_{ij}(t - r_{ij}(t))\bigr) \\
&\qquad + \int_{-\infty}^{t} \mathrm{e}^{-\int_s^t a_{ij}(u)\,\mathrm{d}u}\Biggl[-a_{ij}(s)p_{ij}(s)\bigl(\varphi_{ij}(s - r_{ij}(s)) - \psi_{ij}(s - r_{ij}(s))\bigr)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(s)\bigl(f\bigl(\varphi_{kl}\bigl(s-\tau_{kl}(s)\bigr)\bigr)\varphi_{ij}(s) - f\bigl(\psi_{kl}\bigl(t-\tau_{kl}(s)\bigr)\bigr)\psi_{ij}(s)\bigr) \\
&- \sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(s)\Biggl(\int_0^{+\infty} K_{ij}(u)g\bigl(\varphi_{kl}(s-u)\bigr)\,\mathrm{d}u\,\varphi_{ij}(s) \\
&- \int_0^{+\infty} K_{ij}(u)g\bigl(\psi_{kl}(s-u)\bigr)\,\mathrm{d}u\,\psi_{ij}(s)\Biggr)\Biggr]\,\mathrm{d}s\Biggr|\Biggr\} \\
\leqslant{}& \Biggl\{p_{ij}^{+}\|\varphi-\psi\|_\infty + \int_{-\infty}^{t} \mathrm{e}^{-\int_s^t a_{ij}(u)\,\mathrm{d}u}\Biggl[a_{ij}(s)p_{ij}(s)\|\varphi-\psi\|_\infty \\
&+ \sum_{C_{kl}\in N_r(i,j)} \bigl|B_{ij}^{kl}(s)\bigr|\bigl(\bigl|f\bigl(\varphi_{kl}\bigl(s-\tau_{kl}(s)\bigr)\bigr) - f\bigl(\psi_{kl}\bigl(s-\tau_{kl}(s)\bigr)\bigr)\bigr|\bigl|\varphi_{ij}(s)\bigr| \\
&+ \bigl|f\bigl(\psi_{kl}\bigl(s-\tau_{kl}(s)\bigr)\bigr)\bigr|\bigl|\varphi_{ij}(s)-\psi_{ij}(s)\bigr|\bigr) \\
&+ \sum_{C_{kl}\in N_q(i,j)} \bigl|C_{ij}^{kl}(s)\bigr|\Biggl(\int_0^{+\infty} \bigl|K_{ij}(u)\bigr|\bigl|g\bigl(\varphi_{kl}(s-u)\bigr) - g\bigl(\psi_{kl}(s-u)\bigr)\bigr|\,\mathrm{d}u\,\bigl|\varphi_{ij}(s)\bigr| \\
&+ \int_0^{+\infty} \bigl|K_{ij}(u)\bigr|\bigl|g\bigl(\psi_{kl}(s-u)\bigr)\bigr|\,\mathrm{d}u\,\bigl|\varphi_{ij}(s)-\psi_{ij}(s)\bigr|\Biggr)\Biggr]\,\mathrm{d}s\Biggr\} \\
\leqslant{}& \Biggl\{p_{ij}^{+}\|\varphi-\psi\|_\infty + \int_{-\infty}^{t} \mathrm{e}^{-\int_s^t a_{ij}(u)\,\mathrm{d}u}\Biggl[a_{ij}(s)p_{ij}(s)\|\varphi-\psi\|_\infty \\
&+ \sum_{C_{kl}\in N_r(i,j)} \bigl|B_{ij}^{kl}(s)\bigr|\bigl(\mu\|\varphi\|_\infty + M_f\bigr)\|\varphi-\psi\|_\infty \\
&+ \sum_{C_{kl}\in N_q(i,j)} \bigl|C_{ij}^{kl}(s)\bigr|\int_0^{+\infty} \bigl|K_{ij}(u)\bigr|\,\mathrm{d}u\,\bigl(\gamma\|\varphi\|_\infty + M_g\bigr)\|\varphi-\psi\|_\infty\Biggr]\,\mathrm{d}s\Biggr\} \\
\leqslant{}& \Biggl\{p_{ij}^{+} + \int_{-\infty}^{t} \mathrm{e}^{-\int_s^t a_{ij}(u)\,\mathrm{d}u}\Biggl[a_{ij}(s)p_{ij}(s) + \sum_{C_{kl}\in N_r(i,j)} \bigl|B_{ij}^{kl}(s)\bigr|\bigl(\mu(\omega+I)+M_f\bigr) \\
&+ \sum_{C_{kl}\in N_q(i,j)} \bigl|C_{ij}^{kl}(s)\bigr|\int_0^{+\infty} \bigl|K_{ij}(u)\bigr|\,\mathrm{d}u\,\bigl(\gamma(\omega+I)+M_g\bigr)\Biggr]\,\mathrm{d}s\Biggr\}\|\varphi-\psi\|_\infty \\
\leqslant{}& \Biggl\{p_{ij}^{+} + \int_{-\infty}^{t} \mathrm{e}^{-\int_s^t a_{ij}(u)\,\mathrm{d}u} J_{ij}a_{ij}(s)\,\mathrm{d}s\Biggr\}\|\varphi-\psi\|_\infty, \\
\leqslant{}& \bigl\{p_{ij}^{+} + J_{ij}\bigr\}\|\varphi-\psi\|_\infty \quad \forall t\in\mathbb{R},\ \varphi,\psi\in\Omega.
\end{aligned}
$$

This leads to

$$\|\Pi\varphi - \Pi\psi\|_\infty \leqslant \max_{ij\in J}\bigl(p_{ij}^+ + J_{ij}\bigr)\|\varphi - \psi\|_\infty.$$

Therefore, in view of Theorem 0.3.1 of [6] and $\max_{ij\in J}(p_{ij}^+ + J_{ij}) < 1$, we know that Π owns a sole fixed point $x^* = \{x_{ij}^*\} \in \Omega$ satisfying that

$$\bigl\{x_{ij}^*(t)\bigr\} = x^*(t) = (\Pi x^*)(t) = \bigl\{p_{ij}(t)x_{ij}^*\bigl(t - r_{ij}(t)\bigr)\bigr\} + \bigl\{H_{ij}^{x^*}(t)\bigr\}$$

and

$$\begin{aligned}
x_{ij}^*(t) &= p_{ij}(t)x_{ij}^*\bigl(t - r_{ij}(t)\bigr) + H_{ij}^{x^*}(t)\\
&= p_{ij}(t)x_{ij}^*\bigl(t - r_{ij}(t)\bigr) + \int\limits_{-\infty}^{t} \mathrm{e}^{-\int_s^t a_{ij}(u)\,\mathrm{d}u}\Biggl[-a_{ij}(s)p_{ij}(s)x_{ij}^*\bigl(s - r_{ij}(s)\bigr)\\
&\quad - \sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(s) f\bigl(x_{kl}^*\bigl(s - \tau_{kl}(s)\bigr)\bigr)x_{ij}^*(s)\\
&\quad - \sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(s)\int\limits_0^{+\infty} K_{ij}(u) g\bigl(x_{kl}^*(s-u)\bigr)\,\mathrm{d}u\, x_{ij}^*(s) + I_{ij}(s)\Biggr]\,\mathrm{d}s,
\end{aligned}$$

which, together with (16), results in

$$\begin{aligned}
&\bigl[x_{ij}(t) - p_{ij}(t)x_{ij}^*\bigl(t - r_{ij}(t)\bigr)\bigr]'\\
&\quad = -a_{ij}(t)x_{ij}^*(t) - \sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(t) f\bigl(x_{kl}^*\bigl(t - \tau_{kl}(t)\bigr)\bigr)x_{ij}^*(t)\\
&\qquad - \sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(t)\int\limits_0^{+\infty} K_{ij}(u) g\bigl(x_{kl}^*(t-u)\bigr)\,\mathrm{d}u\, x_{ij}^*(t) + I_{ij}(t).
\end{aligned}$$

This entails that $x^*(t)$ is the almost periodic solution of SICNNs (1).

Eventually, we verify the globally exponential stability of $x^*(t)$.

Denote by $x(t) = \{x_{ij}(t)\}$ a solution of SICNNs (1) incorporating (2), and let

$$z_{ij}(t) = x_{ij}(t) - x_{ij}^*(t), \quad Z_{ij}(t) = z_{ij}(t) - p_{ij}(t)z_{ij}\bigl(t - r_{ij}(t)\bigr), \quad ij \in J.$$

Then, for $ij \in J$,

$$\begin{aligned}
Z_{ij}'(t) &= \bigl[z_{ij}(t) - p_{ij}(t)z_{ij}\bigl(t - r_{ij}(t)\bigr)\bigr]'\\
&= -a_{ij}(t)Z_{ij}(t) - a_{ij}(t)p_{ij}(t)z_{ij}\bigl(t - r_{ij}(t)\bigr)\\
&\quad - \sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(t)\bigl[f\bigl(x_{kl}\bigl(t - \tau_{kl}(t)\bigr)\bigr)x_{ij}(t) - f\bigl(x_{kl}^*\bigl(t - \tau_{kl}(t)\bigr)\bigr)x_{ij}^*(t)\bigr]
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(t)\Bigg[\int_0^{+\infty} K_{ij}(u)g\big(x_{kl}(t-u)\big)\,\mathrm{d}u\, x_{ij}(t) \\
&- \int_0^{+\infty} K_{ij}(u)g\big(x_{kl}^*(t-u)\big)\,\mathrm{d}u\, x_{ij}^*(t)\Bigg].
\end{aligned}
$$

From (S1) and (S2) one can discover a positive number

$$\lambda < \min\Big(\alpha, \min_{ij\in J} a_{ij}^-\Big)$$

satisfying that for all $ij \in J$, there holds $p_{ij}^+ \mathrm{e}^{\lambda r_{ij}^+} < 1$ and

$$
\begin{aligned}
&\sup_{t\in\mathbb{R}}\Bigg\{\lambda - a_{ij}(t) + \Bigg[a_{ij}(t)p_{ij}(t)\frac{\mathrm{e}^{\lambda r_{ij}^+}}{1-p_{ij}^+\mathrm{e}^{\lambda r_{ij}^+}} \\
&\quad + \sum_{C_{kl}\in N_r(i,j)} \big|B_{ij}^{kl}(t)\big|\bigg(M_f\frac{1}{1-p_{ij}^+\mathrm{e}^{\lambda r_{ij}^+}} + \mu(\omega+I)\frac{\mathrm{e}^{\lambda\tau_{kl}^+}}{1-p_{kl}^+\mathrm{e}^{\lambda r_{kl}^+}}\bigg) \\
&\quad + \sum_{C_{kl}\in N_q(i,j)} \big|C_{ij}^{kl}(t)\big| \int_0^{+\infty}\big|K_{ij}(u)\big|\bigg(M_g\frac{1}{1-p_{ij}^+\mathrm{e}^{\lambda r_{ij}^+}} + \gamma(\omega+I)\frac{\mathrm{e}^{\lambda u}}{1-p_{kl}^+\mathrm{e}^{\lambda r_{kl}^+}}\bigg)\,\mathrm{d}u\Bigg]\Bigg\} \\
&\quad < 0. \qquad (18)
\end{aligned}
$$

Set

$$
\begin{aligned}
\|\varphi\|_\xi = \sup_{t\leqslant 0}\max_{ij\in J}\Big|&\big[\varphi_{ij}(t) - p_{ij}(t)\varphi_{ij}\big(t-r_{ij}(t)\big)\big] \\
&- \big[x_{ij}^*(t) - p_{ij}(t)x_{ij}^*\big(t-r_{ij}(t)\big)\big]\Big|.
\end{aligned}
$$

For arbitrary $\epsilon > 0$, one can obtain

$$\|Z(0)\| < (\|\varphi\|_\xi + \epsilon), \qquad (19)$$

thus, one can take a sufficiently large constant $M > 1$ satisfying that

$$\big\|Z(t)\big\| < \big(\|\varphi\|_\xi + \epsilon\big)\mathrm{e}^{-\lambda t} < M\big(\|\varphi\|_\xi + \epsilon\big)\mathrm{e}^{-\lambda t} \quad \forall t \in (-\infty, 0].$$

Hereafter, we validate

$$\big\|Z(t)\big\| < M\big(\|\varphi\|_\xi + \epsilon\big)\mathrm{e}^{-\lambda t} \quad \forall t > 0. \qquad (20)$$

By way of contradiction, there must be $ij \in J$ and $\rho > 0$ obeying that

$$\big|Z_{ij}(\rho)\big| = \big\|Z(\rho)\big\| = M\big(\|\varphi\|_\xi + \epsilon\big)\mathrm{e}^{-\lambda\rho} \qquad (21)$$

and

$$\big\|Z(s)\big\| < M\big(\|\varphi\|_\xi + \epsilon\big)\mathrm{e}^{-\lambda s} \quad \forall s \in (-\infty, \rho).$$

Moreover,

$$\begin{aligned} \mathrm{e}^{\lambda \upsilon}\big|z_{ij}(\upsilon)\big| &\leqslant \mathrm{e}^{\lambda \upsilon}\big|z_{ij}(\upsilon) - p_{ij}(\upsilon)z_{ij}\big(\upsilon - r_{ij}(\upsilon)\big)\big| + \mathrm{e}^{\lambda \upsilon}\big|p_{ij}(\upsilon)z_{ij}\big(\upsilon - r_{ij}(\upsilon)\big)\big| \\ &\leqslant \mathrm{e}^{\lambda \upsilon}\big|Z_{ij}(\upsilon)\big| + p_{ij}^{+}\mathrm{e}^{\lambda r_{ij}^{+}}\mathrm{e}^{\lambda(\upsilon - r_{ij}(\upsilon))}\big|z_{ij}\big(\upsilon - r_{ij}(\upsilon)\big)\big| \\ &\leqslant M\big(\|\varphi\|_\xi + \epsilon\big) + p_{ij}^{+}\mathrm{e}^{\lambda r_{ij}^{+}} \sup_{s \in (-\infty, t]} \mathrm{e}^{\lambda s}\big|z_{ij}(s)\big|, \end{aligned} \tag{22}$$

where $\upsilon \in (-\infty, t]$, $t \in (-\infty, \rho)$, $ij \in J$, which indicates that

$$\mathrm{e}^{\lambda t}\big|z_{ij}(t)\big| \leqslant \sup_{s \in (-\infty, t]} \mathrm{e}^{\lambda s}\big|z_{ij}(s)\big| \leqslant \frac{M(\|\varphi\|_\xi + \epsilon)}{1 - p_{ij}^{+}\mathrm{e}^{\lambda r_{ij}^{+}}} \quad \forall t \in (-\infty, \rho),\ ij \in J. \tag{23}$$

Note that

$$\begin{aligned} &Z_{ij}'(s) + a_{ij}(s)Z_{ij}(s) \\ &\quad = -a_{ij}(s)p_{ij}(s)z_{ij}\big(s - r_{ij}(s)\big) \\ &\qquad - \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(s)\big[f\big(x_{kl}\big(s - \tau_{kl}(s)\big)\big)x_{ij}(s) - f\big(x_{kl}^{*}\big(s - \tau_{kl}(s)\big)\big)x_{ij}^{*}(s)\big] \\ &\qquad - \sum_{C_{kl} \in N_q(i,j)} C_{ij}^{kl}(s)\Bigg[\int_0^{+\infty} K_{ij}(u)g\big(x_{kl}(s-u)\big)\,\mathrm{d}u\, x_{ij}(s) \\ &\qquad - \int_0^{+\infty} K_{ij}(u)g\big(x_{kl}^{*}(s-u)\big)\,\mathrm{d}u\, x_{ij}^{*}(s)\Bigg], \quad s \in [0, t],\ t \in [0, \rho], \end{aligned}$$

which means that

$$\begin{aligned} Z_{ij}(t) = Z_{ij}(0)\mathrm{e}^{-\int_0^t a_{ij}(u)\,\mathrm{d}u} + \int_0^t \mathrm{e}^{-\int_s^t a_{ij}(u)\,\mathrm{d}u}\Bigg\{ &-a_{ij}(s)p_{ij}(s)z_{ij}\big(s - r_{ij}(s)\big) \\ &- \sum_{C_{kl} \in N_r(i,j)} B_{ij}^{kl}(s)\big[f\big(x_{kl}\big(s - \tau_{kl}(s)\big)\big)x_{ij}(s) - f\big(x_{kl}^{*}\big(s - \tau_{kl}(s)\big)\big)x_{ij}^{*}(s)\big] \\ &- \sum_{C_{kl} \in N_q(i,j)} C_{ij}^{kl}(s)\Bigg[\int_0^{+\infty} K_{ij}(u)g\big(x_{kl}(s-u)\big)\,\mathrm{d}u\, x_{ij}(s) \\ &- \int_0^{+\infty} K_{ij}(u)g\big(x_{kl}^{*}(s-u)\big)\,\mathrm{d}u\, x_{ij}^{*}(s)\Bigg]\Bigg\}\,\mathrm{d}s, \quad t \in [0, \rho]. \end{aligned}$$

Consequently, owing to (S0), (S2), (17), (18), (19) and (23), we obtain

$$
\begin{aligned}
&\big|Z_{ij}(\rho)\big| \\
&\quad= \Bigg|Z_{ij}(0)\mathrm{e}^{-\int_0^\rho a_{ij}(u)\,\mathrm{d}u} + \int\limits_0^\rho \mathrm{e}^{-\int_s^\rho a_{ij}(u)\,\mathrm{d}u}\bigg\{-a_{ij}(s)p_{ij}(s)z_{ij}\big(s-r_{ij}(s)\big) \\
&\qquad - \sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}(s)\big[f\big(x_{kl}\big(s-\tau_{kl}(s)\big)\big)x_{ij}(s) - f\big(x_{kl}^*\big(s-\tau_{kl}(s)\big)\big)x_{ij}^*(s)\big] \\
&\qquad - \sum_{C_{kl}\in N_q(i,j)} C_{ij}^{kl}(s)\Bigg[\int\limits_0^{+\infty} K_{ij}(u)g(x_{kl}(s-u))\,\mathrm{d}u\,x_{ij}(s) \\
&\qquad - \int\limits_0^{+\infty} K_{ij}(u)g(x_{kl}^*(s-u))\,\mathrm{d}u\,x_{ij}^*(s)\Bigg]\bigg\}\,\mathrm{d}s\Bigg| \\
&\quad\leqslant \big|Z_{ij}(0)\big|\mathrm{e}^{-\int_0^\rho a_{ij}(u)\,\mathrm{d}u} + \int\limits_0^\rho \mathrm{e}^{-\int_s^\rho a_{ij}(u)\,\mathrm{d}u}\bigg\{\big|a_{ij}(s)p_{ij}(s)\big|\big|z_{ij}\big(s-r_{ij}(s)\big)\big| \\
&\qquad + \sum_{C_{kl}\in N_r(i,j)} \big|B_{ij}^{kl}(s)\big|\big[\big|f\big(x_{kl}\big(s-\tau_{kl}(s)\big)\big)x_{ij}(s) - f\big(x_{kl}\big(s-\tau_{kl}(s)\big)\big)x_{ij}^*(s)\big| \\
&\qquad + \big|f\big(x_{kl}\big(s-\tau_{kl}(s)\big)\big)x_{ij}^*(s) - f\big(x_{kl}^*\big(s-\tau_{kl}(s)\big)\big)x_{ij}^*(s)\big|\big] \\
&\qquad + \sum_{C_{kl}\in N_q(i,j)} \big|C_{ij}^{kl}(s)\big| \int\limits_0^{+\infty} \big|K_{ij}(u)\big|\big[\big|g\big(x_{kl}(s-u)\big)x_{ij}(s) - g\big(x_{kl}(s-u)\big)x_{ij}^*(s)\big| \\
&\qquad + \big|g\big(x_{kl}(s-u)\big)x_{ij}^*(s) - g\big(x_{kl}^*(s-u)\big)x_{ij}^*(s)\big|\big]\,\mathrm{d}u\bigg\}\,\mathrm{d}s \\
&\quad\leqslant \big(\|\varphi\|_\xi+\epsilon\big)\mathrm{e}^{-\lambda\rho}\mathrm{e}^{-\int_0^\rho (a_{ij}(u)-\lambda)\,\mathrm{d}u} \\
&\qquad + \int\limits_0^\rho \mathrm{e}^{-\int_s^\rho (a_{ij}(u)-\lambda)\,\mathrm{d}u}\Bigg\{a_{ij}(s)p_{ij}(s)\frac{\mathrm{e}^{\lambda r_{ij}^+}}{1-p_{ij}^+\mathrm{e}^{\lambda r_{ij}^+}} \\
&\qquad + \sum_{C_{kl}\in N_r(i,j)} \big|B_{ij}^{kl}(s)\big|\Bigg[M_f\frac{1}{1-p_{ij}^+\mathrm{e}^{\lambda r_{ij}^+}} + \mu(\omega+I)\frac{\mathrm{e}^{\lambda\tau_{kl}^+}}{1-p_{kl}^+\mathrm{e}^{\lambda r_{kl}^+}}\Bigg] \\
&\qquad + \sum_{C_{kl}\in N_q(i,j)} \big|C_{ij}^{kl}(s)\big|\int\limits_0^{+\infty}\big|K_{ij}(u)\big|\Bigg[M_g\frac{1}{1-p_{ij}^+\mathrm{e}^{\lambda r_{ij}^+}} + \gamma(\omega+I)\frac{\mathrm{e}^{\lambda u}}{1-p_{kl}^+\mathrm{e}^{\lambda r_{kl}^+}}\Bigg]\,\mathrm{d}u\Bigg\}\,\mathrm{d}s \\
&\qquad \times M\big(\|\varphi\|_\xi+\epsilon\big)\mathrm{e}^{-\lambda\rho} \\
&\quad\leqslant M\big(\|\varphi\|_\xi+\epsilon\big)\mathrm{e}^{-\lambda\rho}\Bigg[\bigg(\frac{1}{M}-1\bigg)\mathrm{e}^{-\int_0^\rho (a_{ij}(u)-\lambda)\,\mathrm{d}u} + 1\Bigg] < M\big(\|\varphi\|_\xi+\epsilon\big)\mathrm{e}^{-\lambda\rho}.
\end{aligned}
$$

This causes a conflict with (21). Consequently, (20) holds. Letting $\epsilon \to 0^+$, we have

$$\left\|Z(t)\right\| \leqslant M\|\varphi\|_\xi \mathrm{e}^{-\lambda t} \quad \forall t > 0. \tag{24}$$

Then, using a similar discussion with (22) and (23), we get from (24) that

$$\mathrm{e}^{\lambda t}\left|z_{ij}(t)\right| \leqslant \sup_{s\in(-\infty,t]} \mathrm{e}^{\lambda s}\left|z_{ij}(s)\right| \leqslant \frac{M\|\varphi\|_\xi}{1 - p_{ij}^+ \mathrm{e}^{\lambda r_{ij}^+}}$$

and

$$\left|z_{ij}(t)\right| \leqslant B_{\varphi,x^*}\mathrm{e}^{-\lambda t} \quad \forall t > 0,\ ij \in J,$$

where $B_{\varphi,x^*} = M\|\varphi\|_\xi/(1 - p_{ij}^+ \mathrm{e}^{\lambda r_{ij}^+})$. This assures Theorem 1. □

Remark 2. In Theorem 1, we firstly set up the positive stability of delayed almost periodic SICNNs with D operator. So far, many achievements on the exponential convergence or stability of delayed cellular neural network models have been revealed, see, e.g., [16, 23, 24, 26] and the related references. However, as far as we know, there is no result exploring the positive almost periodicity of SICNNs with D operator. Our results complement and improve some corresponding ones of the existing publications in [23, 24, 26].

4 Numerical example

Regard the neutral-type SICNNs incorporating D operator:

$$\begin{aligned}
&\left[x_{ij}(t) - \frac{|\sin(i+j)t|}{100} x_{ij}\big(t - \left|\sin(i+j)t\right| - 0.1\big)g\right]' \\
&\quad = -a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} B_{ij}^{kl}\frac{1}{2}\arctan\big(x_{kl}\big(t - \left|\cos(i+j)t\right|\big)\big)x_{ij}(t) \\
&\qquad - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\int_0^{+\infty} \mathrm{e}^{-u}\frac{1}{2}\arctan\big(x_{kl}(t-u)\big)\,\mathrm{d}u\, x_{ij}(t) + I_{ij}(t),
\end{aligned} \tag{25}$$

where $i, j = 1, 2$,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 + 2|\cos 100t| & 0.8 + |\sin 100t| \\ 1 + 1.3|\cos 100t| & 1 + 1.2|\sin 100t| \end{bmatrix},$$

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{pmatrix} 0.01|\cos\sqrt{2}t| & 0.02|\cos\sqrt{3}t| \\ 0.02|\cos\sqrt{3}t| & 0.01|\cos 2t| \end{pmatrix}$$

and

$$\begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix} = \frac{1}{\mathrm{e}^{1/25}}\begin{bmatrix} |\sin 4t| + 1/2 & |\sin 3t| + 1/3 \\ |\sin 2t| + 1/3 & |\sin t| + 1/2 \end{bmatrix}.$$

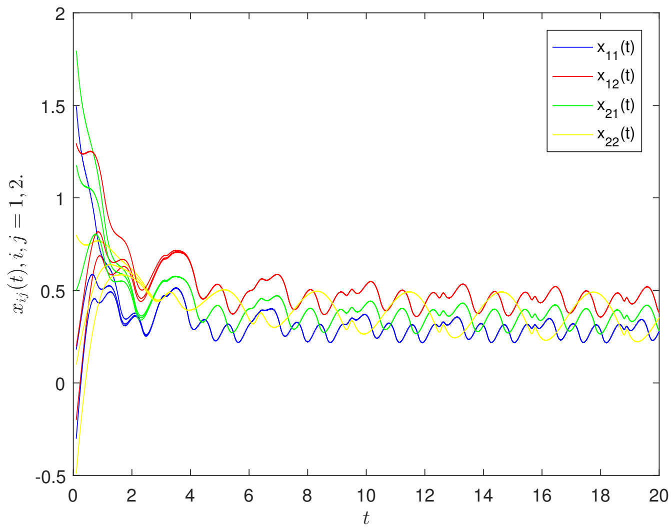

Figure 1. Numerical state vector $x(t)$ for system (25) including three groups initial values: $(0.5 + \cos 2t$, $-0.4 + 2\sin 3t$, $0.8 + \cos t$, $0.5 + 3\sin t)$, $(-0.8 + \cos 2t$, $\sin 2t$, $1.2\cos 2t$, $0.5\sin 2t)$, $(-0.5 + \sin 2t$, $0.3 + \cos t$, $0.4 + \sin t$, $-0.5\cos 2t)$.

Take

$$M_f = M_g = \frac{\pi}{4}, \qquad \mu = \gamma = \frac{1}{2}, \qquad \omega = L = 1,$$
$$\sum_{C_{kl}\in N_1(i,j)} \left|B_{ij}^{kl}(t)\right| = \sum_{C_{kl}\in N_1(i,j)} \left|C_{ij}^{kl}(t)\right| \leqslant 0.06.$$

Obviously, all requirements of Theorem 1 are obeyed in (25). Thus, SICNNs (25) possesses a sole one globally exponentially stable almost periodic solution, which has positiveness (see Fig. 1).

Remark 3. It should be noted that the positive almost periodicity of SICNNs incorporating D operator and mixed delays has not been touched in the previous publications [4, 9]. Thus, the corresponding conclusions of the above mentioned literatures are ineffective to reflect the positive almost periodic stability of SICNNs (25).

5 Conclusions

In this work, we obtain some results involving the existence and global exponential stability of the positive almost periodic solution for a kind of shunting inhibitory cellular neural networks incorporating mixed delays and D operator with the help of some analysis methods and inequality techniques. Because neutral-type operator exists in the neural networks system, the existing methods are no longer applicable to show the positiveness of the almost periodic solutions, we have developed novel techniques and mathematical approaches for overcoming the obstacles coming from neutral-type operator. Lemma 2 is important for the judgment of the prior boundedness of the operator equation. Moreover, numerical examples are worked out to demonstrate the advantages of our results. The strategy adopted in this work can be also applied to explore other types of D operator cellular neural networks, such as neural networks systems involving parameters uncertainties and impulse disturbance, neural networks accompanying neutral-type mixed delays and so on. This is our future investigation direction.

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