Vilnius
University
Press

How to empower Grünwald–Letnikov fractional difference equations with available initial condition?*

Yiheng Wei[a] ![ORCID]**, Jinde Cao**[a,b] ![ORCID]**, Chuang Li**[c] ![ORCID]**, Yangquan Chen**[d] ![ORCID]

[a]School of Mathematics, Southeast University,
Nanjing 211189, China
neudawei@seu.edu.cn

[b]Yonsei Frontier Lab, Yonsei University,
Seoul 03722, South Korea

[c]School of Mechanical and Electrical Engineering,
Hainan University,
Haikou 570208, China

[d]School of Engineering, University of California,
Merced, CA 95343, USA

Abstract. In this paper, the initial condition independence property of Grünwald–Letnikov fractional difference is revealed for the first time. For example, the solution $x(k)$ of equation ${}^{\mathrm{G}}_a\nabla^\alpha_k x(k) = f(x(k))$, $k \geqslant a + 1$, cannot be calculated with initial condition $x(a)$. First, the initial condition independence property is carefully investigated in both time domain and frequency domain. Afterwards, some possible schemes are formulated to make the considered system connect to initial condition. Armed with this information, the concerned property is examined on three modified Grünwald–Letnikov definitions. Finally, results from illustrative examples demonstrate that the developed schemes are sharp.

Keywords: fractional calculus, independence, initial condition, Grünwald–Letnikov definition, dynamic properties.

1 Introduction

Fractional calculus, born at the end of seventeenth century, has provided many hot topics of research in many fields of science and engineering [4, 19]. The reason of the success of the fractional calculus is its nonlocality. One of the possibility is the consideration of discrete time systems, which have the advantages of simple calculation and small singularity [24, 25]. For the details of the most recent advances, one can refer to some excellent monographs [5, 7, 13] and the references cited therein as well.

The Grünwald–Letnikov definition was originally proposed for the fractional derivative (time-domain case). To avoid the inconvenience of using its limit form, the infinitesimal sampling period h was usually set as a finite small value in numerical computation process [10,12,16]. Besides, with the discretization approximation of Grünwald–Letnikov derivative, fractional-order systems can be implemented by FPGA (Field Programmable Gate Array) [15,18]. Along this way, the interpretation of Grünwald–Letnikov derivative was studied [6,17]. The initial value problem was evaluated by [1,3]. Actually, when h is finite, especially when $h = 1$, it does denote the fractional difference [13]. For the discrete case, [14] investigated the system order identification, [9] addressed the system pseudo state estimation, [11,20] discussed the stability analysis issue, and [8] considered the perfect control problem. Due to their efforts, a series of wonderful and meaningful results have been achieved [2].

Despite the wide use of Grünwald–Letnikov fractional difference, it has a major drawback (initial condition independence), which is the restriction on the dynamic behavior or even the stability. For example, by using the definition given in [20], the state response of $_a^{\mathrm{G}}\nabla_k^\alpha x(k) = \lambda x(k) + bu(k)$ is meaningless, which brings many difficulties for analysis and synthesis. To the best of the authors' knowledge, no results on this property have been reported, while it is indeed an essential property to be explored before Grünwald–Letnikov fractional difference can be used as a viable tool. Bearing this in mind, this paper plans to investigate the initial condition independence problem. It should be pointed out that this work will deepen our understanding on Grünwald–Letnikov fractional difference and enlarge its applicability.

The outline of the rest paper is organized as follows. Section 2 introduces some basic knowledge of nabla discrete fractional calculus. Section 3 reveals the initial conditions independent property of Grünwald–Letnikov fractional difference and provides three schemes to empower Grünwald–Letnikov fractional difference equations with available initial condition. Section 4 discusses several relevant definitions briefly and presents some detailed numerical examples to confirm the correctness and effectiveness of the obtained results. Finally, the paper ends with a conclusion in Section 5.

2 Preliminaries

This section recalls some basic definitions and properties on nabla fractional calculus, for more details, we refer the reader to [7,13].

Definition 1. The nth nabla difference and nth nabla sum for a function $x : \mathbb{N}_{a+1-n} \to \mathbb{R}$ are defined respectively as

$$\nabla^n x(k) := \sum_{i=0}^{n} (-1)^i \binom{n}{i} x(k-i), \tag{1}$$

$$_a\nabla_k^{-n} x(k) := \sum_{i=0}^{k-a-1} (-1)^i \binom{-n}{i} x(k-i), \tag{2}$$

where $n \in \mathbb{Z}_+$ is the difference/sum order, $k \in \mathbb{N}_{a+1}$ is the time variable, $a \in \mathbb{R}$ is the initial instant, $\mathbb{N}_{a+1} := \{a+1, a+2, a+3, \dots\}$, $\binom{p}{q} := \Gamma(p+1)/[\Gamma(q+1)\Gamma(p-q+1)]$, and $\Gamma(\cdot)$ is the gamma function.

It can be found that (1) and (2) have a similar form. The distinction lies in that the nabla difference has local memory, and the nabla sum has nonlocal memory. However, to avoid the singularity, i.e., $\Gamma(-n) = \infty$, $n \in \mathbb{N}$, the generalized binomial coefficient can be calculated as alternative expression $\binom{-n}{i} = (-)^i \Gamma(n+i)/[\Gamma(i+1)\Gamma(n)]$. For the special case of $n = 0$, let us make an assumption

$$\nabla^0 x(k) = {}_a\nabla_k^0 x(k) := x(k). \tag{3}$$

Extending the order n in Definition 1 from the positive integers to the positive real number, the following definition can be derived.

Definition 2. The αth Grünwald–Letnikov fractional difference/sum for a function $x : \mathbb{N}_{a+1} \to \mathbb{R}$ is defined as

$$\begin{smallmatrix}G\\a\end{smallmatrix}\nabla_k^\alpha x(k) := \sum_{i=0}^{k-a-1} (-1)^i \binom{\alpha}{i} x(k-i), \tag{4}$$

where $\alpha \in \mathbb{R}$, $k \in \mathbb{N}_{a+1}$, and $a \in \mathbb{R}$.

When the order $\alpha = 0$, $\begin{smallmatrix}G\\a\end{smallmatrix}\nabla_k^\alpha x(k) = x(k)$, $k \in \mathbb{N}_{a+1}$, like (3). When the order $\alpha = -n \in \mathbb{Z}_-$, $\begin{smallmatrix}G\\a\end{smallmatrix}\nabla_k^{-n} x(k)$ in (4) reduces to the classical nabla sum ${}_a\nabla_k^{-n} x(k)$ in (2). When the order $\alpha \in \mathbb{R}_- \setminus \mathbb{Z}_-$, $\begin{smallmatrix}G\\a\end{smallmatrix}\nabla_k^\alpha x(k)$ becomes the Grünwald–Letnikov fractional sum, which is also known as the nabla Riemann–Liouville fractional sum [7]. When the order $\alpha = n \in \mathbb{Z}_+$, $\begin{smallmatrix}G\\a\end{smallmatrix}\nabla_k^n x(k)$ is not identical to $\nabla^n x(k)$ in (1), especially for those $k - a - 1 \leqslant n$, since the upper limit of summation in formula (4) is a time-varying variable $k - a - 1$ instead of the constant n. If the number of $x(k-i)$ in the summation is called as the memory length, it is worth noting that $\begin{smallmatrix}G\\a\end{smallmatrix}\nabla_k^n x(k)$ has a time-varying memory length, while $\nabla^n x(k)$ has a fixed memory length, e.g., $[\begin{smallmatrix}G\\a\end{smallmatrix}\nabla_k^1 x(k)]_{k=a+1} = x(a+1)$, $[\nabla^1 x(k)]_{k=a+1} = x(a+1) - x(a)$, and $\begin{smallmatrix}G\\a\end{smallmatrix}\nabla_k^1 x(k) = \nabla^1 x(k) = x(k) - x(k-1)$, $k \geqslant a+2$. When the order $\alpha \in \mathbb{R}_+ \setminus \mathbb{Z}_+$, $\begin{smallmatrix}G\\a\end{smallmatrix}\nabla_k^\alpha x(k)$ becomes the Grünwald–Letnikov fractional difference. Due to the perfect match in format of fractional difference and fractional sum, the following relationship follows [23]:

$$\begin{smallmatrix}G\\a\end{smallmatrix}\nabla_k^\alpha \begin{smallmatrix}G\\a\end{smallmatrix}\nabla_k^{-\alpha} x(k) = \begin{smallmatrix}G\\a\end{smallmatrix}\nabla_k^{-\alpha} \begin{smallmatrix}G\\a\end{smallmatrix}\nabla_k^\alpha x(k) = x(k), \tag{5}$$

where $\alpha \in (n-1, n)$, $n \in \mathbb{Z}_+$, $k \in \mathbb{N}_{a+1}$ and $a \in \mathbb{R}$.

By combining the rising function $p^{\overline{q}} := \Gamma(p+q)/\Gamma(p)$ with the convolution operation, the Grünwald–Letnikov fractional difference/sum can be rewritten as

$$\begin{smallmatrix}G\\a\end{smallmatrix}\nabla_k^\alpha x(k) = \frac{(k-a)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} * x(k), \tag{6}$$

where $\alpha \in \mathbb{R} \setminus \mathbb{N}$, $k \in \mathbb{N}_{a+1}$ and $u(k) * v(k) := \sum_{i=0}^{k-a-1} u(i+a+1)v(k-i)$.

Given a function $x : \mathbb{N}_{a+1} \to \mathbb{R}$, $a \in \mathbb{R}$, if there exists some $r > 0$ such that $X(s) = \mathscr{N}_a\{x(k)\} := \sum_{k=1}^{+\infty}(1-s)^{k-1}x(k+a)$ converges for $|s-1| < r$, then the following equality holds [22]:

$$\mathscr{N}_a\{{}_a^{\mathrm{G}}\nabla_k^\alpha x(k)\} = s^\alpha X(s), \tag{7}$$

where $|s-1| < \min\{r, 1\}$ and $\alpha \in \mathbb{R}\backslash\mathbb{Z}_+$. Notably, the sufficient and necessary condition for $\lim_{k\to+\infty} x(k) = 0$ is that the multiplicity of pole $s = 0$ for $X(s)$ is less than 1 and all other principal poles of $X(s)$ satisfy $|s-1| > 1$.

It can be observed from (5) and (7) that $\mathscr{N}_a\{{}_a^{\mathrm{G}}\nabla_k^\alpha x(k)\}$ is independent of the initial condition $x(a)$ or $[{}_a^{\mathrm{G}}\nabla_k^{\alpha-i}x(k)]_{k=a}$, $i \in \mathbb{Z}_+$, $i < \alpha + 1$. This property will make it impossible to get $x(k)$ from ${}_a^{\mathrm{G}}\nabla_k^\alpha x(k) = \lambda x(k)$, which means that the state or output feedback control for the system ${}_a^{\mathrm{G}}\nabla_k^\alpha x(k) = f(x(k), u(k))$, $y(k) = g(x(k), u(k))$ is meaningless. *For this reason, the objective of this work* is to analyze this default property and determine a recipe for eliminating this independence.

3 Main results

This section includes two parts, i.e., showing the independence of initial conditions and solving the initial condition independence problem.

3.1 The independent case

Consider the following system with Grünwald–Letnikov definition:

$$\begin{aligned}
{}_a^{\mathrm{G}}\nabla_k^\alpha x(k) &= f(x(k), u(k)), \\
y(k) &= g(x(k), u(k)),
\end{aligned} \tag{8}$$

where $\alpha \in (0,1)$, $k \in \mathbb{N}_{a+1}$, $x(k) \in \mathbb{R}^n$, is the pseudo state, $u(k) \in \mathbb{R}^p$ is the input, $y(k) \in \mathbb{R}^q$ is the output, $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ are linear or nonlinear functions.

Defining $v(k) = f(x(k), u(k))$ and using the property in (5), the first equation of (8) can be rewritten as $x(k) = {}_a^{\mathrm{G}}\nabla_k^{-\alpha}v(k)$, which implies that *when* $u(k) = 0$, $k \in \mathbb{N}_{a+1}$, *is considered,* $x(k)$, $k \in \mathbb{N}_{a+1}$, *cannot be calculated with initial condition* $x(a)$. Inspired by [21], a discrete time frequency distributed model is introduced here: :

$$\begin{aligned}
\nabla z(\omega, k) &= -\omega z(\omega, k) + f(x(k), u(k)), \\
x(k) &= \int_0^{+\infty} \mu_\alpha(\omega)z(\omega, k)\,\mathrm{d}\omega, \\
y(k) &= g(x(k), u(k)),
\end{aligned} \tag{9}$$

where $z(\omega, a) = 0$, $\omega \in [0, +\infty)$. It can be checked that the relation from $u(k)$ to $x(k)$ is equivalent to that of (8). Along this way, different equivalent representation of $1/s^\alpha$ could give different equivalent models of (8). For simplicity, it will not be expanded here. To get

the equivalent relationship, *zero initial condition should be configured for the equivalent model like* (9), which is different from the Riemann–Liouville case and the Caputo case [21].

In system (8), the fractional difference is not calculated directly. To continue, consider the fractional difference operation directly

$$x(k) = {}^{\mathrm{G}}_a\nabla^\alpha_k v(k), \tag{10}$$

where $\alpha \in (0,1)$, $v(k)$ is known, $x(k)$ is to be calculated, and $k \in \mathbb{N}_{a+1}$. Since $s^\alpha = \int_0^{+\infty} \mu_\alpha(\omega)/(1/s+\omega)\,\mathrm{d}\omega = \int_0^{+\infty} \mu_\alpha(\omega)s/(1+\omega s)\,\mathrm{d}\omega$ holds for $s \in \mathbb{C}\setminus\mathbb{R}_-$, $x(k)$ can be calculated by the following systems:

$$_a\nabla^{-1}_k z(\omega, k) = -\omega z(\omega, k) + v(k),$$
$$x(k) = \int\limits_0^{+\infty} \mu_\alpha(\omega)z(\omega, k)\,\mathrm{d}\omega,$$

$$\omega\nabla z(\omega, k) = -z(\omega, k) + \nabla v(k),$$
$$x(k) = \int\limits_0^{+\infty} \mu_\alpha(\omega)z(\omega, k)\,\mathrm{d}\omega,$$

where $\alpha \in (0,1)$, $z(\omega, a) = 0$, $k \in \mathbb{N}_{a+1}$. Note that $x(k)$ *can be accurately calculated without the dependance on initial condition* $v(a)$.

In the previous discussion, the order is assumed to be $\alpha \in (0,1)$. If $\alpha \in (n-1, n)$, $n \in \mathbb{Z}_+$, $^{\mathrm{G}}_a\nabla^\alpha_k x(k) = \nabla^{n-1}{}^{\mathrm{G}}_a\nabla^{\alpha-n+1}_k x(k)$ brings the equivalent description of system (8) as

$$\nabla^{n-1}\sigma(k) = f\big(x(k), u(k)\big),$$
$$\nabla z(\omega, k) = -\omega z(\omega, k) + \sigma(k),$$
$$x(k) = \int\limits_0^{+\infty} \mu_{\alpha-n+1}(\omega)z(\omega, k)\,\mathrm{d}\omega,$$
$$y(k) = g\big(x(k), u(k)\big),$$

where *zero initial conditions* $[\nabla^{i-1}\sigma(k)]_{k=a} = 0$, $i = 1, 2, \ldots, n-1$, and $z(\omega, a) = 0$ are needed. Likewise, the large-order case of (10) can be analyzed.

Remark 1. It can be observed that no matter the system output of (8), and the difference output of (10) can be accurately calculated without considering the initial condition. Even if the order is extended from $0 < \alpha < 1$ to $n-1 < \alpha < n$, the independence of initial conditions is always the same. This point is actually implied in Definition 2. More specially, the calculation of $^{\mathrm{G}}_a\nabla^\alpha_k x(k)$, $k \in \mathbb{N}_{a+1}$, only needs the value of $x(k)$, $k \in \mathbb{N}_{a+1}$.

Up till now, the initial conditions independence property of system (8) *has been discovered.* To further show its influence, let us consider the following linear system governed by

$$
{}^{\mathrm{G}}_a\nabla^\alpha_k x(k) = \lambda x(k) + bu(k) \tag{11}
$$

with $\alpha > 0$, $k \in \mathbb{N}_{a+1}$, and $b \neq 0$. $x(k)$ can be calculated as

$$
x(k) = \mathcal{N}_a^{-1}\left\{\frac{bU(s)}{s^\alpha - \lambda}\right\}, \tag{12}
$$

where $U(s) = \mathcal{N}_a\{u(k)\}$. If the discrete Mittag-Leffler function $\mathcal{F}_{\alpha,\beta}(\lambda, k, a) := \mathcal{N}_a^{-1}\{s^{\alpha-\beta}/(s^\alpha - \lambda)\}$ is introduced here, then one has

$$
x(k) = b\mathcal{F}_{\alpha,\alpha}(\lambda, k, a) * u(k). \tag{13}
$$

It is worth mentioning that in both (12) *and* (13), *only the input response can be found, and there is no state response.* This result is quite coincident with the previous discussion.

Remark 2. One point worth emphasizing is that the exact calculation of $x(k)$ in (8). Two methods will be discussed.

The first one is to solve fractional difference equation directly. By using (4) and (6), it follows

$$
\sum_{i=0}^{k-a-1} \frac{(i+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} x(k-i) = \lambda x(k) + bu(k), \tag{14}
$$

where α, a, λ, b, $u(k)$, $k \in \mathbb{N}_{a+1}$, are known, and $x(k)$, $k \in \mathbb{N}_{a+1}$, is to be calculated. (14) can be rewritten as

$$
x(k) = \frac{1}{1-\lambda}\left[bu(k) - \sum_{i=1}^{k-a-1} \frac{(i+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} x(k-i)\right].
$$

From this the value of $x(k)$ can be calculated sequentially. $(i+1)^{\overline{-\alpha-1}}/\Gamma(-\alpha) = \Gamma(i-\alpha)/(\Gamma(-\alpha)\Gamma(i+1))$, which involves the gamma function. When calculating the value of gamma function using MATLAB code `gamma()`, one has $\Gamma(171) = 7.257415615308056 \cdot 10^{306}$ and $\Gamma(171) = \infty$. To avoid the overflow problem, two available ways can be taken. The one is recursive computation by using the property

$$
\frac{(i+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} = \frac{i^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}\frac{i-\alpha-1}{i}.
$$

The other is equivalent transform by using the relation

$$
\frac{(i+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} = \frac{1}{\Gamma(-\alpha)}e^{\ln(\Gamma(i-\alpha)/\Gamma(i+1))} = \frac{1}{\Gamma(-\alpha)}e^{\ln\Gamma(i-\alpha)-\ln\Gamma(i+1)},
$$

where the code `gammaln()` is helpful. Some other treatment can be made before introducing logarithmic function when $\Gamma(i - \alpha) < 0$.

The second one is to solve fractional sum equation indirectly. By using (5), (11) can be rewritten as

$$
\begin{aligned}
x(k) &= \lambda \, {}^{\mathrm{G}}_a\nabla_k^{-\alpha}x(k) + b \, {}^{\mathrm{G}}_a\nabla_k^{-\alpha}u(k) \\
&= \lambda \sum_{i=0}^{k-a-1} \frac{(i+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} x(k-i) + b \sum_{i=0}^{k-a-1} \frac{(i+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u(k-i) \\
&= \frac{\lambda}{1-\lambda} \sum_{i=1}^{k-a-1} \frac{(i+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} x(k-i) + \frac{b}{1-\lambda} \sum_{i=0}^{k-a-1} \frac{(i+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u(k-i) \\
&= \frac{b}{1-\lambda} u(k) + \frac{1}{1-\lambda} \sum_{i=1}^{k-a-1} \frac{(i+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \big[\lambda x(k-i) + bu(k-i)\big],
\end{aligned}
$$

where $(i+1)^{\overline{\alpha-1}}/\Gamma(\alpha)$ could be calculated like $(i+1)^{\overline{-\alpha-1}}/\Gamma(-\alpha)$, while the unexpected case $\Gamma(i-\alpha) < 0$ will not appear any more.

3.2 The dependent case

The system ${}^{\mathrm{C}}_a\nabla_k^\alpha x(k) = \lambda x(k)$, $\alpha \in (n-1, n)$, $n \in \mathbb{Z}_+$, $k \in \mathbb{N}_{a+1}$, with Caputo fractional difference involves the initial conditions $[\nabla^i x(k)]_{k=a}$, $i = 1, 2, \ldots, n-1$. The system ${}^{\mathrm{R}}_a\nabla_k^\alpha x(k) = \lambda x(k)$, $\alpha \in (n-1, n)$, $n \in \mathbb{Z}_+$, $k \in \mathbb{N}_{a+1}$, under Riemann–Liouville definition should consider initial conditions regarding to the Riemann–Liouville fractional difference $[{}^{\mathrm{R}}_a\nabla_k^{\alpha-i}x(k)]_{k=a}$, $i = 1, 2, \ldots, n-1$, and Riemann–Liouville fractional sum $[{}^{\mathrm{R}}_a\nabla_k^{\alpha-n}x(k)]_{k=a}$. *However,* ${}^{\mathrm{G}}_a\nabla_k^\alpha x(k) = \lambda x(k)$, $k \in \mathbb{N}_{a+1}$, *under Grünwald–Letnikov definition does not depend on initial condition* $x(a)$ *or any other case.* Consequently, the question on how to make fractional-order difference equation connect to initial condition becomes urgent. In this part, four possible schemes are provided.

Scheme 1 ▶ Consider the following system:

$$
{}^{\mathrm{G}}_a\nabla_k^\alpha x(k) = \lambda x(k-m), \tag{15}
$$

where $\alpha > 0$, $k \in \mathbb{N}_{a+1}$, $\lambda \in \mathbb{R}$ and $m \in \mathbb{Z}_+$.

Using the calculating approach in Remark 2 yields

$$
x(k) = \lambda x(k-m) - \sum_{i=1}^{k-a-1} \frac{(i+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} x(k-i), \tag{16}
$$

$$
x(k) = \lambda \sum_{i=0}^{k-a-1} \frac{(i+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} x(k-m-i). \tag{17}
$$

It can be observed that when calculating $x(k)$ for $k \in \mathbb{N}_{a+1}$, the initial conditions $x(a+1-m)$, $x(a+2-m)$, ..., $x(a)$ are all adopted. This scheme just hints how to solve the initial conditions independence issue when applying the Grünwald–Letnikov fractional difference.

Defining $X(s) := \mathcal{N}_\alpha\{x(k)\}$, the time-delay property of the nabla Laplace transform follows [22]

$$\mathcal{N}_\alpha\{x(k-m)\} = (1-s)^m \left[X(s) + \sum_{i=1-m}^{0} (1-s)^{i-1} x(a+i) \right],$$

from which the state response in frequency domain becomes

$$X(s) = \lambda \sum_{i=1}^{m} \frac{(1-s)^{m-i}}{s^\alpha - \lambda(1-s)^m} x(a+1-i). \tag{18}$$

Note that (18) coincides with (16) and (17). System (15) will be asymptotically stable if λ is designed such that the multiplicity of root $s = 0$ is less than 1 or all other the principal roots lie in the region $|s - 1| > 1$.

Scheme 2 ▶ Consider the following system:

$$\begin{matrix}{}^{\mathrm{G}}_{a}\end{matrix}\nabla_k^\alpha x(k+m) = \lambda x(k) \tag{19}$$

with $\alpha > 0$, $k \in \mathbb{N}_{a+1}$, $\lambda \in \mathbb{R}$, and $m \in \mathbb{Z}_+$.

Using the calculating approach in Remark 2 yields

$$x(k+m) = \lambda x(k) - \sum_{i=1}^{k-a-1} \frac{(i+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} x(k+m-i),$$

$$x(k+m) = \lambda \sum_{i=0}^{k-a-1} \frac{(i+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} x(k-i).$$

As such, the calculation of $x(k)$ for $k \in \mathbb{N}_{a+m+1}$ strongly depends on the initial conditions $x(a+1)$, $x(a+2)$, ..., $x(a+m)$.

By applying the following time advance property [22]

$$\mathcal{N}_\alpha\{x(k+m)\} = (1-s)^{-m} \left[X(s) - \sum_{i=1}^{m} (1-s)^{i-1} x(a+i) \right], \tag{20}$$

$X(s) := \mathcal{N}_\alpha\{x(k)\}$ can be expressed as

$$X(s) = \sum_{i=1}^{m} \frac{s^\alpha (1-s)^{i-1}}{s^\alpha - \lambda(1-s)^m} x(a+i).$$

To sum up, in face of the independence of initial conditions for Grünwald–Letnikov fractional difference, the autonomous system ${}^{\mathrm{G}}_a\nabla_k^\alpha x(k) = \lambda x(k)$ cannot work, while system (15) or (19) can be used as a substitute. The stable condition of the system (19) is the same with the system (15).

Remark 3. Combing the two methods, a strange but available system follows ${}^{\mathrm{G}}_a\nabla^\alpha_k x(k) = \lambda x(k+m)$, where $\alpha > 0$, $k \in \mathbb{N}_{a+1}$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $m \in \mathbb{Z}_+$. In this regard, one has the relationships $x(k+m) = (1/\lambda)\sum_{i=1}^{k-a-1}(i+1)^{\overline{-\alpha-1}}/\Gamma(-\alpha)x(k-i)$ and $X(s) = -\lambda\sum_{i=1}^m(1-s)^{i-1}/[s^\alpha(1-s)^m - \lambda]x(a+i)$, which means that the initial conditions $x(a+1)$, $x(a+2)$, ..., $x(a+m)$ are needed to calculate $x(k)$ for $k \in \mathbb{N}_{a+m+1}$. Different from the methods in (15) and (19), this method is not directly applicable to the nonlinear case, such as ${}^{\mathrm{G}}_a\nabla^\alpha_k x(k) = f(x(k+m))$.

Scheme 3 ▶ Afterwards, let us continue to consider the system ${}^{\mathrm{G}}_a\nabla^\alpha_k x(k) = \lambda x(k)$. If we assume $k \in \mathbb{N}_{a+1+m}$, $m \in \mathbb{Z}_+$ and take $x(a+1)$, $x(a+2)$, ..., $x(a+m)$ as the initial condition, the system becomes

$$\, {}^{\mathrm{G}}_a\nabla^\alpha_{k+m} x(k+m) = \lambda x(k+m) \tag{21}$$

with $\alpha > 0$, $k \in \mathbb{N}_{a+1}$, $\lambda \in \mathbb{R}$, and $m \in \mathbb{Z}_+$. From this the initial condition has been exerted successfully.

Using the calculating approach in Remark 2 yields

$$x(k+m) = -\frac{1}{1-\lambda}\sum_{i=1}^{k+m-a-1}\frac{(i+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}x(k+m-i),$$

$$x(k+m) = \frac{1}{1-\lambda}\sum_{i=1}^{k+m-a-1}\frac{(i+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}x(k+m-i).$$

Recalling Definition 2 yields

$$\, {}^{\mathrm{G}}_a\nabla^\alpha_{k+m} x(k+m)$$

$$= \sum_{i=0}^{k+m-a-1}(-1)^i\binom{\alpha}{i}x(k+m-i)$$

$$= \sum_{i=0}^{k-a-1}(-1)^i\binom{\alpha}{i}x(k+m-i) + \sum_{i=k-a}^{k+m-a-1}(-1)^i\binom{\alpha}{i}x(k+m-i)$$

$$= {}^{\mathrm{G}}_a\nabla^\alpha_k x(k+m) + \sum_{i=1}^m\frac{(k-a+i)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}x(m+a+1-i),$$

from which $X(s) := \mathcal{N}_\alpha\{x(k)\}$ can be expressed as

$$X(s) = \sum_{i=1}^m(1-s)^{i-1}x(a+i) - \frac{s^\alpha}{s^\alpha - \lambda}\sum_{i=1}^m(1-s)^{m-i}x(a+m+1-i)$$

$$+ \frac{1}{s^\alpha - \lambda}\sum_{i=1}^m\sum_{j=0}^{i-1}(1-s)^{m-i+j}\frac{(j+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}x(a+m+1-i),$$

where (20) and $\mathcal{N}_\alpha\{(k-a)^{\overline{-\alpha-1}}/\Gamma(-\alpha)\} = s^\alpha$ are adopted.

It can be found that the initial conditions $x(a+1)$, $x(a+2)$, ..., $x(a+m)$ are implied in $X(s)$. The corresponding stability condition becomes that the root of $s^\alpha - \lambda = 0$ lies in the region $|s - 1| > 1$. Compared with systems (15) and (19), it is more convenient to use frequency tool in system (21). However, the inconvenience is that this method is not applicable to the nonlinear case.

Scheme 4 ▶ Along this way, consider the following system:

$$\underset{a-m}{\overset{G}{}}\nabla_k^\alpha x(k) = \lambda x(k) \tag{22}$$

with $\alpha > 0$, $k \in \mathbb{N}_{a+1}$, $\lambda \in \mathbb{R}$, and $m \in \mathbb{Z}_+$.

Using the calculating approach in Remark 2 yields

$$x(k) = -\frac{1}{1-\lambda} \sum_{i=1}^{k+m-a-1} \frac{(i+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} x(k-i),$$

$$x(k) = \frac{1}{1-\lambda} \sum_{i=1}^{k+m-a-1} \frac{(i+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} x(k-i).$$

The calculation of $x(k)$, $k \in \mathbb{N}_{a+1}$, strongly depends on the initial conditions $x(a-m+1)$, $x(a-m+2)$, ..., $x(a)$.

Recalling Definition 2 yields

$$\underset{a-m}{\overset{G}{}}\nabla_k^\alpha x(k) = \sum_{i=0}^{k+m-a-1} (-1)^i \binom{\alpha}{i} x(k-i)$$

$$= \sum_{i=0}^{k-a-1} (-1)^i \binom{\alpha}{i} x(k-i) + \sum_{i=k-a}^{k+m-a-1} (-1)^i \binom{\alpha}{i} x(k-i)$$

$$= \underset{a}{\overset{G}{}}\nabla_k^\alpha x(k) + \sum_{i=1}^{m} \frac{(k-a+i)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} x(a+1-i).$$

Defining $X(s) := \mathscr{N}_\alpha\{x(k)\}$, it follows

$$X(s) = -\frac{s^\alpha}{s^\alpha - \lambda} \sum_{i=1}^{m} (1-s)^{-i} x(a+1-i)$$

$$+ \frac{1}{s^\alpha - \lambda} \sum_{i=1}^{m} \sum_{j=0}^{i-1} (1-s)^{-i+j} \frac{(j+1)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} x(a+1-i),$$

where (20) and $\mathscr{N}_\alpha\{(k-a)^{\overline{-\alpha-1}}/\Gamma(-\alpha)\} = s^\alpha$ are adopted again.

It can be found that the initial conditions $x(a+1-m)$, $x(a+2-m)$, ..., $x(a)$ are implied in $X(s)$. The corresponding stability condition becomes that the root of $s^\alpha - \lambda = 0$ lies in the region $|s - 1| > 1$.

Till now, four available schemes have been presented already. Note that the systems in (15), (19), (21), and (22) only need the initial condition about $x(k)$ not the difference or sum of $x(k)$, which is helpful in practical applications. The nature of them is shifting. Along this way, there are various of methods to introduce the initial condition. The basic principle might be to connect the future signal with the past signal and calculate the unknown value with the known value.

Remark 4. The Grünwald–Letnikov fractional difference/sum adopted in [8,9,11,13,14] is actually $_{a-1}^{\ \ G}\nabla_k^\alpha x(k) = \sum_{i=0}^{k-a}(-1)^i\binom{\alpha}{i}x(k-i)$, $k \in \mathbb{N}_a$, which is a special case of (22) with $m = 1$. In that case, the initial conditions independence can also be solved. When $k = a$, system $_{a-1}^{\ \ G}\nabla_k^\alpha x(k) = \lambda x(k)$, $k \in \mathbb{N}_a$, becomes $x(a) = \lambda x(a)$, which is unexpected. To avoid this confusing phenomenon, the discrete time $k \in \mathbb{N}_{a+1}$ should be applied.

4 Further discussion

In this section, the developed results will be discussed for some variants of discrete Grünwald–Letnikov fractional calculus. Afterwards, three detailed numerical examples are provided to verify the correctness of the proposed theory.

4.1 Extension

In the previous discussion, the basic is assumed. Hereafter, three other definitions will be considered.

Definition 3. For $x : \mathbb{N}_{a+1} \to \mathbb{R}$, its αth tempered Grünwald–Letnikov fractional difference/sum is defined by

$$_a^G\nabla_k^{\alpha,\lambda}x(k) := (1-\lambda)^{a-k}\sum_{i=0}^{k-a-1}(-1)^i\binom{\alpha}{i}(1-\lambda)^{k-i-a}x(k-i),$$

where $\alpha \in \mathbb{R}$, $\lambda \in \mathbb{R}$, $\lambda \neq 1$, $k \in \mathbb{N}_{a+1}$, and $a \in \mathbb{R}$.

By using the nabla Laplace transform, it follows

$$\mathscr{N}_a\{_a^G\nabla_k^{\alpha,\lambda}x(k)\} = \left(\frac{s-\lambda}{1-\lambda}\right)^\alpha X(s). \tag{23}$$

Although a new parameter λ is introduced, there is no item regarding to the initial condition in (23).

Definition 4. For $x : \mathbb{N}_{a+1} \to \mathbb{R}$, its $\alpha(k)$th Grünwald–Letnikov fractional difference/ sum is defined by

$$_a^G\nabla_k^{\alpha(k)}x(k) := \sum_{i=0}^{k-a-1}(-1)^i\binom{\alpha(k)}{i}x(k-i),$$

where $\alpha(k) \in \mathbb{R}$, $k \in \mathbb{N}_{a+1}$, and $a \in \mathbb{R}$.

The elegant formula $\mathcal{N}_a\{{}^{\mathrm{G}}_a\nabla_k^{\alpha(k)}x(k)\} = s^{\alpha(k)}X(s)$ does not hold, while this definition can be rewritten as ${}^{\mathrm{G}}_a\nabla_k^{\alpha(k)}x(k) = \overline{(k-a)^{-\alpha(k)-1}}/\Gamma(-\alpha(k)) * x(k)$. Defining

$$\Pi(s) = \sum_{k=0}^{+\infty}(s-1)^k\binom{\alpha(k+a+1)}{k},$$

it follows

$$\mathcal{N}_a\{{}^{\mathrm{G}}_a\nabla_k^{\alpha(k)}x(k)\} = \Pi(s)X(s).$$

From this, when ${}^{\mathrm{G}}_a\nabla_k^{\alpha(k)}x(k)$, $k \in \mathbb{N}_{a+1}$, is calculated, no more information of $x(k)$, $k < a+1$ is needed. To be more precise, all the fractional differences in Definitions 2, 3, and 4 are independent on the initial condition. Fortunately, by using the schemes in Section 3.2, the initial condition dependence can be attained.

Definition 5. For $x : \mathbb{N}_{a+1-N} \to \mathbb{R}$, its αth Grünwald–Letnikov fractional difference/sum is defined by

$$_{k-N}^{\mathrm{G}}\nabla_k^\alpha x(k) := \sum_{i=0}^N(-1)^i\binom{\alpha}{i}x(k-i),$$

where $\alpha \in \mathbb{R}$, $N \in \mathbb{Z}_+$, $k \in \mathbb{N}_{a+1}$, and $a \in \mathbb{R}$.

Notably, the equality $_{k-N}^{\mathrm{G}}\nabla_k^N x(k) = \nabla^N x(k)$ strongly implies the possible connection to the initial condition. Another equality $_{k-N}^{\mathrm{G}}\nabla_k^\alpha x(k) = {}^{\mathrm{G}}_a\nabla_k^\alpha x(k) - {}^{\mathrm{G}}_a\nabla_{k-N}^\alpha x(k)$ also shows that the calculation of $_{k-N}^{\mathrm{G}}\nabla_k^\alpha x(k)$ for $k \in \mathbb{N}_{a+1}$ depends on the history information $x(k)$, $k < a+1$. Different from the previous definitions, here the memory length is a constant value $N+1$ instead of a time varying value $k-a$. Along this way, both ${}^{\mathrm{G}}_a\nabla_{k+N}^\alpha x(k)$ and ${}^{\mathrm{G}}_k\nabla_{k+N}^\alpha x(k)$ might be the alternative schemes to generate the initial conditions.

4.2 Examples

Example 1. Consider the following five systems:

$$
\begin{aligned}
\text{case 1:}\quad & {}^{\mathrm{G}}_3\nabla_k^{0.6}x_1(k) = \lambda x_1(k-1),\\
\text{case 2:}\quad & {}^{\mathrm{G}}_3\nabla_k^{0.6}x_2(k) = \lambda x_2(k+1),\\
\text{case 3:}\quad & {}^{\mathrm{G}}_3\nabla_k^{0.6}x_3(k+1) = \lambda x_3(k),\\
\text{case 4:}\quad & {}^{\mathrm{G}}_3\nabla_{k+1}^{0.6}x_4(k+1) = \lambda x_4(k+1),\\
\text{case 5:}\quad & {}^{\mathrm{G}}_2\nabla_k^{0.6}x_5(k) = \lambda x_5(k)
\end{aligned}
\tag{24}
$$

with $k \in \mathbb{N}_4$ and $x_1(3) = x_2(4) = x_3(4) = x_4(4) = x_5(3) = \sigma$. For the purpose of plotting, denote $x(k)$ as $x_i(k)$, $i = 1, 2, \ldots, 5$, for each case. To make the system stable, $\lambda = -1.2$, $\lambda = -6$, $\lambda = -0.5$, $\lambda = 2$, and $\lambda = 2$ are selected for cases 1–5, respectively.
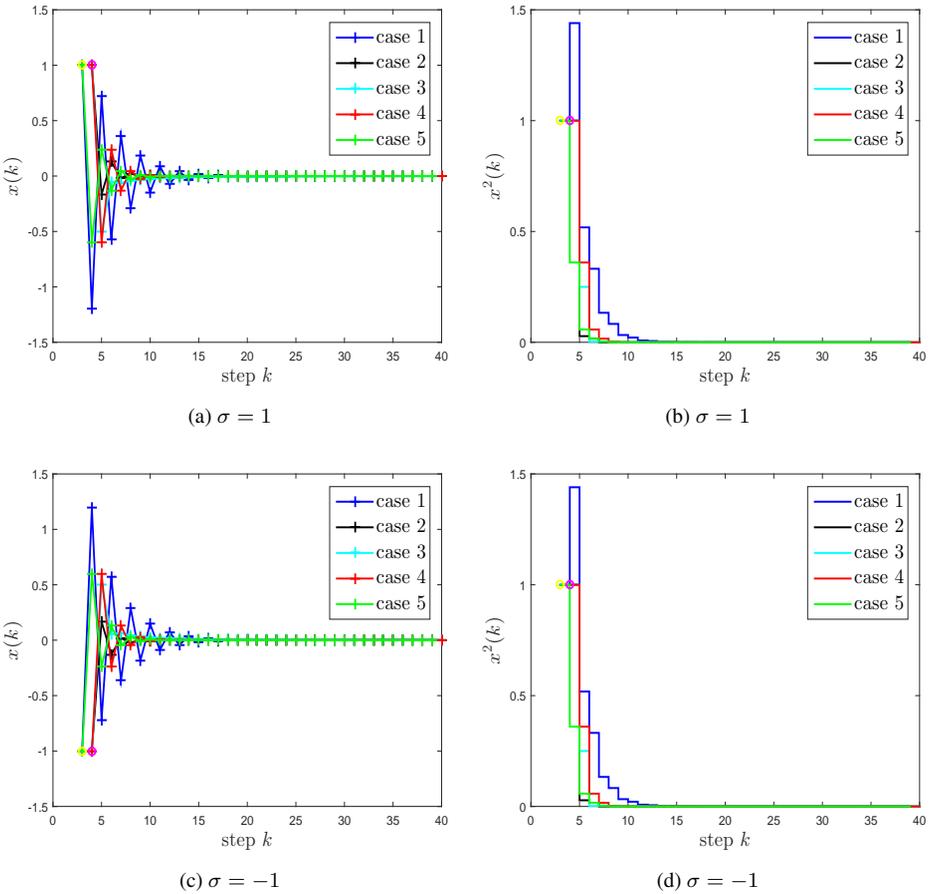
Figure 1. The state response of (24) with different initial conditions.

When the value of σ is assumed as 1, -1 in succession, then the resulting result are plotted in Figs. 1(a)–1(d).

It can be observed that with the shifting principle, the initial conditions are available to calculate $x(k)$, $k \in \mathbb{N}_4$, in cases 1–5. Moreover, for different σ, $x(k)$ can converge to 0 as k increases. In this connection, the results in Section 3.2 have been validated.

Example 2. To evaluate the practicability of the proposed result, the following gradient methods are considered:

$$\text{case 1:} \quad {}_{3}^{\mathrm{G}}\nabla_k^\alpha x_1(k) = -\rho \frac{\mathrm{d}}{\mathrm{d}x} J(x)\bigg|_{x=x_1(k)},$$

$$\text{case 2:} \quad {}_{3}^{\mathrm{G}}\nabla_k^\alpha x_2(k) = -\rho \frac{\mathrm{d}}{\mathrm{d}x} J(x)\bigg|_{x=x_2(k-1)},$$

$$\text{case 3:} \quad {}_{3}^{\mathrm{G}}\nabla_{k}^{\alpha}x_3(k) = -\rho\frac{\mathrm{d}}{\mathrm{d}x}J(x)\bigg|_{x=x_3(k+1)},$$

$$\text{case 4:} \quad {}_{3}^{\mathrm{G}}\nabla_{k}^{\alpha}x_4(k+1) = -\rho\frac{\mathrm{d}}{\mathrm{d}x}J(x)\bigg|_{x=x_4(k)},$$

$$\text{case 5:} \quad {}_{3}^{\mathrm{G}}\nabla_{k+1}^{\alpha}x_5(k+1) = -\rho\frac{\mathrm{d}}{\mathrm{d}x}J(x)\bigg|_{x=x_5(k+1)},$$

$$\text{case 6:} \quad {}_{2}^{\mathrm{G}}\nabla_{k}^{\alpha}x_6(k) = -\rho\frac{\mathrm{d}}{\mathrm{d}x}J(x)\bigg|_{x=x_6(k)}.$$

In the beginning, the quadratic objective function $J(x) = \eta(x - x_m)^2 + J_m$ is considered, where $\eta = 3$, $x_m = 5$, $J_m = -12$ (see Fig. 2). The learning rate ρ is selected as $\rho = 1.2$ for cases 1,3,5,6 and $\rho = 0.16$ for cases 2 and 4. Set $x_2(3) = x_3(4) = x_4(4) = x_5(4) = x_6(3) = \sigma$ for cases 2–6 with $\sigma = 1$. The value of $x_1(4)$ in case 1 is calculated directly. Similarly, denote $x(k)$ as $x_i(k)$ is adopted for plotting. In this case, the optimization results with different α are plotted in Fig. 3(a)–3(f).

These figures clearly show that all the presented cases could find the exact extreme point x_m for $\alpha < 2$. Since the stability domain of (15) or (19) is difficult to obtain and therefore cases 2 and 4 are not included in the case of $\alpha > 2$. For cases 1–6, the overshoot appears when $\alpha > 1$. Notably, the initial value $x(4)$ in case 1 is different from others (see yellow circle and red circle) since the calculation is $x_1(4) = 2(\rho\eta/(2\rho\eta + 1))x_m$, not the setting value 1.

In what follows, the quartic function $J(x) = 3x^4 - 4x^3 - 12x^2$ shown in Fig. 4 is considered. Because the gradient $\mathrm{d}J(x)/\mathrm{d}x = 12x(x - 2)(x + 1)$ is nonlinear, the exact solutions of cases 1, 3, 5, and 6 cannot be calculated directly. Therefore, only cases 2 and 4 are discussed hereinafter. With the following parameters $\alpha = 1.2$, $\rho = 0.016$, and $\sigma = -2.5, -2.4, \dots, 2.5$, the simulation results can be obtained as Figs. 5(a)–5(b).
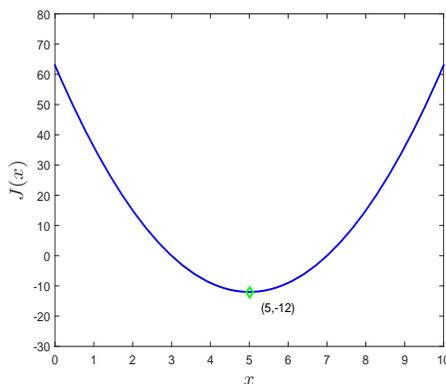

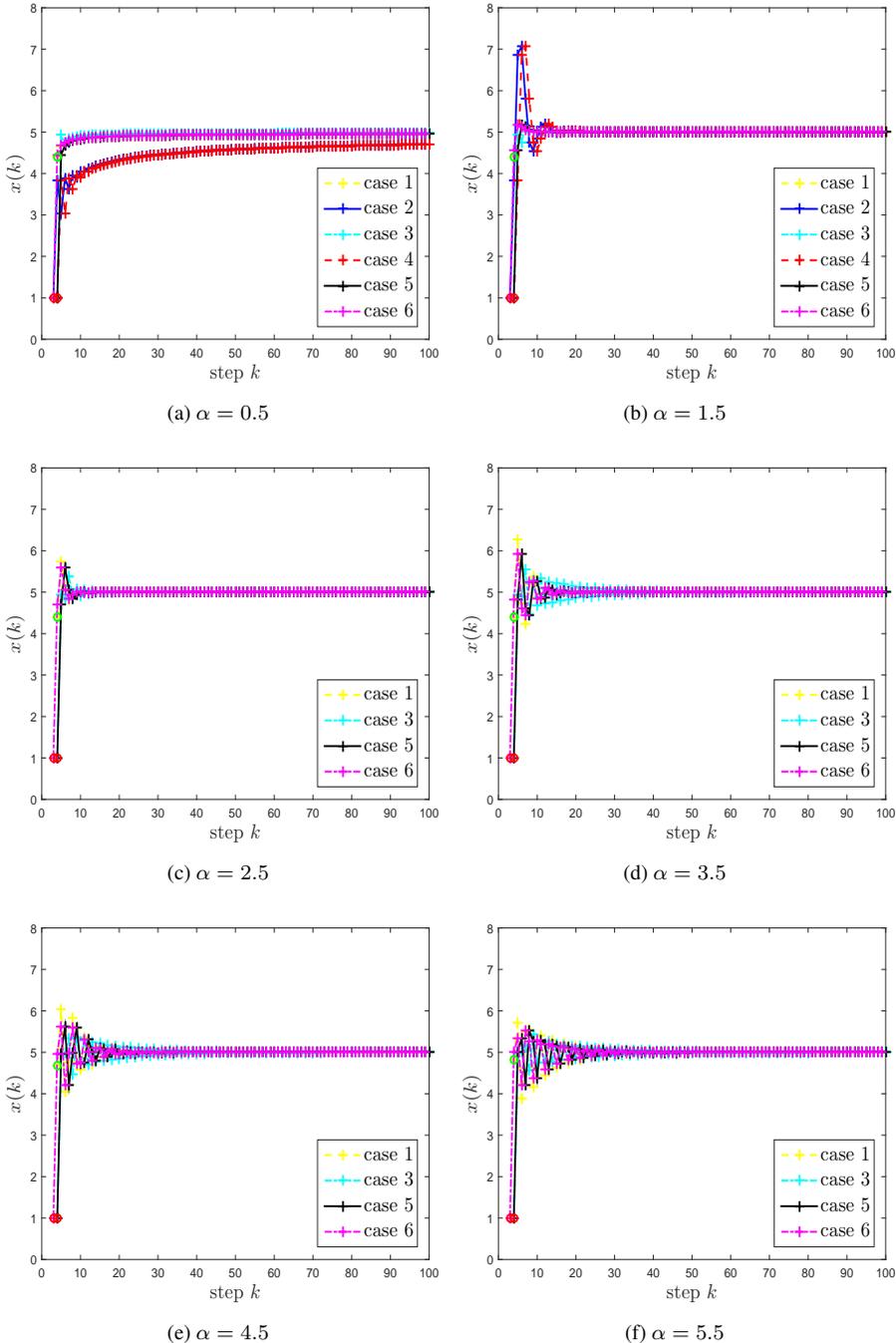**Figure 2.** The quadratic function to be optimized.

(a) $\alpha = 0.5$

(b) $\alpha = 1.5$

(c) $\alpha = 2.5$

(d) $\alpha = 3.5$

(e) $\alpha = 4.5$

(f) $\alpha = 5.5$

Figure 3. The optimization results of a quadratic function with different α.
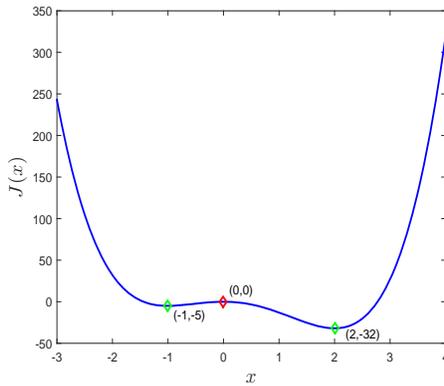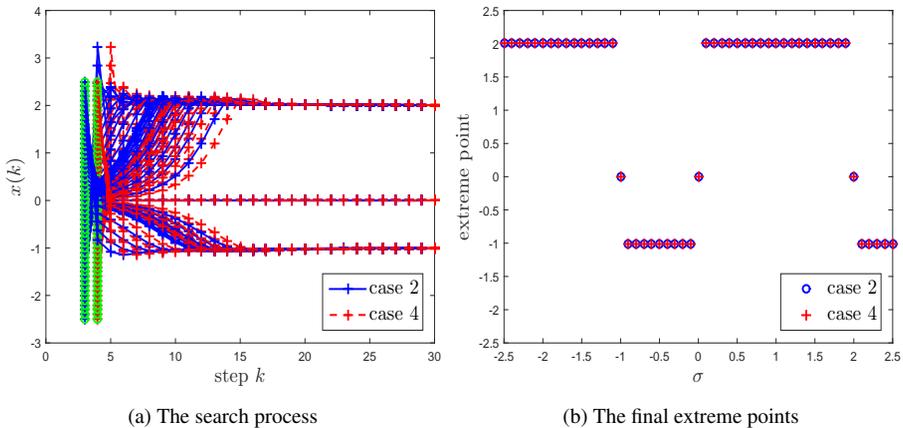
Figure 4. The quartic function to be optimized.

(a) The search process	(b) The final extreme points

Figure 5. The optimization results of a quartic function.

Since $\mathrm{d}J(x)/\mathrm{d}x = 0$, the exact extreme points are $x = -1$, $x = 0$, and $x = 2$. More specially, $x = -1$ is a local minima point, $x = 0$ is maximum point, and $x = 2$ is a global minima point. From Fig. 5(a) it can be found that all the curves converge to the exact extreme points. Likewise, cases 2 and 4 give the similar results. The relationship between the convergent point and the initial condition is clearly shown in Fig. 5(b), which means that the initial conditions independence disappears with the special design.

Example 3. Construct the following fractional-order neural network:

$$
{}^{\mathrm{G}}_{3}\nabla^{\alpha}_{k}x_1(k+1) = -0.05x_2(k) - 0.05x_3(k) + 0.01\tanh\big(x_2(k)\big),
$$
$$
{}^{\mathrm{G}}_{3}\nabla^{\alpha}_{k}x_2(k+1) = 0.05x_1(k) + 0.02x_2(k) + 0.01\tanh\big(x_1(k)\big),
$$
$$
{}^{\mathrm{G}}_{3}\nabla^{\alpha}_{k}x_3(k+1) = 0.1 - 0.2x_3(k) + 0.05x_1(k)x_3(k) + 0.01\tanh\big(x_3(k)\big),
$$

where $x(4) = [1, -1, 0]^{\mathrm{T}}$, $\alpha = 0.98$, $k \in \mathbb{N}_3$.

(a) The phase diagram of $x(k)$
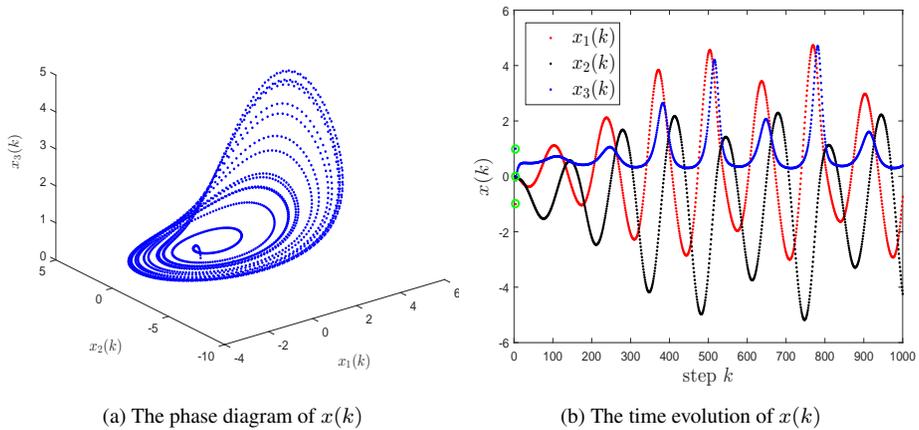
(b) The time evolution of $x(k)$

Figure 6. Simulation results of a chaotic system.

With the given conditions, the phase diagram in the three dimensional space is depicted in Fig. 6(a), and the trajectories of $x(k)$ over time is shown in Fig. 6(b). It could see that the proposed initial condition scheme is effective.

Before ending this section, the main contributions of this paper will be summarized to make it more readability.

  (i) The independence on the initial condition is shown for Grünwald–Letnikov fractional difference equation.
 (ii) The equivalent frequency distributed models for a fractional-order system with Grünwald–Letnikov definition are derived.
(iii) The frequency distributed model is also developed for Grünwald–Letnikov fractional difference calculation.
 (iv) The unavailability of state response for a linear fractional-order system is analyzed.
  (v) The dependent case is considered, which connects the system response with the initial condition.
 (vi) The independence problem is also discussed for some variants of Grünwald–Letnikov definition.

5 Conclusions

In this paper, the initial conditions independent property has been studied for the first time. Inspired by the meaningless system ${}_{a}^{\mathrm{G}}\nabla_{k}^{\alpha}x(k) = \lambda x(k)$, $k \in \mathbb{N}_{a+1}$, the initial value problem on Grünwald–Letnikov fractional difference is discussed from the definition, the distributed frequency model, and the time–domain response. To improve the practicability, four schemes are developed by connecting the initial conditions to the considered system. Apart from this, three illustrative examples are provided to confirm our findings.

It is believed that the proposed principles could greatly enrich the comprehension of Grünwald–Letnikov fractional difference and facilitate its applications. Future research efforts will be directed towards the following topics.

(i) Discuss the stability condition of systems ${}^{\mathrm{G}}_a\nabla^\alpha_k x(k) = \lambda x(k-m)$, ${}^{\mathrm{G}}_a\nabla^\alpha_k x(k) = \lambda x(k+m)$, ${}^{\mathrm{G}}_a\nabla^\alpha_k x(k+m) = \lambda x(k)$ further and analyze the dynamic behaviour of its time–domain system response.

(ii) Extend the related results to other Grünwald–Letnikov based definitions, i.e., the right-hands, the nonsingular kernel case, time-scales case, etc.

(iii) Apply the initial conditions dependent Grünwald–Letnikov difference to practical applications, such as modeling, controlling, filtering, optimization, etc.

References

1. M.S. Abdelouahab, N.E. Hamri, The Grünwald–Letnikov fractional-order derivative with fixed memory length, *Mediterr. J. Math.*, **13**(2):557–572, 2016, https://doi.org/10.1007/s00009-015-0525-3.

2. F.M. Atıcı, S. Chang, J. Jonnalagadda, Grünwald–Letnikov fractional operators: from past to present, *Fract. Differ. Calc.*, **11**(1):147–159, 2021, https://doi.org/10.7153/fdc-2021-11-10.

3. H. Chen, F. Holland, M. Stynes, An analysis of the Grünwald–Letnikov scheme for initial-value problems with weakly singular solutions, *Appl. Numer. Math.*, **139**:52–61, 2019, https://doi.org/10.13140/RG.2.2.36154.24001.

4. Y.Q. Chen, A Collection of Fractional Calculus Books, 2020, http://mechatronics.ucmerced.edu/fcbooks/.

5. J.F. Cheng, *Fractional Difference Equation Theory*, Xiamen Univ. Press, Xiamen, 2011.

6. R. Cioć, Physical and geometrical interpretation of Grünwald–Letnikov differintegrals: Measurement of path and acceleration, *Fract. Calc. Appl. Anal.*, **19**(1):161–172, 2016, https://doi.org/10.1515/fca-2016-0009.

7. C. Goodrich, A.C. Peterson, *Discrete Fractional Calculus*, Springer, Cham, 2015.

8. W.P. Hunek, Perfect control for right-invertible Grünwald–Letnikov plants – An innovative approach to practical implementation, *Fract. Calc. Appl. Anal.*, **22**(2):424–443, 2019, https://doi.org/10.1515/fca-2019-0026.

9. T.Y. Liu, Y.H. Wei, W.D. Yin, Y. Wang, Q. Liang, State estimation for nonlinear discrete–time fractional systems: A Bayesian perspective, *Signal Process.*, **165**:250–261, 2019, https://doi.org/10.1016/j.sigpro.2019.06.037.

10. C.L. MacDonald, N. Bhattacharya, B.P. Sprouse, G.A. Silva, Efficient computation of the Grünwald–Letnikov fractional diffusion derivative using adaptive time step memory, *J. Comput. Phys.*, **297**:221–236, 2015, https://doi.org/10.1016/j.jcp.2015.04.048.

11. D. Mozyrska, P. Oziablo, M. Wyrwas, Stability of fractional variable order difference systems, *Fract. Calc. Appl. Anal.*, **22**(3):807–824, 2019, https://doi.org/10.1515/fca-2019-0044.

12. A.D. Obembe, S.A. Abu-Khamsin, M.E. Hossain, K. Mustapha, Analysis of subdiffusion in disordered and fractured media using a Grünwald–Letnikov fractional calculus model, *Comput. Geosci.*, **22**(5):1231–1250, 2018, `https://doi.org/10.1007/s10596-018-9749-1`.

13. P. Ostalczyk, *Discrete Fractional Calculus: Applications in Control and Image Processing*, World Scientific, Singapore, New Jersey, 2016, `https://doi.org/10.1142/9833`.

14. P. Ostalczyk, D. Sankowski, M. Bąkała, J. Nowakowski, Fractional-order value identification of the discrete integrator from a noised signal. part i, *Fract. Calc. Appl. Anal.*, **22**(1):217–235, 2019, `https://doi.org/10.1515/fca-2019-0014`.

15. A.D. Pano-Azucena, E. Tlelo-Cuautle, J.M. Muñoz-Pacheco, L.G. de la Fraga, FPGA-based implementation of different families of fractional-order chaotic oscillators applying Grünwald–Letnikov method, *Commun. Nonlinear Sci. Numer. Simul.*, **72**:516–527, 2019, `https://doi.org/10.1016/j.cnsns.2019.01.014`.

16. R. Stanisławski, New results in stability analysis for LTI SISO systems modeled by GL-discretized fractional-order transfer functions, *Fract. Calc. Appl. Anal.*, **20**(1):243–259, 2017, `https://doi.org/10.1515/fca-2017-0013`.

17. J.A. Tenreiro Machado, Fractional derivatives: probability interpretation and frequency response of rational approximations, *Commun. Nonlinear Sci. Numer. Simul.*, **14**(9-10):3492–3497, 2009, `https://doi.org/10.1016/j.cnsns.2009.02.004`.

18. M.F. Tolba, L.A. Said, A.H. Madian, A.G. Radwan, FPGA implementation of the fractional order integrator/differentiator: Two approaches and applications, *IEEE Trans. Circuits Syst. I, Regul. Pap.*, **66**(4):1484–1495, 2018, `https://doi.org/10.1109/TCSI.2018.2885013`.

19. A. Tudorache, R. Luca, On a singular Riemann–Liouville fractional boundary value problem with parameters, *Nonlinear Anal. Model. Control*, **26**(1):151–168, 2021, `https://doi.org/10.15388/namc.2021.26.21414`.

20. Y.H. Wei, Y.Q. Chen, T.Y. Liu, Y. Wang, Lyapunov functions for nabla discrete fractional order systems, *ISA Trans.*, **88**:82–90, 2019, `https://doi.org/10.1016/j.isatra.2018.12.016`.

21. Y.H. Wei, Y.Q. Chen, J.C. Wang, Y. Wang, Analysis and description of the infinite-dimensional nature for nabla discrete fractional order systems, *Commun. Nonlinear Sci. Numer. Simul.*, **72**:472–492, 2019, `https://doi.org/10.1016/j.cnsns.2018.12.023`.

22. Y.H. Wei, Y.Q. Chen, Y. Wang, Y.Q. Chen, Some fundamental properties on the sampling free nabla Laplace transform, in *International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, 2019, `https://doi.org/10.1115/DETC2019-97351`.

23. Y.H. Wei, W.D. Yin, Y.T. Zhao, Y. Wang, A new insight into the Grünwald–Letnikov discrete fractional calculus, *J. Comput. Nonlinear Dyn.*, **14**(4):041008, 2019, `https://doi.org/10.1115/1.4042635`.

24. G.C. Wu, T. Abdeljawad, J.L. Liu, D. Baleanu, K.T. Wu, Mittag–Leffler stability analysis of fractional discrete-time neural networks via fixed point technique, *Nonlinear Anal. Model. Control*, **24**(6):919–936, 2019, `https://doi.org/10.15388/NA.2019.6.5`.

25. C.B. Zhai, J. Ren, A fractional q-difference equation eigenvalue problem with $p(t)$-Laplacian operator, *Nonlinear Anal. Model. Control*, **26**(3):482–501, 2021, `https://doi.org/10.15388/namc.2021.26.23055`.