Nonlinear Analysis: Modelling and Control, Vol. 27, No. 4, 609–629
https://doi.org/10.15388/namc.2022.27.26363

Existence of multiple positive solutions for a class of infinite-point singular p-Laplacian fractional differential equation with singular source terms*

Limin Guo[a,1], **Jingbo Zhao**[b,1], **Lianying Liao**[b], **Lishan Liu**[c]

[a]School of Science,
Changzhou Institute of Technology,
Changzhou 213002, Jiangsu, China
guolimin811113@163.com

[b]School of Automotive Engineering,
Changzhou Institute of Technology,
Changzhou 213002, Jiangsu, China
zhaojb@czu.cn

[c]School of Mathematical Sciences,
Qufu Normal University,
Qufu 273165, China

Received: June 1, 2021 / **Revised:** November 25, 2021 / **Published online:** February 22, 2022

Abstract. Based on properties of Green's function and by Avery–Peterson fixed point theorem, the existence of multiple positive solutions are obtained for singular p-Laplacian fractional differential equation with infinite-point boundary conditions, and an example is given to demonstrate the validity of our main results.

Keywords: fractional differential model, multiple positive solutions, Avery–Peterson fixed point theorem, singular problem.

1 Introduction

In this paper, we will devote to considering the following infinite-point singular p-Laplacian fractional differential equation:

$$D_{0+}^{\beta}\left(\varphi_p\left({}^cD_{0+}^{\gamma}u\right)\right)(t) + f\left(t, u(t), u'(t)\right) = 0, \quad 0 < t < 1, \tag{1}$$

*This research was supported by the National Natural Science Foundation of China (12101086, 11871302), Changzhou Science and technology planning project (CJ20210133), and Project of Shandong Province Higher Educational Science and Technology Program (J18KA217).

[1]Corresponding author.

with infinite-point boundary condition

$$\begin{aligned} &u(0) = u'(0) = \cdots = u^{(i-1)}(0) = u^{(i+1)}(0) = \cdots = u^{(n-1)}(0) = 0, \\ &u^{(i)}(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j), \qquad {}^cD_{0+}^{\gamma} u(0) = 0, \\ &\varphi_p\bigl({}^cD_{0+}^{\gamma} u(1)\bigr) = \sum_{i=1}^{\infty} \zeta_j \varphi_p\bigl({}^cD_{0+}^{\gamma} u(\xi_j)\bigr), \end{aligned} \tag{2}$$

where $\beta, \gamma \in \mathbb{R}^+ = [0, +\infty)$, $1 < \beta \leqslant 2$, $n-1 < \gamma \leqslant n$ $(n \geqslant 3)$, $\gamma > i$, p-Laplacian operator φ_p is defined as $\varphi_p(s) = |s|^{p-2}s$, $p, q > 1$, $1/p + 1/q = 1$, and $0 < \eta_i, \zeta_i, \xi_i < 1$ $(i = 1, 2, \dots, \infty)$, $f \in C((0,1) \times \mathbb{R}_+^1 \times \mathbb{R}_+^1, \mathbb{R}_+^1))$ $(\mathbb{R}_+^1 = [0, +\infty))$, and $f(t, x_1, x_2)$ has singularity at $t = 0,\ 1$, $D_{0+}^{\beta} u$ is the standard Riemann–Liouville derivative, ${}^cD_{0+}^{\gamma} u$ is the standard Caputo derivative.

Fractional-order system may have additional attractive feature over the integer-order system. For example, the analytical solutions of the systems

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = a t^{a-1}, \qquad {}^cD_t^{\alpha} x(t) = a t^{a-1}, \quad 0 < \alpha < 1,$$

are $t^a + x(0)$ and $a\Gamma(a) t^{a+\alpha-1}/\Gamma(a+\alpha) + x(0)$, respectively. Obviously, the integer-order system is unstable for $a \in (0,1)$, the fractional dynamic system is stable as $0 < a \leqslant 1-\alpha$. Moreover, fractional-order systems have been shown to be more accurate and realistic than integer-order models, and it also provides an excellent tool to describe the hereditary properties of material and processes, particularly, in viscoelasticity, electrochemistry, porous media, and so on. As a result, there has been a significant development in the study of fractional differential equations in recent years, readers can refer to [2, 4–10, 15–17, 21–24]. Jong [12] studied the following p-Laplacian fractional differential equations:

$$D_{0+}^{\beta}\bigl(\varphi_p\bigl(D_{0+}^{\alpha} u\bigr)\bigr)(t) = f\bigl(t, u(t)\bigr), \quad 0 < t < 1, \tag{3}$$

with m-point boundary condition

$$\begin{aligned} &u(0) = 0, \qquad D_{0+}^{\gamma} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\gamma} u(\eta_i), \\ &D_{0+}^{\alpha} u(0) = 0, \qquad \varphi_p\bigl(D_{0+}^{\alpha} u(1)\bigr) = \sum_{i=1}^{m-2} \zeta_i \varphi_p\bigl(D_{0+}^{\alpha} u(\eta_i)\bigr), \end{aligned} \tag{4}$$

where $1 < \alpha, \beta \leqslant 2, 3 < \alpha + \beta \leqslant 4, 0 < \gamma \leqslant 1, \alpha - \gamma - 1 > 0$, $0 < \eta_i, \zeta_i, \xi_i < 1$ $(i = 1, 2, \dots, \infty)$, $\sum_{i=1}^{m-2} \xi_i \eta_i^{\alpha-\gamma-1} < 1$, $\sum_{i=1}^{m-2} \zeta_i \eta_i^{\beta-1} < 1 < 1$, p-Laplacian operator φ_p is defined as $\varphi_p(s) = |s|^{p-2}s$, $p, q > 1$, $1/p + 1/q = 1$, and $f \in C([0,1] \times (0, +\infty), [0, +\infty))$. The authors obtained the existence and uniqueness of solutions by using the fixed point theorem for mixed monotone operators. Jong [11] obtained the existence and uniqueness of positive solutions by the Banach contraction mapping principle for equation (3), (4). In [20], the author considered following fractional differential equation:

$$D_{0+}^{\alpha}u(t) + g(t)f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1,$$

with infinite-point boundary condition

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j),$$

where $n - 1 < \alpha < n$, $n \geqslant 3$, $i \in [1, n-2]$ is a fixed integer, $\alpha_j \geqslant 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ $(j = 1, 2, \dots)$, f permits singularities with respect to both the time and space variables. According to introducing height functions, the author obtained the existence and multiplicity of positive solution theorems, and Zhang and Zhai obtained the existence and uniqueness of positive solution for this equation in [18]. In [19], Zhang and Liu investigated the following infinite-point fractional differential equation:

$$D_{0+}^{\alpha}u(t) = f\bigl(t, x(t), D_{0^+}^{\alpha-2}u(t), D_{0^+}^{\alpha-1}u(t)\bigr), \quad 0 < t < 1,$$

with infinite-point boundary condition

$$u(0) = 0, \quad D_{0^+}^{\alpha-1}u(0) = \cdots = \sum_{j=1}^{\infty} \alpha_j u(\xi_j), \qquad u^{(i)}(1) = \sum_{i=1}^{\infty} \alpha_j u(\xi_j),$$

where $2 < \alpha \leqslant 3$, $f \in [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ is a Caratheodory function, $\xi_i, \gamma_i \in (0,1)$, and $\{\xi_i\}_{i=1}^{+\infty}$, $\{\gamma_i\}_{i=1}^{+\infty}$ are two monotonic sequence with $\lim_{i\to+\infty} \xi_i = a$, $\lim_{i\to+\infty} \gamma_i = b$, $a, b \in (0,1)$, $\alpha_i, \beta_i \in \mathbb{R}$, $D_{0+}^{\alpha}u$ is the standard Riemann–Liouville derivative. The authors established the existence of at least one solution for this equation by Mawhin's continuation theorem.

Motivated by the excellent results above, in this paper, the existence of multiple positive solutions are obtained for a singular infinite-point p-Laplacian boundary value problems. Compared with [19], the equation in this paper is p-Laplacian fractional differential equation, and the method which we used in this paper is Avery–Peterson fixed point theorem. Compared with [12], fractional derivative is involved in the nonlinear terms for BVP (1), (2), and multiple positive solutions are obtained for the BVP (1), (2).

2 Preliminaries and lemmas

In this section, we introduce definitions and preliminary results, which are used throughout this paper. First, we let $E = C^1[0,1]$ be the Banach space with the maximum norm $\|u\| = \max\{\|u\|_0, \|u'\|_0\}$, where $\|u\|_0 = \max_{t\in[0,1]} |u(t)|$, $\|u'\|_0 = \max_{t\in[0,1]} |u'(t)|$, then we list a condition below to be used later in the paper.

(H0) $f : (0,1) \times \mathbb{R}_+^1 \times \mathbb{R}_+^1 \to \mathbb{R}_+^1$, and for all $t \in (0,1)$, there exists an function $\vartheta(t)$ such that $\vartheta(t) \in L^1(0,1)$ and $f(t, x_0, x_1) \leqslant \vartheta(t)$.

Now, we state some lemmas, which are basic and used in this paper.

Lemma 1. (*See* [13, 16].) *Assume that* $u \in C^n[0,1]$, *then*

$$I_{0+}^{\alpha}\, {}^{c}D_{0+}^{\alpha} u(t) = u(t) - C_1 - C_2 t - \cdots - C_n t^{n-1},$$

where n *is the least integer greater than or equal to* α, $C_i \in \mathbb{R}^1$ $(i = 1, 2, \ldots, n)$.

Lemma 2. (*See* [13, 16].) *Assume that* $u \in C^n[0,1]$, *then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$

where n *is the least integer greater than or equal to* α, $C_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$.

Lemma 3. (*See* [3, *Thm.* 1.2.7].) *Let* $H \subset C^1[J, E]$, *then* H *is a relatively compact set if and only if*

(i) H' *is equicontinuous, and* $H'(t)$ *is a relatively compact set for any* $t \in J$ *on* E;
(ii) *There exists* $t_0 \in J$ *such that* $H(t_0)$ *is a relatively compact set on* E.

Lemma 4. (*See* [1, 14].) *Let* P *be a cone of* E, Φ *and* θ *be nonnegative continuous convex functionals on* P, ϕ *be a nonnegative continuous concave functional on* P, *and* ψ *be a nonnegative continuous functional on* P, $\psi(\mu x) \leqslant \mu\psi(x)$ *for* $0 \leqslant \mu \leqslant 1$ *such that for some positive numbers L and h*, $\phi(x) \leqslant \psi(x)$ *and* $\|x\| \leqslant L\Phi(x)$ *for all* $x \in \overline{P(\Phi, h)}$. *Let* $A : \overline{P(\Phi, h)} \to \overline{P(\Phi, h)}$ *is completely continuous and there exist positive numbers* e, c, d *with* $e < c$ *such that the following conditions are satisfied*:

(S1) $\{x \in P(\Phi, \theta, \phi, c, d, h)\colon \phi(x) > c\} \neq \phi$, $\phi(Ax) > c$ *for* $x \in P(\Phi, \theta, \phi, c, d, h)$;
(S2) $\phi(Ax) > c$ *for* $x \in P(\Phi, \phi, c, h)$, *and* $\theta(Ax) > d$;
(S3) $0 \notin R(\Phi, \psi, e, h)$ *and* $\psi(Ax) < e$ *for* $x \in R(\Phi, \psi, e, h)$ *with* $\psi(x) = e$.

Then A *has at least three fixed points* x_1, x_2, x_3 *such that* $\Phi(x_i) \leqslant h$ *for* $i = 1, 2, 3$, *and* $c < \phi(x_1)$, $e < \psi(x_2)$, $\phi(x_2) < c$, $\psi(x_3) < e$.

Lemma 5. *Given* $y \in L^1[0,1] \cap C(0,1)$, *then the solution of the BVP*

$${}^{c}D_{0+}^{\gamma} u(t) + y(t) = 0, \quad 0 < t < 1, \tag{5}$$

with boundary condition (2) *can be expressed by*

$$u(t) = \int_0^1 G(t,s) y(s)\, \mathrm{d}s, \quad t \in [0,1], \tag{6}$$

where

$$G(t,s) = \frac{1}{\Delta\Gamma(\gamma)} \begin{cases} t^i \Gamma(\gamma) P(s)(1-s)^{\gamma-i-1} - \Delta(t-s)^{\gamma-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^i \Gamma(\gamma) P(s)(1-s)^{\gamma-i-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{7}$$

in which

$$P(s) = \frac{1}{\Gamma(\gamma - i)} - \frac{1}{\Gamma(\gamma)} \sum_{s \leqslant \xi_j} \eta_j \left(\frac{\xi_j - s}{1 - s}\right)^{\gamma-1} (1-s)^i, \tag{8}$$

$$\Delta = i! - \sum_{j=1}^{\infty} \eta_j \xi_j^i. \tag{9}$$

Proof. By means of Lemma 1 we reduce (5) to an equivalent integral equation

$$u(t) = -I_{0+}^{\gamma} y(t) + C_1 + C_2 t + \cdots + C_i t^{i-1} + C_{i+1} t^i + \cdots + C_n t^{n-1}$$

for $C_i \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$. From $u(0) = u'(0) = \cdots = u^{(i-1)}(0) = u^{(i+1)}(0) = \cdots = u^{(n-1)}(0) = 0$ we have $C_j = 0 (j \neq i+1)$. Consequently, we get

$$u(t) = C_{i+1} t^i - I_{0+}^{\gamma} y(t),$$

hence, we have

$$u^{(i)}(t) = i! C_{i+1} - I_{0+}^{\gamma - i} y(t). \tag{10}$$

On the other hand, $u^{(i)}(1) = \sum_{j=1}^{\infty} \eta_j u(\xi_j)$ combining with (10), we get

$$\begin{aligned} C_{i+1} &= \int_0^1 \frac{(1-s)^{\gamma-i-1}}{\Gamma(\gamma - i)\Delta} y(s)\,\mathrm{d}s - \sum_{j=1}^{\infty} \eta_j \int_0^{\xi_j} \frac{(\xi_j - s)^{\gamma-1}}{\Gamma(\gamma)\Delta} y(s)\,\mathrm{d}s \\ &= \int_0^1 \frac{(1-s)^{\gamma-i-1} P(s)}{\Delta} y(s)\,\mathrm{d}s, \end{aligned}$$

where $P(s)$ is as (8), and Δ is as (9). Hence,

$$\begin{aligned} u(t) &= C_{i+1} t^i - I_{0+}^{\gamma} y(t) \\ &= -\int_0^t \frac{\Delta (t-s)^{\gamma-1}}{\Gamma(\gamma)\Delta} y(s)\,\mathrm{d}s + \int_0^1 \frac{(1-s)^{\gamma-i-1} t^i P(s)}{\Delta} y(s)\,\mathrm{d}s. \end{aligned}$$

Therefore, $G(t,s)$ is as (7). By simple calculation we have

$$\frac{\partial G(t,s)}{\partial t} = \frac{1}{\Delta \Gamma(\gamma)} \begin{cases} i t^{i-1} \Gamma(\gamma) P(s)(1-s)^{\gamma-i-1} - (\gamma - 1)\Delta (t-s)^{\gamma-2}, \\ \quad 0 \leqslant s \leqslant t \leqslant 1, \\ i t^{i-1} \Gamma(\gamma) P(s)(1-s)^{\gamma-i-1}, \quad 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$ □

Lemma 6. *Let* $f \in C((0,1] \times (0,+\infty)^2, [0,+\infty))$*, then the BVP* (1), (2) *has a unique solution.*

$$u(t) = \int_0^1 G(t,s) \varphi_q \left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau \right) \mathrm{d}s, \tag{11}$$

where $G(t,s)$ *is as* (6)*, and*

$$H(t,s) = \frac{1}{\overline{\Delta}\Gamma(\beta)} \begin{cases} \Gamma(\beta)\overline{P}(s) t^{\beta-1}(1-s)^{\beta-1} - \overline{\Delta}(t-s)^{\beta-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \Gamma(\beta)\overline{P}(s) t^{\beta-1}(1-s)^{\beta-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{12}$$

in which

$$\overline{\Delta} = 1 - \sum_{i=1}^{\infty} \zeta_i \xi_i^{\beta-1}, \tag{13}$$

$$\overline{P}(s) = \frac{1}{\Gamma(\beta)} - \frac{1}{\Gamma(\beta)} \sum_{s \leqslant \xi_j} \zeta_j \left(\frac{\xi_j - s}{1 - s}\right)^{\beta-1}. \tag{14}$$

Proof. Let $\overline{y} \in C[0,1]$, $v(t) = \varphi_p({}^cD_{0+}^{\gamma} u)(t)$. Consider the boundary value problem

$$D_{0+}^{\beta} v(t) + \overline{y}(t) = 0, \quad 0 < t < 1, \qquad v(0) = 0, \qquad v(1) = \sum_{j=1}^{\infty} \zeta_j v(\xi_j). \tag{15}$$

By means of the Lemma 2 we reduce (15) to an equivalent integral equation

$$v(t) = -I_{0+}^{\beta} \overline{y}(t) + C_1 t^{\beta-1} + C_2 t^{\beta-2} \tag{16}$$

for $C_i \in \mathbb{R}$ $(i = 1, 2)$. From $v(0) = 0$ we have $C_2 = 0$. Consequently, we get

$$v(t) = C_1 t^{\beta-1} - I_{0+}^{\beta} \overline{y}(t).$$

On the other hand, $v(1) = \sum_{j=1}^{\infty} \zeta_j v(\xi_j)$, and combining with (16), we get

$$\begin{aligned} C_1 &= \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)\overline{\Delta}} \overline{y}(s)\,\mathrm{d}s - \sum_{j=1}^{\infty} \zeta_j \int_0^{\xi_j} \frac{(\xi_j - s)^{\beta-1}}{\Gamma(\beta)\overline{\Delta}} \overline{y}(s)\,\mathrm{d}s \\ &= \int_0^1 \frac{(1-s)^{\beta-1}\overline{P}(s)}{\overline{\Delta}} \overline{y}(s)\,\mathrm{d}s, \end{aligned}$$

where $\overline{\Delta}$ is as (13), $\overline{P}(s)$ is as (14). Hence,

$$\begin{aligned} v(t) &= C_1 t^{\beta-1} - I_{0+}^{\beta} y(t) \\ &= -\int_0^t \frac{\overline{\Delta}(t-s)^{\beta-1}}{\Gamma(\beta)\overline{\Delta}} \overline{y}(s)\,\mathrm{d}s + \int_0^1 \frac{(1-s)^{\beta-1} t^{\beta-1} \overline{P}(s)}{\overline{\Delta}} \overline{y}(s)\,\mathrm{d}s. \end{aligned}$$

Therefore, $H(t,s)$ is as (12). □

Lemma 7. *Take $\jmath, \ell \in (0,1)$ with $\jmath < \ell$ such that $\jmath^i \geqslant \ell^{\gamma-1}$, $i\jmath^{i-1} > (\gamma-1)\ell^{\gamma-2}$, then we have*

$$0 \leqslant G(t,s) \leqslant \frac{i}{\Gamma(\gamma - i)} g(s), \quad 0 \leqslant \frac{\partial G(t,s)}{\partial t} \leqslant \frac{i}{\Gamma(\gamma - i)} g(s), \quad t, s \in [0,1],$$

$$G(t,s) \geqslant \hbar g(s), \quad \frac{\partial G(t,s)}{\partial t} \geqslant \hbar g(s), \quad t \in [\jmath, \ell],\ s \in [0,1],$$

where $g(s) = (1-s)^{\gamma-i-1}/\Delta$, $\hbar_1 = \Delta(\jmath^i - \ell^{\gamma-1})/\Gamma(\gamma) < 1$, $\hbar_2 = \Delta(i\jmath^{i-1} - (\gamma-1) \times \ell^{\gamma-2})/\Gamma(\gamma) < 1$. *Then* $\hbar = \min\{\hbar_1, \hbar_2\} < 1$.

Proof. By direct calculation we get $P'(s) \geqslant 0$, $s \in [0,1]$, and so $P(s)$ is nondecreasing with respect to s. For $s \in [0,1]$, $\gamma - 1 > i$, we get

$$\begin{aligned}
\Gamma(\gamma)P(s) &= \frac{\Gamma(\gamma)}{\Gamma(\gamma-i)} - \sum_{s\leqslant\xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\gamma-1} (1-s)^i \\
&\geqslant \frac{\Gamma(\gamma)}{\Gamma(\gamma-i)} - \sum_{j=1}^{\infty} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\gamma-1} (1-s)^i \\
&\geqslant \Gamma(\gamma)P(0) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-i)} - \sum_{j=1}^{\infty} \eta_j \xi_j^{\gamma-1} \\
&= (\gamma-1)(\gamma-2)\cdots(\gamma-i) - \sum_{j=1}^{\infty} \eta_j \xi_j^{\gamma-1} \\
&\geqslant i! - \sum_{j=1}^{\infty} \eta_j \xi_j^{i} = \varDelta,
\end{aligned}$$

and obviously,

$$P(s) = \frac{1}{\Gamma(\gamma-i)} - \frac{1}{\Gamma(\gamma)} \sum_{s\leqslant\xi_j} \eta_j \left(\frac{\xi_j - s}{1-s}\right)^{\gamma-1} (1-s)^i \leqslant \frac{1}{\Gamma(\gamma-i)}, \quad s \in [0,1].$$

Hence, for $t, s \in [0,1]$, $i > \mu$, we have

$$\begin{aligned}
G(t,s) &\leqslant \frac{t^i \Gamma(\gamma)P(s)(1-s)^{\gamma-i-1}}{\Gamma(\gamma)\varDelta} \leqslant \frac{\Gamma(\gamma)P(s)(1-s)^{\gamma-i-1}}{\Gamma(\gamma)\varDelta} \leqslant \frac{i}{\Gamma(\gamma-i)} g(s), \\
\frac{\partial G(t,s)}{\partial t} &\leqslant \frac{it^{i-1}\Gamma(\gamma)P(s)(1-s)^{\gamma-i-1}}{\Gamma(\gamma)\varDelta} \leqslant \frac{i(1-s)^{\gamma-i-1}}{\Gamma(\gamma-i)\varDelta} = \frac{i}{\Gamma(\gamma-i)} g(s).
\end{aligned}$$

Furthermore, for $0 \leqslant s \leqslant t \leqslant 1$, we get

$$\begin{aligned}
G(t,s) &= \frac{t^i \Gamma(\gamma)P(s)(1-s)^{\gamma-i-1} - \varDelta(t-s)^{\gamma-1}}{\varDelta\Gamma(\gamma)} \\
&= \frac{t^i \Gamma(\gamma)P(s)(1-s)^{\gamma-i-1} - \varDelta(t-s)^{\gamma-i-1}(t-s)^i}{\varDelta\Gamma(\gamma)} \geqslant 0,
\end{aligned}$$

and obviously, for $0 \leqslant t \leqslant s \leqslant 1$, we get $G(t,s) \geqslant 0$. On the other hand, for $0 \leqslant s \leqslant t \leqslant 1$, we have

$$\begin{aligned}
&\frac{\partial G(t,s)}{\partial t} \\
&\quad = \frac{it^{i-1}\Gamma(\gamma)P(s)(1-s)^{\gamma-i-1} - \varDelta(\gamma-1)(t-s)^{\gamma-2}}{\varDelta\Gamma(\gamma)} \\
&\quad \geqslant \frac{it^{i-1}\Gamma(\gamma)P(0)(1-s)^{\gamma-i-1} - \varDelta(\gamma-1)(t-s)^{\gamma-i-1}(t-s)^{i-1}}{\varDelta\Gamma(\gamma)}
\end{aligned}$$

$$\begin{aligned}
&\geqslant \frac{(i(\frac{\Gamma(\gamma)}{\Gamma(\gamma-i)}-\sum_{j=1}^{\infty}\eta_j\xi_j^{\gamma-1})-(\gamma-1)(i!-\sum_{j=1}^{\infty}\eta_j\xi_j^{i}))(1-s)^{\gamma-i-1}t^{i-1}}{\Delta\Gamma(\gamma)}\\
&=\frac{(\frac{\Gamma(\gamma)i}{\Gamma(\gamma-i)}-(\gamma-1)i!+(\gamma-1)\sum_{j=1}^{\infty}\eta_j\xi_i^{i}-i\sum_{j=1}^{\infty}\eta_j\xi_j^{\gamma-1})(1-s)^{\gamma-i-1}t^{i-1}}{\Delta\Gamma(\gamma)}\\
&=\frac{1}{\Delta\Gamma(\gamma)}\Biggl((\gamma-1)i\bigl((\gamma-2)(\gamma-3)\cdots(\gamma-i)-(i-1)!\bigr)\\
&\quad+(\gamma-1)\sum_{j=1}^{\infty}\eta_j\xi_j^{i}-i\sum_{j=1}^{\infty}\eta_j\xi_j^{\gamma-1}\Biggr)(1-s)^{\gamma-i-1}t^{i-1}\geqslant 0,
\end{aligned}$$

and $\partial G(t,s)/\partial t \geqslant 0$ obviously holds for $0 \leqslant t \leqslant s \leqslant 1$.

For $t \in [\jmath,\ell]$, we get

$$\begin{aligned}
G(t,s)&=\frac{t^{i}\Gamma(\gamma)P(s)(1-s)^{\gamma-i-1}-\Delta(t-s)^{\gamma-1}}{\Delta\Gamma(\gamma)}\\
&\geqslant\frac{\jmath^{i}\Gamma(\gamma)P(s)(1-s)^{\gamma-i-1}-\Delta(\ell-\ell s)^{\gamma-1}}{\Delta\Gamma(\gamma)}\\
&\geqslant\frac{\Delta(\jmath^{i}(1-s)^{\gamma-i-1}-\ell^{\gamma-1}(1-s)^{\gamma-1})}{\Delta\Gamma(\gamma)}\\
&\geqslant\frac{\Delta(\jmath^{i}-\ell^{\gamma-1})(1-s)^{\gamma-i-1}}{\Delta\Gamma(\gamma)}=\hbar_1 g(s),\quad s\in[0,1],
\end{aligned}$$

and for $t \in [\jmath,\ell]$, we have

$$\begin{aligned}
\frac{\partial G(t,s)}{\partial t}&=\frac{it^{i-1}\Gamma(\gamma)P(s)(1-s)^{\gamma-i-1}-\Delta(\gamma-1)(t-s)^{\gamma-2}}{\Delta\Gamma(\gamma)}\\
&\geqslant\frac{i\jmath^{i-1}\Delta(1-s)^{\gamma-i-1}-\Delta(\gamma-1)(\ell-\ell s)^{\gamma-2}}{\Delta\Gamma(\gamma)}\\
&\geqslant\frac{\Delta(i\jmath^{i-1}-(\gamma-1)\ell^{\gamma-2})(1-s)^{\gamma-i-1}}{\Delta\Gamma(\gamma)}=\hbar_2 g(s),\quad s\in[0,1].
\end{aligned}$$

Therefore, the proof of Lemma 7 is completed. □

Lemma 8. *Let* $\overline{\Delta} > 0$*, then the Green functions defined by* (12) *satisfies*:

(i) $H:[0,1]\times[0,1]\to\mathbb{R}_+^1$ *is continuous and* $H(t,s)>0$ *for all* $t,s\in(0,1)$;
(ii) $t^{\beta-1}H(1,s)\leqslant H(t,s)\leqslant H(1,s)$ *for all* $t,s\in[0,1]$ *in which* $H(1,s)=(1/\overline{\Delta})(\overline{P}(s)-\overline{\Delta}/\Gamma(\beta))(1-s)^{\beta-1}$.

Proof. The proof is similar to Lemma 3 of [20], we omit it here. □

Now we define a cone K on $C^1[0,1]$ and an operator $A: K\to C^1[0,1]$ as follows:

$$K=\Bigl\{u\in C^1[0,1]\colon\ u(t)\geqslant 0,\ u'(t)\geqslant 0,\ t\in[0,1],\ \min_{t\in[\jmath,\ell]}u^{(j)}(t)\geqslant\sigma\|u\|,\ j=0,1\Bigr\},$$

where $0 < \sigma = \hbar\Gamma(\gamma - i)/i < 1$, $0 < \hbar < \hbar_1 = (\jmath^i - \ell^{\gamma-1})/\Gamma(\gamma) < i/\Gamma(\gamma - i)$, $\jmath$ and ℓ are the same as in Lemma 7, and

$$Au(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) f\bigl(\tau, u(\tau), u'(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s.$$

Problems (1), (2) has a positive solution if and only if u is a fixed point of A in K.

3 Main results

Lemma 9. *The operator $A : K \to E$ is continuous.*

Proof. First, by the integrability of f and (H0) we get

$$\begin{aligned}
&Au(t)\\
&\quad= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) f\bigl(\tau, u(\tau), u'(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad\leqslant \frac{i}{\Gamma(\gamma - i)\varDelta}\int_0^1 (1-s)^{\gamma-i-1}\varphi_q\left(\int_0^1 \frac{1}{\overline{\varDelta}\Gamma(\beta)}(1-\tau)^{\beta-1} f\bigl(\tau, u(\tau), u'(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad\leqslant \frac{i}{\Gamma(\gamma - i)\varDelta(\overline{\varDelta}\Gamma(\beta))^{q-1}}\int_0^1 (1-s)^{\gamma-i-1}\varphi_q\left(\int_0^1 \vartheta(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad= \frac{i\varphi_q(\int_0^1 \vartheta(\tau)\,\mathrm{d}\tau)}{\Gamma(\gamma - i)\varDelta(\overline{\varDelta}\Gamma(\beta))^{q-1}}\int_0^1 (1-s)^{\gamma-i-1}\,\mathrm{d}s\\
&\quad= \frac{i\varphi_q(\int_0^1 \vartheta(\tau)\,\mathrm{d}\tau)}{\Gamma(\gamma - i)\varDelta(\overline{\varDelta}\Gamma(\beta))^{q-1}}\frac{(1-s)^{\gamma-i}}{\gamma - i} < +\infty,
\end{aligned}$$

so we have that A is well defined on K. Moreover, it follows from the uniform continuity of $G(t,s)$ on $[0,1]\times[0,1]$ and (H0)

$$\begin{aligned}
&\bigl|Au(t_2) - Au(t_1)\bigr|\\
&\quad\leqslant \int_0^1 \bigl|G(t_2,s) - G(t_1,s)\bigr|\varphi_q\left(\frac{1}{\overline{\varDelta}\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1} f\bigl(\tau, u(\tau), u'(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s\\
&\quad\leqslant \frac{\varphi_q(\int_0^1 \vartheta(\tau)\,\mathrm{d}\tau)}{(\overline{\varDelta}\Gamma(\beta))^{q-1}}\int_0^1 \bigl|G(t_2,s) - G(t_1,s)\bigr|\,\mathrm{d}s.
\end{aligned}$$

Thus, we have that $Au \in C[0,1]$, $u \in K$. Furthermore, by the uniform continuity of $\partial G(t,s)/\partial t$, for $t, s \in [0,1]$, we get

$$(Au)'(t) = \int_0^1 \frac{\partial G(t,s)}{\partial t}\phi_q\Biggl(\int_0^1 H(s,\tau)f\bigl(\tau,u(\tau),u'(\tau)\bigr)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \in C[0,1].$$

Let $u_n, u \in K$, $u_n \to u$ in $C^1[0,1]$. Since $G(t,s)$, $\partial G(t,s)/\partial t$ are uniformly continuous, there exists $M > 0$ such that

$$\max\biggl\{G(t,s), \frac{\partial G(t,s)}{\partial t}\biggr\} \leqslant M, \quad t,s \in [0,1].$$

On the other hand, since $u_n \to u$ in $C^1[0,1]$, there exists $A > 0$ such that $\|u_n\| \leqslant A$ $(n = 1,2,\dots)$, and then $\|u\| \leqslant A$. Furthermore, by (H0) we have

$$\begin{aligned}\int_0^1 H(s,\tau)f\bigl(\tau,u(\tau),u'(\tau)\bigr)\,\mathrm{d}\tau &\leqslant \frac{1}{\overline{\Delta}\Gamma(\beta)}\int_0^1 (1-\tau)^{\beta-1}f\bigl(\tau,u(\tau),u'(\tau)\bigr)\,\mathrm{d}\tau \\ &= \frac{1}{(\overline{\Delta}\Gamma(\beta))^{q-1}}\int_0^1 (1-\tau)^{\beta-1}\vartheta(\tau)\,\mathrm{d}\tau,\end{aligned}$$

so, we have

$$\begin{aligned}\varphi_q\Biggl(\int_0^1 H(s,\tau)f(\tau,x_0,x_1)\,\mathrm{d}\tau\Biggr) &\leqslant \frac{1}{(\overline{\Delta}\Gamma(\beta))^{q-1}}\varphi_q\Biggl(\int_0^1 (1-\tau)^{\beta-1}\vartheta(\tau)\,\mathrm{d}\tau\Biggr) \\ &= M_0, \end{aligned} \tag{17}$$

hence, for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $s_1, s_2 \in [0,1]$, $x_0^1, x_0^2, x_1^1, x_1^2 \in [0,A]$, $|s_1 - s_2| < \delta$, $|x_0^1 - x_0^2| < \delta$, $|x_1^1 - x_1^2| < \delta$, by (17) and Lebesgue controls convergence theorem, we have

$$\Biggl|\varphi_q\Biggl(\int_0^1 H(s_1,\tau)f(\tau,x_0^1,x_1^1)\,\mathrm{d}\tau\Biggr) - \varphi_q\Biggl(\int_0^1 H(s_2,\tau)f(\tau,x_0^2,x_1^2)\,\mathrm{d}\tau\Biggr)\Biggr| < \varepsilon. \tag{18}$$

By $\|u_n - u\| \to 0$, for the above $\delta > 0$, there exists N such that for all $n > N$, we get $|u_n(t) - u(t)$, $|u_n'(t) - u'(t)| \leqslant \|u_n - u\| < \delta$ for any $t \in [0,1]$. Hence, for any $t \in [0,1]$, $n > N$, by (18) we obtain

$$\begin{aligned}&\Biggl|\varphi_q\Biggl(\int_0^1 H(s,\tau)f\bigl(\tau,u_n(\tau),u_n'(\tau)\bigr)\,\mathrm{d}\tau\Biggr) - \varphi_q\Biggl(\int_0^1 H(s,\tau)f\bigl(\tau,u(\tau),u'(\tau)\,\mathrm{d}\tau\bigr)\Biggr)\Biggr| \\ &\quad < \varepsilon. \end{aligned} \tag{19}$$

Thus, for $n > N$, $t \in [0,1]$, by (19) we have

$$\begin{aligned}
&\bigl|(Au_n)(t) - (Au)(t)\bigr| \\
&\quad = \Biggl| \int_0^1 G(t,s)\varphi_q\Biggl(\int_0^1 H(s,\tau) f\bigl(\tau, u_n(\tau), u_n'(\tau)\bigr)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \\
&\qquad - \int_0^1 G(t,s)\varphi_q\Biggl(\int_0^1 H(s,\tau) f\bigl(\tau, u(\tau), u'(\tau)\bigr)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s\Biggr| \\
&\quad = \Biggl| \int_0^1 G(t,s)\Biggl(\varphi_q\Biggl(\int_0^1 H(s,\tau) f\bigl(\tau, u_n(\tau), u_n'(\tau)\bigr)\,\mathrm{d}\tau\Biggr) \\
&\qquad - \varphi_q\Biggl(\int_0^1 H(s,\tau) f\bigl(\tau, u(\tau), u'(\tau)\bigr)\,\mathrm{d}\tau\Biggr)\Biggr)\,\mathrm{d}s\Biggr| \\
&\quad \leqslant M\int_0^1 \varepsilon\,\mathrm{d}s \leqslant M\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&\bigl|(Au_n)'(t) - (Au)'(t)\bigr| \\
&\quad = \Biggl| \int_0^1 \frac{\partial G(t,s)}{\partial t}\varphi_q\Biggl(\int_0^1 H(s,\tau) f\bigl(\tau, u_n(\tau), u_n'(\tau)\bigr)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \\
&\qquad - \int_0^1 \frac{\partial G(t,s)}{\partial t}\varphi_q\Biggl(\int_0^1 H(s,\tau) f\bigl(\tau, u(\tau), u'(\tau)\,\mathrm{d}\tau\bigr)\Biggr)\,\mathrm{d}s\Biggr| \\
&\quad = \Biggl| \int_0^1 \frac{\partial G(t,s)}{\partial t}\Biggl(\varphi_q\Biggl(\int_0^1 H(s,\tau) f\bigl(\tau, u_n(\tau), u_n'(\tau)\bigr)\,\mathrm{d}\tau\Biggr) \\
&\qquad - \varphi_q\Biggl(\int_0^1 H(s,\tau) f\bigl(\tau, u(\tau), u'(\tau)\bigr)\,\mathrm{d}\tau\Biggr)\Biggr)\,\mathrm{d}s\Biggr| \\
&\quad \leqslant M\varepsilon,
\end{aligned}$$

and hence, we get $\|Au_n - Au\|_0 \to 0$, $\|(Au_n)' - (Au)'\|_0 \to 0$ $(n \to \infty)$. That is, $\|Au_n - Au\| \to 0$ $(n \to \infty)$, namely, A is continuous in the space E. □

Lemma 10. *$A : P \to P$ is completely continuous.*

Proof. From Lemma 7 we have $(Au)(t), (Au)'(t) \geqslant 0$, $t \in [0,1]$, and

$$\begin{aligned}
\max_{t\in[0,1]}(Au)(t) &= \max_{t\in[0,1]} \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\leqslant \int_0^1 \max_{t\in[0,1]} G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\right)\mathrm{d}s \\
&\leqslant \int_0^1 \frac{i}{\Gamma(\gamma - i)} g(s)\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s, \\
\max_{t\in[0,1]}(Au)'(t) &= \max_{t\in[0,1]} \int_0^1 \frac{\partial G(t,s)}{\partial t}\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\leqslant \int_0^1 \max_{t\in[0,1]} \frac{\partial G(t,s)}{\partial t}\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\leqslant \int_0^1 \frac{i}{\Gamma(\gamma - i)} g(s)\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s,
\end{aligned}$$

thus, $\|Au\|_0$, $\|(Au)'\|_0 \leqslant \int_0^1 (i/\Gamma(\gamma - i)) g(s)\varphi_q(\int_0^1 H(s,\tau) f(\tau, u(\tau), u'(\tau))\,\mathrm{d}\tau)\mathrm{d}s$, consequently,

$$\begin{aligned}
\|Au\| &= \max\big\{\|Au\|_0, \big\|(Au)'\big\|_0\big\} \\
&\leqslant \int_0^1 \frac{i}{\Gamma(\gamma - i)} g(s)\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s.
\end{aligned}$$

On the other hand, for all $u \in P$, $t \in [\jmath, \ell]$, by Lemma 7 we have

$$\begin{aligned}
(Au)(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\geqslant \int_0^1 \hbar_1 g(s)\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\geqslant \frac{\hbar_1}{\frac{i}{\Gamma(\alpha - i)}} \int_0^1 \frac{i}{\Gamma(\gamma - i)} g(s)\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\geqslant \frac{\hbar_1}{\frac{i}{\Gamma(\alpha - i)}} \int_0^1 G(t,s)\varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s,
\end{aligned}$$

$$
\begin{aligned}
(Au)'(t) &= \int_0^1 \frac{\partial G(t,s)}{\partial t} \varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\geqslant \int_0^1 \hbar_2 g(s) \varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\geqslant \frac{\hbar_2}{\frac{i}{\Gamma(\alpha-i)}} \int_0^1 \frac{i}{\Gamma(\gamma-i)} g(s) \varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s, \\
&\geqslant \frac{\hbar_2}{\frac{i}{\Gamma(\alpha-i)}} \int_0^1 \frac{\partial G(t,s)}{\partial t} \varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\min\big\{Au(t), (Au)'(t)\big\} \\
&\quad\geqslant \frac{\min\{\hbar_1, \hbar_2\}}{\frac{i}{\Gamma(\alpha-i)}} \int_0^1 \frac{i}{\Gamma(\alpha-i)} g(s) \varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad\geqslant \frac{\hbar\Gamma(\alpha-i)}{i}\|Au\| = \sigma\|Au\|.
\end{aligned}
$$

Thus, $A(P) \subset P$.

Now we will prove that AV is relatively compact for bounded $V \subset K$. Since V is bounded, there exists $D > 0$ such that for any $u \in V$, $\|u\| \leqslant D$. For $t \in [0,1]$, $u \in V$, we have

$$
\begin{aligned}
\big|Au(t)\big| &= \left|\int_0^1 G(t,s) \varphi_q\left(\int_0^1 H(s,\tau) f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\right| \\
&\leqslant \frac{i}{\Gamma(\gamma-i)\varDelta} \int_0^1 (1-s)^{\gamma-i-1} \varphi_q\left(\int_0^1 \frac{1}{\overline{\varDelta}\Gamma(\beta)}(1-\tau)^{\beta-1} f\big(\tau, u(\tau), u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\leqslant \frac{i}{\Gamma(\gamma-i)\varDelta(\overline{\varDelta}\Gamma(\beta))^{q-1}} \int_0^1 (1-s)^{\gamma-i-1} \varphi_q\left(\int_0^1 \vartheta(\tau)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&= \frac{i\varphi_q(\int_0^1 \vartheta(\tau)\,\mathrm{d}\tau)}{\Gamma(\gamma-i)\varDelta(\overline{\varDelta}\Gamma(\beta))^{q-1}} \frac{(1-s)^{\gamma-i}}{\gamma-i} < +\infty.
\end{aligned}
$$

Similarly, for $t \in [0,1]$, $u \in V$, we derive

$$
\big|(Au)'(t)\big| \leqslant \frac{i\varphi_q(\int_0^1 \vartheta(\tau)\,\mathrm{d}\tau)}{\Gamma(\gamma-i)\varDelta(\overline{\varDelta}\Gamma(\beta))^{q-1}} \frac{(1-s)^{\gamma-i}}{\gamma-i},
$$

which shows that AV is bounded. Next, we will verify that $(AV)'$ is equicontinuous. Let $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $u \in V$, we get

$$\begin{aligned}
&\left|(Au)'(t_2) - (Au)'(t_1)\right| \\
&\quad \leqslant \left|\left(t_2^{i-1} - t_1^{i-1}\right)\right| \\
&\qquad \times \int_0^1 \frac{i\Gamma(\gamma)P(s)(1-s)^{\gamma-i-1}}{\Delta\Gamma(\gamma)} \varphi_q\left(\int_0^1 H(s,\tau) f\left(\tau, u(\tau), u'(\tau)\right) \mathrm{d}\tau\right) \mathrm{d}s \\
&\qquad + \left|\frac{1}{\Gamma(\gamma-1)} \int_0^{t_2} (t_2 - s)^{\gamma-2} \varphi_q\left(\int_0^1 H(s,\tau) f\left(\tau, u(\tau), u'(\tau)\right) \mathrm{d}\tau\right) \mathrm{d}s\right. \\
&\qquad \left. - \frac{1}{\Gamma(\gamma-1)} \int_0^{t_1} (t_1 - s)^{\gamma-2} \varphi_q\left(\int_0^1 H(s,\tau) f\left(\tau, u(\tau), u'(\tau)\right) \mathrm{d}\tau\right) \mathrm{d}s\right| \\
&\quad \leqslant \left|(t_2^{i-1} - t_1^{i-1})\right| \\
&\qquad \times \frac{i\Gamma(\gamma)}{\Delta\Gamma(\gamma)} \int_0^1 \frac{(1-s)^{\gamma-i-1}}{\Gamma(\gamma-i)} \varphi_q\left(\int_0^1 \frac{1}{\overline{\Delta}\Gamma(\beta)} (1-\tau)^{\beta-1} f\left(\tau, u(\tau), u'(\tau)\right) \mathrm{d}\tau\right) \mathrm{d}s \\
&\qquad + \frac{M_0}{\Gamma(\gamma-1)} \left[\int_0^{t_2} (t_2 - s)^{\gamma-2} \mathrm{d}s - \int_0^{t_1} (t_1 - s)^{\gamma-2} ds\right] \\
&\quad \leqslant \frac{i\varphi_q(\int_0^1 \vartheta(\tau)\,\mathrm{d}\tau)}{\Delta\Gamma(\gamma-i)(\overline{\Delta}\Gamma(\beta))^{q-1}} \left|\left(t_2^{i-1} - t_1^{i-1}\right)\right| + \frac{M_0}{\Gamma(\gamma-1)} \int_0^1 (1-s)^{\gamma-2}\,\mathrm{d}s \\
&\qquad \times \left|t_2^{\gamma-1} - t_1^{\gamma-1}\right|.
\end{aligned}$$

Furthermore,

$$\int_0^t (t-s)^{\gamma-2} \mathrm{d}s = t^{\gamma-1} \int_0^1 (1-s)^{\gamma-2} \mathrm{d}s.$$

Thus, we obtain

$$\begin{aligned}
\left|(Au)'(t_2) - (Au)'(t_1)\right| \leqslant{}& \frac{i\varphi_q(\int_0^1 \vartheta(\tau)\,\mathrm{d}\tau)}{\Delta\Gamma(\gamma-i)(\overline{\Delta}\Gamma(\beta))^{q-1}} \left|\left(t_2^{i-1} - t_1^{i-1}\right)\right| \\
&+ \frac{C}{\Gamma(\gamma-1)} \int_0^1 (1-s)^{\gamma-2}\,\mathrm{d}s \cdot \left|t_2^{\gamma-1} - t_1^{\gamma-1}\right|, \quad u \in V.
\end{aligned}$$

From above and the uniform continuity of t^{i-1}, $t^{\alpha-2}$, and together with Lemma 3, we can derive that AV is relatively compact in $C^1[0,1]$, and so we get that $A : K \to K$ is completely continuous. □

Let φ, θ be nonnegative continuous convex functionals on P, ϕ be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then for nonnegative numbers e, c, d, h, we define the following convex sets:

$$P(\varphi,h)=\{x\in P \mid \varphi(x)<h\},$$
$$P(\varphi,\phi,c,h)=\{x\in P \mid \phi(x)\geqslant c,\ \varphi(x)\leqslant h\},$$
$$P(\varphi,\theta,\phi,c,d,h)=\{x\in P \mid c\leqslant\phi(x),\ \theta(x)\leqslant d,\ \varphi(x)\leqslant h\},$$
$$R(\varphi,\psi,e,h)=\{x\in P \mid e\leqslant\psi(x),\ \varphi(x)\leqslant h\}.$$

We will apply the following fixed point theorem of Avery and Peterson to solve problem (1), (2).

Let the convex functions $\psi(u)=\theta(u)=\Phi(u)=\|u\|$ on P, and define a concave function $\phi(u)=\min\{\min_{t\in[\jmath,\ell]}|u(t)|,\min_{t\in[\jmath,\ell]}|u'(t)|\}$, where $\jmath$, ℓ are the same as in Lemma 7.

Theorem 1. *Assume that there exist positive numbers e, c, d, h with $c>e$, $d>\max\{1/\hbar,e^{1-\jmath/2}\}c$, $h>rc/(\hbar Q)$, and $h\geqslant d$ such that*

- (H1) $\phi_q(\int_0^1 f(t,x,y)\,\mathrm{d}t)<h/r$ *for* $(t,x,y)\in[0,1]\times[0,h]^2$;
- (H2) $\phi_q(\int_0^1 H(1,t)f(t,x,y))\,\mathrm{d}s\geqslant c/(\hbar Q)$ *for* $(t,x,y)\in[\jmath,\ell]\times[c,d]^2$;
- (H3) $\phi_q(\int_0^1 f(t,x,y)\,\mathrm{d}t)<e/r$ *for* $(t,x,y)\in[0,1]\times[0,e]^2$, *where* $r=i\int_0^1 g(s)\,\mathrm{d}s/(\Gamma(\gamma-i)(\overline{\Delta}\Gamma(\beta))^{q-1})$, $Q=\int_\jmath^\ell s^{(q-1)(\beta-1)}g(s)\,\mathrm{d}s$.

Then problem (1), (2) *has at least three fixed points u_1, u_2, u_3 satisfying*

$$\|u_i\|\leqslant h,\quad i=1,2,3, \tag{20}$$

and

$$c<\min\Bigl\{\min_{t\in[\jmath,\ell]}|u_1(t)|,\min_{t\in[\jmath,\ell]}|u_1'(t)|\Bigr\},\qquad e<\|u_2\|, \tag{21}$$

$$\min\Bigl\{\min_{t\in[\jmath,\ell]}|u_2(t)|,\min_{t\in[\jmath,\ell]}|u_2'(t)|\Bigr\}<c,\qquad \|u_3\|<e. \tag{22}$$

Proof. Let $u\in\overline{P(\varphi,h)}$. By condition (H1) we get

$$\begin{aligned}\|Au\|_0&=\max_{t\in[0,1]}|Au(t)|\\&\leqslant\frac{i}{\Gamma(\gamma-i)}\int_0^1 g(s)\varphi_q\biggl(\int_0^1 H(s,\tau)f(\tau,u(\tau),u'(\tau))\,\mathrm{d}\tau\biggr)\,\mathrm{d}s\\&=\frac{i}{\Gamma(\gamma-i)(\overline{\Delta}\Gamma(\beta))^{q-1}}\int_0^1 g(s)\,\mathrm{d}s\cdot\frac{h}{r}\leqslant h,\end{aligned}$$

$$
\begin{aligned}
\left\|(Au)'\right\|_0 &= \max_{t\in[0,1]} \left|(Au)'(t)\right| \\
&= \int_0^1 \frac{\partial G(t,s)}{\partial t}\varphi_q\left(\int_0^1 H(s,\tau) f\bigl(\tau,u(\tau),u(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\leqslant \frac{i}{\Gamma(\gamma-i)(\overline{\Delta}\Gamma(\beta))^{q-1}}\int_0^1 g(s)\varphi_q\left(\int_0^1 f\bigl(\tau,u(\tau),u'(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&= \frac{i}{\Gamma(\gamma-i)(\overline{\Delta}\Gamma(\beta))^{q-1}}\int_0^1 g(s)\,\mathrm{d}s\cdot\frac{h}{r}\leqslant h.
\end{aligned}
$$

Consequently, we obtain $\Phi(Au) = \|Au\| \leqslant h$. This, together with Lemmas 9 and 10, means that $A : \overline{P(\Phi,h)} \to \overline{P(\Phi,h)}$ is completely continuous.

Take $u(t) = ce^{t-0.5\jmath}$, $t \in [0,1]$. By simple calculation we have that $u \in P$, $\|u\| < d$, and $\phi(u) > c$, and so $\{u \in P(\Phi,\theta,\phi,c,d,h)\colon c < \phi(u)\} \neq \emptyset$. For $u \in P(\Phi,\theta,\phi,c,d,h)$, by (H2) we get

$$
\begin{aligned}
\phi(Au) &= \min\Bigl\{\min_{t\in[\jmath,\ell]}\left|Au(t)\right|, \min_{t\in[\jmath,\ell]}\left|(Au)'(t)\right|\Bigr\} \\
&\geqslant \hbar\int_0^1 g(s)s^{(q-1)(\beta-1)}\varphi_q\left(\int_0^1 H(1,\tau) f\bigl(\tau,u(\tau),u'(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&> \hbar\int_\jmath^\ell g(s)s^{(q-1)(\beta-1)}\,\mathrm{d}s\,\frac{c}{\hbar Q} = c,
\end{aligned}
$$

which shows that condition (S1) is satisfied.

Take $u \in P(\Phi,\phi,c,h)$ and $\|Au\| > d$. Since $Au \in P$, we obtain

$$
\phi(Au) = \min\Bigl\{\min_{t\in[\jmath,\ell]}\left|Au(t)\right|, \min_{t\in[\jmath,\ell]}\left|(Au)'(t)\right|\Bigr\} \geqslant \sigma\|Au\| \geqslant \sigma d > c,
$$

which implies that condition (S2) holds.

Next, we will verify that condition (S3) holds. For $\psi(0) = 0$, we have $0 \in R(\Phi,\psi,e,h)$. Let $u \in R(\Phi,\psi,e,h)$ and $\psi(u) = \|u\| = e$, by (H3) we get

$$
\begin{aligned}
\|Au\|_0 &= \max_{t\in[0,1]}\left|Au(t)\right| \\
&\leqslant \int_0^1 \frac{i}{\Gamma(\gamma-i)} g(s)\varphi_q\left(\int_0^1 H(s,\tau) f\bigl(\tau,u(\tau),u(\tau)\bigr)\,\mathrm{d}\tau\right)\mathrm{d}s
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \frac{i}{\Gamma(\gamma-i)(\overline{\Delta}\Gamma(\beta))^{q-1}}\int_0^1 g(s)\,\mathrm{d}s\cdot\varphi_q\left(\int_0^1 f\big(\tau,u(\tau),u'(\tau)\big)\,\mathrm{d}\tau\right)\\
&< \frac{e}{r}\frac{i}{\Gamma(\gamma-i)(\overline{\Delta}\Gamma(\beta))^{q-1}}\int_0^1 g(s)\,\mathrm{d}s \leqslant e
\end{aligned}$$

and

$$\begin{aligned}
\big\|(Au)'\big\|_0 &= \max_{t\in[0,1]}\big|(Au)'(t)\big|\\
&\leqslant \int_0^1 \frac{i}{\Gamma(\gamma-i)} g(s)\varphi_q\left(\int_0^1 H(s,\tau)f\big(\tau,u(\tau),u'(\tau)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\\
&= \frac{i}{\Gamma(\gamma-i)(\overline{\Delta}\Gamma(\beta))^{q-1}}\int_0^1 g(s)\,\mathrm{d}s\cdot\varphi_q\left(\int_0^1 f\big(\tau,u(\tau),u'(\tau)\big)\,\mathrm{d}\tau\right)\\
&< \frac{e}{r}\frac{i}{\Gamma(\gamma-i)(\overline{\Delta}\Gamma(\beta))^{q-1}}\int_0^1 g(s)\,\mathrm{d}s \leqslant e.
\end{aligned}$$

Consequently, we have $\psi(Au) = \|Au\| < e$. Thus, condition (S3) holds.

By Lemma 4 we get that (1), (2) has at least three positive solutions u_1, u_2, u_3 satisfying (20)–(22). □

4 An example

Consider the following infinite-point p-Laplacian fractional differential equations:

$$\begin{aligned}
&D_{0+}^{3/2}\big(\varphi_3\big({}^cD_{0+}^{7/2}u\big)\big)(t) + f\big(t,u(t),u'(t)\big) = 0, \quad 0<t<1,\\
&u(0)=u'(0)=u'''(0)=0, \qquad u''(1)=\sum_{j=1}^{\infty}\eta_j u(\xi_j),\\
&{}^cD_{0+}^{7/2}u(0)=0, \qquad \varphi_3\big({}^cD_{0+}^{7/2}u(1)\big)=\sum_{i=1}^{\infty}\zeta_j\varphi_3\big({}^cD_{0+}^{7/2}u(\xi_j)\big),
\end{aligned} \tag{23}$$

where

$$f(t,x,y)=\begin{cases}
\frac{1}{4\pi t^{1/2}(1-t)^{1/2}}(x^2+y^2)^2, & (t,x,y)\in(0,1]\times[0,\frac12]\times[0,\frac12],\\
\frac{499^2}{4\pi t^{1/2}(1-t)^{1/2}}(\sqrt[6]{x}+\sqrt[6]{y})^2, & (t,x,y)\in(0,1]\times[1,20]\times[1,20],\\
\frac{499^2}{4\pi t^{1/2}(1-t)^{1/2}}, & (t,x,y)\in(0,1]\times[100,\infty)\times[100,\infty).
\end{cases}$$

Clearly,

$$f(t,x,y) \leqslant \frac{499^3}{4\pi t^{1/2}(1-t)^{1/2}} = \vartheta(t)$$

and

$$\int_0^1 \vartheta(t)\,\mathrm{d}t = \frac{499^3}{4\pi}\int_0^1 t^{-1/2}(1-t)^{-1/2}\,\mathrm{d}t = \frac{499^3}{4\pi}B\left(\frac{1}{2},\frac{1}{2}\right),$$

so, $\psi(t)$ is integrable, condition (H0) holds.

We take $\jmath = 3/15$, $\ell = 4/15$, $\varsigma_j = 2/(3j^{7/2})$, $\eta_j = 1/(2j^2)$, $\xi_j = 1/j$, by simple calculation we have

$$\Delta = i! - \sum_{j=1}^{\infty}\eta_j\xi_j^i \approx 1.4589,$$

$$\overline{\Delta} = 1 - \sum_{i=1}^{\infty}\zeta_i\xi_i^{\beta-1} = 1 - \frac{2}{3}\sum_{j=1}^{\infty}\frac{1}{j^{7/2}j^{1/2}} \approx 0.7215,$$

$$g(s) = \frac{(1-s)^{\alpha-i-1}}{\Delta} = \frac{1}{1.4589}(1-s)^{1/2}.$$

Hence, we have

$$\begin{aligned} r &= \frac{i}{\Gamma(\gamma-i)(\overline{\Delta}\Gamma(\beta))^{q-1}}\int_0^1 g(s)\,\mathrm{d}s \\ &= \frac{2}{\Gamma(\frac{7}{2}-2)(0.7215\Gamma(\frac{3}{2}))^{1/2}}\int_0^1 \frac{1}{1.4589}(1-s)^{1/2}\,\mathrm{d}s \\ &= \frac{8}{3\times 1.4589\times(0.7215\times 0.5)^{1/2}}\frac{1}{\pi^{3/4}} \approx \frac{4.3039}{\pi^{3/4}}, \end{aligned}$$

$$L = \int_{\jmath}^{\ell} g(s)\,\mathrm{d}s = \frac{1}{1.4589}\frac{2}{3}\left(\left(\frac{4}{5}\right)^{3/2} - \left(\frac{11}{15}\right)^{3/2}\right),$$

$$\begin{aligned} Q &= \int_{\jmath}^{\ell} s^{(q-1)(\beta-1)}g(s)\,\mathrm{d}s = \int_{3/15}^{4/15} s^{1/2\cdot 1/2}\frac{1}{1.4589}(1-s)^{1/2}\,\mathrm{d}s \\ &= 0.6854\int_{3/15}^{4/15} s^{1/4}(1-s)^{1/2}\,\mathrm{d}s = 0.0278, \end{aligned}$$

$$\hbar_1 = \Delta\frac{\jmath^i - \ell^{\alpha-1}}{\Gamma(\gamma)} \approx 0.1938\frac{1}{\sqrt{\pi}} \approx 0.1093 < 1,$$

$$\hbar_2 = \frac{\Delta(i\jmath^{i-1} - (\gamma-1)\ell^{\alpha-2})}{\Gamma(\gamma)} \approx 0.4336\frac{1}{\sqrt{\pi}} \approx 0.2446 < 1,$$

and as $\hbar_1 < \hbar_2 < 1$, $\hbar = \hbar_1$. Let $e = 1/2$, $c = 1$, $d = 20$, $h = 400$. Then for $(t, x, y) \in [0, 1] \times [0, 400]^2$, we have

$$\varphi_q\left(\int_0^1 f(t,x,y)\,\mathrm{d}t\right) = \varphi_{3/2}\left(\int_0^1 \frac{499^2}{4\pi t^{1/2}(1-t)^{1/2}}\,\mathrm{d}t\right) = \sqrt{\frac{499^2}{4\pi}B\left(\frac{1}{2},\frac{1}{2}\right)}$$
$$= 124.75 < \frac{h}{r} \approx 219.23,$$

so condition (H1) of Theorem 1 hold. For $(t, x, y) \in [0, 1] \times [1, 20] \times [1, 20]$, by MATLAB software we have

$$\varphi_q\left(\int_0^1 H(1,t)f(t,x,y)\,\mathrm{d}t\right)$$
$$\geqslant \varphi_{3/2}\left(\int_0^1 \frac{1}{\overline{\Delta}}\left(\overline{P}(t) - \frac{\overline{\Delta}}{\Gamma(\beta)}\right)(1-t)^{\beta-1}\frac{499^2}{\pi t^{1/2}(1-t)^{1/2}}\,\mathrm{d}t\right) \approx 345.56$$
$$\geqslant \frac{1}{0.1093 \times 0.6854 \int_{3/15}^{4/15} s^{1/4}(1-s)^{1/2}\,\mathrm{d}s} = \frac{c}{\hbar Q} \approx 333.33,$$

therefore, condition (H2) of Theorem 1 hold. By the same method with proofing (H1) we get (H3) hold, so all the conditions of Theorem 1 hold. Hence, the BVP (23) has at least three positive solutions u_1, u_2, u_3 satisfying $u_i \leqslant 600$, $i = 1, 2, 3$, and $1 < \phi(u_1)$, $1/2 < \|u_2\|$, $\phi(u_2) < 1$, $\|u_3\| < 1/2$.

Acknowledgment. The authors would like to thank the referee for his/her valuable comments and suggestions.

References

1. R.I. Avery, J. Henderson, Three positive fixed points of nonlinear operators on ordered Banach spaces, *Comput. Math. Appl.*, **42**(3):313–322, 2001, https://doi.org/10.1016/S0898-1221(01)00156-0.
2. A. Cabada, Z. Hamdi, Nonlinear fractional differential equations with integral boundary value conditions, *Appl. Math. Comput.*, **228**(2012):251–257, 2014, https://doi.org/10.7153/fdc-2019-09-05.
3. D. Guo, Y. Cho, J. Zhu, *Partial Ordering Methods in Nonlinear Problems*, Nova, New York, 2004.
4. L. Guo, L. Liu, Unique iterative positive solutions for a singular p-Laplacian fractional differential equation system with infinite-point boundary conditions, *Bound. Value Probl.*, **2019**:113, 2019, https://doi.org/10.1186/s13661-019-1227-8.

5. L. Guo, L. Liu, Y. Feng, Uniqueness of iterative positive solutions for the singular infinite-point p-Laplacian fractional differential system via sequential technique, *Nonlinear Anal. Model. Control*, **25**(5), 2020, https://doi.org/10.15388/namc.2020.25.19277.
6. L. Guo, L. Liu, Y. Wang, Maximal and minimal iterative positive solutions for p-Laplacian Hadamard fractional differential equations with the derivative term contained in the nonlinear term, *AIMS Math.*, **6**(11):12583–12598, 2021, https://doi.org/10.3934/math.2021725.
7. L. Guo, L. Liu, Y. Wu, Iterative unique positive solutions for singular p-Laplacian fractional differential equation system with several parameters, *Nonlinear Anal. Model. Control*, **23**(2): 182–203, 2018, https://doi.org/10.15388/NA.2018.2.3.
8. L. Guo, L. Liu, Y. Wu, Y.M. Zou, Blow-up and global solutions for some parabolic systems under nonlinear boundary conditions, *J. Korean Math. Soc.*, **56**(4):1017–1029, 2019, https://doi.org/org/10.4134/JKMS.j180539.
9. P. Hentenryck, R. Bent, E. Upfal, *An introduction to the Fractional Calculus and Fractional Differential Equations* , Wiley, New York, 1993.
10. M. Jleli, B. Samet, Existence of positive solutions to an arbitrary order fractional differential equation via a mixed monotone operator method, *Nonlinear Anal. Model. Control*, **20**(3):367–376, 2015, https://doi.org/10.15388/NA.2015.3.4.
11. K.S. Jong, Existence and uniqueness of positive solutions of a kind of multi-point boundary value problems for nonlinear fractional differential equations with p-Laplacian operator, *Mediterr. J. Math.*, **15**(3):129, 2018, https://doi.org/10.1007/s00009-018-1155-3.
12. K.S. Jong, H.C. Choi, Y.H. Ri, Existence of positive solutions of a class of multi-point boundary value problems for p-Laplacian fractional differential equations with singular source terms, *Commun. Nonlinear Sci. Numer. Simul.*, **72**:272–281, 2019, https://doi.org/10.1016/j.cnsns.2018.12.021.
13. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Math. Stud., Vol. 204, Elsevier, Amsterdam, 2006.
14. R.W. Leggett, L.R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.*, **28**(4):673–505, 1979, https://doi.org/10.1016/S0898-1221(01)00156-0.
15. D. Ma, Positive solutions of multi-point boundary value problem of fractional differential equation, *Arab. J. Math. Sci.*, **21**(2):225–236, 2015, https://doi.org/10.1016/j.ajmsc.2014.11.001.
16. I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.
17. Y. Wang, L. Liu, X. Zhang, Y. Wu, Positive solutions of an abstract fractional semipositone differential system model for bioprocesses of HIV infection, *Appl. Math. Comput.*, **258**(C): 312–324, 2015, https://doi.org/10.1016/j.amc.2015.01.080.
18. C. Zhai, L. Wang, Some existence,uniqueness results on positive solutions for a fractional differential equation with infinite-point boundary conditions, *Nonlinear Anal. Model. Control*, **22**(4):566–577, 2017, https://doi.org/10.15388/NA.2017.4.10.
19. W. Zhang, W. Liu, Existence of solutions for fractional differential equations with infinite point boundary conditions at resonance, *Bound. Value Probl.*, **2018**(1):36, 2018, https://doi.org/10.1186/s13661-018-0954-6.

20. X. Zhang, Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions, *Appl. Math. Lett.*, **39**:22–27, 2015, https://doi.org/10.1016/j.aml.2014.08.008.
21. X. Zhang, L. Liu, Y. Wu, B. Wiwatanapataphee, The spectral analysis for a singular fractional differential equation with a signed measure, *Appl. Math. Comput.*, **257**(C):252–263, 2015, https://doi.org/10.1016/j.amc.2014.12.068.
22. X. Zhang, Z. Shao, Q. Zhong, Positive solutions for semipositone $(k, n-k)$ conjugate boundary value roblems with singularities on space variables, *Appl. Math. Lett.*, **217**(16):50–57, 2017, https://doi.org/10.1016/j.aml.2017.04.007.
23. X. Zhang, Q. Zhong, Uniqueness of solution for higher-order fractional differential equations with conjugate type integral conditions, *Fract. Calc. Appl. Anal.*, **20**(6):1471–1484, 2017, https://doi.org/10.1515/fca-2017-0077.
24. X. Zhang, Q. Zhong, Triple positive solutions for nonlocal fractional differential equations with singularities both on time and space variables, *Appl. Math. Lett.*, **80**:12–19, 2018, https://doi.org/10.1016/j.aml.2017.12.022.