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Relative controllability of multiagent systems with pairwise different delays in states*

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Abstract. In this manuscript, relative controllability of leader–follower multiagent systems with pairwise different delays in states and fixed interaction topology is considered. The interaction topology of the group of agents is modeled by a directed graph. The agents with unidirectional information flows are selected as leaders, and the others are followers. Dynamics of each follower obeys a generic time-invariant delay differential equation, and the delays of agents, which satisfy a specified condition, are different one another because of the degeneration or burn-in of sensors. With a neighbor-based protocol steering, the dynamics of followers become a compact form with multiple delays. Solution of the multidelayed system without pairwise matrices permutation is obtained by improving the method in the references, and relative controllability is established via Gramian criterion. Further rank criterion of a single delay system is dealt with. Simulation illustrates the theoretical deduction.

Keywords: multiagent systems, relative controllability, multiple time delays, solution.

1 Introduction

The cluster behaviors of multiagent systems are hot topics because of the wide applications of them, such as unmanned air vehicles, satellite formation, underwater robot, etc. Cooperative control of distributed multiagent systems is concerning with the control and operate capabilities with limited processing abilities, locally sensed information, and limited intercomponent communications achieving a collective goal [29]. Consensus of multiagent systems, which relies on a neighbor-based protocol to achieve a common interesting objective [9, 14, 31, 38, 40], is a typical instance of cooperative control. The

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factors like time delays [8, 28, 36, 37] and switching topology [6, 25], widely existing in the application of formation control [3, 39], flocking [1], and others, are considered while dealing with the consensus of multiagent systems.

An inevitable problem of multiagent systems is the controllability, which determines whether we can control and operate the multiagent systems by assigning suitable leaders in the group of agents. Tanner [33] derives the controllability criterion of multiagent systems with a leader and reveals the relation between the communication topology and the controllability. Liu and Chu [16] present the controllability of multiagent systems with switching topology and point out the relationship between the controllability and connectivity. Ji and Wang [12] generalize the control problems into the system with time delays in state and switching topology. Tian et al. [34] deal with the controllability of multiagent systems with periodically switching topologies and switching leaders who reveal that the switching-leader controllability is equivalent to multiple-leaders controllability. Other literature is paying attention to the reflection of graph-theoretic notions on the properties of multiagent systems (see [27] etc.).

Structural controllability firstly defined and researched by Lin [15] is introduced into the multiagent systems by Liu [20] who proves that the multiagent systems with switching topology is structurally controllable if and only if the union graph of interaction topology is connected. More literatures around the controllability of multiagent systems it is suggested referring to [10, 11, 17–19, 21, 32, 35].

Time delay is ubiquitous in intelligent network, digital communication, unmanned aerial vehicle, etc. (see more in [5, 8, 12, 19]). Comparing with the classical controllability, it is more appropriate to consider the relative controllability for time-delay systems because the latter can exactly describe the influence of the delay on the controllability (see more in [13]). We call the leader–follower multiagent systems relatively controllable if, for an arbitrary initial function on the delayed interval, there exist piecewise continuous control functions, which adjust the leaders' trajectories such that the states of the followers can be steered to any terminal ones in a finite time.

For relative controllability, there is abundant literature (see in [2, 13, 26]). Khusainov [13] presents a solution of the delayed system by constructing the delayed exponential matrix and establishes the rank criterion of relative controllability. Pospíšil [26] investigates the relative controllability of linear delayed neutral differential system by using the Legendre polynomials.

Relative controllability of multiagent systems with two delays in state is considered in [30] in which Gramian and rank criteria are established, respectively. With reference to [30], this paper considers the relative controllability of multiagent systems with pairwise different delays in states and fixed communication topology. Some agents with unidirectional information flows are selected as leaders, which act as external steering inputs. With a neighbor-based protocol steering, the multiagent systems are transformed into a system with multiple delays. Further, the solution of this system without pairwise matrices permutation is obtained by improving the method in [22, 24]. Based on this, Gramian and rank criteria are established, respectively. An example is dealt with to illustrate the theorem deduction. The contribution of this paper lies in establishing a framework of judging the controllability of multiagent systems with multiple delays.

One of the difficulties lies in constructing the solution of the multidelayed system without matrices pairwise permutations.

This paper is organized as follows. In Section 2, we present some basic knowledge of graph theory. In Section 3, we formulate the problems and explore the solution of multiagent systems. Controllability is tackled in Section 4, and simulation is shown in Section 5, respectively.

2 Preliminaries

Denote by $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{G})$ a weighted digraph of N nodes, $\mathcal{V}$ the set of nodes v_i with $i = 1, \dots, N$, $\mathcal{E}$ the set of the directed edges with $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and $\mathcal{G}$ a weighted adjacency matrix. A directed edge $\varepsilon_{ij} \in \mathcal{E}$ is an ordered pair of nodes (v_i, v_j) with v_i and v_j called parent and child nodes, respectively. In a digraph $\mathfrak{G}$, $\varepsilon_{ij} \in \mathcal{E}$ means that node v_j can obtain information from v_i but might not inversely. The elements of the adjacency matrix $\mathcal{G} = [a_{ij}]$ are defined by $a_{ij} > 0$ if $\varepsilon_{ij} \in \mathcal{E}$ or zero otherwise. The set of neighbors of node v_i is denoted by $\mathcal{N}_i = \{v_j \in \mathcal{V}\colon \varepsilon_{ij} \in \mathcal{E}\}$. The Laplacian matrix $\mathcal{L} \in R^{N \times N}$ is defined by

$$\mathcal{L} = (\mathcal{L}_{ij})_{N \times N}, \quad \mathcal{L}_{ij} = \begin{cases} \sum_{j \neq i}^{N} a_{ij}, & i = j, \\ -a_{ij}, & i \neq j. \end{cases}$$

More properties of the Laplacian matrix $\mathcal{L}$ can be found in [7].

In what follows, we denote by N and L the positive integers, $\varTheta$, I, and θ the corresponding dimensional zero matrix, unit matrix, and zero vector, respectively, and $\mathbb{R}^n$ the n-dimensional Euclidean space.

3 Formulation

In some application the dynamics of agents may exhibit different time delays because the degeneration or burn-in of sensors. With reference to [30], in what follows, we will continuous to consider the relative controllability of a group of agents with pairwise different delays in states and directly fixed interaction topology.

Suppose that the multiagent systems are consisting of $N + L$ agents, and interaction topology of the systems is modeled by a weighted digraph $\mathfrak{G}$, each node of the graph representing an agent and the set of nodes represented by $\mathcal{V} = \{v_1, \dots, v_N, \dots, v_{N+L}\}$. Further, suppose that $v_{N+1}, \dots, v_{N+L}$, information flows of which are unidirectional, are selected as leaders. The rest labeled by $v_1, \dots, v_N$ are followers. Dynamics of the followers obey the following generic time-invariant delay differential equations:

$$\dot{x}_i(t) = A_i x_i(t) + B_i x_i(t - \tau_i) + C_i u_i(t), \quad i = 1, \dots, N, \tag{1}$$

where $x_i \in \mathbb{R}^n$, A_i, B_i, and C_i are the parameter matrices of appropriate dimensions, u_i is the steering input, $i = 1, \dots, N$, and τ_j is corresponding delay of v_j, which satisfies $\tau_j < \tau_{j+1} < \tau_j + \tau_1$, $j = 1, \dots, N - 1$. Whereas dynamics of the leaders are assumed in any form as long as they are controllable. Interactions among agents are realized through the following relative protocol:

$$u_i(t) = K \sum_{v_k \in \mathcal{N}_i} w_{ik}\big(x_i(t) - x_k(t)\big) + P \sum_{k=1}^{L} a_{ik}\delta_{ik}\big(y_k(t) - x_i(t)\big), \quad i = 1, \dots, N, \tag{2}$$

where K and P are the gain matrices of appropriate dimensions, $\mathcal{N}_i$ is the neighbour of v_i, w_{ik} is the coupled weight of followers and their neighbors, $y_k(t) = x_{N+k}(t)$, $k = 1, \dots, L$, is the output of leaders, which acts as exogenous control input of followers, a_{ik} is the coupled weight of leader and follower, and δ_{ik} is equal to one if the leader v_{N+k} has information flow towards the follower v_i directly or zero else, $k = 1, \dots, L$. Under (2), (1) becomes

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}_1 x(t-\tau_1) + \tilde{B}_2 x(t-\tau_2) + \cdots + \tilde{B}_N x(t-\tau_N) + \tilde{C}u(t), \tag{3}$$

where $x(t) = (x_1^{\mathrm{T}}(t), \dots, x_N^{\mathrm{T}}(t))^{\mathrm{T}}$, $u(t) = (y_1^{\mathrm{T}}(t), \dots, y_L^{\mathrm{T}}(t))^{\mathrm{T}}$, $\tilde{A} = \tilde{A}_1 + \tilde{A}_2 - \tilde{A}_3$ with

$$\tilde{A}_1 = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_N \end{bmatrix}, \qquad \tilde{A}_2 = \begin{bmatrix} l_{11}C_1K & \cdots & l_{1N}C_1K \\ \vdots & \ddots & \vdots \\ l_{N1}C_NK & \cdots & l_{NN}C_NK \end{bmatrix}$$

and

$$\tilde{A}_3 = \begin{bmatrix} \sum_{k=1}^{L} a_{1k}\delta_{1k}C_1P & & \\ & \ddots & \\ & & \sum_{k=1}^{L} a_{Nk}\delta_{Nk}C_NP \end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix} a_{11}\delta_{11}C_1P & \cdots & a_{1L}\delta_{1L}C_1P \\ \vdots & \ddots & \vdots \\ a_{N1}\delta_{N1}C_NP & \cdots & a_{NL}\delta_{NL}C_NP \end{bmatrix},$$

and $\tilde{B}_i$ is a block matrix with the ith block of main diagonal being B_i or zero else.

Remark 1. System (3) contains multiple delays and does not enjoy the matrices pairwise permutable. It is a hot topic around the well-posedness and controllability of the delay differential equations (see in [13, 22, 23]). Khusainov et al. [13] present the explicit solution of the linear delay system by constructing a delayed matrix exponential function and establish a criterion for the relative controllability of the system with pure delay. Mahmudov [22] presents a delayed perturbation of Mittag-Leffler-type matrix function and solves the linear nonhomogeneous fractional delay system. Medved' et al. [23] generate the results of Khusainov and Shuklin and establish a multidelayed exponential function to solve the multidelayed system with pairwise matrices permutation. With reference to [30], we will construct the solution of (3) without matrices pairwise permutations by improving the methods in [22, 23].

3.1 Solution

With reference to [22, 23], firstly, we introduce the following matrix sequence:

$$\begin{aligned} Q_0(s) &= Q_k(-\tau_1) = \Theta, \\ Q_1(0) &= I, \\ Q_{k+1}(s) &= \tilde{A}Q_k(s) + \tilde{B}_1 Q_k(s-\tau_1), \end{aligned} \tag{4}$$

where $k = 0, 1, \dots$, $s = j\tau_1$, $j = 0, 1, \dots$. Further, introduce the following matrix function:

$$X_1(t) = \begin{cases} \Theta, & -\infty < t < -\tau_1, \\ \Theta, & -\tau_1 \leqslant t < 0, \\ \sum_{i=0}^{\infty} \sum_{j=0}^{p-1} Q_{i+1}(j\tau_1) \frac{(t-j\tau_1)^i}{\Gamma(i+1)}, & (p-1)\tau_1 \leqslant t < p\tau_1, \end{cases} \tag{5}$$

where p is a positive integer, and $\Gamma(\cdot)$ is the gamma function defined by

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} \mathrm{e}^{-t}\,\mathrm{d}t \quad (x > 0).$$

If $0 \leqslant t \leqslant \tau_1$, we have

$$X_1(t) = \sum_{i=0}^{\infty} \frac{\tilde{A}^i t^i}{\Gamma(i+1)} =: \mathrm{e}^{\tilde{A}t}.$$

With reference to [23], for $i = 2, \dots, N$, construct the following function:

$$\Phi_{\tau_i}(t) = \begin{cases} \Theta, & -\infty < t < -\tau_i, \\ X_{i-1}(t+\tau_i), & -\tau_i \leqslant t < 0, \\ X_{i-1}(t+\tau_i) + \int_0^t X_{i-1}(t-s_1)\tilde{B}_i X_{i-1}(s_1)\,\mathrm{d}s_1 \\ \quad + \int_{\tau_i}^t \int_{\tau_i}^{s_1} X_{i-1}(t-s_1)\tilde{B}_i X_{i-1}(s_1-s_2)\tilde{B}_i X_{i-1}(s_2-\tau_i)\,\mathrm{d}s_2\,\mathrm{d}s_1 \\ \quad + \dots + \int_{(k-1)\tau_i}^t \int_{(k-1)\tau_i}^{s_1} \cdots \int_{(k-1)\tau_i}^{s_{k-1}} X_{i-1}(t-s_1)\tilde{B}_i X_{i-1}(s_1-s_2) \\ \quad \times \tilde{B}_i \cdots X_{i-1}(s_{k-1}-s_k)\tilde{B}_i X_{i-1}(s_k-(k-1)\tau_i)\,\mathrm{d}s_k \cdots \mathrm{d}s_2\,\mathrm{d}s_1, \\ \qquad (k-1)\tau_i \leqslant t < k\tau_i, \end{cases} \tag{6}$$

where $X_{j-1}(t) = \Phi_{\tau_{j-1}}(t-\tau_{j-1})$, $j = 3, \dots, N+1$, and $X_1(t)$ is equal to (5). For $X_j(t)$ with $j = N$, we have the following lemma hold.

Lemma 1. *$X_N(t)$ is a solution of the following matrix equation*:

$$\begin{aligned} \dot{X}_N(t) &= \tilde{A}X_N(t) + \tilde{B}_1 X_N(t-\tau_1) + \tilde{B}_2 X_N(t-\tau_2) \\ &\quad + \dots + \tilde{B}_N X_N(t-\tau_N) \end{aligned}$$

with

$$X_N(0) = I, X_N(t) = \Theta, \quad t \in [-\tau_N, 0).$$

Proof. With reference to [22,30], $X_i(t) = \varPhi_{\tau_i}(t-\tau_i)$, $i = 1, 2$, is the respective solution of

$$\begin{aligned}
\dot{X}_1(t) &= \tilde{A}X_1(t) + \tilde{B}_1 X_1(t-\tau_1),\\
\dot{X}_2(t) &= \tilde{A}X_2(t) + \tilde{B}_1 X_2(t-\tau_1) + \tilde{B}_2 X_2(t-\tau_2),
\end{aligned}$$

which satisfies $X_i(0) = I$, $X_i(t) = \varTheta$, $i = 1, 2$, $t \in [-\tau_2, 0)$.

Suppose that $X_i(t) = \varPhi_{\tau_i}(t-\tau_i)$, $2 < i \leqslant N-1$, is a solution of

$$\dot{X}_i(t) = \tilde{A}X_i(t) + \tilde{B}_1 X_i(t-\tau_1) + \tilde{B}_2 X_i(t-\tau_2) + \cdots + \tilde{B}_i X_i(t-\tau_i)$$

with $X_i(0) = I$, $X_i(t) = \varTheta$, $t \in [-\tau_i, 0)$. Based on (6), we know that for $t < 0$, $t - \tau_{i+1} < -\tau_{i+1}$, $X_{i+1}(t) = \varPhi_{\tau_{i+1}}(t-\tau_{i+1}) = \varTheta$, and $X_{i+1}(0) = \varPhi_{\tau_{i+1}}(-\tau_{i+1}) = X_i(0) = I$. For $0 < t < \tau_{i+1}$, we have

$$X_{i+1}(t) = \varPhi_{\tau_{i+1}}(t-\tau_{i+1}) = X_i(t).$$

Thus, it holds that

$$\begin{aligned}
\dot{X}_{i+1}(t) = \dot{X}_i(t) &= \tilde{A}X_i(t) + \tilde{B}_1 X_i(t-\tau_1) + \tilde{B}_2 X_i(t-\tau_2) + \cdots\\
&\quad + \tilde{B}_i X_i(t-\tau_i)\\
&= \tilde{A}X_{i+1}(t) + \tilde{B}_1 X_{i+1}(t-\tau_1) + \tilde{B}_2 X_{i+1}(t-\tau_2) + \cdots\\
&\quad + \tilde{B}_i X_{i+1}(t-\tau_i).
\end{aligned}$$

Further, for $-\tau_{i+1} < t - \tau_{i+1} < 0$, we have

$$X_{i+1}(t-\tau_{i+1}) = \varPhi_{\tau_{i+1}}(t-2\tau_{i+1}) = \varTheta.$$

Thus, it holds for $0 < t < \tau_{i+1}$.

For $k\tau_{i+1} \leqslant t < (k+1)\tau_{i+1}$, take the derivative of X_{i+1} and arrange it to get

$$\dot{X}_{i+1}(t) = \tilde{A}X_{i+1}(t) + \tilde{B}_1\hat{X}_1(t) + \cdots + \tilde{B}_i\hat{X}_i(t) + \tilde{B}_{i+1}\hat{X}_{i+1}(t),$$

where

$$\begin{aligned}
\hat{X}_j(t) &= X_i(t-\tau_j) + \int_0^{t-\tau_{i+1}} X_i(t-\tau_{i+1}-s_1-\tau_j)\tilde{B}_{i+1}X_i(s_1)\,\mathrm{d}s_1 + \cdots\\
&\quad + \int_{(k-1)\tau_{i+1}}^{t-\tau_{i+1}} \int_{(k-1)\tau_{i+1}}^{s_1} \cdots \int_{(k-1)\tau_{i+1}}^{s_{k-1}} X_i(t-\tau_{i+1}-s_1-\tau_j)\\
&\quad \times \tilde{B}_{i+1}X_i(s_1-s_2)\cdots\tilde{B}_{i+1}X_i(s_{k-1}-s_k)\\
&\quad \times \tilde{B}_{i+1}X_i\bigl(s_k-(k-1)\tau_{i+1}\bigr)\,\mathrm{d}s_k\cdots\mathrm{d}s_2\,\mathrm{d}s_1, \quad j = 1, \ldots, i,
\end{aligned}$$

and

$$\begin{aligned}\hat{X}_{i+1}(t) &= X_i(t-\tau_{i+1}) + \int\limits_{\tau_{i+1}}^{t-\tau_{i+1}} X_i(t-\tau_{i+1}-s_1)\tilde{B}_{i+1}X_i(s_1-\tau_{i+1})\,\mathrm{d}s_1 + \cdots \\ &\quad + \int\limits_{(k-1)\tau_{i+1}}^{t-\tau_{i+1}} \int\limits_{(k-1)\tau_{i+1}}^{s_1} \cdots \int\limits_{(k-1)\tau_{i+1}}^{s_{k-2}} X_i(t-\tau_{i+1}-s_1) \\ &\quad \times \tilde{B}_{i+1}X_i(s_1-s_2)\cdots\tilde{B}_{i+1}X_i(s_{k-2}-s_{k-1})\tilde{B}_{i+1} \\ &\quad \times X_i\bigl(s_{k-1}-(k-1)\tau_{i+1}\bigr)\,\mathrm{d}s_{k-1}\cdots\mathrm{d}s_2\,\mathrm{d}s_1.\end{aligned}$$

From $j\tau_{i+1} \leqslant s_1 \leqslant t-\tau_{i+1}$ we have $-\tau_r \leqslant t-\tau_{i+1}-s_1-\tau_r \leqslant t-\tau_{i+1}-j\tau_{i+1}-\tau_r$. From $k\tau_{i+1} \leqslant t < (k+1)\tau_{i+1}$ we have $(k-j-1)\tau_{i+1}-\tau_r \leqslant t-\tau_{i+1}-j\tau_{i+1}-\tau_r < (k-j)\tau_{i+1}-\tau_r$. Thus, we obtain that $-\tau_r \leqslant t-\tau_{i+1}-s_1-\tau_r \leqslant t-\tau_{i+1}-j\tau_{i+1}-\tau_r < (k-j)\tau_{i+1}-\tau_r$, $j=0,1,\dots,k-1$, $r=1,2,\dots,i$. Further, we have for $-\tau_r \leqslant t-\tau_{i+1}-s_1-\tau_r < 0$, $t-\tau_{i+1}-\tau_r < s_1 \leqslant t-\tau_{i+1}$, $X_i(t-\tau_{i+1}-s_1-\tau_r) = \Theta$, $r=1,2,\dots,i$. For $0 \leqslant t-\tau_{i+1}-s_1-\tau_r \leqslant t-\tau_{i+1}-j\tau_{i+1}-\tau_r$, $j\tau_{i+1} \leqslant s_1 \leqslant t-\tau_{i+1}-\tau_r$, $X_i(t-\tau_{i+1}-s_1-\tau_r) \neq \Theta$, $j=0,1,\dots,k-1$, $r=1,2,\dots,i+1$. Thus, we simplify $\hat{X}_j(t)$ as

$$\begin{aligned}\hat{X}_j(t) &= X_i(t-\tau_j) + \int\limits_{0}^{t-\tau_{i+1}-\tau_j} X_i(t-\tau_{i+1}-s_1-\tau_j)\tilde{B}_{i+1}X_i(s_1)\,\mathrm{d}s_1 + \cdots \\ &\quad + \int\limits_{(k-1)\tau_{i+1}}^{t-\tau_{i+1}-\tau_j} \int\limits_{(k-1)\tau_{i+1}}^{s_1} \cdots \int\limits_{(k-1)\tau_{i+1}}^{s_{k-1}} X_i(t-\tau_{i+1}-s_1-\tau_j)\tilde{B}_{i+1}X_i(s_1-s_2)\cdots \\ &\quad \times \tilde{B}_{i+1}X_i(s_{k-1}-s_k)\tilde{B}_{i+1}X_i\bigl(s_k-(k-1)\tau_{i+1}\bigr)\,\mathrm{d}s_k\cdots\mathrm{d}s_2\,\mathrm{d}s_1 \\ &= X_{i+1}(t-\tau_j), \quad j=1,\dots,i.\end{aligned}$$

Further, making a change of variable $s_j' = s_j - \tau_{i+1}$, $j=1,\dots,k-1$, we have

$$\begin{aligned}\hat{X}_{i+1}(t) &= X_i(t-\tau_{i+1}) + \int\limits_{0}^{t-2\tau_{i+1}} X_i(t-2\tau_{i+1}-s_1)\tilde{B}_{i+1}X_i(s_1)\,\mathrm{d}s_1 + \cdots \\ &\quad + \int\limits_{(k-2)\tau_{i+1}}^{t-2\tau_{i+1}} \int\limits_{(k-2)\tau_{i+1}}^{s_1} \cdots \int\limits_{(k-2)\tau_{i+1}}^{s_{k-2}} X_i(t-2\tau_{i+1}-s_1)\tilde{B}_{i+1}X_i(s_1-s_2)\cdots \\ &\quad \times \tilde{B}_{i+1}X_i(s_{k-2}-s_{k-1})\tilde{B}_{i+1}X_i\bigl(s_{k-1}-(k-2)\tau_{i+1}\bigr)\,\mathrm{d}s_{k-1}\cdots\mathrm{d}s_2\,\mathrm{d}s_1 \\ &= X_{i+1}(t-\tau_{i+1}).\end{aligned}$$

Thus, we have

$$\begin{aligned}\dot{X}_{i+1}(t) = \tilde{A}X_{i+1}(t) + \tilde{B}_1 X_{i+1}(t-\tau_1) + \cdots + \tilde{B}_i X_{i+1}(t-\tau_i) \\ + \tilde{B}_{i+1} X_{i+1}(t-\tau_{i+1}),\end{aligned}$$

which implies the assumption is held and the proof is completed. □

Lemma 2. *The homogeneous problem*

$$\begin{aligned}\dot{x}(t) &= \tilde{A}x(t) + \tilde{B}_1 x(t-\tau_1) + \tilde{B}_2 x(t-\tau_2) + \cdots \\ &\quad + \tilde{B}_N x(t-\tau_N), \qquad (7)\\ x(t) &= \varphi(t), \quad t \in [-\tau_N, 0], \qquad (8)\end{aligned}$$

has a solution of the form

$$\begin{aligned}x(t) &= X_N\big(t+\tau_1+\delta(t)\big)\varphi\big(-\tau_1-\delta(t)\big) \\ &\quad + \int_{-\tau_1}^{0} X_N\big(t-s+\delta(t)\big)\big(\varphi'\big(s-\delta(t)\big) - \tilde{A}\varphi\big(s-\delta(t)\big)\big)\,\mathrm{d}s, \qquad (9)\end{aligned}$$

where

$$\delta(t) = \begin{cases} 0, & t \in [-\tau_1, \infty), \\ \tau_1, & t \in [-\tau_2, -\tau_1), \\ \dots, & \\ \tau_{N-1}, & t \in [-\tau_N, -\tau_{N-1}), \\ 0, & t \in (-\infty, -\tau_N). \end{cases}$$

Proof. From Lemma 1 it is obtained that (9) solves (7). Next, we verify that (9) satisfies (8). For $-\tau_{i+1} \leqslant t < -\tau_i$, $i = 0, 1, 2, \dots, N-1$, $\delta(t) = \tau_i$. Thus, $-\tau_{i+1}+\tau_1+\tau_i \leqslant t+\tau_1+\delta(t) < \tau_1$. From $\tau_i \leqslant \tau_{i+1} \leqslant \tau_1+\tau_i$ we further arrive at $0 \leqslant -\tau_{i+1}+\tau_1+\tau_i \leqslant t+\tau_1+\delta(t) < \tau_1$ and $-\tau_j \leqslant -\tau_{i+1}+\tau_1+\tau_i-\tau_j \leqslant t+\tau_1+\delta(t)-\tau_j < \tau_1-\tau_j$, $j = 2, 3, \dots, N$. Thus

$$\begin{aligned}X_N\big(t+\tau_1+\delta(t)\big) &= \Phi_{\tau_N}(t+\tau_1+\tau_i-\tau_N) = X_{N-1}\big(t+\tau_1+\delta(t)\big) \\ &= \cdots \\ &= X_2(t+\tau_1+\tau_i) = \Phi_{\tau_2}(t+\tau_1+\tau_i-\tau_2) \\ &= X_1(t+\tau_1+\tau_i) = \mathrm{e}^{\tilde{A}(t+\tau_1+\tau_i)}.\end{aligned}$$

Besides, from $-\tau_1 \leqslant s \leqslant 0$, $t+\tau_i \leqslant t-s+\delta(t) \leqslant t+\tau_1+\tau_i$. Again, from $-\tau_{i+1}+\tau_i \leqslant t+\tau_i < 0$ we have $-\tau_{i+1}+\tau_i \leqslant t+\tau_i \leqslant t-s+\delta(t) \leqslant t+\tau_1+\tau_i < \tau_i$. For $t+\tau_i \leqslant t-s+\delta(t) < 0$, $t+\tau_i \leqslant s \leqslant 0$, $X_N(t-s+\delta(t)) = \Theta$. For $-\tau_1 \leqslant s \leqslant t+\tau_i$, $X_N(t-s+\delta(t)) = \mathrm{e}^{\tilde{A}(t-s+\tau_i)}$.

Thus, we obtain that

$$\begin{aligned}
x(t) &= \mathrm{e}^{\tilde{A}(t+\tau_1+\tau_i)}\varphi(-\tau_1-\tau_i) \\
&\quad + \int_{-\tau_1}^{t+\tau_i} \mathrm{e}^{\tilde{A}(t-s+\tau_i)}\varphi'(s-\tau_i)\,\mathrm{d}s - \int_{\tau_1}^{t+\tau_i} \mathrm{e}^{\tilde{A}(t-s+\tau_i)}\tilde{A}\varphi(s-\tau_i)\,\mathrm{d}s \\
&= \mathrm{e}^{\tilde{A}(t+\tau_1+\tau_i)}\varphi(-\tau_1-\tau_i) + \varphi(t) - \mathrm{e}^{\tilde{A}(t+\tau_1+\tau_i)}\varphi(-\tau_1-\tau_i) \\
&= \varphi(t).
\end{aligned}$$

The proof is completed. □

Lemma 3. *System* (3) *with $x(t)=\theta$, $t\in[-\tau_N,0]$, has a solution of the following form*:

$$\tilde{x}(t) = \int_0^t X_N(t-s)\tilde{C}u(s)\,\mathrm{d}s.$$

Proof. Suppose (3) with $x(t)=\theta$, $t\in[-\tau_N,0]$, has a solution of the form

$$\tilde{x}(t) = \int_0^t X_N(t-s)g(s)\,\mathrm{d}s. \tag{10}$$

Taking the derivative of (10) with respect to t and following from Lemma 1 to yield

$$\begin{aligned}
\dot{\tilde{x}}(t) &= g(t) + \tilde{A}\int_0^t X_N(t-s)g(s)\,\mathrm{d}s + \tilde{B}_1\int_0^t X_N(t-s-\tau_1)g(s)\,\mathrm{d}s \\
&\quad + \cdots + \tilde{B}_N\int_0^t X_N(t-s-\tau_N)g(s)\,\mathrm{d}s.
\end{aligned}$$

For $0\leqslant s\leqslant t$, we have $-\tau_i\leqslant t-s-\tau_i\leqslant t-\tau_i$, $i=1,\dots,N$. Thus, for $s\in(t-\tau_i,\,t]$, $\tilde{x}(t-s-\tau_i)=\theta$. For $s\in[0,t-\tau_i]$, $\tilde{x}(t-s-\tau_i)\neq\theta$, $i=1,\dots,N$. Further, we obtain

$$\begin{aligned}
\dot{\tilde{x}}(t) &= g(t) + \tilde{A}\int_0^t X_N(t-s)g(s)\,\mathrm{d}s + \tilde{B}_1\int_0^{t-\tau_1} X_N(t-s-\tau_1)g(s)\,\mathrm{d}s \\
&\quad + \cdots + \tilde{B}_N\int_0^{t-\tau_N} X_N(t-s-\tau_N)g(s)\,\mathrm{d}s \\
&= g(t) + \tilde{A}\tilde{x}(t) + \tilde{B}_1\tilde{x}(t-\tau_1) + \cdots + \tilde{B}_N\tilde{x}(t-\tau_N).
\end{aligned}$$

Comparing it with (3), we obtain $g(t)=\tilde{C}u(t)$. This completes the proof. □

Lemma 4. *System* (3) *with initial data* (8) *has a solution of the form*

$$x(t) = X_N\big(t+\tau_1+\delta(t)\big)\varphi\big(-\tau_1-\delta(t)\big) + \int_{-\tau_1}^{0} X_N\big(t-s+\delta(t)\big)\big(\varphi'\big(s-\delta(t)\big) - \tilde{A}\varphi\big(s-\delta(t)\big)\big)\,\mathrm{d}s + \int_{0}^{t} X_N(t-s)\tilde{C}u(s)\,\mathrm{d}s. \tag{11}$$

Proof. It follows from Lemmas 2–3 that solution of (3) with initial data (8) is (11). This completes the proof. □

4 Controllability

In this section, relative controllability of multiagent systems will be considered. Firstly, we present the definition of it.

Definition 1. System (3) is called relatively controllable if, for any initial vector function $\varphi(t)$, $t \in [-\tau_N, 0]$, and final state x_1, there exists a terminal time $t_1 > 0$ and a measurable function $u^*(t)$ such that system (3) has a solution $x^*(t)$ on $[-\tau_N, t_1]$, which satisfies $x^*(t_1) = x_1$ and $x^*(t) \equiv \varphi(t)$, $t \in [-\tau_N, 0]$.

4.1 Gramian criterion

Given some $t_1 > 0$, construct the following matrix:

$$G(0,t_1) = \int_{0}^{t_1} X_N(t_1-s)\tilde{C}\tilde{C}^{\mathrm{T}}X_N^{\mathrm{T}}(t_1-s)\,\mathrm{d}s. \tag{12}$$

Denote

$$\eta = X_N(t_1+\tau_1)\varphi(-\tau_1) + \int_{-\tau_1}^{0} X_N(t_1-s)\big(\varphi'(s) - \tilde{A}\varphi(s)\big)\,\mathrm{d}s. \tag{13}$$

For the controllability of (3), we have the following theorem hold.

Theorem 1. *System* (3) *is relatively controllable if and only if there exists a* $t_1 > 0$ *such that* (12) *is nonsingular.*

Proof. Sufficiency. Suppose that (12) is nonsingular for some $t_1 > 0$. For any terminal state x_1 and any initial function $x(t) = \varphi(t)$, $t \in [-\tau_N, 0]$, construct the following control input:

$$u^*(s) = \tilde{C}^{\mathrm{T}}X_N^{\mathrm{T}}(t_1-s)G^{-1}(0,t_1)(x_1-\eta).$$

From Lemmas 2–4 the solution of (3) always has the form of (11), which automatically satisfies the initial condition, thus we have

$$\begin{aligned} x^*(t_1) &= \eta + \int_0^{t_1} X_N(t_1 - s)\tilde{C}\tilde{C}^{\mathrm{T}} X_N^{\mathrm{T}}(t_1 - s)\,\mathrm{d}s\, G^{-1}(0,t_1)(x_1 - \eta) \\ &= \eta + G(0,t_1)G^{-1}(0,t_1)(x_1 - \eta) = x_1, \end{aligned}$$

which implies that system (3) is relatively controllable.

Necessity. Suppose that system (3) is relatively controllable, but (12) is singular. There exists a nonzero vector $\tilde{x}$ such that $\tilde{x}^{\mathrm{T}} G(0,t_1)\tilde{x} = 0$. Thus, we have that

$$\begin{aligned} \int_0^{t_1} \left\|\tilde{x}^{\mathrm{T}} X_N(t_1 - s)\tilde{C}\right\|^2 \mathrm{d}s &= \int_0^{t_1} \tilde{x}^{\mathrm{T}} X_N(t_1 - s)\tilde{C}\tilde{C}^{\mathrm{T}} X_N^{\mathrm{T}}(t_1 - s)\tilde{x}\,\mathrm{d}s \\ &= \tilde{x}^{\mathrm{T}} G(0,t_1)\tilde{x} = 0. \end{aligned}$$

Further, we arrive at

$$\tilde{x}^{\mathrm{T}} X_N(t_1 - s)\tilde{C} = \theta, \quad s \in [0,t_1].$$

System (3) being relatively controllable, we know that for an arbitrary initial function $x(t) = \varphi(t)$, $t \in [-\tau_N, 0]$, and the given terminal states $\tilde{x}$ and θ, there exist measurable control functions such that

$$x^*(t_1) = \eta + \int_0^{t_1} X_N(t_1 - s)\tilde{C}u_1^*(s)\,\mathrm{d}s = \tilde{x},$$

$$x^*(t_1) = \eta + \int_0^{t_1} X_N(t_1 - s)\tilde{C}u_2^*(s)\,\mathrm{d}s = \theta.$$

Thus, we obtain

$$\int_0^{t_1} X_N(t_1 - s)\tilde{C}\big(u_1^*(s) - u_2^*(s)\big)\,\mathrm{d}s = \tilde{x}.$$

Further, it yields

$$\tilde{x}^{\mathrm{T}}\tilde{x} = \int_0^{t_1} \tilde{x}^{\mathrm{T}} X_N(t_1 - s)\tilde{C}\big(u_1^*(s) - u_2^*(s)\big)\,\mathrm{d}s = 0,$$

which implies that $\tilde{x} = \theta$. This contradicts with the assumption that $\tilde{x}$ is a nonzero vector. Thus, (12) is nonsingular. The proof is completed. $\square$

4.2 Rank criterion

Next, we consider the rank criterion of relative controllability for the system with single delay.

For $\tau_1 = \tau_j$, $j = 2, \dots, N$, system (3) degenerates into

$$\dot{x}(t) = \tilde{A}x(t) + \bar{B}x(t - \tau_1) + \tilde{C}u(t), \tag{14}$$

where $\bar{B}$ is the sum of $\tilde{B}_j$, $j = 1, \dots, N$. With reference to [22], solution of (14) with initial data (8) is degenerated into the form

$$\begin{aligned} x(t) = X_1(t + \tau_1)\varphi(-\tau_1) + \int_0^t X_1(t - s)\tilde{C}u(s)\,\mathrm{d}s \\ + \int_{-\tau_1}^0 X_1(t - s)\bigl(\varphi'(s) - \tilde{A}\varphi(s)\bigr)\,\mathrm{d}s, \end{aligned}$$

where $X_1(\cdot)$ is defined by (5) with the matrix sequence (4) replaced by

$$Q_{k+1}(s) = \tilde{A}Q_k(s) + \bar{B}Q_k(s - \tau_1). \tag{15}$$

In what follows, we will use $Y(\cdot)$ to replace $X_1(\cdot)$ for the simplicity of notation.

Lemma 5. *The derivative of $Y(\cdot)$ up to any kth order can be represented as*

$$Y^{(k)}(t) = \sum_{j=0}^{k} Q_{k+1}(j\tau_1)Y(t - j\tau_1). \tag{16}$$

Proof. It is trivial for $k = 1$. Suppose that (16) holds for any integer k. Then for $k + 1$, we have

$$\begin{aligned} Y^{(k+1)}(t) &= \sum_{j=0}^{k} Q_{k+1}(j\tau_1)\bigl(AY(t - j\tau_1) + \bar{B}Y(t - j\tau_1 - \tau_1)\bigr) \\ &= Q_{k+1}(0)\tilde{A}Y(t) + Q_{k+1}(k\tau_1)\bar{B}Y(t - k\tau_1 - \tau_1) \\ &\quad + \sum_{j=1}^{k} \bigl(Q_{k+1}(j\tau_1)\tilde{A} + Q_{k+1}(j\tau_1 - \tau_1)\bar{B}\bigr)Y(t - j\tau_1) \\ &= \sum_{j=0}^{k+1} Q_{k+2}(j\tau_1)Y(t - j\tau_1), \end{aligned}$$

which implies that (16) holds for any positive integer k, and the proof is completed. □

Next, we present the result of the rank criterion for system (14) without matrices pairwise permutation.

Theorem 2. *If* $\operatorname{rank}(\hat{Q}) = Nn$, *then system* (14) *is relatively controllable for some* t_1, *where*

$$\begin{aligned} \hat{Q} = \bigl[&Q_1(0)\tilde{C},\ Q_2(0)\tilde{C},\ \dots,\ Q_{Nn+1}(0)\tilde{C}, \\ &Q_2(\tau_1)\tilde{C},\ Q_3(\tau_1)\tilde{C},\ \dots,\ Q_{Nn+1}(\tau_1)\tilde{C},\ \dots, Q_{Nn+1}(Nn\tau_1)\tilde{C}\bigr]. \end{aligned} \tag{17}$$

Proof. Assume that $\operatorname{rank}(\hat{Q}) = Nn$, whereas system (14) is uncontrollable. Then from Theorem 1 we know there exists a nonzero vector $\bar{x}$ such that

$$\bar{x}^{\mathrm{T}} Y(t)\tilde{C} = \theta, \quad t \in [0, t_1]. \tag{18}$$

Taking the derivative of $Y(\cdot)$ in (18) up to any order and from Lemma 5 we have

$$\bar{x}^{\mathrm{T}} Y^{(k)}(t)\tilde{C} = \sum_{j=0}^{k} \bar{x}^{\mathrm{T}} Q_{k+1}(j\tau_1) Y(t - j\tau_1)\tilde{C} = \theta, \quad t \in [0, t_1]. \tag{19}$$

Taking $t = 0$ in (19), we have

$$\bar{x}^{\mathrm{T}} Q_{k+1}(0)\tilde{C} = \theta, k = 0, 1, 2, \dots . \tag{20}$$

Continuous take $t = (j-1)\tau_1$ and suppose that

$$\bar{x}^{\mathrm{T}} Q_{k+1}((j-1)\tau_1)\tilde{C} = \theta \tag{21}$$

holds, where $j = 1, 2, \dots, k$. For $t = j\tau_1$, we have

$$\begin{aligned}
&\bar{x}^{\mathrm{T}} Y^{(k)}(j\tau_1)\tilde{C} \\
&\quad= \sum_{i=0}^{\infty} \bar{x}^{\mathrm{T}} Q_{k+1}(0) Q_{i+1}(0)\tilde{C} \frac{(j\tau_1)^i}{\Gamma(i+1)} \\
&\qquad+ \sum_{i=0}^{\infty} \bar{x}^{\mathrm{T}} \bigl(Q_{k+1}(0) Q_{i+1}(\tau_1) + Q_{k+1}(\tau_1) Q_{i+1}(0)\bigr)\tilde{C} \frac{((j-1)\tau_1)^i}{\Gamma(i+1)} + \cdots \\
&\qquad+ \sum_{i=0}^{\infty} \bar{x}^{\mathrm{T}} \bigl(Q_{k+1}(0) Q_{i+1}\bigl((j-1)\tau_1\bigr) + Q_{k+1}(\tau_1) Q_{i+1}\bigl((j-2)\tau_1\bigr) + \cdots \\
&\qquad\qquad+ Q_{k+1}\bigl((j-1)\tau_1\bigr) Q_{i+1}(0)\bigr)\tilde{C} \frac{\tau_1^i}{\Gamma(i+1)} + \bar{x}^{\mathrm{T}} Q_{k+1}(j\tau_1)\tilde{C} \\
&\quad= \theta.
\end{aligned}$$

From the definition of the matrix sequence in (15) we know that $Q_{k+1}(j\tau_1)$ is nothing but a combination of $\tilde{A}$ and $\bar{B}$ in a stack with k positions, where j matrices $\bar{B}$ are inserted into j positions, and $k-j$ matrices $\tilde{A}$ are inserted into $k-j$ positions, total ways of which are $C_k^j = k!/(j!(k-j)!)$. Thus, for $Q_{k+i+1}(j\tau_1)$, we regard it as a combination of $\tilde{A}$ and $\bar{B}$ in a stack with $k+i$ positions: we separate the stack into two parts with the former part being k positions, and the latter one being i positions. The first way is that all the j matrices $\bar{B}$ are inserted into the latter i positions, and the $k-j$ matrices $\tilde{A}$ are inserted into the remained $k-j$ positions. The second way is that $j-1$ matrices $\bar{B}$ are inserted into the latter i positions, the remained one $\bar{B}$ is inserted into the former k positions, and the $k-j$ matrices $\tilde{A}$ are inserted into the $k-j$ positions. Following this process until the j matrices $\bar{B}$ are all inserted into the former k positions, we obtain that

$$\begin{aligned}
Q_{k+i+1}(j\tau_1) = {}& Q_{k+1}(0) Q_{i+1}(j\tau_1) + Q_{k+1}(\tau_1) Q_{i+1}\bigl((j-1)\tau_1\bigr) + \cdots \\
&+ Q_{k+1}(j\tau_1) Q_{i+1}(0)
\end{aligned}$$

by using the stepwise principle of combination, where $j = 1, 2, \dots, k + i$. Thus, we further arrive at

$$\begin{aligned}\bar{x}^{\mathrm{T}} Y^{(k)}(j\tau_1)\tilde{C} &= \sum_{i=0}^{\infty} \bar{x}^{\mathrm{T}} Q_{k+i+1}(0)\tilde{C}\frac{(j\tau_1)^i}{\Gamma(i+1)} \\ &\quad + \sum_{i=0}^{\infty} \bar{x}^{\mathrm{T}} Q_{k+i+1}(\tau_1)\tilde{C}\frac{((j-1)\tau_1)^i}{\Gamma(i+1)} + \cdots \\ &\quad + \sum_{i=0}^{\infty} \bar{x}^{\mathrm{T}} Q_{k+i+1}(j\tau_1)\tilde{C}\frac{\tau_1^i}{\Gamma(i+1)} + \bar{x}^{\mathrm{T}} Q_{k+1}(j\tau_1)\tilde{C} \\ &= \theta.\end{aligned}$$

From the assumption we obtain that $\bar{x}^{\mathrm{T}} Q_{k+1}(j\tau_1)\tilde{C} = \theta$, which implies that (21) holds. Rearrange (20) and (21) to yield

$$\begin{aligned}&\bar{x}^{\mathrm{T}}\big[Q_1(0)\tilde{C},\ Q_2(0)\tilde{C},\ \dots,\ Q_2(\tau_1)\tilde{C},\ Q_3(\tau_1)\tilde{C},\ \dots, \\ &\quad Q_{k+1}(k\tau_1)\tilde{C},\ \dots\big] = \theta,\end{aligned} \tag{22}$$

which implies that

$$\begin{aligned}&\bar{x}^{\mathrm{T}}\big[Q_1(0)\tilde{C},\ Q_2(0)\tilde{C},\ \dots,\ Q_{k+1}(0)\tilde{C}, Q_2(\tau_1)\tilde{C},\ Q_3(\tau_1)\tilde{C},\ \dots, \\ &\quad Q_{k+1}(\tau_1)\tilde{C},\ \dots,\ Q_{k+1}(k\tau_1)\tilde{C}\big] \\ &\quad =: \bar{x}^{\mathrm{T}}\bar{Q} = \theta\end{aligned} \tag{23}$$

for any finite integer $k \geqslant Nn$ because the solution of (22) is a solution of (23). This implies that $\bar{Q}$ is row linearly dependent, thus we have $\operatorname{rank}(\bar{Q}) < Nn$, which contradicts with the assumption. Thus, for the relative controllability of system (14), we need $\operatorname{rank}(\hat{Q}) = Nn$. □

Remark 2. If $\tilde{A}\bar{B} = \bar{B}\tilde{A}$, then (17) is degenerated into

$$\begin{aligned}\hat{Q} = \big[&\tilde{C},\ \tilde{A}\tilde{C},\ \dots,\ \tilde{A}^{Nn}\tilde{C},\ \bar{B}\tilde{C},\ \tilde{A}\bar{B}\tilde{C},\ \dots,\ \tilde{A}^{Nn-1}\bar{B}\tilde{C}, \\ &\bar{B}^2\tilde{C},\ \tilde{A}\bar{B}^2\tilde{C},\ \dots,\ \tilde{A}^{Nn-2}\bar{B}^2\tilde{C},\ \dots,\ \bar{B}^{Nn}\tilde{C}\big].\end{aligned}$$

Further, if $\bar{B} = \Theta$, then (17) is degenerated into

$$\hat{Q} = \big[\tilde{C},\ \tilde{A}\tilde{C},\ \dots,\ \tilde{A}^{Nn}\tilde{C}\big];$$

if $\tilde{A} = \Theta$, then (17) is degenerated into

$$\hat{Q} = \big[\tilde{C},\ \bar{B}\tilde{C},\ \dots,\ \bar{B}^{Nn}\tilde{C}\big].$$

5 Simulation

In this section an example of leader–follower multiagent systems will be considered to verify the theorem deduction. To simplify the problem, we assume that the system consists of 4 agents, and we show the interaction topology in Fig. 1, where the one labeled by 0 is assigned as leader, and the others are followers. The dynamics of the followers obey the following delay differential equations:

$$\begin{aligned}\dot{x}_1(t) &= a_1x_1(t) + b_1x_1(t-\tau_1) + c_1u_1(t),\\ \dot{x}_2(t) &= a_2x_2(t) + b_2x_2(t-\tau_2) + c_2u_2(t),\\ \dot{x}_3(t) &= a_3x_3(t) + b_3x_3(t-\tau_3) + c_3u_3(t),\end{aligned} \tag{24}$$

where $x_i \in R$, $i = 1, 2, 3$. Taking values as $\tau_1 = 2$, $\tau_2 = 3$, $\tau_3 = 4$, $t_1 = 10$, and $x_1 = [-100, -50, -40]^{\mathrm{T}}$, we know that the delays of each agent satisfy the condition in (1). Thus, for an initial vector function $x(t) = [10\cos(5t), -40\sin(\pi t), 50\exp(-2t)]^{\mathrm{T}}$ and protocol (2), system (24) always has a solution in the form of (11). Denote that $\tilde{\delta}(t) = t_1 - \tau_3 - t$. From Theorem 1 we know if (12) is nonsingular, system (24) is relative controllable. For $0 \leqslant t < \tau_3$, the control input function is

$$u^*(t) = \tilde{C}^{\mathrm{T}} X_2^{\mathrm{T}}(t_1 - t)G^{-1}(0, t_1)(x_1 - \eta),$$

where η is defined by (13). For $\tau_3 \leqslant t < 2\tau_3$, the control input is

$$\begin{aligned}u^*(t) = {}& \tilde{C}^{\mathrm{T}} X_2^{\mathrm{T}}(t_1 - t)G^{-1}(0, t_1)(x_1 - \eta)\\ &+ \tilde{C}^{\mathrm{T}} \int_0^{\tilde{\delta}(t)} X_2^{\mathrm{T}}(s_1)\tilde{B}_3^{\mathrm{T}} X_2^{\mathrm{T}}\big(\tilde{\delta}(t) - s_1\big)\,\mathrm{d}s_1\, G^{-1}(0, t_1)(x_1 - \eta).\end{aligned}$$

For $2\tau_3 \leqslant t < 3\tau_3$, the control input is

$$\begin{aligned}u^*(t) = {}& \tilde{C}^{\mathrm{T}} X_2^{\mathrm{T}}(t_1 - t)G^{-1}(0, t_1)(x_1 - \eta)\\ &+ \tilde{C}^{\mathrm{T}} \int_0^{\tilde{\delta}(t)} X_2^{\mathrm{T}}(s_1)\tilde{B}_3^{\mathrm{T}} X_2^{\mathrm{T}}\big(\tilde{\delta}(t) - s_1\big)\,\mathrm{d}s_1\, G^{-1}(0, t_1)(x_1 - \eta)\\ &+ \tilde{C}^{\mathrm{T}} \int_{\tau_3}^{\tilde{\delta}(t)} \int_{\tau_3}^{s_1} X_2^{\mathrm{T}}(s_2 - \tau_3)\tilde{B}_3^{\mathrm{T}} X_2^{\mathrm{T}}(s_1 - s_2)\tilde{B}_3^{\mathrm{T}} X_2^{\mathrm{T}}\big(\tilde{\delta}(t) - s_1\big)\,\mathrm{d}s_1\\ &\times G^{-1}(0, t_1)(x_1 - \eta).\end{aligned}$$

Other parameters are taken values as: $a_1 = 0.2$, $a_2 = 0.5$, $a_3 = 0.8$, $b_1 = 0.18$, $b_2 = 0.3$, $b_3 = 0.5$, $c_1 = 0.8$, $c_2 = 0.1$, $c_3 = 0.7$, $w_{12} = 1.2$, $w_{13} = 0.21$, $w_{21} = 1.8$, $w_{32} = 0.12$, $p_{10} = 0.24$, $p_{20} = 0.2$. We have that (12) is nonsingular, thus system (24) is relative controllable. Results of simulation are presented in Figs. 2–4.

From Figs. 2–4 we know that all the trajectories of the followers achieve the given terminal state x_1 in a finite time under the steering of leader, which verifies the theory deduction.

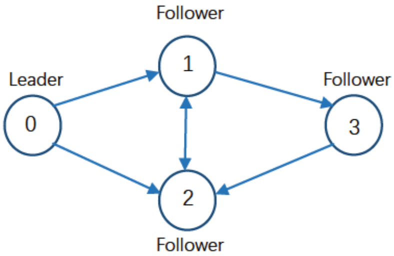

Figure 1. The communication topology of multiagent systems.

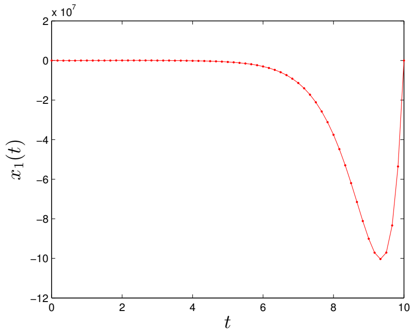

Figure 2. State of the first agent.

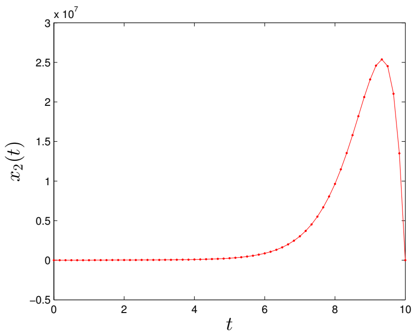

Figure 3. State of the second agent.

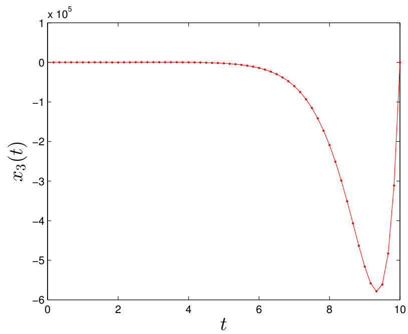

Figure 4. State of the third agent.

6 Conclusion

This paper considers the relative controllability of multiagent systems with pairwise different delays in states. Based on a neighbor-based interaction protocol, the multiagent systems are transformed into a multidelayed system, and solution of it is obtained by improving the methods in [22, 23] without the pairwise matrices permutation. Following from the solution, Gramian criterion of relative controllability is established, and rank criterion is also yielded for the single-delayed system without pairwise matrices permutation. This work guarantees that we can further explore the iterative learning control of the delayed multiagent systems (see more in [4]).

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