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In almost all practical problems involving ordinary annuities, the payments and interest conversions are made either annually, semi-annually, quarterly or monthly. In all such cases the following method of treatment will be found simple and direct.

All such annuities may be classified under three types: (1) when payment interval and interest-conversion interval coincide; (2) when payments are made several times during one interest-conversion interval; (3) when the payment interval contains several interest conversions.

The following uniform notation will be used throughout:

i = interest rate per interest-conversion interval,

n = number of interest conversions in the term of the annuity,

m = number of interest conversions per payment period (when greater than 1),

p = number of payments per interest-conversion period (when greater than 1),

$s_{\overline{n|}}$ = amount of annuity of 1 per period for n periods,

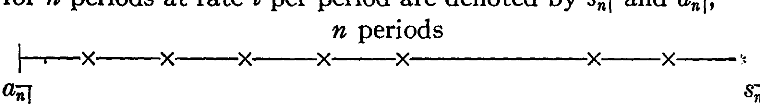
$a_{\overline{n|}}$ = present value of annuity of 1 per period for n periods,

$j_{(p)} = p[(1+i)^{\frac{1}{p}} - 1]$,

R = periodic payment of annuity.

An *ordinary* annuity is a series of *equal* payments made at the *ends* of equal successive intervals of time. The *amount* of the annuity is defined as the sum of the compound amounts of the annuity payments at the date of the last payment. The *present value* of the annuity is defined as the sum of the present values of the annuity payments at a date one period before the first payment is made.

The amount and present value of an annuity of 1 per period for n periods at rate i per period are denoted by $s_{\overline{n|}}$ and $a_{\overline{n|}}$,



respectively. The present value of any sum due in n periods is found by multiplying the sum by the discount factor v^n, where $v = \frac{1}{1+i}$. From the definitions of the amount and present value of an annuity, it is evident that the present value of the annuity may be found from the amount by discounting the amount back n periods, so that

(a) $$a_{\overline{n}|} = s_{\overline{n}|} \cdot \ v^n.$$

Formulas are first derived for the amount and present value of annuities with periodic payment 1, then for any periodic payment, R, the amount and present value may be found by multiplying these results by R.

Case I. The interval between payments is the same as the interval between interest conversions.

For example, payments made annually and interest convertible annually; or payments made quarterly and interest convertible quarterly.

We wish to derive a formula for the amount of an ordinary annuity of 1 per period for n periods at interest rate i per period; let us denote it by x. Suppose unit principal \$1 to be invested at compound interest for n periods at rate i per period. This investment will produce an interest payment of i at the end of each period for n periods, forming an ordinary annuity of periodic payment of i per period for n periods. Since each unit of capital shares equally in the interest increase, the amount of this annuity of i per period at the end of the n periods will be $i.x$. Thus, the original investment of 1 results in an interest accumulation at the end of the term of n periods of $i.x$. Now, on the other hand, the original investment of 1 at compound interest for n periods at rate i per period will give a compound amount of $(1+i)^n$, by the fundamental formula for compound interest. The compound interest earned on the investment will be this compound amount diminished by the original principal 1, or $(1+i)^n-1$. Thus, from this standpoint, the interest accumulation on the investment will be $(1+i)^n-1$. Equating these two expressions for the value of the interest earned by the investment of 1, we have

$$i.x = (1+i)^n - 1.$$

Solving this simple equation for x, by dividing both sides by i, we get

$$x = \frac{(1+i)^n - 1}{i}.$$

Since $s_{\overline{n}|}$ denotes the amount of an annuity of 1 per period for n periods, we have

(1) $$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}.$$

If we substitute this expression for $s_{\overline{n|}}$ in formula (a), it becomes

$$a_{\overline{n|}} = s_{\overline{n|}} \cdot v^n = \frac{(1+i)^n - 1}{i} \cdot v^n = \frac{(1+i)^n \cdot v^n - v^n}{i} = \frac{1-v^n}{i},$$

since $v^n = \frac{1}{(1+i)^n}$. Hence,

(2) $$a_{\overline{n|}} = \frac{1-v^n}{i}.$$

The amount and present value of an annuity of periodic payment R per period for n periods at rate i per period are then given by the formulas

(3) $$S = Rs_{\overline{n|}} \text{ (at rate } i),$$

(4) $$A = Ra_{\overline{n|}} \text{ (at rate } i).$$

The values of $s_{\overline{n|}}$ and $a_{\overline{n|}}$ at all current rates of interest can be found in annuity tables.*

Example 1. Find the amount and present value of an annuity of \$500 per year for 10 years with interest at 5% per annum.

Taking an annual basis, $n = 10$ yearly periods, $i = 5\%$ per year, and the periodic payment $R = \$500$ per year; the required amount is then

$$S = 500\, s_{\overline{10|}} \text{ (at } 5\%) = 500 \times 12.5778925 = \$6,288.946,$$
$$A = 500\, a_{\overline{10|}} \text{ (at } 5\%) = 500 \times 7.7217349 = \$3,860.867.$$

Example 2. Find the amount and present value of an annuity of \$1,200 per year payable monthly for 4 years when interest is at 6% convertible monthly.

With a month as the basic period, we take $n = 48$ monthly periods, $i = \frac{1}{2}\%$ per month, and $R = \$100$ per month. Then

$$S = 100\, s_{\overline{48|}} \text{ (at } \tfrac{1}{2}\%) = 100 \times 54.09783 = \$5,409.783,$$
$$A = 100\, a_{\overline{48|}} \text{ (at } \tfrac{1}{2}\%) = 100 \times 42.58032 = \$4,258.032.$$

Case II. There are p annuity payments made during each interest conversion period.

For example, payments made semi-annually and interest converted annually; or payments made monthly and interest converted semi-annually.

We wish to find a formula for the amount of an annuity of 1 per interest conversion period, payable p times per period, for n such periods at rate i per period. Let this amount be denoted by x.

*The most complete set of annuity tables is contained in Glover's *Compound-interest Tables and Seven-place Logarithms.* George Wahr, publisher, Ann Arbor, Michigan.

Just as in case I, suppose \$1 is invested at compound interest for n interest conversion periods at rate i per period. The compound amount earned by this investment during each p^{th} part of an interest period is $(1+i)^{\frac{1}{p}}$, by definition of compound amount for a fractional part of a period. The interest earned each p^{th} part of a period will then be $(1+i)^{\frac{1}{p}}-1$, and the total interest accumulation for one interest conversion period will be $p[(1+i)^{\frac{1}{p}}-1]$; this expression is denoted by $j_{(p)}$. Just as in case I the investment earned i each period, so now the investment earns $j_{(p)}$ each interest period, forming an annuity of periodic payment $j_{(p)}$ for n periods at rate i per period. Hence, the amount of the annuity resulting from the interest on our original investment will be $j_{(p)}\cdot x$. But on the other hand, just as in case I, the compound interest earned by the original investment is $(1+i)^n-1$. Equating these two values for the total interest accumulation, we have

$$j_{(p)}\cdot\ x=(1+i)^n-1.$$

Solving this simple equation for x, by dividing both sides by $j_{(p)}$, we get

$$x=\frac{(1+i)^n-1}{j_{(p)}}.$$

To adapt this formula to a form suitable for use with the annuity tables, we break up the fraction on the right into a product of two fractions thus:

$$\frac{(1+i)^n-1}{j_{(p)}}=\frac{(1+i)^n-1}{i}\cdot\ \frac{i}{j_{(p)}}.$$

But the first fraction on the right is exactly $s_{\overline{n}|}$, by formula (1) so that

$$x=s_{\overline{n}|}\cdot\ \frac{i}{j_{(p)}}.$$

Hence, the amount of an annuity of periodic payment R per period, payable p times per period, for n periods at rate i per period is given by the formula

$$S=R\cdot s_{\overline{n}|}\cdot\ \frac{i}{j_{(p)}}. \tag{5}$$

By use of relation (a), we see that

$$A=R\cdot a_{\overline{n}|}\cdot\ \frac{i}{j_{(p)}}. \tag{6}$$

is the formula for the present value of an annuity of R per period, payable p times per period, for n periods at rate i per period.

The values of $s_{\overline{n}|}$ and $a_{\overline{n}|}$ and of $\frac{i}{j_{(p)}}$ can be found directly in annuity tables.

Example 3. Find the amount and present value of an annuity of \$2,000 per annum payable quarterly for 5 years with interest at 7% per annum.

Choose the interest-conversion period of a year as the basic period, then $n=5$ yearly periods, $i=7\%$ per period, the periodic payment $R=\$2,000$ per year, and the number of payments per interest-conversion period is $p=4$. Hence, by formulas (5) and (6), the required values are

$$S = 2000\ s_{\overline{5}|} \cdot \frac{i}{j_{(4)}} \text{ (at } 7\%) = 2000 \times 5.7507390 \times 1.0258800$$
$$= \$11{,}799.136,$$
$$A = 2000\ a_{\overline{5}|} \cdot \frac{i}{j_{(4)}} \text{ (at } 7\%) = 2000 \times 4.1001974 \times 1.0258800$$
$$= \$8{,}412.621.$$

Example 4. What is the amount and present value of an annuity of \$800 per year payable monthly for 3½ years when interest is at 6% convertible quarterly?

In this case, we choose as our basic period the interest-conversion period of one quarter. Then the term of the annuity is $n=14$ quarterly periods, $i=1\frac{1}{2}\%$ per period, $R=\$200$ per period, and $p=3$ since there are three monthly payments in each quarterly period. Hence,

$$S = 200\ s_{\overline{14}|} \cdot \frac{i}{j_{(3)}} \text{ (at } 1\tfrac{1}{2}\%) = 200 \times 15.450382 \times 1.0049983$$
$$= \$3{,}105.476,$$
$$A = 200\ a_{\overline{14}|} \cdot \frac{i}{j_{(3)}} \text{ (at } 1\tfrac{1}{2}\%) = 200 \times 12.543382 \times 1.0049983$$
$$= \$2{,}521.216.$$

Case III. Interest is convertible m times during the interval between payments.

For example, payments made annually and interest convertible semi-annually; or payments made quarterly and interest convertible monthly.

We wish first to find the amount of an annuity of 1 per payment period for n interest-conversion periods when interest is allowed at rate i per interest-conversion period.

Suppose we assume that payments of 1 are made at the ends of successive interest-conversion periods for the n periods at rate i per period; the amount of this annuity will be $s_{\overline{n}|}$ (at rate i) (by case I). But this assumption of payments of 1 each interest-conversion period would lead at the end of a payment period of m interest periods to an amount of $s_{\overline{m}|}$ (at rate i); our original assumption would then give a periodic payment of $s_{\overline{m}|}$ at the end of each payment period instead of a payment of 1 as it should be. The result $s_{\overline{n}|}$ (at rate i) is therefore $s_{\overline{m}|}$ times too large; hence the required amount is $\frac{s_{\overline{n}|}}{s_{\overline{m}|}}$ (at rate i), which may be written

$$s_{\overline{n}|} \cdot \frac{1}{s_{\overline{m}|}} \text{ (at rate } i\text{)}.$$

The formula for the amount of an annuity of periodic payment R per payment period for n interest-conversion periods at rate i per interest period is therefore

$$S = R \cdot s_{\overline{n}|} \cdot \frac{1}{s_{\overline{m}|}} \text{ (at rate } i\text{)}, \tag{7}$$

where m is the number of interest conversions in one payment interval. By use of relation (a), we obtain the corresponding formula for the present value:

$$A = R \cdot a_{\overline{n}|} \cdot \frac{1}{s_{\overline{m}|}} \text{ (at rate } i\text{)}. \tag{8}$$

The values of $s_{\overline{n}|}$, $a_{\overline{n}|}$ and of $\frac{1}{s_{\overline{m}|}}$ can be found from annuity tables.

Example 5. Find the amount and present value of an annuity of \$450 per year payable annually for 8 years at 6% convertible quarterly.

Here the number of interest-conversion periods in the term of the annuity is $n = 32$ quarters, $i = 1\frac{1}{2}\%$ per quarterly period, $m = 4$, $R = \$450$ per year. Hence, by formulas (7) and (8),

$$S = 450\ s_{\overline{32}|} \cdot \frac{1}{s_{\overline{4}|}} \text{ (at } 1\tfrac{1}{2}\%) = 450 \times 40.688288 \times 0.244445$$
$$= \$4{,}475.718,$$

$$A = 450\ a_{\overline{32}|} \cdot \frac{1}{s_{\overline{4}|}} \text{ (at } 1\tfrac{1}{2}\%) = 450 \times 25.267139 \times 0.244445$$
$$= \$2{,}779.392.$$

Example 6. To how much will an annuity of \$1,400 per annum payable semi-annually with interest at 6% convertible monthly amount in 6 years?

The term of this annuity contains $n=72$ interest-conversion periods, the interest rate per conversion period is $i=\frac{1}{2}\%$, $m=6$, and the periodic payment is $R=\$700$ per half year. The required amount is therefore

$$S=700\ s_{\overline{72}|}\cdot\frac{1}{s_{\overline{6}|}}\ (\text{at } \tfrac{1}{2}\%)=700\times 86.408856\times 0.164595$$
$$=\$9{,}955.756.$$

If the interest rate i is not given in the annuity tables, substitute the values of $s_{\overline{n}|}$, $a_{\overline{n}|}$, from (1) and (2), and $j_{(p)}=p[(1+i)^{\frac{1}{p}}-1]$ in the other formulas, and calculate by the use of logarithms.

It may be noted that the preceding method of treatment does not explicitly involve the use of equivalent effective and nominal rates of interest, as do all previous discussions of the subject; also that this method involves a minimum of algebraic technique.