Longtime behavior for a generalized Cahn–Hilliard system with fractional operators

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Abstract

In this contribution, we deal with the longtime behavior of the solutions to the fractional variant of the Cahn–Hilliard system, with possibly singular potentials, that we have recently investigated in the paper *Well-posedness and regularity for a generalized fractional Cahn–Hilliard system*. More precisely, we study the ω-limit of the phase parameter y and characterize it completely. Our characterization depends on the first eigenvalues $\lambda_1 \geq 0$ of one of the operators involved: if $\lambda_1 > 0$, then the chemical potential μ vanishes at infinity and every element y_ω of the ω-limit is a stationary solution to the phase equation; if instead $\lambda_1 = 0$, then every element y_ω of the ω-limit satisfies a problem containing a real function μ_∞ related to the chemical potential μ. Such a function μ_∞ is nonunique and time dependent, in general, as we show by an example. However, we give sufficient conditions for μ_∞ to be uniquely determined and constant.

1 Introduction

The paper [10] investigates the abstract evolutionary system

$$\partial_t y + A^{2r}\mu = 0, \tag{1.1}$$

$$\tau \partial_t y + B^{2\sigma} y + f'(y) = \mu + u, \tag{1.2}$$

$$y(0) = y_0, \tag{1.3}$$

where A^{2r} and $B^{2\sigma}$, with $r > 0$ and $\sigma > 0$, denote fractional powers in the spectral sense of the unbounded linear operators A and B, respectively, which are supposed to be densely defined in $H := L^2(\Omega)$, with $\Omega \subset \mathbb{R}^3$, selfadjoint, and monotone. The above system is a generalization of the Cahn–Hilliard system (namely, the *nonviscous* system or the *viscous* one, depending on whether $\tau = 0$ or $\tau > 0$), which models a phase separation process taking place in the container Ω. The unknown functions y and μ stand for the *order parameter* and the *chemical potential*, respectively, while u is a given source term. Moreover, f denotes a double-well potential, for which typical and physically significant examples are the so-called *classical regular potential*, the *logarithmic double-well potential*, and the *double obstacle potential*, which are given, in this order, by

$$f_{reg}(r) := \frac{1}{4}\left(r^2 - 1\right)^2, \quad r \in \mathbb{R}, \tag{1.4}$$

$$f_{log}(r) := \left((1+r)\ln(1+r) + (1-r)\ln(1-r)\right) - c_1 r^2, \quad r \in (-1, 1), \tag{1.5}$$

$$f_{2obs}(r) := -c_2 r^2 \quad \text{if } |r| \leq 1 \quad \text{and} \quad f_{2obs}(r) := +\infty \quad \text{if } |r| > 1. \tag{1.6}$$

Here, the constants c_i in (1.5) and (1.6) satisfy $c_1 > 1$ and $c_2 > 0$, so that f_{log} and f_{2obs} are nonconvex. In cases like (1.6), one has to split f into a nondifferentiable convex part $\widehat{\beta}$ (the indicator function of $[-1, 1]$, in the present example) and a smooth perturbation $\widehat{\pi}$. Accordingly, one has to

replace the derivative of the convex part by the subdifferential and interpret (1.2) as a differential inclusion or, equivalently, as a variational inequality involving $\widehat{\beta}$ rather than its subdifferential. Actually, the latter has been done in [10], and we do the same in this paper.

Fractional versions of the Cahn–Hilliard system have been considered by several authors and are the subject of a number of recent papers. As for references regarding well-posedness and related problems, a rather large list of citations is given in [10]. Here we recall some literature dealing with the asymptotic behavior of the solutions. Indeed, one can find a number of results in this direction both for the standard Cahn–Hilliard equations and for variants of them. The latter are obtained, e.g., by adding viscosity or memory contributions as well as convective terms; another possibility is coupling (1.1)–(1.3) with other equations, like heat type equations or fluid dynamics equations, or introducing non–local–in–space terms; finally, one can replace the classical Neumann boundary conditions by other ones, e.g., the dynamic boundary conditions. Without any claim of completeness, by starting from [37], we can quote, e.g., [1, 2, 5, 7–9, 19–23, 30, 33, 34] for the study of the trajectories and related topics, and [12–18, 22, 24–29, 31, 36] for the existence of global or exponential attractors and their properties. However, if nonlocal terms are considered in these papers, they are not defined as fractional powers in the spectral sense of the operators involved. On the contrary, our framework is followed in [6], where the longtime behavior of the solutions to a fractional version of the Allen–Cahn equation is studied.

Let us come to the content of this paper. Our aim is studying the ω-limit (in a suitable topology) of the component y of the solution to a proper weak version of problem (1.1)–(1.3). The characterization we give (Theorem 2.2) depends on the first eigenvalue λ_1 of the operator A. If $\lambda_1 > 0$, then $\mu(t)$ tends to zero as t approaches infinity, and every element y_ω of the ω-limit is a stationary solution, i.e., it solves the equation

$$B^{2\sigma} y_\omega + f'(y_\omega) = u_\infty, \tag{1.7}$$

at least in a weak sense, where u_∞ is the limit of $u(t)$ as t tends to infinity. If, instead, $\lambda_1 = 0$, then the element y_ω satisfies a weaker property, namely, a weak form of the equation

$$B^{2\sigma} y_\omega + f'(y_\omega) = u_\infty + \mu_\infty(t) \quad \text{for a.a. } t \in (0, +\infty), \tag{1.8}$$

for at least one function $\mu_\infty \in L^\infty_{loc}([0, +\infty))$. We also show that, in the general case, the characterization (1.8) is the best possible (see Example 2.3): μ_∞ is nonconstant and nonunique, in general, and $\mu(t)$ does not converge at infinity. On the other hand, we give sufficient conditions on f and on the solution that ensure that the function μ_∞ is unique and constant and that (1.8) holds in the strong sense (see Proposition 2.4).

2   Statement of the problem and results

In this section, we state precise assumptions and notations and present our results. Our framework is the same as in [10], and we briefly recall it here, for the reader's convenience. First of all, the open set $\Omega \subset \mathbb{R}^3$ is assumed to be bounded, connected and smooth. We use the notation

$$H := L^2(\Omega) \tag{2.1}$$

and denote by $\| \cdot \|$ and $(\,\cdot\,,\,\cdot\,)$ the standard norm and inner product of H. As for the operators involved in our system, we postulate that

$$A : D(A) \subset H \to H \quad \text{and} \quad B : D(B) \subset H \to H \quad \text{are}$$

unbounded, monotone, selfadjoint linear operators with compact resolvents. $\tag{2.2}$

We denote by $\{\lambda_j\}$ and $\{\lambda'_j\}$ the nondecreasing sequences of the eigenvalues and by $\{e_j\}$ and $\{e'_j\}$ the (complete) systems of the corresponding orthonormal eigenvectors, that is,

$$Ae_j = \lambda_j e_j, \quad Be'_j = \lambda'_j e'_j, \quad \text{and} \quad (e_i, e_j) = (e'_i, e'_j) = \delta_{ij}, \quad \text{for } i, j = 1, 2, \ldots, \tag{2.3}$$

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \quad \text{and} \quad 0 \leq \lambda'_1 \leq \lambda'_2 \leq \ldots, \quad \text{with} \quad \lim_{j \to \infty} \lambda_j = \lim_{j \to \infty} \lambda'_j = +\infty. \tag{2.4}$$

The power A^r of A with an arbitrary positive real exponent r is given by

$$A^r v = \sum_{j=1}^{\infty} \lambda_j^r (v, e_j) e_j \quad \text{for } v \in V_A^r, \tag{2.5}$$

where

$$V_A^r := D(A^r) = \left\{ v \in H : \sum_{j=1}^{\infty} |\lambda_j^r (v, e_j)|^2 < +\infty \right\}. \tag{2.6}$$

In principle, we could endow V_A^r with the standard graph norm in order to make V_A^r a Hilbert space. However, we will choose an equivalent Hilbert structure later on. In the same way, for $\sigma > 0$, we define the power B^σ of B. For its domain, we use the notation

$$V_B^\sigma := D(B^\sigma), \quad \text{with the norm } \| \cdot \|_{B, \sigma} \text{ associated to the inner product}$$
$$(v, w)_{B, \sigma} := (v, w) + (B^\sigma v, B^\sigma w) \quad \text{for } v, w \in V_B^\sigma. \tag{2.7}$$

Accordingly, we introduce a space with a negative exponent. We set

$$V_A^{-r} := (V_A^r)^* \quad \text{for } r > 0 \tag{2.8}$$

and use the symbol $\langle \cdot, \cdot \rangle_{A, r}$ for the duality pairing between V_A^{-r} and V_A^r. We also identify H with a subspace of V_A^{-r} in the usual way, i.e., such that

$$\langle v, w \rangle_{A, r} = (v, w) \quad \text{for every } v \in H \text{ and } w \in V_A^r. \tag{2.9}$$

At this point, we can start listing our assumptions. First of all,

$$r \text{ and } \sigma \text{ are fixed positive real numbers and } \tau \in [0, 1] \text{ is fixed as well.} \tag{2.10}$$

As for the linear operators, we postulate, besides (2.2), that

$$\text{either} \quad \lambda_1 > 0 \quad \text{or} \quad 0 = \lambda_1 < \lambda_2 \text{ and } e_1 \text{ is a constant;} \tag{2.11}$$

$$\text{if} \quad \lambda_1 = 0, \text{ then the constant functions belong to } V_B^\sigma. \tag{2.12}$$

In [10] some remarks are given on the above assumptions. Moreover, it is shown that an equivalent Hilbert structure on V_A^r is obtained by taking the norm defined by

$$\|v\|_{A, r}^2 := \begin{cases} \|A^r v\|^2 = \displaystyle\sum_{j=1}^{\infty} |\lambda_j^r (v, e_j)|^2 & \text{if } \lambda_1 > 0, \\[4mm] |(v, e_1)|^2 + \|A^r v\|^2 = |(v, e_1)|^2 + \displaystyle\sum_{j=2}^{\infty} |\lambda_j^r (v, e_j)|^2 & \text{if } \lambda_1 = 0. \end{cases} \tag{2.13}$$

We notice that the term (v, e_1) appearing in (2.13) in the case $\lambda_1 = 0$ is proportional to the mean value of v

$$\text{mean } v := \frac{1}{|\Omega|} \int_\Omega v, \tag{2.14}$$

since e_1 is a constant by (2.11). In particular, we have the Poincaré type inequality

$$\|v\| \le C_P \|A^r v\| \quad \text{for every } v \in V_A^r \text{ with } \operatorname{mean} v = 0, \text{ if } \lambda_1 = 0. \tag{2.15}$$

For the nonlinearity f appearing in our system, we split it as $f = \widehat{\beta} + \widehat{\pi}$ and postulate the following properties (which are fulfilled by all of the important potentials (1.4)–(1.6)):

$$\widehat{\beta} : \mathbb{R} \to [0, +\infty] \quad \text{is convex, proper, and l.s.c., with} \quad \widehat{\beta}(0) = 0; \tag{2.16}$$

$$\widehat{\pi} : \mathbb{R} \to \mathbb{R} \quad \text{is of class } C^1 \text{ with a Lipschitz continuous first derivative}; \tag{2.17}$$

$$\text{it holds} \quad \liminf_{|s| \nearrow +\infty} \frac{\widehat{\beta}(s) + \widehat{\pi}(s)}{s^2} > 0. \tag{2.18}$$

We set, for convenience,

$$\beta := \partial \widehat{\beta}, \quad \pi := \widehat{\pi}', \quad L_\pi = \text{the Lipschitz constant of } \pi, \quad \text{and} \quad L_\pi' := L_\pi + 1. \tag{2.19}$$

Moreover, we term $D(\widehat{\beta})$ and $D(\beta)$ the effective domains of $\widehat{\beta}$ and β, respectively, and notice that β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$.

At this point, we can state the problem under investigation, and we do it on the half-line $t \ge 0$, due to the subject of the present paper. The data are required to satisfy

$$u \in W_{loc}^{1,1}([0, +\infty); H) \quad \text{and} \quad \partial_t u \in L^1(0, +\infty; H). \tag{2.20}$$

$$y_0 \in V_B^\sigma \quad \text{and} \quad \widehat{\beta}(y_0) \in L^1(\Omega). \tag{2.21}$$

$$\text{If } \lambda_1 = 0, \text{ then} \quad m_0 := \operatorname{mean} y_0 \quad \text{belongs to the interior of } D(\beta). \tag{2.22}$$

A solution to our system is a pair (y, μ) fulfilling the regularity requirements

$$y \in L^\infty(0, T; V_B^\sigma), \quad \partial_t y \in L^2(0, T; V_A^{-r}) \quad \text{and} \quad \tau \partial_t y \in L^2(0, T; H), \tag{2.23}$$

$$\mu \in L^2(0, T; V_A^r), \tag{2.24}$$

$$\widehat{\beta}(y) \in L^1(\Omega \times (0, T)), \tag{2.25}$$

for every $T > 0$, and satisfying the following weak formulation of the equations (1.1)–(1.3):

$$\langle \partial_t y(t), v \rangle_{A,r} + (A^r \mu(t), A^r v) = 0 \quad \text{for every } v \in V_A^r \text{ and for a.a. } t \in (0, +\infty), \tag{2.26}$$

$$\big(\tau \partial_t y(t), y(t) - v\big) + \big(B^\sigma y(t), B^\sigma(y(t) - v)\big)$$
$$+ \int_\Omega \widehat{\beta}(y(t)) + \big(\pi(y(t)) - u(t), y(t) - v\big) \le \big(\mu(t), y(t) - v\big) + \int_\Omega \widehat{\beta}(v)$$
$$\text{for every } v \in V_B^\sigma \text{ and for a.a. } t \in (0, +\infty), \tag{2.27}$$

$$y(0) = y_0. \tag{2.28}$$

We remark that, if $\lambda_1 = 0$, then $A^r(1) = 0$ by (2.11), so that (2.26) implies that

$$\frac{d}{dt} \int_\Omega y(t) = 0 \quad \text{for a.a. } t \in (0, +\infty), \quad \text{i.e.,}$$
$$\operatorname{mean} y(t) = m_0 \quad \text{for every } t \in [0, +\infty). \tag{2.29}$$

The well-posedness result stated below was proved in [10] under a different assumption on u. Namely, in studying the problem on the finite time interval $(0, T)$, it was assumed that $u \in H^1(0, T; H)$, while (2.20) only implies that $u \in W^{1,1}(0, T; H)$. However, we point out that our assumption is sufficient to obtain the same result. We will give some explanation on this in the next section.

Theorem 2.1. *Let the assumptions* (2.2), (2.10)–(2.12) *and* (2.16)–(2.18) *on the structure of the system, and* (2.20)–(2.22) *on the data, be fulfilled. Then there exists a pair* (y, μ) *satisfying* (2.23)–(2.25) *and solving problem* (2.26)–(2.28). *Moreover, the component* y *of the solution is uniquely determined.*

In [10, Rem. 4.1], sufficient conditions were given that ensure uniqueness also for μ. However, the aim of this paper is the study of the longtime behavior of the component y alone. The rather weak regularity conditions (2.23) imply that

$$y : [0, +\infty) \to V_B^\sigma \quad \text{is weakly continuous.}$$

This enables us to the define the following (possibly empty) ω-limit set

$$\omega = \omega(y_0, u) := \{y_\omega \in V_B^\sigma : y(t_n) \to y_\omega \quad \text{weakly in } V_B^\sigma \text{ for some } \{t_n\} \nearrow +\infty\}. \qquad (2.30)$$

Here is our result, which holds under the additional assumption that $u(t)$ has a limit u_∞ as t tends to infinity in the sense of the forthcoming (2.31). The second part of the statement distinguishes two cases regarding the first eigenvalue λ_1 of A. If λ_1 is positive, then every element of the ω-limit is a stationary solution in the sense specified below; if instead $\lambda_1 = 0$, then the elements of the ω-limit just satisfy a weaker property.

Theorem 2.2. *Let the assumptions* (2.2), (2.10)–(2.12) *and* (2.16)–(2.18) *on the structure of the system, and* (2.20)–(2.22) *on the data, be fulfilled. In addition, assume that there is some* $u_\infty \in H$ *such that*

$$u - u_\infty \in L^2(0, +\infty; H), \qquad (2.31)$$

and let (y, μ) *be a solution to* (2.26)–(2.28) *according to Theorem 2.1. Then the ω-limit* (2.30) *is nonempty. Moreover, it is characterized as follows:*

i) If $\lambda_1 > 0$, then every element $y_\omega \in \omega$ satisfies

$$\left(B^\sigma y_\omega, B^\sigma(y_\omega - v)\right) + \int_\Omega \widehat{\beta}(y_\omega) + \left(\pi(y_\omega) - u_\infty, y_\omega - v\right) \leq \int_\Omega \widehat{\beta}(v)$$
$$\text{for every } v \in V_B^\sigma. \qquad (2.32)$$

ii) If $\lambda_1 = 0$, then, for every element $y_\omega \in \omega$, there exists some $\mu_\infty \in L_{loc}^\infty([0, +\infty))$ such that

$$\left(B^\sigma y_\omega, B^\sigma(y_\omega - v)\right) + \int_\Omega \widehat{\beta}(y_\omega) + \left(\pi(y_\omega) - u_\infty, y_\omega - v\right)$$
$$\leq \left(\mu_\infty(t), y_\omega - v\right) + \int_\Omega \widehat{\beta}(v)$$
$$\text{for every } v \in V_B^\sigma \text{ and for a.a. } t \in (0, +\infty). \qquad (2.33)$$

In (2.31), u_∞ obviously denotes the function $[0, +\infty) \ni t \mapsto u_\infty \in H$ rather than the element $u_\infty \in H$. In the right-hand side of (2.33), $\mu_\infty(t)$ denotes the constant function $\Omega \ni x \mapsto \mu_\infty(t)$ rather than the real value $\mu_\infty(t)$. Conventions of this type will be used also in the following.

The part $ii)$ of the above result seems to be rather poor. Nevertheless, this characterization is the best possible for the general case, that is, one can neither expect uniqueness for μ_∞, nor further properties for it, as the following example shows. Notice that assuming that A and B are particularly good operators does not help at all.

Example 2.3. Let the operators A and B satisfy the hypotheses of Theorem 2.2, and assume that $\lambda_1 = 0$. Moreover, let us choose $\widehat{\pi} = 0$ and $\widehat{\beta}$ given by

$$\widehat{\beta}(r) := r^2 + |r| \quad \text{for } r \in \mathbb{R}.$$

Then (2.16)–(2.18) are satisfied. But β is multivalued, since $\beta(0) = \text{sign}(0) = [-1, 1]$. Thus, if we take $y_0 = 0$, $u = 0$, and any function $\bar{\mu} \in L^\infty(0, +\infty)$ satisfying $|\bar{\mu}(t)| \leq 1$ for a.a. $t \in (0, +\infty)$, then a solution (y, μ) to problem (2.26)–(2.28) is given by the formulas $y(x, t) = 0$ and $\mu(x, t) = \bar{\mu}(t)$. Indeed, (y, μ) trivially solves the first equation (2.26) (since μ is space independent), as well as (2.28); moreover, the variational inequality (2.27) is solved in the stronger form

$$\tau \partial_t y + B^{2\sigma} y + \xi + \pi(y) = \mu + u \quad \text{with} \quad \xi \in \beta(y),$$

since we can take $\xi = \mu$ (we have $\bar{\mu}(t) \in [-1, 1] = \beta(0)$, indeed). So, the only element y_ω of the ω-limit is $y_\omega = 0$, while we have lots of possible μ_∞'s, namely, the set of such functions coincides with the set of the admissible functions termed $\bar{\mu}$ before.

On the contrary, under further conditions on β and on the solution, the characterization in the case $\lambda_1 = 0$ can be improved. Here are the new requirements:

$$D(\beta) \text{ is an open interval, and } \beta \text{ is a single-valued } C^1 \text{ function.} \tag{2.34}$$

There exists a compact interval $[a, b] \subset D(\beta)$ such that
$$y(x, t) \in [a, b] \text{ for a.a. } (x, t) \in \Omega \times (0, +\infty). \tag{2.35}$$

$$V_B^\sigma \cap L^\infty(\Omega) \text{ is dense in } V_B^\sigma. \tag{2.36}$$

The above assumptions (with (2.35) only in a given finite time interval $(0, T)$) have been introduced in the paper [11]. One of the motivations was the derivation of the strong form of (2.27), i.e.,

$$\tau \, \partial_t y + B^{2\sigma} y + \beta(y) + \pi(y) = \mu + u. \tag{2.37}$$

Precisely, it has been proved that $y \in L^2(0, T; V_B^{2\sigma})$ and that (2.37) is satisfied almost everywhere (see Rem. 3.5 and the subsequent lines of [11], where some comments on (2.34)–(2.36) were given as well). Here, we point out that the proof of the derivation of (2.37) also holds true for the half–line $t \geq 0$ if (2.35) is assumed. We use (2.34)–(2.36) in the result stated below.

Proposition 2.4. *In addition to the assumptions of Theorem 2.2, suppose that* (2.34)–(2.36) *are satisfied. Then the function* μ_∞ *appearing in* (2.33) *is uniquely determined and constant. Moreover,* $y_\omega \in V_B^{2\sigma}$, *and the pair* (y_ω, μ_∞) *satisfies the equation*

$$B^{2\sigma} y_\omega + \beta(y_\omega) + \pi(y_\omega) = \mu_\infty + u_\infty \quad \text{a.e. in } \Omega. \tag{2.38}$$

Theorem 2.2 and Proposition 2.4 will be proved in the last section. In the next one, we establish some auxiliary global estimates. To this end, we also recall the approximation and the discretization of problem (2.26)–(2.28) given in [10].

Notation 2.5. In the remainder of the paper, we will use the same small letter c for (possibly) different constants that depend only on the structure of our system (but τ) and on the assumptions on the data. When some final time T is considered, the symbol c_T denotes (possibly different) constants that depend on T in addition. On the contrary, precise constants we could refer to are treated in a different way (see, e.g., the forthcoming (3.1), where greek and capital letters are used).

3 Global estimates

The proof of Theorem 2.2 is based on some global–in–time a priori estimates, which we derive in this section by starting from the approximating and discrete problems introduced in [10]. Thus, some recalls are needed.

The approximation of problem (2.26)–(2.28) by a more regular one relies on the use of the Moreau–Yosida regularizations $\widehat{\beta}_\lambda$ and β_λ of $\widehat{\beta}$ and β at the level $\lambda > 0$ (see, e.g., [4, p. 28 and p. 39]). We notice that, by accounting for (2.18), the inequalities

$$\widehat{\beta}_\lambda(s) + \widehat{\pi}(s) \geq \alpha\, s^2 - C \geq -C' \tag{3.1}$$

hold true for some positive constants α, C, C', every $s \in \mathbb{R}$, and every sufficiently small $\lambda > 0$. In case the reader aims to check (3.1), we suggest the use of the following representation of $\widehat{\beta}_\lambda$, namely

$$\widehat{\beta}_\lambda(s) := \inf_{r \in \mathbb{R}} \left\{ \frac{1}{2\lambda}|r - s|^2 + \widehat{\beta}(r) \right\} = \frac{1}{2\lambda}\big|s - J_\lambda(s)\big|^2 + \widehat{\beta}\big(J_\lambda(s)\big), \tag{3.2}$$

where $J_\lambda : \mathbb{R} \to \mathbb{R}$ denotes the resolvent operator associated to β, that is, $J_\lambda(s)$ is defined as the unique solution to the multi-equation

$$J_\lambda(s) + \lambda\beta(J_\lambda(s)) \ni s \equiv J_\lambda(s) + \lambda\beta_\lambda(s) \quad \text{for all } s \in \mathbb{R}.$$

Indeed, by combining (2.16)–(2.18), which imply

$$\widehat{\beta}(s) + \widehat{\pi}(s) \geq 2\alpha\, s^2 - c \quad \text{for all } s \in \mathbb{R}$$

and for some constant $\alpha > 0$, along with (3.2) and the Taylor formula with integral remainder to estimate the difference $\widehat{\pi}(s) - \widehat{\pi}(J_\lambda(s))$, one can arrive at (3.1).

The approximating problem on any finite time integral $(0, T)$ is obtained by replacing $\widehat{\beta}$ in (2.27) by $\widehat{\beta}_\lambda$, namely,

$$\langle \partial_t y^\lambda(t), v \rangle_{A,r} + (A^r \mu^\lambda(t), A^r v) = 0 \quad \text{for every } v \in V_A^r \text{ and for a.a. } t \in (0, T), \tag{3.3}$$

$$\big(\tau \partial_t y^\lambda(t), y^\lambda(t) - v\big) + \big(B^\sigma y^\lambda(t), B^\sigma(y^\lambda(t) - v)\big)$$

$$+ \int_\Omega \widehat{\beta}_\lambda(y^\lambda(t)) + \big(\pi(y^\lambda(t)) - u(t), y^\lambda(t) - v\big)$$

$$\leq \big(\mu^\lambda(t), y^\lambda(t) - v\big) + \int_\Omega \widehat{\beta}_\lambda(v) \quad \text{for every } v \in V_B^\sigma \text{ and for a.a. } t \in (0, T), \tag{3.4}$$

$$y^\lambda(0) = y_0. \tag{3.5}$$

In principle, the regularity required for the solution (y^λ, μ^λ) is still given by (2.23)–(2.25). However, due to the Lipschitz continuity of β_λ, (2.25) can be improved. Namely, (2.23) implies $\beta_\lambda(y^\lambda) \in L^2(0, T; H)$. Using this and the fact that $\widehat{\beta}_\lambda$ is differentiable and β_λ is its derivative, one sees that, in place of (3.4), one can equivalently consider the pointwise variational equation

$$\big(\tau \partial_t y^\lambda(t), v\big) + \big(B^\sigma y^\lambda(t), B^\sigma v\big) + \big(\beta_\lambda(y^\lambda(t)) + \pi(y^\lambda(t)) - u(t), v\big) = \big(\mu^\lambda(t), v\big)$$

$$\text{for every } v \in V_B^\sigma \text{ and for a.a. } t \in (0, T). \tag{3.6}$$

In [10], it was shown that the above problem is well-posed and that its unique solution (y^λ, μ^λ) converges to a solution (y, μ) to problem (2.26)–(2.28) in the weak topology associated with the regularity

requirements, essentially. Moreover, the solution (y^λ, μ^λ) is obtained as the limit of suitable interpolant functions constructed by starting from the solution to a proper discrete problem. For the reader's convenience, we recall both the notation for the interpolants and the discrete problem.

Let N be a positive integer and Z be one of the spaces H, V_A^r, V_B^σ. We set $h := T/N$ and $I_n := ((n-1)h, nh)$ for $n = 1, \ldots, N$. Given $z = (z_0, z_1, \ldots, z_N) \in Z^{N+1}$, the piecewise constant and piecewise linear interpolants

$$\overline{z}_h \in L^\infty(0, T; Z), \quad \underline{z}_h \in L^\infty(0, T; Z) \quad \text{and} \quad \widehat{z}_h \in W^{1,\infty}(0, T; Z)$$

are defined by setting

$$\overline{z}_h(t) = z^n \quad \text{and} \quad \underline{z}_h(t) = z^{n-1} \quad \text{for a.a. } t \in I_n, \ n = 1, \ldots, N, \tag{3.7}$$

$$\widehat{z}_h(0) = z_0 \quad \text{and} \quad \partial_t \widehat{z}_h(t) = \frac{z^{n+1} - z^n}{h} \quad \text{for a.a. } t \in I_n, \ n = 1, \ldots, N. \tag{3.8}$$

The discrete problem consists in finding two $(N+1)$-tuples $(y^0, \ldots, y^N)$ and $(\mu^0, \ldots, \mu^N)$ satisfying

$$y^0 = y_0, \quad \mu^0 = 0, \quad (y^1, \ldots, y^N) \in (V_B^{2\sigma})^N \quad \text{and} \quad (\mu^1, \ldots, \mu^N) \in (V_A^{2r})^N, \tag{3.9}$$

and solving

$$\frac{y^{n+1} - y^n}{h} + \mu^{n+1} + A^{2r}\mu^{n+1} = \mu^n, \tag{3.10}$$

$$\tau \frac{y^{n+1} - y^n}{h} + (L_\pi' I + B^{2\sigma} + \beta_\lambda + \pi)(y^{n+1}) = L_\pi' y^n + \mu^{n+1} + u^{n+1}, \tag{3.11}$$

for $n = 0, 1, \ldots, N - 1$, where $I : H \to H$ is the identity, L_π' is given by (2.19), and

$$u^n := u(nh) \quad \text{for } n = 0, 1, \ldots, N. \tag{3.12}$$

Precisely, it has been proved that such a discrete problem is uniquely solvable. Moreover, as just said, some of the interpolants defined above by starting from the discete solution converge to the solution (y^λ, μ^λ) to the regularized problem (3.3)–(3.5).

Now, we start estimating. It is understood that the assumptions of Theorem 2.2 are in force. In particular, every constant c we introduce will depend only on these assumptions. We closely follow the lines of [10]. However, we modify the argument a little and obtain estimates that are uniform with respect to T. In doing this modification, we also avoid using the regularity condition $\partial_t u \in L^2(0, T; H)$, which was supposed in [10], and just owe to the regularity $\partial_t u \in L^1(0, T; H)$ (but uniformly with respect to T in the sense of (2.20) in order to obtain a global–in–time estimate). Since this is the only point of [10] where the $L^2(0, T; H)$ regularity for $\partial_t u$ is accounted for, the well-posedness result in Theorem 2.1 holds under our assumption (2.20), as announced before its statement.

First uniform estimate. We test (3.10) and (3.11) (by taking the scalar product in H) by $h\mu^{n+1}$ and $y^{n+1} - y^n$, respectively, and add the resulting identities. Noting an obvious cancellation, we obtain the equation

$$h(\mu^{n+1} - \mu^n, \mu^{n+1}) + h(A^{2r}\mu^{n+1}, \mu^{n+1}) + \frac{\tau}{h} \|y^{n+1} - y^n\|^2$$
$$+ (B^{2\sigma}y^{n+1}, y^{n+1} - y^n) + \left((L_\pi' I + \beta_\lambda + \pi)(y^{n+1}), y^{n+1} - y^n\right)$$
$$= L_\pi'(y^n, y^{n+1} - y^n) + (u^{n+1}, y^{n+1} - y^n).$$

Now, we observe that the function $r \mapsto \frac{L'_\pi}{2} r^2 + \widehat{\beta}_\lambda(r) + \widehat{\pi}(r)$ is convex on $\mathbb{R}$, since $\widehat{\beta}_\lambda$ is convex and $|\pi'| \leq L_\pi$. Thus, we have that

$$\big((L'_\pi I + \beta_\lambda + \pi)(y^{n+1}), y^{n+1} - y^n\big)$$
$$\geq \frac{L'_\pi}{2} \|y^{n+1}\|^2 + \int_\Omega \big(\widehat{\beta}_\lambda(y^{n+1}) + \widehat{\pi}(y^{n+1})\big) - \frac{L'_\pi}{2} \|y^n\|^2 - \int_\Omega \big(\widehat{\beta}_\lambda(y^n) + \widehat{\pi}(y^n)\big).$$

We easily deduce that

$$\frac{h}{2} \|\mu^{n+1}\|^2 + \frac{h}{2} \|\mu^{n+1} - \mu^n\|^2 - \frac{h}{2} \|\mu^n\|^2 + h\|A^r \mu^{n+1}\|^2$$
$$+ \frac{\tau}{h} \|y^{n+1} - y^n\|^2 + \frac{1}{2} \|B^\sigma y^{n+1}\|^2 + \frac{1}{2} \|B^\sigma(y^{n+1} - y^n)\|^2 - \frac{1}{2} \|B^\sigma y^n\|^2$$
$$+ \frac{L'_\pi}{2} \|y^{n+1}\|^2 + \int_\Omega \big(\widehat{\beta}_\lambda(y^{n+1}) + \widehat{\pi}(y^{n+1})\big) - \frac{L'_\pi}{2} \|y^n\|^2 - \int_\Omega \big(\widehat{\beta}_\lambda(y^n) + \widehat{\pi}(y^n)\big)$$
$$\leq -\frac{L'_\pi}{2}\big(\|y^n\|^2 - \|y^{n+1}\|^2 + \|y^{n+1} - y^n\|^2\big) + (u^{n+1}, y^{n+1} - y^n).$$

Then, we first rearrange and then sum up for $n = 0, \ldots, k-1$ with $k \leq N$, employing summation by parts in the last term. We thus arrive at the inequality

$$\frac{h}{2} \|\mu^k\|^2 + \sum_{n=0}^{k-1} \frac{h}{2} \|\mu^{n+1} - \mu^n\|^2 + \sum_{n=0}^{k-1} h\|A^r \mu^{n+1}\|^2$$
$$+ \tau \sum_{n=0}^{k-1} h\left\|\frac{y^{n+1} - y^n}{h}\right\|^2 + \frac{1}{2} \|B^\sigma y^k\|^2 - \frac{1}{2} \|B^\sigma y_0\|^2 + \sum_{n=0}^{k-1} \frac{1}{2} \|B^\sigma(y^{n+1} - y^n)\|^2$$
$$+ \int_\Omega \big(\widehat{\beta}_\lambda(y^k) + \widehat{\pi}(y^k)\big) - \int_\Omega \big(\widehat{\beta}(y_0) + \widehat{\pi}(y_0)\big) + \frac{L'_\pi}{2} \sum_{n=0}^{k-1} \|y^{n+1} - y^n\|^2$$
$$\leq (u^k, y^k) - (u^1, y_0) - \sum_{n=1}^{k-1}(u^{n+1} - u^n, y^n). \tag{3.13}$$

Next, we observe that (3.1) implies that

$$\int_\Omega \big(\widehat{\beta}_\lambda(y^k) + \widehat{\pi}(y^k)\big) \geq \frac{1}{2} \int_\Omega \big(\widehat{\beta}_\lambda(y^k) + \widehat{\pi}(y^k)\big) + \frac{\alpha}{2} \|y^k\|^2 - c$$

for every sufficiently small $\lambda > 0$ and that the integrals are bounded from below. Moreover, we differently deal with the right-hand side of (3.13) with respect to [10]. Namely, we estimate it as follows:

$$(u^k, y^k) - (u^1, y_0) - \sum_{n=1}^{k-1}(u^{n+1} - u^n, y^n)$$
$$\leq \frac{\alpha}{4} \|y^k\|^2 + \frac{1}{\alpha} \|u^k\|^2 + \|u^1\| \|y_0\| + \sum_{n=1}^{k-1} h\left\|\frac{u^{n+1} - u^n}{h}\right\| \|y^n\|.$$

At this point, we combine (3.13) with the inequalities just obtained and apply the discrete Gronwall-Bellman lemma given in [35, Thm. 1] by observing that

$$\|u^k\| \leq \|u(0)\| + \|\partial_t u\|_{L^1(0,+\infty;H)}, \quad \text{and} \quad \sum_{n=1}^{k-1} h\left\|\frac{u^{n+1} - u^n}{h}\right\| \leq \|\partial_t u\|_{L^1(0,+\infty;H)},$$

and that the above norm of $\partial_t u$ is finite by (2.20). We obtain the estimate

$$h \|\mu^k\|^2 + \sum_{n=0}^{k-1} h \|\mu^{n+1} - \mu^n\|^2 + \sum_{n=0}^{k-1} h \|A^r \mu^{n+1}\|^2 + \tau \sum_{n=0}^{k-1} h \left\| \frac{y^{n+1} - y^n}{h} \right\|^2$$

$$+ \|y^k\|_{B,\sigma}^2 + \sum_{n=0}^{k-1} \|B^\sigma(y^{n+1} - y^n)\|^2 + \int_\Omega \left(\widehat{\beta}_\lambda(y^k) + \widehat{\pi}(y^k) \right) + \sum_{n=0}^{k-1} \|y^{n+1} - y^n\|^2$$

$$\leq c \quad \text{for } k = 0, \ldots, N. \tag{3.14}$$

In terms of the interpolants (see also [10, Prop. 3.9]), by neglecting the first contribution and recalling that $\mu^0 = 0$, we have that

$$\|\overline{\mu}_h - \underline{\mu}_h\|_{L^2(0,T;H)} + \|A^r \overline{\mu}_h\|_{L^2(0,T;H)} + \|A^r \underline{\mu}_h\|_{L^2(0,T;H)}$$

$$+ \|\underline{y}_h\|_{L^\infty(0,T;V_B^\sigma)} + \|\overline{y}_h\|_{L^\infty(0,T;V_B^\sigma)} + \|\widehat{y}_h\|_{L^\infty(0,T;V_B^\sigma)}$$

$$+ h^{-1/2}\|B^\sigma(\overline{y}_h - \underline{y}_h)\|_{L^2(0,T;H)} + \tau^{1/2}\|\partial_t \widehat{y}_h\|_{L^2(0,T;H)}$$

$$+ \|\widehat{\beta}_\lambda(\overline{y}_h) + \widehat{\pi}(\overline{y}_h)\|_{L^\infty(0,T;L^1(\Omega))} + h^{-1/2}\|\overline{y}_h - \widehat{y}_h\|_{L^2(0,T;H)} \leq c. \tag{3.15}$$

Second uniform estimate. By observing that (3.10) implies $\partial_t \widehat{y}_h + \overline{\mu}_h + A^{2r}\overline{\mu}_h = \underline{\mu}_h$, whence also

$$\int_0^T \left(\partial_t \widehat{y}_h(t), v(t) \right) dt = \int_0^T \left((\underline{\mu}_h - \overline{\mu}_h)(t), v(t) \right) dt - \int_0^T \left(A^r \overline{\mu}_h(t), A^r v(t) \right) dt$$

$$\leq c \left(\|\underline{\mu}_h - \overline{\mu}_h\|_{L^2(0,T;H)} + \|A^r \overline{\mu}_h\|_{L^2(0,T;H)} \right) \|v\|_{L^2(0,T;V_A^r)}$$

for every $v \in L^2(0, T; V_A^r)$, we deduce that

$$\|\partial_t \widehat{y}_h\|_{L^2(0,T;V_A^{-r})} \leq c \left(\|\underline{\mu}_h - \overline{\mu}_h\|_{L^2(0,T;H)} + \|A^r \overline{\mu}_h\|_{L^2(0,T;H)} \right).$$

Hence, from (3.15) we infer that

$$\|\partial_t \widehat{y}_h\|_{L^2(0,T;V_A^{-r})} \leq c. \tag{3.16}$$

Basic estimate. We recall that estimates (3.15)–(3.16) hold for every $N > 1$, every sufficiently small $\lambda > 0$, and every $T > 0$. Now, we owe to the convergence results of [10]. We deduce that

$$\|y^\lambda\|_{L^\infty(0,T;V_B^\sigma)} + \|\partial_t y^\lambda\|_{L^2(0,T;V_A^{-r})} + \tau^{1/2}\|\partial_t y^\lambda\|_{L^2(0,T;H)} + \|A^r \mu^\lambda\|_{L^2(0,T;H)} \leq c.$$

Since c is independent of both λ and T, at the limit as $\lambda \searrow 0$ we conclude that

$$y \in L^\infty(0, +\infty; V_B^\sigma), \quad \partial_t y \in L^2(0, +\infty; V_A^{-r}), \quad \text{and } A^r \mu \in L^2(0, +\infty; H), \tag{3.17}$$

$$\partial_t y \in L^2(0, +\infty; H) \quad \text{if } \tau > 0. \tag{3.18}$$

4 Longtime behavior

This section is devoted to the proofs of our results on the longtime behavior. We start with the proof of Theorem 2.2.

First part. Since y belongs to $L^\infty(0, +\infty; V_B^\sigma)$ by the first conclusion of (3.17), we deduce that the ω-limit ω given by (2.30) is nonempty. Thus, the first sentence of our result is established. Let us come to the second part.

Second part, first case. We first assume that $\lambda_1 > 0$. We pick an arbitrary element $y_\omega \in \omega$ and a sequence $\{t_n\}$ as in (2.30), and we prove that y_ω is a stationary solution in the sense of (2.32). To this end, we define the functions y_n, μ_n, and u_n, on $(0, +\infty)$ by setting, for a.a. $t \in (0, +\infty)$,

$$y_n(t) := y(t + t_n), \quad \mu_n(t) := \mu(t + t_n), \quad \text{and} \quad u_n(t) := u(t + t_n).$$

We notice that (2.20) and (2.31) imply that

$$\|u_n\|_{L^\infty(0,+\infty;H)} \leq c, \quad \text{and} \quad u_n - u_\infty \to 0 \quad \text{strongly in } L^2(0, +\infty; H). \tag{4.1}$$

Moreover, from (3.17) we clearly deduce that

$$\|y_n\|_{L^\infty(0,+\infty;V_B^\sigma)} \leq c, \tag{4.2}$$
$$\partial_t y_n \to 0, \quad \text{strongly in } L^2(0, +\infty; V_A^{-r}), \tag{4.3}$$
$$A^r \mu_n \to 0 \quad \text{strongly in } L^2(0, +\infty; H), \tag{4.4}$$

whence also

$$\mu_n \to 0 \quad \text{strongly in } L^2(0, +\infty; V_A^r), \tag{4.5}$$

since $\lambda_1 > 0$. In addition, we have that

$$\partial_t y_n \to 0 \quad \text{strongly in } L^2(0, +\infty; H) \quad \text{if } \tau > 0. \tag{4.6}$$

By weak-star compactness, we deduce from (4.2) that there exists some element $y_\infty \in L^\infty(0, +\infty; V_B^\sigma)$ such that

$$y_n \to y_\infty \quad \text{weakly star in } L^\infty(0, +\infty; V_B^\sigma), \tag{4.7}$$

at least for a (not relabeled) subsequence. Now, we fix an arbitrary time $T > 0$ and look for the problem solved by y_∞ on $(0, T)$. It is clear that (y_n, μ_n) satisfies the variational inequality

$$\left(\tau \partial_t y_n(t), y_n(t) - v\right) + \left(B^\sigma y_n(t), B^\sigma(y_n(t) - v)\right)$$
$$+ \int_\Omega \widehat{\beta}(y_n(t)) + \left(\pi(y_n(t)) - u_n(t), y_n(t) - v\right) \leq \left(\mu_n(t), y_n(t) - v\right) + \int_\Omega \widehat{\beta}(v)$$
$$\text{for every } v \in V_B^\sigma \text{ and for a.a. } t \in (0, T), \tag{4.8}$$

as well as its integrated version

$$\int_0^T \left(\tau \partial_t y_n(t), y_n(t) - v(t)\right) dt + \int_0^T \left(B^\sigma y_n(t), B^\sigma(y_n(t) - v(t))\right) dt$$
$$+ \int_Q \widehat{\beta}(y_n) + \int_0^T \left(\pi(y_n(t)) - u_n(t), y_n(t) - v(t)\right) dt$$
$$\leq \int_0^T \left(\mu_n(t), y_n(t) - v(t)\right) dt + \int_Q \widehat{\beta}(v) \quad \text{for every } v \in L^2(0, T; V_B^\sigma). \tag{4.9}$$

Now, we want to let n tend to infinity in (4.9). First, by (4.8) with $v = 0$, we have that

$$\|\widehat{\beta}(y_n(t))\|_{L^1(\Omega)} \leq \|B^\sigma y_n(t)\|^2 + \int_\Omega \widehat{\beta}(y_n(t))$$
$$\leq \left(\|\tau \partial_t y_n(t)\| + \|\pi(y_n(t))\| + \|u_n(t)\| + \|\mu_n(t)\|\right) \|y_n(t)\| \quad \text{for a.a. } t \in (0, T).$$

So, by accounting for the Lipschitz continuity of π, and owing to (4.1)–(4.5), we obtain that

$$\|\widehat{\beta}(y_n)\|_{L^2(0,T;L^1(\Omega))} \leq c_T \,. \tag{4.10}$$

On the other hand, by recalling (4.2), (4.3) and the compact embedding $V_B^\sigma \subset H$ that follows from (2.2), we can apply [32, Sect. 8, Cor. 4] and deduce that

$$y_n \to y_\infty \quad \text{strongly in } C^0([0,T];H). \tag{4.11}$$

We infer that $\pi(y_n)$ converges to $\pi(y_\infty)$ in the same topology since π is Lipschitz continuous. In order to deal with the nonlinearity $\widehat{\beta}$, we notice that we can assume that $y_n \to y_\infty$ a.e. in $\Omega \times (0,T)$ so that, by lower semicontinuity, we deduce the inequality

$$\int_Q \widehat{\beta}(y_\infty) \leq \liminf_{n\to\infty} \int_Q \widehat{\beta}(y_n)$$

where the last term is finite by (4.10). As (4.7) also implies that

$$\int_0^T \|B^\sigma y_\infty(t)\|^2 \, dt \leq \liminf_{n\to\infty} \int_0^T \|B^\sigma y_n(t)\|^2 \, dt \,,$$

and since the second statement in (4.1) yields that $u_n \to u_\infty$ strongly in $L^2(0,T;H)$, from (4.9) and (4.7) it follows that y_∞ satisfies the variational inequality

$$\int_0^T \left(B^\sigma y_\infty(t), B^\sigma(y_\infty(t) - v(t)) \right) dt + \int_Q \widehat{\beta}(y_\infty)$$
$$+ \int_0^T \left(\pi(y_\infty(t)) - u_\infty, y_\infty(t) - v(t) \right) dt$$
$$\leq \int_Q \widehat{\beta}(v) \qquad \text{for every } v \in L^2(0,T;V_B^\sigma). \tag{4.12}$$

Equivalently, y_∞ fulfills

$$\left(B^\sigma y_\infty(t), B^\sigma(y_\infty(t) - v) \right) + \int_\Omega \widehat{\beta}(y_\infty(t)) + \left(\pi(y_\infty(t)) - u_\infty, y_\infty(t) - v \right) \leq \int_\Omega \widehat{\beta}(v)$$
$$\text{for every } v \in V_B^\sigma \text{ and for a.a. } t \in (0,T). \tag{4.13}$$

At this point, we can easily conclude. In view of (4.3) and (4.7), we have that

$$\partial_t y_\infty = 0 \,, \quad \text{whence } y_\infty \text{ takes a constant value } \bar{y} \in V_B^\sigma \text{ on } [0,T].$$

On the other hand, $y_n(0)$ converges to $y_\infty(0)$ in H by (4.11). Thus, $y_n(0)$ converges to $\bar{y}$ in H. As $y_n(0) = y(t_n)$ converges weakly to y_ω in V_B^σ by assumption, we conclude that $\bar{y} = y_\omega$, that is,

$$y_\infty(t) = y_\omega \quad \text{for every } t \in [0,T]. \tag{4.14}$$

Therefore, (4.13) becomes (2.32).

Second part, second case. Assume now that $\lambda_1 = 0$. Coming back to the proof just concluded, we see that the assumption $\lambda_1 > 0$ has been used just to obtain (4.5), its consequence (4.10), and to make μ_n disappear in the limiting inequality (2.32). Therefore, the same argument essentially applies in the case $\lambda_1 = 0$ (with the modifications that are needed to prove (2.33) instead of (2.32)),

provided we can derive a convergence property for μ_n (in place of (4.5), which should be false now) and (4.10). To this end, we recall assumption (2.22) and notice that it implies the existence of some $\delta > 0$ such that $m_0 \pm \delta$ belong to $D(\beta)$. Then, as v in (4.8), we choose the convex combination $\frac{1}{2}(m_0 \pm \delta) + \frac{1}{2} y_n(t)$ (thus, with values in $D(\widehat{\beta})$), which gives $y_n(t) - v = \frac{1}{2}(y_n(t) - m_0 \mp \delta)$. Thanks to the convexity of $\widehat{\beta}$, we obtain for a.a. $t \in (0, T)$

$$
\begin{aligned}
&\frac{\tau}{2}\left(\partial_t y_n(t), y_n(t) - m_0 \mp \delta\right) + \frac{1}{2}\left(B^\sigma y_n(t), B^\sigma(y_n(t) - m_0 \mp \delta)\right) \\
&\quad + \int_\Omega \widehat{\beta}(y_n(t)) + \frac{1}{2}\left(\pi(y_n(t)) - u_n(t), y_n(t) - m_0 \mp \delta\right) \\
&\leq \frac{1}{2}\left(\mu_n(t), y_n(t) - m_0 \mp \delta\right) + \int_\Omega \widehat{\beta}\left(\tfrac{1}{2}(m_0 \pm \delta) + \tfrac{1}{2} y_n(t)\right) \\
&\leq \frac{1}{2}\left(\mu_n(t), y_n(t) - m_0 \mp \delta\right) + \frac{1}{2}\int_\Omega \widehat{\beta}(y_n(t)) + \frac{1}{2}\int_\Omega \widehat{\beta}(m_0 \pm \delta).
\end{aligned}
$$

By multiplying by 2 and rearranging, we deduce that

$$
\begin{aligned}
&\pm\delta\left(\mu_n(t), 1\right) + \|B^\sigma y_n(t)\|^2 + \int_\Omega \widehat{\beta}(y_n(t)) \\
&\leq -\tau\left(\partial_t y_n(t), y_n(t) - m_0 \mp \delta\right) + \left(B^\sigma y_n(t), B^\sigma(m_0 \pm \delta)\right) \\
&\quad - \left(\pi(y_n(t)) - u_n(t), y_n(t) - m_0 \mp \delta\right) + \int_\Omega \widehat{\beta}(m_0 \pm \delta) \\
&\quad + \left(\mu_n(t), y_n(t) - m_0\right). \tag{4.15}
\end{aligned}
$$

Now, we recall the conservation property (2.29) and note that the Poincaré type inequality (2.15) is valid since $\lambda_1 = 0$. We thus have that for a.a. $t \in (0, T)$ it holds

$$
\begin{aligned}
\left(\mu_n(t), y_n(t) - m_0\right) &= \left(\mu_n(t) - \operatorname{mean}\mu_n(t), y_n(t) - m_0\right) \\
&\leq c\,\|A^r(\mu_n(t) - \operatorname{mean}\mu_n(t))\|\,\|y_n(t) - m_0\| = c\,\|A^r\mu_n(t)\|\,\|y_n(t) - m_0\|,
\end{aligned}
$$

so that we can use (4.4) in the right-hand side of (4.15). By also accounting for (4.1), (4.2) and (4.6), we deduce that the function

$$
t \mapsto \delta\left|\left(\mu_n(t), 1\right)\right| + \int_\Omega \widehat{\beta}(y_n(t))
$$

is bounded in $L^2(0, T)$, uniformly with respect to n. In particular, (4.10) holds also in this case. Moreover, the mean value of μ_n is estimated in $L^2(0, T)$ so that the definition (2.13) of the norm in V_A^r and (4.4) imply that μ_n is bounded in $L^2(0, T; V_A^r)$. Therefore, we have, at least for a subsequence,

$$
\mu_n \to \mu_\infty \quad \text{weakly in } L^2(0, T; V_A^r), \tag{4.16}
$$

which is the desired convergence property to be established in place of (4.5). At this point, we repeat the argument used in the case $\lambda_1 > 0$ provided that we modify (4.13), since we have (4.16) instead of (4.5). In place of that variational inequality, we obtain the following one:

$$
\begin{aligned}
&\left(B^\sigma y_\infty(t), B^\sigma(y_\infty(t) - v)\right) + \int_\Omega \widehat{\beta}(y_\infty(t)) + \left(\pi(y_\infty(t)) - u_\infty, y_\infty(t) - v\right) \\
&\leq \left(\mu_\infty(t), y_\infty(t) - v\right) + \int_\Omega \widehat{\beta}(v) \\
&\qquad \text{for every } v \in V_B^\sigma \text{ and for a.a. } t \in (0, T). \tag{4.17}
\end{aligned}
$$

On the contrary, (4.14) holds true with the same proof also in the present case. Finally, (4.4) implies that $A^r \mu_\infty(t) = 0$ for a.a. $t \in (0, T)$, i.e., that μ_∞ is space independent since $\lambda_1 = 0$ (cf. (2.11)). Therefore, (4.17) becomes

$$\left(B^\sigma y_\omega, B^\sigma(y_\omega - v) \right) + \int_\Omega \widehat{\beta}(y_\omega) + \left(\pi(y_\omega) - u_\infty, y_\omega - v \right)$$

$$\leq \left(\mu_\infty(t), y_\omega - v \right) + \int_\Omega \widehat{\beta}(v)$$

$$\text{for every } v \in V_B^\sigma \text{ and for a.a. } t \in (0, T). \tag{4.18}$$

Hence, as (4.18) holds for arbitrary values of T, this implies the validity of (2.33) with a proper function $\mu_\infty \in L_{loc}^\infty([0, +\infty))$. Indeed, let us denote for $m = 0, 1, \dots$ by μ_∞^m the function μ_∞ which satisfies (4.18) with $T = m$, and we construct μ_∞ on $(0, +\infty)$ by setting $\mu_\infty(t) := \mu_\infty^m(t)$ for a.a. $t \in (m, m + 1)$, for $m = 0, 1, \dots$. Then, it turns out that $\mu_\infty \in L_{loc}^\infty([0, +\infty))$, and the inequality (2.33) clearly holds. This completes the proof of Theorem 2.2. $\qquad\qquad\square$

Proof of Proposition 2.4. We come back to the proof just concluded, keeping its notation. More precisely, we consider the part of the proof in the first case $i)$ that also holds for the second one. For the first part of Proposition 2.4, it is sufficient to show that, for every $T \in (0, +\infty)$, the function μ_∞ is unique and constant on $(0, T)$. So, we fix an arbitrary $T > 0$. By virtue of the results of [11] summarized just before the statement we are considering, we can replace the variational inequality (2.27) by the equation (2.37), so that (4.8) can be written in the strong form

$$\tau \, \partial_t y_n + B^{2\sigma} y_n + \beta(y_n) + \pi(y_n) = \mu_n + u_n \quad \text{a.e. in } \Omega \times (0, T). \tag{4.19}$$

Now, we observe that our assumption (2.35) obviously implies that y_n takes its values in $[a, b]$. From (4.11) and the Lipschitz continuity of $\beta + \pi$ in $[a, b]$, we thus infer that

$$(\beta + \pi)(y_n) \to (\beta + \pi)(y_\infty) \quad \text{strongly in } C^0([0, T]; H).$$

On the other hand, by comparison in (4.19), we see that $B^{2\sigma} y_n$ is in fact bounded in $L^2(0, T; H)$. Therefore, the limiting function y_∞ belongs to $V_B^{2\sigma}$ and satisfies the equation

$$B^{2\sigma} y_\infty + \beta(y_\infty) + \pi(y_\infty) = \mu_\infty + u_\infty \quad \text{a.e. in } \Omega \times (0, T).$$

But we already know that y_∞ takes the constant value y_ω. Therefore, $y_\omega \in V_B^{2\sigma}$, and we have that

$$B^{2\sigma} y_\omega + \beta(y_\omega) + \pi(y_\omega) = \mu_\infty(t) + u_\infty \quad \text{for a.a. } t \in (0, T).$$

By comparison, we conclude that μ_∞ is unique and time independent, thus constant, and the above equation becomes (2.38). This completes the proof.

References

[1] H. Abels, M. Wilke, Convergence to equilibrium for the Cahn–Hilliard equation with a logarithmic free energy, *Nonlinear Anal.* **67** (2007), 3176-3193.

[2] G. Akagi, G. Schimperna, A. Segatti, Fractional Cahn–Hilliard, Allen–Cahn and porous medium equations, *J. Differential Equations* **261** (2016), 2935-2985.

[3] V. Barbu, "Nonlinear Differential Equations of Monotone Type in Banach Spaces", Springer, London, New York, 2010.

[4] H. Brezis, "Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert", North-Holland Math. Stud. **5**, North-Holland, Amsterdam, 1973.

[5] R. Chill, E. Fašangová, J. Prüss, Convergence to steady state of solutions of the Cahn–Hilliard and Caginalp equations with dynamic boundary conditions, *Math. Nachr.* **279** (2006), 1448-1462.

[6] P. Colli, G. Gilardi, Well-posedness, regularity and asymptotic analyses for a fractional phase field system, *Asymptot. Anal.*, to appear (see also preprint arXiv:1806.04625 [math.AP] (2018), pp. 1-34).

[7] P. Colli, G. Gilardi, Ph. Laurençot, A. Novick-Cohen, Uniqueness and long-time behavior for the conserved phase-field system with memory, *Discrete Contin. Dynam. Systems* **5** (1999), 375-390.

[8] P. Colli, G. Gilardi, P. Podio-Guidugli, J. Sprekels, Well-posedness and long-time behaviour for a nonstandard viscous Cahn–Hilliard system, *SIAM J. Appl. Math.* **71** (2011), 1849-1870.

[9] P. Colli, G. Gilardi, J. Sprekels, On the longtime behavior of a viscous Cahn–Hilliard system with convection and dynamic boundary conditions, *J. Elliptic Parabol. Equ.* **4** (2018), 327-347.

[10] P. Colli, G. Gilardi, J. Sprekels, Well-posedness and regularity for a generalized fractional Cahn–Hilliard system, *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, to appear (see also preprint arXiv:1804.11290 [math.AP] (2018), pp. 1-36).

[11] P. Colli, G. Gilardi, J. Sprekels, Optimal distributed control of a generalized fractional Cahn–Hilliard system, *Appl. Math. Optim.* doi:10.1007/s00245-018-9540-7 (see also preprint arXiv:1807.03218 [math.AP] (2018), pp. 1-36).

[12] M. Efendiev, H. Gajewski, S. Zelik, The finite dimensional attractor for a 4th order system of the Cahn–Hilliard type with a supercritical nonlinearity, *Adv. Differential Equations* **7** (2002), 1073-1100.

[13] M. Efendiev, A. Miranville, S. Zelik, Exponential attractors for a singularly perturbed Cahn–Hilliard system. *Math. Nachr.* **272** (2004), 11-31.

[14] C. G. Gal, Exponential attractors for a Cahn–Hilliard model in bounded domains with permeable walls, *Electron. J. Differential Equations* (2006), No. 143, 23 pp.

[15] C. Gal, Well-posedness and long time behavior of the non-isothermal viscous Cahn–Hilliard equation with dynamic boundary conditions, *Dyn. Partial Differ. Equ.* **5** (2008), 39-67.

[16] C. G. Gal, Non-local Cahn–Hilliard equations with fractional dynamic boundary, *European J. Appl. Math.* **28** (2017), 736-788.

[17] C. G. Gal, M. Grasselli, Asymptotic behavior of a Cahn–Hilliard–Navier–Stokes system in 2D, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27** (2010), 401-436.

[18] C. G. Gal, M. Grasselli, Instability of two-phase flows: a lower bound on the dimension of the global attractor of the Cahn–Hilliard–Navier–Stokes system, *Phys. D* **240** (2011), 629-635.

[19] G. Gilardi, A. Miranville, G. Schimperna, Long-time behavior of the Cahn–Hilliard equation with irregular potentials and dynamic boundary conditions, *Chin. Ann. Math. Ser. B* **31** (2010), 679-712.

[20] G. Gilardi, E. Rocca, Well-posedness and long-time behaviour for a singular phase field system of conserved type, *IMA J. Appl. Math.* **72** (2007), 498-530.

[21] G. Gilardi, J. Sprekels, Asymptotic limits and optimal control for the Cahn–Hilliard system with convection and dynamic boundary conditions, *Nonlinear Anal.* **178** (2019), 1-31.

[22] M. Grasselli, H. Petzeltová, G. Schimperna, Asymptotic behavior of a nonisothermal viscous Cahn–Hilliard equation with inertial term, *J. Differential Equations* **239** (2007), 38-60.

[23] J. Jiang, H. Wu, S. Zheng, Well-posedness and long-time behavior of a non-autonomous Cahn–Hilliard–Darcy system with mass source modeling tumor growth, *J. Differential Equations* **259** (2015), 3032-3077.

[24] D. Li, C. Zhong, Global attractor for the Cahn–Hilliard system with fast growing nonlinearity, *J. Differential Equations* **149** (1998), 191-210.

[25] A. Miranville, Long-time behavior of some models of Cahn–Hilliard equations in deformable continua, *Nonlinear Anal.* **2** (2001), 273-304.

[26] A. Miranville, Asymptotic behavior of the Cahn–Hilliard–Oono equation, *J. Appl. Anal. Comput.* **1** (2011), 523-536.

[27] A. Miranville, Asymptotic behavior of a generalized Cahn–Hilliard equation with a proliferation term, *Appl. Anal.* **92** (2013), 1308-1321.

[28] A. Miranville, S. Zelik, Robust exponential attractors for Cahn–Hilliard type equations with singular potentials, *Math. Methods Appl. Sci.* **27** (2004), 545-582.

[29] A. Miranville, S. Zelik, Exponential attractors for the Cahn–Hilliard equation with dynamic boundary conditions, *Math. Models Appl. Sci.* **28** (2005), 709-735.

[30] J. Prüss, V. Vergara, R. Zacher, Well-posedness and long-time behaviour for the non-isothermal Cahn–Hilliard equation with memory, *Discrete Contin. Dyn. Syst.* **26** (2010), 625-647.

[31] A. Segatti, On the hyperbolic relaxation of the Cahn–Hilliard equation in 3D: approximation and long time behaviour, *Math. Models Methods Appl. Sci.* **17** (2007), 411-437.

[32] J. Simon, Compact sets in the space $L^p(0, T; B)$, *Ann. Mat. Pura Appl. (4)* **146** (1987), 65-96.

[33] X.-M. Wang, H. Wu, Long-time behavior for the Hele–Shaw–Cahn–Hilliard system, *Asymptot. Anal.* **78** (2012), 217-245.

[34] H. Wu, S. Zheng, Convergence to equilibrium for the Cahn–Hilliard equation with dynamic boundary conditions, *J. Differential Equations* **204** (2004), 511-531.

[35] C.-C. Yeh, Discrete inequalities of the Gronwall-Bellman type in n independent variables, *J. Math. Anal. Appl.* **105** (1985), 322-332.

[36] X. Zhao, C. Liu, On the existence of global attractor for 3D viscous Cahn–Hilliard equation, *Acta Appl. Math.* **138** (2015), 199-212.

[37] S. Zheng, Asymptotic behavior of solution to the Cahn–Hilliard equation, *Appl. Anal.* **23** (1986), 165-184.