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Precise Laplace asymptotics for singular stochastic PDEs: The case of 2D gPAM



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ABSTRACT

We implement a Laplace method for the renormalised solution to the generalised 2D Parabolic Anderson Model (gPAM) driven by a small spatial white noise. Our work rests upon Hairer’s theory of regularity structures which allows to generalise classical ideas of Azencott and Ben Arous on path space as well as Aida and Inahama and Kawabi on rough path space to the space of models. The technical cornerstone of our argument is a Taylor expansion of the solution in the noise intensity parameter: We prove precise bounds for its terms and the remainder and use them to estimate asymptotically irrelevant terms to arbitrary order. While most of our arguments are not specific to gPAM, we also outline how to adapt those that are.

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0. Introduction

We start this introduction with a review of the history of (precise) Laplace asymptotics and mention some of their applications (subsection 0.1). Our contribution is the development of the Laplace method for the generalised Parabolic Anderson Model (gPAM), a singular stochastic PDE, so we collect its well-posedness and large deviation results in subsection 0.2. We then state our main results in full detail in subsection 0.3 and indicate how they may be generalised to cover other singular stochastic PDEs (subsection 0.4). Finally, we discuss the organisation of this article in subsection 0.5.

0.1. History, relation to other work, and applications

In 1774, Laplace [68]¹ himself devised the method that now bears his name to calculate asymptotics as $\varepsilon \rightarrow 0$ of integrals

$$\int_D f(x) \exp\left(-\frac{F(x)}{\varepsilon^2}\right) dx$$

¹ The quoted article is an English translation of the original 1774 article in French. The latter is properly referenced in the article we quoted.

for intervals $D = [a, b]$ and functions F with a unique minimum inside D . It was generalised later to cover (some) domains $D \subseteq \mathbb{R}^d$ for $d > 1$ as well and quickly found its way into the canon of classical asymptotic analysis. In parallel to the rapid developments in mathematical physics, Laplace’s method gained further traction: it inspired asymptotic approximation techniques that were successfully applied in functional space. More precisely, they allowed to study newly emerging *path integrals* of the form

$$\mathfrak{J}_D(\varepsilon) := \int_D f(x) \exp\left(-\frac{F(x)}{\varepsilon^2}\right) d\mathbb{P}\left(\frac{x}{\varepsilon}\right)$$

as popularised by Glimm and Jaffe [40] when $\varepsilon \rightarrow 0$, where D is a Borel subset of a Banach space \mathcal{X} , f and F are real-valued functions on \mathcal{X} , and \mathbb{P} is a probability measure on $\mathcal{B}(\mathcal{X})$.² In this direction, pioneering work was done by Feynman [32] in his research on quantum mechanics in the 1960’s.

Probabilists joined the venture in the 1960’s. Building on earlier works of Cramer, Varadhan’s theory of *large deviations* [79] provides a powerful framework to set up Laplace’s method in infinite-dimensional spaces and, in turn, allowed to calculate log-asymptotics of $\mathfrak{J}_{\mathcal{X}}(\varepsilon)$ as $\varepsilon \rightarrow 0$. A large deviation principle for rescaled Wiener measure was given in [75], as were precise asymptotic results for Wiener integrals of the form (0.1), with $X^\varepsilon = \varepsilon w$ in notation introduced below. We fix the following notation: For a family $(X^\varepsilon : \varepsilon \in [0, 1])$ of \mathcal{X} -valued random variables that satisfies a large deviation principle (LDP), we set³

$$J(\varepsilon) := \mathbb{E} \left[\exp\left(-\frac{F(X^\varepsilon)}{\varepsilon^2}\right) \right] \tag{0.1}$$

where $F : \mathcal{X} \rightarrow \mathbb{R}$ is assumed to have $N + 3$ Fréchet derivatives for some $N \geq 0$. *Precise* Laplace asymptotics, eponymous for this article, assert that J has an expansion of type

$$J(\varepsilon) = \exp(-c\varepsilon^{-2})(a_0 + \varepsilon a_1 + \dots + \varepsilon^N a_N + o(\varepsilon^N)) \quad \text{as } \varepsilon \rightarrow 0 \tag{0.2}$$

with explicitly given coefficients $c, a_0, \dots, a_N \in \mathbb{R}$. The hypotheses (H1) to (H4) on F – detailed in Theorem 1 below – under which this statement is true have found their definite form in the works of Ben Arous [11] on path space $\mathcal{X} = \mathcal{C}([0, 1]; \mathbb{R}^d)$: Building on foundational work by Azencott [8] (in the elliptic case), he establishes (0.2) for finite-dimensional diffusions

$$dX_t^\varepsilon = \varepsilon \sigma(X_t^\varepsilon) dw_t + b(\varepsilon, X_t^\varepsilon) dt, \quad X_0^\varepsilon = x, \tag{0.3}$$

² One also requires sufficient regularity of f , F , and ∂D , the boundary of D .

³ The restriction to $f \equiv 1$ is for simplicity only. See also Remark 0.5 below.

where w is an r -dimensional Brownian motion (BM) and σ and b satisfy suitable boundedness and differentiability assumptions. All of the previous statements, and many more, are surveyed in an extensive review article by Piterbarg and Fatalov [72]. We further note that sharp Laplace asymptotics (0.2) for some classical stochastic PDEs have been established by Rovira and Tindel [73,74]. See also Albeverio et al. [1,7].

The advent of pathwise solution techniques initiated by Lyons’s theory of *rough paths* [69] has brought new momentum to the Laplace method. Starting with Aida [3], the Japanese school of stochastic analysis has spearheaded these developments: Inahama [56,55] and then Inahama and Kawabi [53,51,52] consider the Stratonovich version of (0.3) in infinite-dimensional Banach spaces in which the classical theory of stochastic differential equations (SDEs) breaks down. They bypass this shortcoming by referring to rough paths theory – which *does* work in infinite dimensions – and obtain the expansion (0.2) in this context. In another article, Inahama [57] has considered (0.3) for $w = w^H$, an r -dimensional fractional Brownian motion (fBM) of Hurst parameter $H \in (1/4, 1/2]$ with rough path lift \mathbf{W}^H . More precisely, he considers the solution X^ε of the rough differential equation (RDE)

$$dX_t^\varepsilon = \varepsilon\sigma(X_t^\varepsilon) d\mathbf{W}_t^H + b(\varepsilon, X_t^\varepsilon) dt, \quad X_0^\varepsilon = 0. \tag{0.4}$$

When b and σ are sufficiently smooth with bounded derivatives, he establishes (0.2) under (essentially) the same hypotheses (H1) to (H4) for $\mathcal{X} = \mathcal{C}^{q-\text{var}}([0, 1]; \mathbb{R}^d)$ and $q > H^{-1}$.

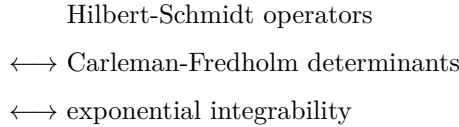
In recent years, insights from rough paths theory have contributed to important breakthroughs in stochastic PDEs: In the seminal article [44], Hairer derives an *intrinsic* local solution to the KPZ equation and then developed his ideas into the theory of *regularity structures* [45], a far-reaching generalisation of rough paths (RP) theory. — In this article, we consider a popular example that Hairer’s theory allows to treat – the generalised 2D Parabolic Anderson Model (gPAM) formally given by the equation

$$\partial_t \hat{u}^{(\varepsilon)} = \Delta \hat{u}^{(\varepsilon)} + g(\hat{u}^{(\varepsilon)}) (\varepsilon \xi - \varepsilon^2 \text{, , } \infty \text{ } g'(\hat{u}^{(\varepsilon)})), \quad \hat{u}(0, \cdot) = u_0,$$

where ξ is a spatial white noise (SWN) – and establish the expansion (0.2) for $X^\varepsilon = \hat{u}^{(\varepsilon)}$. See subsection 0.2 for a detailed discussion of gPAM and its large deviation behaviour and subsection 0.3 for a precise statement of our results. In subsection 0.4, we will explain our focus on gPAM, point out the difficulties in taking the solutions of other singular stochastic PDEs for X^ε , and indicate how to overcome them.

We emphasise that the arguments presented in this article immediately generalise Inahama’s result [57] for the specific driver \mathbf{W}^H in (0.4) to *any* Gaussian rough path [37,38]. On the surface, the restriction in Inahama’s work is due to its heavy reliance on a variation embedding of \mathcal{H}^H , the Cameron-Martin space of w^H , due to Friz and Victoir [36] – but ultimately can be traced further: the embedding allows him to identify a certain

operator (below denoted by $D^2(F \circ \Phi \circ \mathcal{L})(h)$) as Hilbert-Schmidt and then resort to the classical correspondence



for elements in the second Wiener chaos, see for example Janson [61, Chap. 6]. In contrast, our argument is solely based on large deviations⁴ and therefore bypasses this correspondence completely, see subsection 3.2 for details and subsection 0.5 for an outline of the argument. In turn, we do not have to rely on the variation embedding specific to \mathcal{H}^H .

A Laplace method, in spirit of Azencott [9], think asymptotic evaluation of $\mathbb{P}(X_1^\varepsilon \in A)$, rather than $\mathbb{E} [e^{-F(X^\varepsilon)/\varepsilon^2}]$ as in [8,11], on rough path type spaces has also been used in [31] to analyze option price asymptotics of *rough volatility*, cf. [15] and also Section 14.5 in [33] for context. Related “pre-rough” works, also with applications in finance, include Kusuoka and Osajima [65,71].

Finally, we thank Ismael Bailleul for pointing us to his recent work with Masato Hoshino [18, Sec. 8] which establishes the smoothness of the abstract solution map to singular stochastic PDEs (our Theorem 2.26 in the context of gPAM) in the greatest generality that the regularity structures framework allows. This is an important first step towards obtaining so-called “Stochastic Taylor expansions” (see Theorem 2 below) for *any* subcritical singular stochastic PDE. However, it does not resolve the issue of getting quantitative remainder estimates (as in Theorem 2 (i) for gPAM) in greatest generality. In addition, the setting of Bailleul and Hoshino does not deal with Cameron-Martin perturbations $\xi \mapsto \xi + h$ of the noise which Theorem 1 relies on. See sec. 0.4 below for a sketch of the technical problem that these perturbations induce.

Applications. Beside being interesting in their own right, precise Laplace asymptotics have applications in various fields of stochastic analysis and mathematical physics.

In that regard, we mention a series of articles by Aida [2–6] concerned with *semi-classical analysis*. More specifically, he studies the lowest eigenvalue $E_0(\lambda)$, or ground state, of a Schrödinger-type operator $-L_{\lambda,V} = -L + V_\lambda$ on an abstract Wiener space (B, H, μ) . When L is the Ornstein-Uhlenbeck operator on $L^2(B, \mu)$ and $V_\lambda = \lambda V(\lambda^{-1/2} \cdot)$ a rescaled potential on B , asymptotics like (0.2) are instrumental in his work for proving a lower bound for the so-called semi-classical limit⁵ of $E_0(\lambda)$ as $\lambda \rightarrow \infty$.

Another major field of application are precise *heat kernel asymptotics* in the small time limit. Consider the Stratonovich SDE given by

⁴ Ben Arous [11] also used a large deviation argument. See Remark 3.5 for further comments in this direction.

⁵ In general, semi-classical physics is concerned with the behaviour of quantum mechanical equations as Planck’s constant \hbar goes to 0. This can be phrased as an equivalent problem for which a parameter λ tends to ∞ .

$$dX_t^\varepsilon = \varepsilon \sum_{i=1}^r V_i(X_t^\varepsilon) \circ dw_t^i + \varepsilon^2 V_0(X_t^\varepsilon) dt, \quad X_0^\varepsilon = z \in \mathbb{R}^d, \tag{0.5}$$

where w is an r -dim. BM and V_0, \dots, V_r are smooth vector fields (VF) on \mathbb{R}^d satisfying Hörmander’s bracket condition. Under these assumptions, it is well-known that X_t^1 has a smooth density $p_t(z, \cdot)$ w.r.t. λ^d , the d -dim. Lebesgue measure, for each $t > 0$. From an analytical viewpoint, the generator $\varepsilon^2 \mathcal{L}$ of X^ε (where $\mathcal{L} = V_0 + 1/2 \sum_{i=1}^r V_i^2$) gives rise to the parabolic PDE $\partial_t f = \mathcal{L} f$ of which $p_t(z, \cdot)$ is a fundamental solution – hence the terminology *heat kernel*. Ben Arous [10], building on earlier works by Azencott, Bismut and other, obtains the asymptotic expansion, outside the cutlocus,

$$p_t(z, \bar{z}) = t^{-d/2} \exp\left(-\frac{D(z, \bar{z})}{2t}\right) \left(a_0(z, \bar{z}) + ta_1(z, \bar{z}) + \dots + t^N a_N(z, \bar{z}) + o(t^N)\right) \tag{0.6}$$

as $t \rightarrow 0$

with the following rough strategy of proof: first, Brownian scaling implies that $X_\tau^\varepsilon = X_{\varepsilon^2 \tau}^1$ in distribution, so the small time problem can be transformed into a small noise problem. Secondly, the *Laplace method* is employed to expand the Fourier transform of $p_{\varepsilon^2 \tau}(z, \cdot)$. Fourier inversion then leads to the expansion (0.6), which is where one uses non-degeneracy of the Malliavin matrix. See also Watanabe, Takanobu [80,78] and Kusuoka–Stroock [67] for some representative works on the applications of Malliavin calculus to precise asymptotics of large classes of Wiener functionals.

In the Young regime, more precisely when $w = w^H$ in (0.5) is a fBM with Hurst index $H > 1/2$, Baudoin–Hairer [17] established the existence of a smooth density. Baudoin–Ouyang [20] as well as Inahama [60] then obtained precise heat kernel asymptotics for said density.

These results have subsequently been generalised to the rough regime by Inahama, first for $H \in (1/3, 1/2]$ under ellipticity [59], followed by Inahama and Naganuma [54] under Hörmander’s condition and for $H \in (1/4, 1/2]$, which is the decisive regime for which this can be expected [21].

In the case of singular stochastic PDEs, the existence of densities has been settled by Cannizzaro, Friz, and Gassiat [23] for gPAM and by Gassiat and Labbé [41] for the Φ_3^4 equation. Schönbauer [76] has then generalised these results in the most general regularity structures framework. In all these singular SPDE works, the obstacle in obtaining smoothness is the lack of moment estimates for the inverse of the Malliavin matrix. (In rough paths / SDE context, this was overcome in [26,25]; a meaningful extension to the singular SPDE setting remains an open problem.)

Our work, in a similar setting as [23], constitutes a first example of a family of measures on infinite-dimensional space, well-defined only through singular SPDE theory, for which large deviations can be extended to full Laplace asymptotics. Techniques from singular stochastic PDEs have recently been used to construct the Φ_3^4 measure [47,16] and the Sine-Gordon measure, for which large deviation results (in the semi-classical limit) were

recently shown in [13]; it is likely that the methods of this paper will also be of relevance in establishing precise Laplace asymptotics in these models.

0.2. The generalised parabolic Anderson model and its large deviations

Formally, 2D gPAM⁶ is given by the stochastic PDE

$$\begin{cases} \partial_t u^\varepsilon(t, x) &= \Delta u^\varepsilon(t, x) + g(u^\varepsilon(t, x))\varepsilon\xi(x), \\ u^\varepsilon(0, x) &= u_0(x) \end{cases}, \quad (t, x) \in [0, \infty) \times \mathbb{T}^2, \quad (\text{gPAM}_\varepsilon)$$

where the initial datum $u_0 \in \mathcal{C}^\eta(\mathbb{T}^2)$ for $\eta \in (1/2, 1)$ is fixed, $g \in \mathcal{C}_b^{N+7}(\mathbb{R}, \mathbb{R})$ with $N \geq 0^7$, $\varepsilon \in I := [0, 1]$, and ξ is the centred Gaussian field with Cameron-Martin space $\mathcal{H} := L^2(\mathbb{T}^2)$ and formal covariance structure $\rho(x, y) = \delta(x - y)$ known as spatial white noise (SWN). The covariance correctly suggests that SWN is a highly singular object: it can be proved that $\xi \in \mathcal{C}^{-\frac{d}{2}-\kappa}(\mathbb{T}^d)$ a.s. for each $\kappa > 0$ and spatial dimension d . From Schauder theory, one then expects u^ε (and a fortiori $g(u^\varepsilon)$) to be in $\mathcal{C}^{2-\frac{d}{2}-\kappa}$. As is well-known from harmonic analysis, however, the product $g(u^\varepsilon)\xi$ is well-defined if and only if

$$\left(2 - \frac{d}{2} - \kappa\right) + \left(-\frac{d}{2} - \kappa\right) = 2 - d - 2\kappa > 0,$$

which fails in case $d = 2$ and thus proves (gPAM_ε) ill-posed. At the expense of modifying the equation (gPAM_ε) in a *renormalisation procedure*, this malaise has been overcome with Hairer’s advent of *regularity structures*.⁸ In our context, [45, Thm. 1.11] reads:

Theorem 0.1. Fix $\varepsilon > 0$ and let $\xi_\delta := \xi * \rho_\delta$, where $\rho_\delta := \delta^{-2}\rho(\delta^{-1}\cdot)$ is a spatially rescaled version of a fixed mollifier ρ . Then, there exists a choice of constants $(\mathfrak{c}_\delta : \delta > 0)$ and a stopping time $T^\varepsilon > 0$ such that for each $u_0 \in \mathcal{C}^\eta(\mathbb{T}^2)$ and each $T > 0$, the classical solution $\hat{u}_{\xi_\delta}^\varepsilon$ to the renormalised equation

$$\begin{cases} \partial_t \hat{u}_{\xi_\delta}^\varepsilon &= \Delta \hat{u}_{\xi_\delta}^\varepsilon + g(\hat{u}_{\xi_\delta}^\varepsilon) (\varepsilon\xi_\delta - \varepsilon^2\mathfrak{c}_\delta g'(\hat{u}_{\xi_\delta}^\varepsilon)) \\ \hat{u}_{\xi_\delta}^\varepsilon(0, \cdot) &= u_0, \end{cases} \quad (\widehat{\text{gPAM}}_\varepsilon)$$

converges on $\{T^\varepsilon > T\}$ in probability, as $\delta \rightarrow 0$, to a unique limit

⁶ For further motivations of linear PAM as well as its relation to (spatially) discrete models, the reader is directed to the monograph of König [66].

⁷ For Laplace asymptotics, assuming differentiability of order $N + 3$ is commonplace. The additional “+4” is due to regularity structure (and reminiscent of rough paths) theory, see Remark A.36 for details.

⁸ *Paracontrolled calculus* developed by Gubinelli, Imkeller, and Perkowski [39] provides an alternative solution theory.

$$\hat{u}^\varepsilon \in \mathcal{X}_T := \{u \in \mathcal{C}([0, T], \mathcal{C}^\eta(\mathbb{T}^2)) : u(0, \cdot) = u_0\}$$

which is independent of the mollifier ρ . \spadesuit

The regularity structures set-up to be used in this article will be introduced in Appendix A. Its main idea is the following lifting procedure: at first, one constructs a finite number of explicit stochastic objects taylored to the structure of $(\text{gPAM}_\varepsilon)$; their collection is called the *canonical lift* $\mathcal{Z}^{\varepsilon\xi_\delta} = \mathcal{L}(\varepsilon\xi_\delta)$ of the driving noise $\varepsilon\xi_\delta$ to the *model space* \mathcal{M} . Secondly, one establishes the continuity of the solution map $\Phi : \mathcal{M} \rightarrow \mathcal{X}_T$ and, thirdly, builds a renormalised version $\hat{\mathcal{Z}}^{\varepsilon\xi_\delta} \in \mathcal{M}$ from $\mathcal{Z}^{\varepsilon\xi_\delta}$ such that $\hat{u}_\varepsilon^\varepsilon = \Phi(\hat{\mathcal{Z}}^{\varepsilon\xi_\delta})$. At last, one proves that the limit

$$\delta_\varepsilon \hat{\mathcal{Z}} := \lim_{\delta \rightarrow 0} \hat{\mathcal{Z}}^{\varepsilon\xi_\delta} = \lim_{\delta \rightarrow 0} \delta_\varepsilon \hat{\mathcal{Z}}^{\varepsilon\xi_\delta}$$

exists in probability in \mathcal{M} – and finally *defines* $\hat{u}^\varepsilon := \Phi(\delta_\varepsilon \hat{\mathcal{Z}})$. Hairer carried out this programme for the equation $(\text{gPAM}_\varepsilon)$, amongst others, in the seminal article [45]. In a series of papers [19,24,14], it has subsequently been developed into a black-box theory to cover a large class of singular stochastic PDEs.

With the (local in time) existence of solutions secured, Hairer and Weber have subsequently investigated the large deviation behaviour of $(\hat{u}^\varepsilon : \varepsilon \in I)$. More precisely, they obtained the following result [50, Thm. 4.4], where “ u^\spadesuit ” denotes a “graveyard path”:

Theorem 0.2. *The family $(\hat{u}^\varepsilon : \varepsilon \in I)$ satisfies a large deviation principle (LDP) in $\mathcal{X}_T \cup \{u^\spadesuit\}$ with good rate function (RF) \mathcal{J} , where $\hat{u}^\varepsilon := u^\spadesuit$ if $T^\varepsilon < T$.⁹ \spadesuit*

Remark 0.3. Actually, Hairer and Weber have dealt with the Φ_3^4 equation rather than $(\text{gPAM}_\varepsilon)$ but their methods generalise to all equations that fall into the framework of Hairer’s earlier work [45]. For the reader’s convenience, we will spell out their argument for the case of $(\text{gPAM}_\varepsilon)$ in Appendix C. A more detailed formulation of Theorem 0.2 is provided in Theorem C.2. Regarding explosion times, see Appendix B for details. \spadesuit

Large deviations provide a first answer concerning the asymptotics of J in (0.1) on logarithmic scale: for $X^\varepsilon := \hat{u}^\varepsilon$ and $F : \mathcal{X}_T \rightarrow \mathbb{R}$ continuous and bounded, Varadhan’s lemma [29, Thm. 2.1.10] implies that

$$J(\varepsilon) = \exp\left(-\frac{\inf_{\mathcal{X}_T}(F + \mathcal{J}) + o(1)}{\varepsilon^2}\right), \quad \varepsilon \rightarrow 0. \tag{0.7}$$

We will significantly sharpen (0.7) by obtaining the *full expansion*.

⁹ In case $T > T^\varepsilon$, it seems more natural to solve the stochastic PDE up to time T^ε and only set $\hat{u}^\varepsilon(t, \cdot) := u^\spadesuit$ for $t \in (T^\varepsilon, T]$, where “ u^\spadesuit ” is a graveyard *state*. However, we follow Hairer and Weber [50] in discarding all such trajectories in the first place by sending them to the graveyard path u^\spadesuit .

0.3. Main results

Let $h \in \mathcal{H}$ and consider the equation

$$(\partial_t - \Delta)\hat{u}_{\xi_\delta;h}^\varepsilon = g(\hat{u}_{\xi_\delta;h}^\varepsilon)(\varepsilon\xi_\delta + h - \varepsilon^2\mathbf{c}_\delta g'(\hat{u}_{\xi_\delta;h}^\varepsilon)), \quad \hat{u}_{\xi_\delta;h}^\varepsilon(0, \cdot) = u_0, \tag{0.8}$$

a shifted version of $(\widehat{\text{gPAM}}_\varepsilon)$. In analogy with Theorem 0.1, we set $\hat{u}_h^\varepsilon := \lim_{\delta \rightarrow 0} \hat{u}_{\xi_\delta;h}^\varepsilon$ (in prob. in \mathcal{X}_T).

Our main result, eponymous for this work, reads as follows:

Theorem 1 (Precise Laplace asymptotics). *Let $g \in \mathcal{C}_b^{N+7}$. For $T > 0$, we assume that $F : \mathcal{X}_T \rightarrow \mathbb{R}$ satisfies the following hypotheses:*

- (H1) $F \in \mathcal{C}_b(\mathcal{X}_T; \mathbb{R})$, i.e. F is continuous and bounded from \mathcal{X}_T into \mathbb{R} .
- (H2) The functional \mathcal{F} given by

$$\mathcal{F} := (F \circ \Phi \circ \mathcal{L}) + \mathcal{I} : \mathcal{H} \rightarrow (-\infty, +\infty] \tag{0.9}$$

attains its unique minimum¹⁰ at $\mathbf{h} \in \mathcal{H}$. Here, \mathcal{I} denotes Schilder’s rate function, i.e.

$$\mathcal{I}(h) = \frac{\|h\|_{\mathcal{H}}^2}{2} \mathbf{1}_{h \in \mathcal{H}} + \infty \mathbf{1}_{h \notin \mathcal{H}}. \tag{0.10}$$

- (H3) The functional F is $N + 3$ times Fréchet differentiable in a neighbourhood \mathcal{N} of $(\Phi \circ \mathcal{L})(\mathbf{h})$. In addition, there exist constants M_1, \dots, M_{N+3} such that

$$|D^{(k)}F(v)[y, \dots, y]| \leq M_k \|y\|_{\mathcal{X}_T}^k, \quad k = 1, \dots, N + 3, \tag{0.11}$$

holds for all $v \in \mathcal{N}$ and $y \in \mathcal{X}_T$.

- (H4) The minimiser \mathbf{h} is non-degenerate. More precisely, $D^2\mathcal{F}|_{\mathbf{h}} > 0$ in the sense that for all $h \in \mathcal{H} \setminus \{0\}$ we have

$$D^2\mathcal{F}|_{\mathbf{h}}[h, h] > 0. \tag{0.12}$$

¹⁰ In analogy with previous works on Laplace asymptotics, for example Ben Arous [11, sec. 1.4], we emphasise that the theorem remains valid if there are *finitely many* minimisers: One simply localises around each of them and proceeds with the local analysis. The value of the coefficients a_0, \dots, a_N , however, will of course change in that case.

Then, we have the precise Laplace asymptotics

$$\begin{aligned}
 J(\varepsilon) &= \mathbb{E} \left[\exp \left(-\frac{F(\hat{u}^\varepsilon)}{\varepsilon^2} \right) \mathbf{1}_{T^\varepsilon > T} \right] \\
 &= \exp \left(-\frac{\mathcal{F}(\mathbf{h})}{\varepsilon^2} \right) (a_0 + \varepsilon a_1 + \dots + \varepsilon^N a_N + o(\varepsilon^N)) \quad \text{as } \varepsilon \rightarrow 0,
 \end{aligned}
 \tag{0.13}$$

where

$$a_0 = \mathbb{E} \left[\exp \left(-\frac{1}{2} \hat{Q}_{\mathbf{h}} \right) \right], \quad \hat{Q}_{\mathbf{h}} := \partial_\varepsilon^2|_{\varepsilon=0} F(\hat{u}_{\mathbf{h}}^\varepsilon).$$

The other coefficients $(a_k)_{k=1}^N \in \mathbb{R}$ are given explicitly in section 3.3, more specifically in Proposition 3.6. \spadesuit

Remark 0.4. Observe that, under our hypothesis (H2), the factor $\exp(-\varepsilon^{-2} \mathcal{F}(\mathbf{h}))$ in formula (0.13) coincides with the RHS of (0.7). In other words, we recover the previously known large deviation factor. \spadesuit

Remark 0.5. In many works on precise Laplace asymptotics, the functional J is slightly more general than (0.1), namely

$$J(\varepsilon) = \mathbb{E} \left[f(X^\varepsilon) \exp \left(-\frac{F(X^\varepsilon)}{\varepsilon^2} \right) \mathbf{1}_{T^\varepsilon > T} \right],$$

where $f : \mathcal{X}_T \rightarrow \mathbb{R}$ is continuous, bounded, as well as $(N + 1)$ times Fréchet differentiable and positive in a neighbourhood of $(\Phi \circ \mathcal{L})(\mathbf{h})$. Since the presence of f does not add conceptual difficulty, we work with $f \equiv 1$ to streamline the presentation. However, our work can easily be amended for general f . \spadesuit

Remark 0.6. Note that $\hat{Q}_{\mathbf{h}}$ is in the second inhomogeneous Wiener chaos with non-trivial components only in \mathcal{H}_k for $k \in \{0, 2\}$. As such, it admits the representation $\hat{Q}_{\mathbf{h}} = \mathbb{E} [\hat{Q}_{\mathbf{h}}] + I_2(A)$ for some Hilbert-Schmidt operator A and I_2 the second Wiener-Itô isometry. From general principles (see [61, Thm. 6.2] and [77, Thm. 9.2]), this immediately implies that

$$a_0 = e^{-1/2 \mathbb{E}[\hat{Q}_{\mathbf{h}}]} \det_2(\text{Id}_{\mathcal{H}} + A)^{-1/2}
 \tag{0.14}$$

where \det_2 denotes the Carleman-Fredholm determinant. In recent works on precise Laplace asymptotics such as [52], it is shown (in our notation) that

$$A[k, k] = \frac{1}{2} Q_{\mathbf{h}}(\mathcal{L}(k)), \quad \mathbb{E} [\hat{Q}_{\mathbf{h}}] = \text{Tr}(A - \tilde{A}) + c$$

for any $k \in \mathcal{H}$, some explicit $c \in \mathbb{R}$, and some explicit Hilbert-Schmidt operator \tilde{A} that renders $A - \tilde{A}$ trace-class. Such information is interesting from the viewpoint of semi-classical analysis but requires considerably more effort

- (1) due to the need for renormalisation (recall that \hat{Q}_h is *quadratic* in ξ) and
- (2) since one needs more than just Cameron-Martin regularity of the minimiser h (see our sketch in sec. 0.4 below) to show that $A - \tilde{A}$ is trace-class.

These problems are addressed in great detail in a recent follow-up article [64] of the second named author. \spadesuit

Along the way, and of paramount importance in the proof of Theorem 1, we obtain a formula that is commonly referred to as (*stochastic*) *Taylor expansion*.¹¹ We believe that this result is of independent interest and therefore consider it the second contribution of this paper:

Theorem 2 (*Taylor expansion in the noise intensity*). *Let $h \in \mathcal{H}$ and $g \in \mathcal{C}_b^{\ell+4}$ for some $\ell \geq 1$. Then, for each $T > 0$ there exists an $\varepsilon_0 = \varepsilon_0(T) > 0$ such that with $I_0 := [0, \varepsilon_0)$, the function*

$$\hat{u}_h^\bullet : I_0 \rightarrow \mathcal{X}_T, \quad \varepsilon \mapsto \hat{u}_h^\varepsilon \tag{0.15}$$

is ℓ times Fréchet differentiable. By Taylor’s formula, the expansion

$$\hat{u}_h^\varepsilon = w_h + \sum_{m=1}^{\ell-1} \frac{\varepsilon^m}{m!} \hat{u}_h^{(m)} + \hat{R}_{h,\varepsilon}^{(\ell)}, \quad \varepsilon \in I_0, \tag{0.16}$$

then holds with the following summands:

- (1) The term $w_h := \Phi(\mathcal{L}(h))$ is the unique solution to the deterministic PDE

$$(\partial_t - \Delta)w_h = g(w_h)h, \quad w_h(0, \cdot) = u_0. \tag{0.17}$$

- (2) The terms $\hat{u}_h^{(m)}$, $m \in [\ell - 1]$,¹² and $\hat{R}_{h,\varepsilon}^{(\ell)}$ are given by

$$\hat{u}_h^{(m)} := \partial_\varepsilon^m |_{\varepsilon=0} \hat{u}_h^\varepsilon, \quad \hat{R}_{h,\varepsilon}^{(\ell)} := \int_0^1 \frac{(1-s)^{\ell-1}}{(\ell-1)!} \partial_r^\ell |_{r=s\varepsilon} [\hat{u}_h^r] ds,$$

¹¹ Actually, the expansion is fully deterministic and the term “stochastic” only historic: It was accurate in the work of Azencott [8] but became outdated by the emergence of pathwise techniques. This was already remarked by Inahama and Kawabi [52] in the context of rough paths. Henceforth we refrain from using the now misleading term “stochastic”.

¹² We use the notation $[n] := \{1, \dots, n\}$ for any $n \in \mathbb{N}$ throughout the article.

In addition, the terms in the expansion (0.16) have the following properties:¹³

- (i) The terms $u_h^{(m)}(\mathbf{Z})$ and the remainder $R_{h,\varepsilon}^{(\ell)}(\mathbf{Z})$ are continuous functions of the model \mathbf{Z} . In addition, the estimates¹⁴

$$\|u_h^{(m)}(\mathbf{Z})\|_{\mathcal{X}_T} \lesssim (1 + \|\mathbf{Z}^-\|)^m, \quad \|R_{h,\varepsilon}^{(\ell)}(\mathbf{Z})\|_{\mathcal{X}_T} \lesssim \varepsilon^\ell (1 + \|\mathbf{Z}^-\|)^\ell \tag{0.18}$$

hold: The first one uniformly over all $h \in \mathcal{H}$ with $\|h\|_{\mathcal{H}} \leq \rho$, the second additionally over all $(\varepsilon, \mathbf{Z}^-) \in I \times \mathcal{M}_-$ with $\varepsilon \|\mathbf{Z}^-\| < \rho$, both for each $\rho > 0$.

- (ii) The terms $u_h^{(m)}(\mathbf{Z})$ are m -homogeneous w.r.t. dilation in the model \mathbf{Z} , that is

$$\varepsilon^m u_h^{(m)}(\mathbf{Z}) = u_h^{(m)}(\delta_\varepsilon \mathbf{Z}).$$

In addition, the terms $\hat{u}_h^{(m)}$ are limits of $\hat{u}_{\xi_\delta, h}^{(m)}$ as $\delta \rightarrow 0$ in the spirit of Theorem 0.1 above. See Proposition 2.47 below for the linear stochastic PDEs that $\hat{u}_{\xi_\delta, h}^{(m)}$ satisfy. \spadesuit

0.4. Generalisation to other singular stochastic PDEs

In a series of major recent works, Hairer and co-authors [19,24,14] have developed regularity structures into a black-box theory that encompasses a large class of stochastic PDEs. In principle, the arguments which we will use in the proofs of Theorems 1 and 2 apply to any stochastic PDE that is covered by the afore-mentioned articles.

One reason we concentrate on gPAM in this article is that it allows for a relatively simple analysis via regularity structures, so we can emphasise the main principles that our arguments are based upon.

Another reason is that Cameron-Martin shifts $\xi \mapsto \xi + h$ of the driving noise of gPAM, as needed in (0.8), have already been implemented on model space \mathcal{M} via the translation operator T_h in [23, sec. 3.4]. Its well-posedness is a complicated *analytical* problem, in general, because it requires products of elements in Besov spaces of different L^p -scale to exist. Even in the case of gPAM, building a solution theory for the shifted equation (0.8) poses the problem to define

$$\mathbb{H} \curvearrowright := (G\xi)h, \quad G\xi \in \mathcal{C}^{1-\kappa} = B_{\infty, \infty}^{1-\kappa}, \quad h \in \mathcal{H} = B_{2,2}^0 \hookrightarrow \mathcal{C}^{-1}$$

¹³ We formulate the estimates w.r.t. a generic model \mathbf{Z} and the corresponding quantities $u_h^\varepsilon(\mathbf{Z})$ and $u_h^{(m)}(\mathbf{Z})$: see Definition 2.33 on p. 43 below for their precise definition. The expansion in (0.16) then holds for $\varepsilon < \varepsilon_0(T, \mathbf{Z})$. Since $\hat{u}_h^\varepsilon = u_h^\varepsilon(\hat{\mathbf{Z}})$, $\hat{u}_h^{(m)} = u_h^{(m)}(\hat{\mathbf{Z}})$, and $\varepsilon_0(T) = \varepsilon_0(T, \hat{\mathbf{Z}})$, this is consistent with the first part of the theorem where $\mathbf{Z} = \hat{\mathbf{Z}}$.

¹⁴ By “ \lesssim ” we denote estimates up to a constant. If we wish to specify its dependence on a parameter α , we write “ \lesssim_α ”. The “norm” $\|\cdot\|$ appearing on the RHS is the *homogeneous* norm for a minimal admissible model $\mathbf{Z}^- \in \mathcal{M}_-$, see subsection A.2 in the appendix for details. For now, one may think of the homogeneous rough path norm $\|\mathbf{X}\|_\alpha = \|X\|_\alpha + \|\mathbb{X}\|_{2\alpha;2}^{1/2}$ for an α -Hölder, 2-step rough path $\mathbf{X} = (X, \mathbb{X})$.

where $G\xi$ denotes the solution to the linearised equation ($g \equiv 1$); Cannizzaro, Friz, and Gassiat [23, Lem. A.2] have managed to obtain the necessary estimates “by hand”. However, more complicated trees that already appear for the shifted Φ_3^4 or KPZ equation, for example

$$\begin{aligned} \tau &:= \text{tree diagram} \rightsquigarrow \mathbf{\Pi}\tau := (Gh)(G\xi)G((G\xi)(G\xi)(Gh)), \\ \sigma &:= \text{tree diagram} \rightsquigarrow \mathbf{\Pi}\sigma := [G_x((G_x\xi)(G_xh))]^2, \end{aligned}$$

render ad hoc estimates unwieldy. Instead, analytic arguments need to be complemented by insights into the algebraic make-up of the underlying regularity structure.

Schönbauer [76], in contrast, approached the problem from a different perspective. By an ingenious “doubling trick”, similar in spirit to previous arguments by Inahama [58] in the rough paths context, he is able to leverage the *probabilistic* estimates of Chandra and Hairer [24]; thereby, he manages to prove [76, Thm. 3.11] that the action of T_h on BPHZ lifts of smooth functions is well-defined and continuous almost surely. While this argument applies in the most general setting we referenced in the beginning of this section, it falls short of proving T_h well-defined and continuous on *generic* models. The latter is crucial for the localisation procedure, see sec. 1 below, to work beyond gPAM.

We propose to follow a completely different route using the *structural assumptions on the Laplace functionals* that we consider. As the reader will observe later, we only ever want to translate ξ in direction \mathbf{h} , the minimiser of the functional \mathcal{F} given in (H2) above. A priori, it is clear that \mathbf{h} is in \mathcal{H} (otherwise $\mathcal{J}(\mathbf{h}) = +\infty$) — but it is known from variational calculus that minimisers often enjoy better regularity properties, see for example [63, Thm. 14.4.1]. The strategy there – establish some initial regularity and then bootstrap using Schauder estimates – informs our reasoning. Under the premise that $g \in C_b^\infty(\mathbb{R})$ and $u_0 \in C^\infty(\mathbb{T}^2)$, we sketch the argument that $\mathbf{h} \in C^\infty(\mathbb{T}^2)$ for gPAM, see [64, Thm. 2] for a precise statement. We stress that this renders the translation operator T_h well-defined *immediately* without any appeal to advanced analytic estimates.

(1) By first-order optimality, $D\mathcal{F}|_{\mathbf{h}} \equiv 0$. Equivalently, for all $k \in \mathcal{H}$ we have

$$\langle \mathbf{h}, k \rangle_{\mathcal{H}} = DF|_{w_{\mathbf{h}}}(v_{\mathbf{h},k}), \quad v_{\mathbf{h},k} := \partial_\varepsilon|_{\varepsilon=0} w_{\mathbf{h}+\varepsilon k} \tag{0.19}$$

where $w_{\mathbf{h}}$ is the solution to (0.17).

(2) The directional derivative $v_{\mathbf{h},k}$ satisfies the *linear*, inhomogeneous PDE

$$(\partial_t - \Delta)v_{\mathbf{h},k} = g'(w_{\mathbf{h}})v_{\mathbf{h},k}\mathbf{h} + g(w_{\mathbf{h}})k, \quad v_{\mathbf{h},k}(0, \cdot) = 0. \tag{0.20}$$

If \mathbf{h} actually were in H^γ , $\gamma \geq 0^{15}$, we expect $w_{\mathbf{h}} \in \mathcal{C}_T H^{\gamma+1^-}$ by the regularising properties of the heat semigroup. In turn, we can then take $k \in H^{-\theta}$ for some $\theta > \gamma$ and still make sense of (0.20) and its solution $v_{\mathbf{h},k}$. Even more so, we can establish the estimate

$$\|v_{\mathbf{h},k}\|_{\mathcal{C}_T^\gamma} \lesssim_T \|k\|_{H^{-\theta}}, \quad \mathcal{C}_T^\gamma := \mathcal{C}_T H^{-\gamma+\frac{1}{2}} \tag{0.21}$$

via Gronwall’s inequality.

- (3) We extend the linear, bounded operator $DF|_{w_{\mathbf{h}}}$ from \mathcal{X}_T to \mathcal{C}_T^γ by a density argument, preserving its norm. Altogether, combining (0.19) with the estimate (0.21), we obtain the estimate

$$|\langle \mathbf{h}, k \rangle_{\mathcal{H}}| \lesssim_T \|DF|_{w_{\mathbf{h}}}\|_{\mathcal{L}(\mathcal{X}_T, \mathbb{R})} \|k\|_{H^{-\theta}}$$

for $k \in C^\infty$ and infer that $\mathbf{h} \in H^\theta$ by duality.

- (4) Since $\theta > \gamma$, the minimiser \mathbf{h} has gained regularity in this procedure. Bootstrapping the previous steps implies that $\mathbf{h} \in C^\infty$.

We expect that this argument can be adapted to cover the time-dependency of the Cameron-Martin space $L^2([0, T] \times \mathbb{T}^3)$ for the Φ_3^4 equation and, similarly, for other singular stochastic PDEs. Investigating this strategy further is left for future work.

Once $T_{\mathbf{h}}$ is made sense of as outlined, the arguments presented in the article at hand can easily be generalised beyond gPAM.

0.5. Organisation of the article

In this subsection, we discuss the organisation of the article at hand and indicate by triangles “▷” in which (sub-)sections the hypotheses from Theorem 1 are used. We also highlight where we use the large deviation principle (LDP) for $(\delta_\varepsilon \hat{\mathbf{Z}}^- : \varepsilon \in I)$ formulated in Theorem C.1. Note that the boldface terms can be read as an outline for the proof of Theorem 1.

- **(1) Localisation.** In section 1, we localise the functional J to a ρ -neighbourhood of $\mathcal{L}_-(\mathbf{h})$, the lift of the minimiser \mathbf{h} to the space of minimal models \mathcal{M}_- . By the Cameron-Martin theorem, we then recenter the localised functional J_ρ to the neighbourhood $\{\varepsilon \|\hat{\mathbf{Z}}^-\| < \rho\}$, at the expense of changing $(F \circ \Phi^-)(\delta_\varepsilon \hat{\mathbf{Z}}^-)$ to the function $\tilde{F}_{\Phi^-}(\mathbf{h}, \varepsilon) := (F \circ \Phi^-)(T_{\mathbf{h}} \delta_\varepsilon \hat{\mathbf{Z}}^-) - F(w_{\mathbf{h}}) + \varepsilon \xi(\mathbf{h})$.¹⁶ ▷ (H1), (LDP)

¹⁵ Recall that $\mathbf{h} \in \mathcal{H} \equiv H^0$, so the statement is true when $\gamma = 0$.

¹⁶ Note that $(F \circ \Phi^-)(\delta_\varepsilon \hat{\mathbf{Z}}^-) = F(\hat{u}^\varepsilon)$ and $(F \circ \Phi^-)(T_{\mathbf{h}} \delta_\varepsilon \hat{\mathbf{Z}}^-) = F(\hat{u}_{\mathbf{h}}^\varepsilon)$.

- **(2) Taylor expansion in the noise intensity.**¹⁷ In Corollary 2.1, we obtain a Taylor expansion of the functional $F(\hat{u}_h^\varepsilon) = (F \circ \Phi^-)(T_h \delta_\varepsilon \hat{\mathcal{Z}}^-)$ up to order $N + 2$ for small ε . In Corollary 2.3, we then see that the Taylor terms and the remainder depend continuously on $\hat{\mathcal{Z}}$ and provide estimates for both. In addition, we show that the former are homogeneous in ε . This result follows easily from Theorem 2 whose proof we prepare throughout the whole section 2. \triangleright (H3)
The proof of Theorem 1 continues in section 3. In order to better grasp its structure, the reader might want to skip the rest of section 2 in a first reading.
- **(2.1) Abstract fixed-point problem.** In subsection 2.1, we set up the fixed-point problem for U^ε , the abstract representation of \hat{u}_h^ε in a suitable space of modelled distributions. The choice of the latter requires care: although the natural candidate $\mathcal{D}^{\gamma,\eta}(T_h \delta_\varepsilon \hat{\mathcal{Z}})$ has a Banach structure, it is unsuited for taking derivatives in ε , for it is a different space for each $\varepsilon \in I$. Instead, we study a family of fixed point (FP) equations parametrised by ε whose solutions¹⁸ U^ε live in one fixed Banach space $\mathcal{D}_U^{\gamma,\eta}(E_h \hat{\mathcal{Z}})$. We then prove that this choice is consistent: both U^ε and U^ε are reconstructed to \hat{u}_h^ε w.r.t. their underlying models (Proposition 2.13).
- **(2.2) Fréchet differentiability of the fixed-point map.** The main result of subsection 2.2 is Theorem 2.26, in which we prove Fréchet differentiability of order $N + 3$ for the abstract fixed-point map \mathfrak{J} that sends ε to U^ε . In the same theorem, we derive explicit FP equations for the derivatives $\mathfrak{J}^{(m)} := \partial_\varepsilon^m \mathfrak{J}$ and establish their continuous model dependence. Our main tool here is the Implicit Function Theorem (IFT), the applicability of which is verified in Lemmas 2.21 and 2.24.
- **(2.3) Taylor expansion in the modelled distribution space.** In Corollary 2.28, we obtain a Taylor expansion of U^ε in the Banach space $\mathcal{D}_U^{\gamma,\eta}(E_h \hat{\mathcal{Z}})$. In subsection 2.3, we analyse its terms and connect them to their counterparts in Theorem 2. In particular, we obtain estimates for the Taylor terms (subsection 2.3.1) and the remainder (subsection 2.3.2) by means of an abstract version of Duhamel’s principle (Proposition 2.35).
- **(2.4) Proof of Taylor expansion.** In subsection 2.4 we synthesise the results of subsections 2.1 to 2.3 to give a proof of Theorem 2.
- **(2.5) Stochastic PDEs for the Taylor terms.** We derive stochastic PDEs for the Taylor terms $\hat{u}_{\xi_s, h}^{(m)}$ from Theorem 2. Their limits $\hat{u}_h^{(m)}$ as $\delta \rightarrow 0$ appear in the formulas for the coefficients a_0, \dots, a_N .
- **(3) Local analysis in the vicinity of the minimiser.** Section 3 revolves around the study of $\bar{F}_\Phi^-(h, \varepsilon)$. The constant term in the Taylor-like expansion for $(F \circ \Phi^-)(T_h \delta_\varepsilon \hat{\mathcal{Z}}^-)$ compensates the term $-F(w_h)$. \triangleright (H2), (H4)

¹⁷ For a graphical overview of this section’s content – including technical details – see also Fig. 2 on p. 32.

¹⁸ Green colour indicates that we are working in the extended gPAM regularity structure, see sec. A.1.

- **(3.1) First-order optimality annihilates the linear term.** In subsection 3.1, we prove that the sum of $\varepsilon\xi(\mathbf{h})$ and the linear term in the expansion for $(F \circ \Phi^-)(T_{\mathbf{h}}\delta_{\varepsilon}\hat{\mathbf{Z}}^-)$ vanishes because \mathbf{h} is a minimiser of \mathcal{F} (Proposition 3.1). \triangleright (H2)
- **(3.2) Exponential integrability of the quadratic term.** In subsection 3.2, we establish the exponential integrability of the quadratic term $-1/2\hat{Q}_{\mathbf{h}}$ in the afore-mentioned expansion of $(F \circ \Phi^-)(T_{\mathbf{h}}\delta_{\varepsilon}\hat{\mathbf{Z}}^-)$ (Proposition 3.2). We stress that, in contrast to previous works, we do not have to resort to the connection of second Wiener chaos, trace-class operators, and Carleman-Fredholm determinants. Instead, we use a large deviations argument for $(-1/2Q_{\mathbf{h}}(\delta_{\varepsilon}\hat{\mathbf{Z}}^-) : \varepsilon \in I)$, itself valid by continuity of $Q_{\mathbf{h}}$ in the model, and then conclude by non-degeneracy of \mathbf{h} . \triangleright (H4), (LDP)
- **(3.3) The coefficients in the asymptotic expansion.** In subsection 3.3, we decompose J_{ρ} in four steps that finally lead to the asymptotic expansion claimed in eq. (0.13): Its coefficients a_0, \dots, a_N are finite thanks to the exponential integrability of $-1/2\hat{Q}_{\mathbf{h}}$ (Proposition 3.6). We close our proof of Theorem 1 by showing that the four remainder terms created in the decomposition procedure are of order $o(\varepsilon^N)$. Our main tool here is a Fernique-type theorem for the minimal BPHZ model $\hat{\mathbf{Z}}^-$.
- **(A) Background knowledge on regularity structures.** In appendix A we collect the prerequisites of Hairer's theory of regularity structures that are necessary to follow the presentation in this article.
- **(A.1) The original and extended gPAM regularity structure.** In subsection A.1, we briefly recapitulate the algebraic structure necessary to treat $\text{gPAM}_{\varepsilon}$, that is: the original regularity structure, built by Hairer, along with its extended counterpart that was introduced to accommodate Cameron-Martin shifts.
- **(A.2) Admissible models.** Section A.2 deals with (minimal) admissible models and the analytical operations that act on them. After repeating the notions of extension and translation, we introduce the dilation operator and study its properties. In particular, we see that it dovetails nicely with the homogeneous norm on minimal model space. In the end, a short summary of BPHZ renormalisation on model space is presented.
- **(A.3) Modelled distributions.** In section A.3, we recapitulate the notion of a modelled distribution and how (the lifts of) non-linear functions act upon them. We then study Fréchet differentiability of such lifts, obtain a version of Taylor's theorem, and see how modelled distributions behave w.r.t. dilation.
- **(B) Deterministic gPAM and explosion times.** In Appendix B, we prove that the solutions to deterministic gPAM w_h from (0.17) exist globally when the driver h is a Cameron-Martin function. In addition, we provide information on the explosion time T^{ε} of \hat{u}^{ε} from Theorem 0.1 and that of \hat{u}_h^{ε} from (0.15). To quantify the latter, we assume boundedness of $\varepsilon\|\hat{\mathbf{Z}}^-\|$, a consequence of the localisation procedure in section 1.
- **(C) Large Deviations.** Appendix C is, in essence, a recapitulation of Hairer's and Weber's work for the stochastic Allen-Cahn equation in the case of gPAM. Most importantly, in Theorem C.1 we see that the family $(\delta_{\varepsilon}\hat{\mathbf{Z}}^- : \varepsilon \in I)$ satisfies a LDP

on the space \mathcal{M}_- of minimal admissible models. This allows us to give a complete proof of Theorem 0.2, formulated with all the details in Theorem C.2.

- **(D) Fernique’s theorem for the minimal BPHZ model.** In Appendix D, we prove that $\|\hat{\mathbf{Z}}^-\|$, the homogeneous norm of the minimal BPHZ model $\hat{\mathbf{Z}}^-$, has Gaussian tails (Theorem D.1) – as needed for our estimates in subsection 3.3 to work.

► From here on, we assume that the reader is familiar with the content of appendix A. Throughout the whole article, we use “♠” (as opposed to “♠”) as an “end of statement”-sign when a complete proof of the statement in question can be found in [34], the extended arXiv version of the article at hand.

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1. Localisation around the minimiser and Cameron-Martin

We set $F_{\Phi}^- := F \circ \Phi \circ \mathcal{E}$ and denote by $\mathcal{L}_-(\mathbf{h})$ the lift of the minimiser $\mathbf{h} \in \mathcal{H}$ to \mathcal{M}_- . In addition, for an $\|\cdot\|$ -open neighbourhood \mathcal{O} of $\mathcal{L}_-(\mathbf{h})$, we decompose

$$J(\varepsilon) \equiv \mathbb{E} \left[\exp \left(-\frac{F_{\Phi}^-(\delta_{\varepsilon} \hat{\mathbf{Z}}^-)}{\varepsilon^2} \right); T^{\varepsilon} > T \right] = J_{\mathcal{O}}(\varepsilon) + J_{\mathcal{O}^c}(\varepsilon) \tag{1.1}$$

with

$$J_A(\varepsilon) := \mathbb{E} \left[\exp \left(-\frac{F_{\Phi}^-(\delta_{\varepsilon} \hat{\mathbf{Z}}^-)}{\varepsilon^2} \right); T^{\varepsilon} > T, \delta_{\varepsilon} \hat{\mathbf{Z}}^- \in A \right], \quad A \in \{\mathcal{O}, \mathcal{O}^c\}. \tag{1.2}$$

Our aim in this section is to prove that the essential contribution to $J(\varepsilon)$ comes from the vicinity \mathcal{O} of $\mathcal{L}_-(\mathbf{h})$. In other words, we want to qualify the contribution of $J_{\mathcal{O}^c}$ to J as asymptotically irrelevant. To this end, we appropriately adapt the proof of [11, Lem. 1.32] and [55, Lem. 8.2]. However, these authors rule out explosion in finite time. In contrast, Azencott [8] does account for this phenomenon, so the proof of the proposition that follows is also informed by his work.

Proposition 1.1. For each $\|\cdot\|$ -open neighbourhood \mathcal{O} of $\mathcal{L}_-(\mathbf{h})$, there exists some $d > a := \mathcal{F}(\mathbf{h})$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, we have

$$J_{\mathcal{O}^c}(\varepsilon) \leq \exp\left(-\frac{d}{\varepsilon^2}\right). \tag{1.3}$$

♠

Proof. Since $\mathcal{I}_{\mathcal{M}_-}$ is a good rate function (RF), it has compact sublevel sets, i.e. for each $k > 0$, the set

$$L_k := \{\mathbf{Z}^- \in \mathcal{M}_- : \mathcal{I}_{\mathcal{M}_-}(\mathbf{Z}^-) \leq k\} \tag{1.4}$$

is compact in \mathcal{M}_- . For $\delta > 0$, we further denote by $L_k(\delta) := \{\mathbf{Z}^- \in \mathcal{M}_- : \|\mathbf{Z}^-; L_k\| < \delta\}$ the open δ -fattening of L_k .

Since the set $L_k \cap \mathcal{O}^c$ is compact in \mathcal{M}_- as well, the function $\mathcal{F}^- := \mathcal{F} \circ \mathcal{E} = F_{\Phi}^- + \mathcal{I}_{\mathcal{M}_-}$ attains its minimum $a_1 > a$ on this set by continuity of F, Φ , and \mathcal{E} and lower semi-continuity (l.s.c.) of $\mathcal{I}_{\mathcal{M}_-}$ as a RF. Let now $a_2 \in (a, a_1)$; then, there exists $\delta_0 > 0$ such that

$$\mathcal{F}^-(\mathbf{Z}^-) > a_2 \quad \text{for } \mathbf{Z}^- \in \overline{L_k(\delta_0)} \cap \mathcal{O}^c \tag{1.5}$$

and thus for each $\mathbf{Z}^- \in \overline{L_k(\delta)} \cap \mathcal{O}^c$ with $\delta \in (0, \delta_0)$.

▷ The inequality in (1.5) can be seen as follows: By l.s.c. of \mathcal{F}^- , the set $\{\mathcal{F}^- > a_2\}$ is open and since $a_2 < a_1$, we have $L_k \cap \mathcal{O}^c \subseteq \{\mathcal{F}^- > a_2\}$. Hence, $L_k \subseteq \{\mathcal{F}^- > a_2\} \cup \mathcal{O}$ and since $\mathcal{C} := [\{\mathcal{F}^- > a_2\} \cup \mathcal{O}]^c$ is closed, L_k is compact, and $L_k \cap \mathcal{C} = \emptyset$, the function

$$\text{dist}(\cdot; \mathcal{C}) : L_k \ni \mathbf{Z}^- \mapsto \text{dist}(\mathbf{Z}^-; \mathcal{C}) := \|\mathbf{Z}^-; \mathcal{C}\| > 0$$

attains its strictly positive minimum on L_k . Therefore, we may find $\delta_0 > 0$ such that

$$\overline{L_k(\delta_0)} \subseteq \{\mathcal{F}^- > a_2\} \cup \mathcal{O},$$

which implies that $\overline{L_k(\delta_0)} \cap \mathcal{O}^c \subseteq \{\mathcal{F}^- > a_2\}$. ◁

We now fix some $\delta \in (0, \delta_0)$ and split the analysis of $J_{\mathcal{O}^c}(\varepsilon)$ into two parts:

Part 1: Controlling $J_{\mathcal{O}^c}(\varepsilon)$ on $L_k(\delta)^c \cap \mathcal{O}^c$.

By assumption (H1), the estimate

$$\begin{aligned} & \mathbb{E} \left[\exp\left(-\frac{F_{\Phi}^-(\delta_{\varepsilon} \hat{\mathbf{Z}}^-)}{\varepsilon^2}\right); T^{\varepsilon} > T, \delta_{\varepsilon} \hat{\mathbf{Z}}^- \in L_k(\delta)^c \cap \mathcal{O}^c \right] \\ & \leq \exp\left(\frac{\|F\|_{\infty}}{\varepsilon^2}\right) \mathbb{P}(\delta_{\varepsilon} \hat{\mathbf{Z}}^- \in L_k(\delta)^c) \end{aligned} \tag{1.6}$$

holds. From Theorem C.1, we know that the family $\{\mathbb{P}_\varepsilon := \mathbb{P} \circ (\delta_\varepsilon \hat{\mathbf{Z}}^-)^{-1} : \varepsilon > 0\}$ satisfies a LDP on \mathcal{M}_- , so we apply the LDP upper bound to the closed set $L_k(\delta)^c$ to obtain

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbb{P}_\varepsilon (L_k(\delta)^c) \leq -\inf\{\mathcal{I}_{\mathcal{M}_-}(\mathbf{Z}^-) : \mathbf{Z}^- \in L_k(\delta)^c\} \leq -k$$

by definition of $L_k \subseteq L_k(\delta)$. We may thus find $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$ we have

$$\mathbb{P}_\varepsilon (L_k(\delta)^c) \leq \exp\left(-\frac{k}{2\varepsilon^2}\right).$$

Choosing $k > 0$ such that $-d_1 \equiv -d_1(k) := \|F\|_\infty - k/2 < 0$, we further estimate (1.6) by

$$\text{(RHS)} \leq \exp\left(-\frac{d_1}{\varepsilon^2}\right) \tag{1.7}$$

for $\varepsilon \in (0, \varepsilon_1)$. This same k is fixed for the rest of the proof.

Part 2: Controlling $J_{\mathcal{O}^c}(\varepsilon)$ on $L_k(\delta) \cap \mathcal{O}^c$.

We now define the function $F_1^-(\mathbf{Z}^-) := F_{\Phi}^-(\mathbf{Z}^-)$ for $\mathbf{Z}^- \in \overline{L_k(\delta)} \cap \mathcal{O}^c$ and $F_1^-(\mathbf{Z}^-) := c > a_2$ otherwise. This function is l.s.c. and bounded, for F is bounded by assumption (H1). We may thus apply [29, Lem. 2.1.8] in conjunction with Lemma B.3 to infer that

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon^2 \log \mathbb{E} \left[\exp\left(-\frac{F_1^-(\delta_\varepsilon \hat{\mathbf{Z}}^-)}{\varepsilon^2}\right); T^\varepsilon > T \right] \leq -\inf(F_1 + \mathcal{I}_{\mathcal{M}}) \stackrel{(1.5)}{\leq} -a_2.$$

Thus, there exists $a_3 \in (a, a_2)$ and $\varepsilon_2 > 0$ such that for all $\varepsilon \in (0, \varepsilon_2)$, we have

$$\mathbb{E} \left[\exp\left(-\frac{F_1^-(\delta_\varepsilon \hat{\mathbf{Z}}^-)}{\varepsilon^2}\right); T^\varepsilon > T \right] \leq \exp\left(-\frac{a_3}{\varepsilon^2}\right). \tag{1.8}$$

Now observe that

$$\begin{aligned} & \mathbb{E} \left[\exp\left(-\frac{F_{\Phi}^-(\delta_\varepsilon \hat{\mathbf{Z}}^-)}{\varepsilon^2}\right); T^\varepsilon > T, \delta_\varepsilon \hat{\mathbf{Z}}^- \in L_k(\delta) \cap \mathcal{O}^c \right] \\ & \leq \mathbb{E} \left[\dots; T^\varepsilon > T, \delta_\varepsilon \hat{\mathbf{Z}}^- \in \overline{L_k(\delta)} \cap \mathcal{O}^c \right] \\ & \equiv \mathbb{E} \left[\exp\left(-\frac{F_1^-(\delta_\varepsilon \hat{\mathbf{Z}}^-)}{\varepsilon^2}\right); T^\varepsilon > T, \delta_\varepsilon \hat{\mathbf{Z}}^- \in \overline{L_k(\delta)} \cap \mathcal{O}^c \right] \stackrel{(1.8)}{\leq} \exp\left(-\frac{a_3}{\varepsilon^2}\right). \end{aligned}$$

Combining these two steps, we find that $|J_{\mathcal{O}^c}(\varepsilon)| \leq \exp(-\varepsilon^{-2}d)$ for all $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 := \varepsilon_1 \wedge \varepsilon_2$ and $d := d_1 + a_3 > a_3 > a$. This completes the proof. \square

By the previous proposition, it suffices to consider the functional

$$J_\rho(\varepsilon) := \mathbb{E} \left[\exp \left(-\frac{F_\Phi^-(\delta_\varepsilon \hat{\mathbf{Z}}^-)}{\varepsilon^2} \right); T^\varepsilon > T, \|T_{-h} \delta_\varepsilon \hat{\mathbf{Z}}^-\| < \rho \right]. \tag{1.9}$$

for some $\rho > 0$ which we will later choose sufficiently small. Note that $J_\rho(\varepsilon) \equiv J_{\mathcal{O}}(\varepsilon)$ for the specific choice¹⁹

$$\mathcal{O} \equiv \mathcal{O}(h, \rho) := \{\mathbf{Z}^- \in \mathcal{M}_- : \|T_{-h} \mathbf{Z}^-\| < \rho\}.$$

Instead of tracking the h -dependency of $J_\rho(\varepsilon)$ via the set \mathcal{O} , we want to encode it by appropriately changing the functional $F_\Phi \rightsquigarrow \tilde{F}_\Phi(h, \cdot)$. As usual, this amounts to a straightforward application of the Cameron-Martin theorem on the following *abstract Wiener space* (B, \mathcal{H}, μ) , the concept of which has been introduced by Gross [42]:

- the Hilbert space $\mathcal{H} := L^2(\mathbb{T}^2)$,
- the centred Gaussian measure μ with Cameron-Martin space \mathcal{H} , and
- the Banach space B , the closure of $\mathcal{C}^\infty(\mathbb{T}^2)$ in the Hölder-Besov space $\mathcal{C}^{-1-\kappa}(\mathbb{T}^2)$ for any $\kappa > 0$.

Note that under μ , the space white noise ξ is the canonical process.

Proposition 1.2. *In the same setting as before, there exists $\rho_0 = \rho_0(T) > 0$ such that*

$$J_\rho(\varepsilon) = \exp \left(-\frac{\mathcal{F}(h)}{\varepsilon^2} \right) \mathbb{E} \left[\exp \left(-\frac{\tilde{F}_\Phi^-(h, \varepsilon)}{\varepsilon^2} \right); \varepsilon \|\hat{\mathbf{Z}}^-\| < \rho \right] \tag{1.10}$$

for all $\rho \in (0, \rho_0)$, where

$$\tilde{F}_\Phi^-(h, \varepsilon) := F_\Phi^-(T_h \delta_\varepsilon \hat{\mathbf{Z}}^-) - F_\Phi^-(\mathcal{L}_-(h)) + \varepsilon \xi(h). \tag{1.11}$$

\spadesuit

Proof. Recall that white noise ξ is the identity on the Gaussian measure space (B, μ) , that is $\xi(\omega) = \omega$ for $\omega \in B$. Also recall that $T^\varepsilon = T_\infty^-(\delta_\varepsilon \hat{\mathbf{Z}}^-)$. From the Cameron-Martin theorem [28, Prop. 2.26] on the abstract Wiener space (B, \mathcal{H}, μ) , we know that the push-forward measure $\mu_h^\varepsilon := (T_{-h/\varepsilon})_* \mu$ has density

¹⁹ Note that the set \mathcal{O} in the equation that follows is open by continuity of the translation operator T_{-h} on $(\mathcal{M}_-, \|\cdot\|)$, a consequence of Proposition A.14 and Lemma A.9.

$$\frac{d\mu_\mathbf{h}^\varepsilon}{d\mu}(\omega) = \exp\left(-\frac{1}{\varepsilon^2} [\mathcal{J}(\mathbf{h}) + \varepsilon\omega(\mathbf{h})]\right).$$

where $\omega(\mathbf{h}) = I(\mathbf{h})(\omega)$ with the Paley-Wiener map I . We then find that

$$\begin{aligned} & J_\rho(\varepsilon) \\ &= \mathbb{E} \left[\exp\left(-\frac{F_\Phi^-(\delta_\varepsilon \hat{\mathbf{Z}}^-)}{\varepsilon^2}\right); T_\infty^-(\delta_\varepsilon \hat{\mathbf{Z}}^-) > T, \|T_{-\mathbf{h}}\delta_\varepsilon \hat{\mathbf{Z}}^-\| < \rho \right] \\ &= \int_B \exp\left(-\frac{F_\Phi^-(T_{\mathbf{h}}\delta_\varepsilon \hat{\mathbf{Z}}^-(\omega - \mathbf{h}/\varepsilon))}{\varepsilon^2}\right) \mathbf{1}_{T_\infty^-(T_{\mathbf{h}}\delta_\varepsilon \hat{\mathbf{Z}}^-(\omega - \mathbf{h}/\varepsilon)) > T} \mathbf{1}_{\|\delta_\varepsilon \hat{\mathbf{Z}}^-(\omega - \mathbf{h}/\varepsilon)\| < \rho} \mu(d\omega) \\ &= \exp\left(-\frac{F_\Phi^-(\mathcal{L}_-(\mathbf{h})) + \mathcal{J}(\mathbf{h})}{\varepsilon^2}\right) \mathbb{E} \left[\exp\left(-\frac{\tilde{F}_\Phi^-(\mathbf{h}, \varepsilon)}{\varepsilon^2}\right); T_\infty^-(T_{\mathbf{h}}\delta_\varepsilon \hat{\mathbf{Z}}^-) > T, \|\delta_\varepsilon \hat{\mathbf{Z}}^-\| < \rho \right] \\ &= \exp\left(-\frac{\mathcal{F}(\mathbf{h})}{\varepsilon^2}\right) \mathbb{E} \left[\exp\left(-\frac{\tilde{F}_\Phi^-(\mathbf{h}, \varepsilon)}{\varepsilon^2}\right); T_\infty^-(T_{\mathbf{h}}\delta_\varepsilon \hat{\mathbf{Z}}^-) > T, \varepsilon\|\hat{\mathbf{Z}}^-\| < \rho \right]. \end{aligned} \tag{1.12}$$

Note that we have used Lemma A.29 (with $h := -\mathbf{h}$) for the second equality. Since $T_\infty^{\mathbf{h}} = +\infty$ (see Remark B.2), any $T > 0$ satisfies $T < T_\infty^{\mathbf{h}}$. Choosing $\rho_0 > 0$ as in Lemma B.4, we have

$$\{\varepsilon\|\hat{\mathbf{Z}}^-\| < \rho\} \subseteq \{T_\infty^-(T_{\mathbf{h}}\delta_\varepsilon \hat{\mathbf{Z}}^-) > T\} \tag{1.13}$$

for all $\rho \in (0, \rho_0)$, which establishes the claim. \square

2. Taylor expansion in the noise intensity parameter

The application of the Cameron-Martin theorem in the last section prompts us to analyse the function $\tilde{F}_\Phi^-(\mathbf{h}, \cdot)$ further. In this direction, the first and most important step is the local expansion of

$$\Phi^-(T_{\mathbf{h}}\delta_\varepsilon \hat{\mathbf{Z}}^-) \equiv (\Phi \circ \mathcal{E})(T_{\mathbf{h}}\delta_\varepsilon \hat{\mathbf{Z}}^-) \equiv \Phi(T_{\mathbf{h}}\delta_\varepsilon \hat{\mathbf{Z}}) = \Phi(T_{\mathbf{h}}\delta_\varepsilon \hat{\mathbf{Z}}) = \hat{u}_\mathbf{h}^{(\varepsilon)}(\hat{\mathbf{Z}})$$

in ε . This is the content of Theorem 2 given in subsection 0.3 of the introduction.

The local expansion of $F_\Phi^-(T_{\mathbf{h}}\delta_\varepsilon \hat{\mathbf{Z}}^-)$ in ε that we need for the analysis of $\tilde{F}_\Phi^-(\mathbf{h}, \cdot)$ is an easy corollary of that theorem. While Theorem 2 is stated for general $g \in \mathcal{C}^{\ell+4}$, $\ell \geq 1$, the specific regularity assumptions on g (and F) for the study of precise Laplace asymptotics impose $\ell = N + 3$.

Corollary 2.1 (Taylor expansion of $F(\hat{u}_\mathbf{h}^\bullet)$). For each $\varepsilon \in I_0$ we have

$$F_\Phi(T_{\mathbf{h}}\delta_\varepsilon \hat{\mathbf{Z}}) \equiv F(\hat{u}_\mathbf{h}^\varepsilon) = F_\mathbf{h}^{(0)} + \sum_{m=1}^{N+2} \frac{\varepsilon^m}{m!} \hat{F}_\mathbf{h}^{(m)} + \hat{R}_\mathbf{h}^{F; \varepsilon; N+3} \tag{2.1}$$

where $F_h^{(0)} := F(w_h) \equiv F_\Phi(\mathcal{L}(h))$,

$$\hat{F}_h^{(m)} \equiv F_h^{(m)}(\hat{Z}) := \partial_\varepsilon^m |_{\varepsilon=0} F(u_h^\varepsilon(\hat{Z})), \quad m \in [N + 2],$$

and $\hat{R}_h^{F;\varepsilon;N+3} \equiv R_h^{F;\varepsilon;N+3}(\hat{Z})$ is implicitly defined from (2.1). We also set $\hat{Q}_h := \hat{F}_h^{(2)}$. ♠

Remark 2.2. Observe that the preceding corollary was formulated for the specific element $h \in \mathcal{H}$. However, h does not enter the proof by its property (H2) of being a minimiser of \mathcal{F} . Instead, we will need the assumptions in (H3), that is: the differentiability of F in the neighbourhood \mathcal{N} of $w_h \equiv (\Phi \circ \mathcal{L})(h)$ and the bounds (0.11) for its derivatives. ♠

Proof. By Theorem 2, $\varepsilon \mapsto \hat{u}_h^\varepsilon$ is differentiable on $I_0 = [0, \varepsilon_0)$ and thus continuous. Therefore, with possibly smaller $\varepsilon_0 > 0$, we have $\hat{u}_h^\varepsilon \in \mathcal{N}$ for all $\varepsilon \in I_0$. Assumption (H3) and Theorem 2 then immediately imply that $F \circ \hat{u}_h^\bullet \in \mathcal{C}^{N+3}(I_0, \mathbb{R})$, so the expansion in (2.1) follows from Taylor’s theorem. □

We collect the properties of the Taylor terms and the remainder in the corollary that follows.

Corollary 2.3. *In the setting of Corollary 2.1, the following assertions hold for $m = 1, \dots, N + 2$:*

- (i) *The terms $F_h^{(m)}(Z)$ and the remainder $R_h^{F;\varepsilon;N+3}(Z)$ are continuous functions of the model Z . In addition, we have the estimates*

$$|\hat{F}_h^{(m)}| \lesssim (1 + \|\hat{Z}^-\|)^m, \quad |\hat{R}_h^{F;\varepsilon;(N+3)}| \lesssim_\rho \varepsilon^{N+3} (1 + \|\hat{Z}^-\|)^{N+3} \tag{2.2}$$

where the second bound requires $\varepsilon \|\hat{Z}^-\| < \rho$ for some $\rho > 0$ to be valid.

- (ii) *The terms $\hat{F}_h^{(m)} \equiv F_h^{(m)}(\hat{Z})$ are m -homogeneous w.r.t. dilation of the model \hat{Z} , that is*

$$\varepsilon^m F_h^{(m)}(\hat{Z}) = F_h^{(m)}(\delta_\varepsilon \hat{Z}) \tag{2.3}$$

for $\varepsilon \in I_0$. In particular, $\varepsilon^2 Q_h(\hat{Z}) = Q_h(\delta_\varepsilon \hat{Z})$. ♠

For the proof of the corollary, in particular part (i), we need some auxiliary formulas. They have already appeared in the literature before, cf. [52, pp. 306/307], and use Rior-dan’s Formula, a version of the well-known Faa di Bruno formula (see the review paper by Johnson [62]).

Lemma 2.4. *For $k, n \in \mathbb{N}$, let $S_k^n := \{\mathbf{i} \in \mathbb{N}_{\geq 1}^k : |\mathbf{i}| = n\}$. Then, for all $\varepsilon \in I_0$ the equalities*

$$\hat{F}_h^{(m)} = m! \sum_{k=1}^m \frac{1}{k!} \sum_{\mathbf{i} \in S_k^m} D^{(k)} F|_{w_h} \left(\frac{1}{i_1!} \hat{u}_h^{(i_1)}, \dots, \frac{1}{i_k!} \hat{u}_h^{(i_k)} \right), \tag{2.4}$$

$$\begin{aligned} \hat{R}_h^{F;\varepsilon;(N+3)} &= \sum_{k=1}^{N+2} \frac{1}{k!} \sum_{\substack{\mathbf{i} \in [N+3]^k \\ |\mathbf{i}| \geq N+3}} D^{(k)} F|_{w_h} \left(\frac{1}{i_1!} \hat{R}_{h,\varepsilon}^{(N+3)}(i_1), \dots, \hat{R}_{h,\varepsilon}^{(N+3)}(i_k) \right) \\ &+ \int_0^1 \frac{(1-s)^{N+2}}{(N+2)!} D^{(N+3)} F(w_h + s\hat{R}_{h,\varepsilon}^{(1)}) (\hat{R}_{h,\varepsilon}^{(1)}, \dots, \hat{R}_{h,\varepsilon}^{(1)}) ds \end{aligned} \tag{2.5}$$

hold with

$$\hat{R}_{h,\varepsilon}^{(N+3)}(m) := \begin{cases} \varepsilon^m \hat{u}_h^{(m)} & \text{if } m \in [N+2], \\ \hat{R}_{h,\varepsilon}^{(N+3)} & \text{if } m = N+3, \end{cases} \tag{2.6}$$

the terms on the RHS of which are defined in Theorem 2. In particular, we find $\hat{F}_h^{(1)} = DF|_{w_h}(\hat{u}_h^{(1)})$ and

$$\hat{Q}_h = DF|_{w_h}[\hat{u}_h^{(2)}] + D^2 F|_{w_h}[\hat{u}_h^{(1)}, \hat{u}_h^{(1)}]. \tag{2.7}$$



Proof of Corollary 2.3. Assertion (ii) is a direct consequence of Theorem 2(ii) combined with (2.4).

We turn to assertion (i). In conjunction with assumption (H3), eq. (0.11), the explicit representation for $\hat{F}_h^{(m)}$ derived in Lemma 2.4 allows for the estimate

$$|\hat{F}_h^{(m)}| \lesssim_m \sum_{k=1}^m M_k \sum_{\mathbf{i} \in S_k^m} \prod_{\ell=1}^k \|\hat{u}_h^{(i_\ell)}\|_{\mathcal{X}_T} \lesssim_T \sum_{k=1}^m M_k \# [S_k^m] (1 + \|\hat{Z}^-\|)^m \lesssim_m (1 + \|\hat{Z}^-\|)^m. \tag{2.8}$$

In the penultimate estimate, we have applied the estimates from Theorem 2(i). As for the remainder, note that $\{w_h + s\hat{R}_{h,\varepsilon}^{(1)} : s \in [0, 1]\} \subseteq \mathcal{N}$ for $\varepsilon \in I_0$, for it is the line segment connecting w_h and \hat{u}_h^ε . Hence, the integral term in (2.5) may be estimated analogously to (2.8). The same is true for the sum in (2.5), except that one uses the additional estimate

$$\varepsilon^{|\mathbf{i}|} (1 + \|\hat{Z}^-\|)^{|\mathbf{i}|} \leq (1 + \rho)^{|\mathbf{i}| - (N+3)} \varepsilon^{N+3} (1 + \|\hat{Z}^-\|)^{N+3}$$

for multi-indices \mathbf{i} with $|\mathbf{i}| \geq N+3$, as readily implied by $\varepsilon \in [0, 1]$ and $\varepsilon \|\hat{Z}^-\| < \rho$. \square

In the remainder of this section, we will prepare the proof of Theorem 2 to be given in subsection 2.4. If the reader is willing to momentarily believe the validity of that

theorem, she may want to proceed with section 3 on page 59 in a first reading. In that section, we use the Corollaries 2.1 and 2.3 to continue the proof of the precise Laplace asymptotics as claimed in Theorem 1.

2.1. The abstract fixed-point problem

Consider the following abstract fixed-point problems for $Z \in \mathcal{M}$ and $Z \in \mathcal{M}$:

$$U = \mathcal{P}^Z (G(U)\mathfrak{I}) + \mathcal{T}Pu_0 \quad \rightarrow \quad U \in \mathcal{D}_U^{\gamma,\eta}(Z), \tag{2.9}$$

$$U = \mathcal{P}^Z (G(U)[\mathfrak{I} + \mathfrak{I}]) + \mathcal{T}Pu_0 \quad \rightarrow \quad U \in \mathcal{D}_U^{\gamma,\eta}(Z), \tag{2.10}$$

$$U^\varepsilon = \mathcal{P}^Z (G(U^\varepsilon)[\varepsilon\mathfrak{I} + \mathfrak{I}]) + \mathcal{T}Pu_0 \quad \rightarrow \quad U^\varepsilon \in \mathcal{D}_U^{\gamma,\eta}(Z), \tag{2.11}$$

Remark 2.5. Note that the model enters these equations implicitly via the abstract convolution kernels \mathcal{P}^Z and \mathcal{P}^Z , respectively, and thereby characterises the right space of modelled distributions we seek solutions in. Our main case of interest is $Z := E_h \hat{Z} \in \mathcal{M}$ where $h \in \mathcal{H}$ is the minimiser from assumption (H2) and $\hat{Z} \in \mathcal{M}$ the BPHZ model. For $h \in \mathcal{H}$ we suppress the dependence on the specific model $Z \in \mathcal{M}$ and simply write

$$\mathcal{P}^h = \mathcal{P}(T_h Z; \cdot), \quad \mathcal{P}^{e_h} = \mathcal{P}(E_h Z; \cdot). \quad \spadesuit$$

The solvability of the first two fixed-point equations was proved by Cannizzaro, Friz, and Gassiat in [23, Prop. 3.23 & 3.25]. Regarding (2.11), we define the abstract, parameter-dependent FP map

$$\mathcal{M}^Z : I \times \mathcal{D}_U^{\gamma,\eta}(Z) \rightarrow \mathcal{D}_U^{\gamma,\eta}(Z), \quad I := [0, 1], \tag{2.12}$$

for $Z \in \mathcal{M}$ by

$$\mathcal{M}^Z(\varepsilon, Y) := \mathcal{P}^Z (G^\varepsilon(Y)) + \mathcal{T}Pu_0, \quad G^\varepsilon(Y) := G(Y)[\varepsilon\mathfrak{I} + \mathfrak{I}]. \tag{2.13}$$

Next, we revisit and amend some variants of the notion of *local Lipschitz continuity* as introduced by Hairer in [45, sec. 7.3] and slightly rephrased by Schönbauer [76, Def. 3.18].

Definition 2.6. Let $\mathcal{V}, \tilde{\mathcal{V}}$ be two sectors²⁰ in \mathcal{F} and fix $\gamma \geq \bar{\gamma} > 0$ as well as $\eta, \bar{\eta} \in \mathbb{R}$.

- Given a model $Z \in \mathcal{M}$, we say that a map $F : \mathcal{D}_\mathcal{V}^{\gamma,\eta}(Z) \rightarrow \mathcal{D}_\mathcal{V}^{\bar{\gamma},\bar{\eta}}(Z)$ is *locally Lipschitz continuous* if for any compact set $K \subset D$ and any $R > 0$ one has the bound

²⁰ See [45, Def. 2.5]. In addition, the domain D is given as $D := I \times \mathbb{T}^2$ for some interval $I \subseteq \mathbb{R}^+$, cf. Remark A.32 in the appendix.

$$\|F(Y) - F(\bar{Y})\|_{\bar{\gamma}, \bar{\eta}, K} \lesssim \|Y - \bar{Y}\|_{\gamma, \eta, K} \tag{2.14}$$

uniformly over all $Y, \bar{Y} \in \mathcal{D}_V^{\gamma, \eta}(\mathcal{Z})$ with $\|Y\|_{\gamma, \eta, K} \vee \|\bar{Y}\|_{\gamma, \eta, K} \leq R$.

- Let $F^{\mathcal{Z}} : \mathcal{D}_V^{\gamma, \eta}(\mathcal{Z}) \rightarrow \mathcal{D}_V^{\bar{\gamma}, \bar{\eta}}(\mathcal{Z})$ be locally Lipschitz for any $\mathcal{Z} \in \mathcal{M}$. We say that $F = (F^{\mathcal{Z}})_{\mathcal{Z} \in \mathcal{M}}$ is *strongly locally Lipschitz continuous* if for any $\mathcal{Z} \in \mathcal{M}$ there exists a neighbourhood \mathcal{B} of \mathcal{Z} in \mathcal{M} such that for any compact set $K \subseteq D$ and $R > 0$ one has the bound

$$\|F^{\mathcal{Z}}(Y); F^{\bar{\mathcal{Z}}}(\bar{Y})\|_{\bar{\gamma}, \bar{\eta}, K} \lesssim \|Y; \bar{Y}\|_{\gamma, \eta, K} + \|\mathcal{Z}; \bar{\mathcal{Z}}\|_{\gamma, \bar{K}} \tag{2.15}$$

uniformly over all $\bar{\mathcal{Z}} \in \mathcal{B}$ and $Y \in \mathcal{D}_V^{\gamma, \eta}(\mathcal{Z}), \bar{Y} \in \mathcal{D}_V^{\gamma, \eta}(\bar{\mathcal{Z}})$ such that $\|Y\|_{\gamma, \eta, K, \mathcal{Z}} \vee \|\bar{Y}\|_{\gamma, \eta, K, \bar{\mathcal{Z}}} \leq R$. Here, \bar{K} denotes the 1-fattening of K , i.e. $\bar{K} = \{z \in D : \text{dist}(z, K) \leq 1\}$.

- A family $F = (F^\varepsilon)_{\varepsilon \in I}$ of strongly locally Lipschitz maps $F^\varepsilon = (F^{\varepsilon, \mathcal{Z}})_{\mathcal{Z} \in \mathcal{M}}$ is called *uniformly strongly locally Lipschitz continuous* if the bound (2.15) holds uniformly in ε for each F^ε . \spadesuit

The following lemma characterises the Lipschitz properties of the functions $(G^\varepsilon)_{\varepsilon \in I}$ introduced in (2.13).

Lemma 2.7. *The family $(G^\varepsilon)_{\varepsilon \in I}$ is uniformly strongly locally Lipschitz.* \spadesuit

Proof. Let $\alpha := \text{deg}(\mathfrak{I}) = -1 - \kappa$. For any $\zeta > 0$, we have $\mathfrak{I}, \mathfrak{I} \in \mathcal{D}_V^{\zeta, \zeta}$ where $\mathcal{V} = \langle \mathfrak{I}, \mathfrak{I} \rangle$ is a sector of reg. α . Then the same is true for $f_\varepsilon := \varepsilon \mathfrak{I} + \mathfrak{I}$ for any $\varepsilon \in I$. Recall that for $(\mathcal{Z}, Y) \in \mathcal{M} \times \mathcal{D}_U^{\gamma, \eta}$ we have $G(Y) \in \mathcal{D}_U^{\gamma, \eta}(\mathcal{Z})$ by [45, Prop. 6.13] because the sector \mathcal{U} is function-like. Since the pair $(\mathcal{V}, \mathcal{U})$ is $\bar{\gamma}$ -regular (see [45, Def. 4.6]) with $\bar{\gamma} := \gamma + \alpha$, with $\bar{\eta} := \eta + \alpha$ it follows from [45, Prop. 6.12] that $G^\varepsilon(\mathcal{D}_U^{\gamma, \eta}(\mathcal{Z})) \subseteq \mathcal{D}_U^{\bar{\gamma}, \bar{\eta}}(\mathcal{Z})$ for each $\mathcal{Z} \in \mathcal{M}$.

In order to check the estimate (2.15), let $\mathcal{Z}, \bar{\mathcal{Z}}$, and \bar{Y} be as required. Note that $\alpha = \min \mathcal{A}$, so by def. of the structure group \mathcal{G} , we have $\Gamma f_\varepsilon = f_\varepsilon = \bar{\Gamma} f_\varepsilon$ for any $\Gamma, \bar{\Gamma} \in \mathcal{G}$. Hence $\|f_\varepsilon^{\mathcal{Z}}; f_\varepsilon^{\bar{\mathcal{Z}}}\|_{\zeta, \zeta, K} = 0$. Again, [45, Prop. 6.12] then implies that

$$\begin{aligned} \|G^{\varepsilon, \mathcal{Z}}(Y); G^{\varepsilon, \bar{\mathcal{Z}}}(\bar{Y})\|_{\bar{\gamma}, \bar{\eta}, K} &\lesssim \|G^{\mathcal{Z}}(Y); G^{\bar{\mathcal{Z}}}(\bar{Y})\|_{\gamma, \eta, K} + \|f_\varepsilon^{\mathcal{Z}}; f_\varepsilon^{\bar{\mathcal{Z}}}\|_{\zeta, \zeta, K} + \|\mathcal{Z}; \bar{\mathcal{Z}}\|_{\gamma, \bar{K}} \\ &\lesssim_{K, R} \|Y, \bar{Y}\|_{\gamma, \eta, K} + \|\mathcal{Z}; \bar{\mathcal{Z}}\|_{\gamma, \bar{K}}, \end{aligned} \tag{2.16}$$

where the last estimate is a consequence of [49, Prop. 3.11] and the boundedness assumption on Y, \bar{Y} . \square

We can now answer the question of solvability of (2.11) affirmatively. The following corollary is a simple consequence of [45, Thm. 7.8] which applies by the previous lemma.

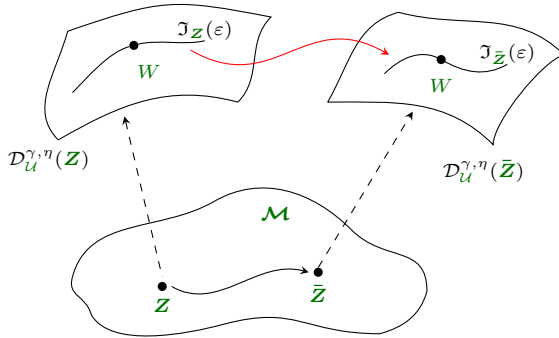


Fig. 1. A schematic depiction of the space $\mathcal{M} \times \mathcal{D}_U^{\gamma,\eta}$. The “fibres” $\mathcal{D}_U^{\gamma,\eta}(Z)$ are Banach spaces that vary when the underlying model $Z \in \mathcal{M}$ changes.

Corollary 2.8. For each $Z \in \mathcal{M}$ there exists a time $\bar{T} > 0$ such that for each $\varepsilon \in I$ there exists a unique fixed point $U^\varepsilon(Z) \in \mathcal{D}_U^{\gamma,\eta;\bar{T}}(Z)$ to (2.11). In addition, the map

$$\mathfrak{J}_Z : I \rightarrow \mathcal{D}_U^{\gamma,\eta;\bar{T}}(Z), \quad \mathfrak{J}_Z(\varepsilon) := U^\varepsilon(Z) \tag{2.17}$$

satisfies the bound

$$\|\mathfrak{J}_Z(\varepsilon); \mathfrak{J}_{\bar{Z}}(\varepsilon)\|_{\gamma,\eta,K} \lesssim_{\bar{T}} \|Z; \bar{Z}\|_{\gamma;\bar{K}}, \tag{2.18}$$

uniformly over all $\varepsilon \in I$ and all $Z, \bar{Z} \in \mathcal{M}$ with $\|Z\|_{\gamma;\bar{K}} \vee \|\bar{Z}\|_{\gamma;\bar{K}} \leq R$, for each $R > 0$ and all compact sets $K \subseteq (0, \bar{T}) \times \mathbb{T}^2$. In particular, \mathfrak{J}_Z depends continuously on Z . \spadesuit

Remark 2.9. If it is clear from the context what the underlying model Z is, we will just write U^ε and \mathfrak{J} , respectively. \spadesuit

Remark 2.10. One might be inclined to think that continuity of \mathfrak{J}_Z in Z is automatic from the Implicit Function Theorem (IFT). We want to emphasise that this is *not* the case due to the structure of the “fibred” space $\mathcal{M} \times \mathcal{D}_U^{\gamma,\eta}$ (see Fig. 1). To the best of the authors’ knowledge, the *geometry* of it has not yet been fully understood. However, we point to the work of Ghani Varzaneh, Riedel, and Scheutzow [43] who introduce the notion of a *measurable field of Banach spaces* to characterise the (in spirit related) space of controlled rough paths. \spadesuit

Notation 2.11. We will use the following notations from [23, Prop. 3.25]. Note that we suppress the dependence on the initial condition $u_0 \in \mathcal{C}^\eta(\mathbb{T}^2)$ that is fixed throughout the paper.

- We denote the solution map to (2.9) by \mathcal{S} , that is:

$$\mathcal{S} : \mathcal{M} \rightarrow \mathcal{D}_U^{\gamma,\eta}(Z), \quad Z \mapsto U = U(Z), \tag{2.19}$$

- We introduce the map \mathcal{S}_{tr} as

$$\mathcal{S}_{tr} : \mathcal{H} \times \mathcal{M} \rightarrow \mathcal{D}_{\mathcal{U}}^{\gamma, \eta}(T_h \mathbf{Z}), \quad \mathcal{S}_{tr}(h, \mathbf{Z}) := \mathcal{S}(T_h \mathbf{Z}). \tag{2.20}$$

- We denote by \mathcal{S} the solution map to (2.10), that is:

$$\mathcal{S} : \mathcal{M} \rightarrow \mathcal{D}_{\mathcal{U}}^{\gamma, \eta}(\mathbf{Z}), \quad \mathbf{Z} \mapsto U = U(\mathbf{Z}), \tag{2.21}$$

- We introduce the map \mathcal{S}_{ex} as

$$\mathcal{S}_{ex} : \mathcal{H} \times \mathcal{M} \rightarrow \mathcal{D}_{\mathcal{U}}^{\gamma, \eta}(E_h \mathbf{Z}), \quad \mathcal{S}_{ex}(h, \mathbf{Z}) := \mathcal{S}(E_h \mathbf{Z}). \tag{2.22}$$

For fixed $\varepsilon \in I$, we add two more definitions to this list.

- We denote by \mathcal{S}^ε the solution map to (2.11), that is:

$$\mathcal{S}^\varepsilon : \mathcal{M} \rightarrow \mathcal{D}_{\mathcal{U}}^{\gamma, \eta}(\mathbf{Z}), \quad \mathbf{Z} \mapsto U^\varepsilon = U^\varepsilon(\mathbf{Z}), \tag{2.23}$$

- We introduce the map $\mathcal{S}_{ex}^\varepsilon$ as

$$\mathcal{S}_{ex}^\varepsilon : \mathcal{H} \times \mathcal{M} \rightarrow \mathcal{D}_{\mathcal{U}}^{\gamma, \eta}(E_h \mathbf{Z}), \quad \mathcal{S}_{ex}^\varepsilon(h, \mathbf{Z}) := \mathcal{S}^\varepsilon(E_h \mathbf{Z}). \tag{2.24}$$

By definition, $\mathcal{S}^\varepsilon(\mathbf{Z}) = \mathfrak{I}_{\mathbf{Z}}(\varepsilon)$: The map \mathcal{S}^ε just emphasises the dependence on the model \mathbf{Z} . \spadesuit

A family of fixed-point equations

Recall that our object of interest, introduced below eq. (0.8) in the introduction, can also be written as

$$\hat{u}_h^\varepsilon(\hat{\mathbf{Z}}) = \Phi(T_h \delta_\varepsilon \hat{\mathbf{Z}}) = \mathcal{R}(\mathcal{S}_{tr}(h, \delta_\varepsilon \hat{\mathbf{Z}})) = \lim_{\delta \rightarrow 0} \mathcal{R}(\mathcal{S}_{tr}(h, M_\delta \mathbf{Z}^{\varepsilon \delta})). \tag{2.25}$$

In order to expand this quantity in ε we would like to differentiate the map $\varepsilon \mapsto \mathcal{S}_{tr}(h, \delta_\varepsilon \hat{\mathbf{Z}})$ but immediately run into a problem: the space $\mathcal{D}_{\mathcal{U}}^{\gamma, \eta}(T_h \delta_\varepsilon \hat{\mathbf{Z}})$ of modelled distributions that $\mathcal{S}_{tr}(h, \delta_\varepsilon \hat{\mathbf{Z}})$ lives in depends on ε itself. This is due to the “fibred” structure of $\mathcal{M} \times \mathcal{D}_{\mathcal{U}}^{\gamma, \eta}$, see Fig. 1.

We need the following lemma that is inspired by [23, Prop. 3.25] to remediate this problem.

Lemma 2.12. *Let $\mathbf{Z} \in \mathcal{M}$ be an admissible model, $h \in \mathcal{H}$, and $\varepsilon \in I$. Then,*

$$(\partial_\varepsilon \circ \mathfrak{t}_1) [\mathcal{S}_{tr}(h, \delta_\varepsilon \mathbf{Z})] = \mathcal{S}_{ex}^\varepsilon(h, \mathbf{Z}). \tag{2.26}$$

\spadesuit

Proof. From [23, Prop. 3.25], we know that

$$\mathcal{S}_{ex}(h, \bar{\mathbf{Z}}) = \mathbf{t}_1(\mathcal{S}_{tr}(h, \bar{\mathbf{Z}})),$$

for each $\bar{\mathbf{Z}} \in \mathcal{M}$, so by choosing $\bar{\mathbf{Z}} = \delta_\varepsilon \mathbf{Z}$ we only need to prove that

$$\mathfrak{d}_\varepsilon [\mathcal{S}_{ex}(h, \delta_\varepsilon \mathbf{Z})] = \mathcal{S}_{ex}^\varepsilon(h, \mathbf{Z}).$$

We denote the RHS by U^ε and the LHS by $\mathfrak{d}_\varepsilon(U_\varepsilon)$, where $U_\varepsilon := \mathcal{S}_{ex}(h, \delta_\varepsilon \mathbf{Z})$. We then observe that

$$\begin{aligned} \mathcal{P}^{E_h \mathbf{Z}}(G(\mathfrak{d}_\varepsilon U_\varepsilon)[\varepsilon \mathbf{1} + \mathbf{1}]) &= \mathcal{P}^{E_h \mathbf{Z}}(\mathfrak{d}_\varepsilon(G(U_\varepsilon)[\mathbf{1} + \mathbf{1}])) \\ &= \mathfrak{d}_\varepsilon [\mathcal{P}^{\delta_\varepsilon E_h \mathbf{Z}}(G(U_\varepsilon)[\mathbf{1} + \mathbf{1}])] = \mathfrak{d}_\varepsilon [\mathcal{P}^{E_h \delta_\varepsilon \mathbf{Z}}(G(U_\varepsilon)[\mathbf{1} + \mathbf{1}])], \end{aligned}$$

where the last and the penultimate equality are due to Lemma A.21 and Lemma A.38(iii), respectively. Note that we have also used that \mathfrak{d}_ε commutes with multiplication and application of G . It follows that

$$\begin{aligned} \mathfrak{d}_\varepsilon U_\varepsilon &= \mathfrak{d}_\varepsilon [\mathcal{P}^{E_h \delta_\varepsilon \mathbf{Z}}(G(U_\varepsilon)[\mathbf{1} + \mathbf{1}]) + \mathcal{T}P u_0] \\ &= \mathcal{P}^{E_h \mathbf{Z}}(G(\mathfrak{d}_\varepsilon U_\varepsilon)[\varepsilon \mathbf{1} + \mathbf{1}]) + \mathcal{T}P u_0, \end{aligned}$$

from which we infer that $\mathfrak{d}_\varepsilon U_\varepsilon$ solves (2.11) (w.r.t. $\mathbf{Z} := E_h \mathbf{Z}$). Hence, $U^\varepsilon = \mathfrak{d}_\varepsilon U_\varepsilon$ by uniqueness. \square

The following proposition links the fixed-point equation (2.9) with driving model $T_h \delta_\varepsilon \hat{\mathbf{Z}} \in \mathcal{M}$ to the family of fixed-point problems in (2.11) where the driving model now is $E_h \hat{\mathbf{Z}} \in \mathcal{M}$. Note that the latter does not suffer from ε -dependency of the solution space $\mathcal{D}^{\gamma, \eta}(E_h \hat{\mathbf{Z}})$, the problem we set out to solve.

Proposition 2.13. *Let $\varepsilon \in I$, $h \in \mathcal{H}$, and $\mathbf{Z} \in \mathcal{M}$. Then,*

$$\mathcal{R}^{T_h \delta_\varepsilon \mathbf{Z}} \mathcal{S}_{tr}(h, \delta_\varepsilon \mathbf{Z}) = \mathcal{R}^{E_h \mathbf{Z}} \mathcal{S}_{ex}^\varepsilon(h, \mathbf{Z}). \tag{2.27}$$

In particular, choosing the BPHZ model $\hat{\mathbf{Z}} \in \mathcal{M}$ leads to

$$\hat{u}_h^\varepsilon(\hat{\mathbf{Z}}) = \mathcal{R}^{E_h \hat{\mathbf{Z}}} \mathcal{S}_{ex}^\varepsilon(h, \hat{\mathbf{Z}}) \equiv \mathcal{R}^{E_h \hat{\mathbf{Z}}} \mathcal{J}^{E_h \hat{\mathbf{Z}}}(\varepsilon). \quad \spadesuit$$

Proof. We combine [23, Lem. 3.22(ii)], which states that $\mathcal{R}^{T_h \bar{\mathbf{Z}}} = \mathcal{R}^{E_h \bar{\mathbf{Z}}} \circ \mathbf{t}_1$ for each $\bar{\mathbf{Z}} \in \mathcal{M}$, and Lemma A.38(ii) to obtain the equality

$$\begin{aligned} \mathcal{R}^{T_h \delta_\varepsilon \mathbf{Z}} \mathcal{S}_{tr}(h, \delta_\varepsilon \mathbf{Z}) &= \mathcal{R}^{E_h \delta_\varepsilon \mathbf{Z}} \mathbf{t}_1(\mathcal{S}_{tr}(h, \delta_\varepsilon \mathbf{Z})) \\ &= \mathcal{R}^{E_h \mathbf{Z}}(\mathfrak{d}_\varepsilon \circ \mathbf{t}_1)(\mathcal{S}_{tr}(h, \delta_\varepsilon \mathbf{Z})) \end{aligned}$$

where we have also used that E_h and δ_ε commute, see Lemma A.21. The claim now follows from Lemma 2.12. \square

In Corollary 2.8, we have proved *local* existence of the FP eq. (2.11) for each $Z \in \mathcal{M}$. More precisely, we have obtained $\bar{T} = \bar{T}(Z) > 0$ such that $U^\varepsilon(Z)$ exists up to time \bar{T} , uniformly over $\varepsilon \in I$.

In general, however, the existence time $T_\infty(\varepsilon, Z)$ of $U^\varepsilon(Z)$ does depend on ε .²¹ Recall that we have started with a fixed time horizon $T \in (0, T_\infty^h \wedge T_0)$, so we need to find the right subset of $I \times \mathcal{M}$ for which existence up to time T is ensured. We need the following corollary towards this end.

Corollary 2.14. *For each $\varepsilon \in I$ and $Z \in \mathcal{M}$, we have $T_\infty(\varepsilon, E_h Z) = T_\infty(T_h \delta_\varepsilon Z)$. \spadesuit*

Proof. Since $U^\varepsilon(E_h Z) = \mathcal{S}_{ex}^\varepsilon(h, Z)$ and $T_\infty(T_h \delta_\varepsilon Z)$ is the explosion time of $S_{tr}(h, \delta_\varepsilon Z)$, the claim follows immediately from Lemma 2.12. \square

With this result, the following corollary was proved in the last part of the proof of Proposition 1.2.

Corollary 2.15. *For $T > 0$ as before, choose $\rho_0 = \rho_0(T) > 0$ as in Lemma B.4. Then, for any $\rho \in (0, \rho_0]$ and $(\varepsilon, Z^-) \in I \times \mathcal{M}_-$ with $\varepsilon \|Z^-\| < \rho$, we have $T_\infty(\varepsilon, E_h Z) > T$ for the extension $Z = \mathcal{E}(Z^-)$ of Z^- . \spadesuit*

For fixed T and Z , it makes sense to consider the *maximal* $\varepsilon \in I$ that satisfies the conditions in the previous corollary.

Definition 2.16. For $T > 0$ and $\rho_0 = \rho_0(T) > 0$ as in Lemma B.4, we define

$$\varepsilon_0(T, Z) := \frac{\rho_0(T)}{\|Z^-\|} \wedge 1 \quad \left(\frac{1}{0} := \infty \right) \tag{2.28}$$

Note that $\varepsilon_0(T, Z) = \sup\{\varepsilon \in I : \varepsilon \|Z^-\| < \rho_0(T)\}$. \spadesuit

A few remarks before we summarise our findings thus far.

Remark 2.17. It is immediate to see that $\varepsilon_0(T, \cdot) \circ \mathcal{E}$ is continuous on $(\mathcal{M}_-, \|\cdot\|)$ and, by Lemma A.9, on $(\mathcal{M}_-, \|\cdot\|)$. Therefore, $\varepsilon_0(T, \cdot)$ is continuous on $(\mathcal{M}, \|\cdot\|)$ by continuity of \mathcal{E} , cf. Theorem A.10; in particular, we have

$$\liminf_{\delta \rightarrow 0} \varepsilon_0(T, \hat{Z}^{\varepsilon_\delta}) = \varepsilon_0(T, \hat{Z}) \quad \text{as} \quad \lim_{\delta \rightarrow 0} \hat{Z}^{\varepsilon_\delta} = \hat{Z}.$$

²¹ Think of the deterministic ODE $\dot{x}_t = \varepsilon x_t^2$ with I.C. $x_0 > 0$. Since $x_t = x_0(1 - \varepsilon x_0 t)^{-1}$, it has explosion time $T(\varepsilon) = (\varepsilon x_0)^{-1}$, which is uniformly bounded from below by x_0^{-1} for $\varepsilon \in [0, 1]$. Hence, any $\bar{T} \leq x_0^{-1}$ would do.

In light of Corollary 2.15, this means that the renormalised solutions $\hat{u}_{\xi\delta,h}^{(\varepsilon)}$ and their limit $\hat{u}_h^{(\varepsilon)}$ as $\delta \rightarrow 0$ exist up to time T , uniformly over all $\varepsilon < \varepsilon_0(T, \hat{\mathbf{Z}})$. Since $T > 0$ is fixed, the previous observation justifies to fix $\varepsilon_0 := \varepsilon_0(T, \hat{\mathbf{Z}})$ and $I_0 := [0, \varepsilon_0)$ for the rest of the article. From here on, we only consider $\varepsilon \in I_0$, so non-explosion is guaranteed both for the approximate and the limiting equations. \spadesuit

Remark 2.18. Aligning with the nature of our problem, we will only consider models $\mathbf{Z} \in \mathcal{M}$ that satisfy $\mathbf{Z} = E_h \mathbf{Z}$ for some $\mathbf{Z} \in \mathcal{M}$ from here on. In light of the previous remark, the solution U^ε w.r.t. $\mathbf{Z} = E_h \hat{\mathbf{Z}}$ exists up to time T for all $\varepsilon \in I_0$. Disregarding such questions of existence, one could work with any $\mathbf{Z} \in \mathcal{M}$ at the expense of replacing $T \rightsquigarrow \bar{T} = \bar{T}(\mathbf{Z})$ again, cf. Corollary 2.8. \spadesuit

Remark 2.19 (*The case $\varepsilon = 0$.*). Recall that our problem reduces to the *deterministic* PDE (0.17), namely

$$(\partial_t - \Delta)w_h = g(w_h)h, \quad w_h(0, \cdot) = u_0,$$

in case $\varepsilon = 0$. This equation is represented by (2.11) in case ε_0 , so that $\mathcal{R}(E_h \hat{\mathbf{Z}}, U^0) = w_h$ as one might reasonably expect. Now recall that $U^0(E_h \hat{\mathbf{Z}})$ takes values in the sector $\mathcal{U} = \langle \mathbf{1}, \mathfrak{J}, \mathfrak{J}, X \rangle$ of which only one symbol (namely \mathfrak{J}) will be mapped to a non-deterministic distribution via $\hat{\Pi}$ (namely, a re-centered version of $G * \xi$). Compared with general ansatz (A.24) for $Y \in \mathcal{D}_{\mathcal{U}}^{\gamma,\eta}(E_h \mathbf{Z})$, one then quickly finds from (2.11) with $\varepsilon = 0$ that $\langle U^0(E_h \hat{\mathbf{Z}}), \mathfrak{J} \rangle = 0$, which is consistent with our intuition. We gain even more information: since for any $\mathbf{Z} \in \mathcal{M}$,

- the interpretation of $\mathbf{1}$ and X is determined by admissibility of \mathbf{Z} and
- that of \mathfrak{J} by the extension operator E_h

this means that $U^0(E_h \mathbf{Z})$ is actually *independent* of \mathbf{Z} .²² Henceforth, we will therefore use the symbol W for any $U^0(E_h \mathbf{Z})$ with explicit representation given by

$$W(z) = w_h(z)\mathbf{1} + \phi_{\mathfrak{J}}^W(z)\mathfrak{J} + \langle \phi_X^W(z), X \rangle. \tag{2.29}$$

Note that $W = \mathfrak{J}_{E_h \hat{\mathbf{Z}}}(0)$ is exactly the constant term in the Taylor expansion of $\tilde{\mathfrak{J}}_{E_h \hat{\mathbf{Z}}}(\varepsilon)$ that we are after. With $W := U(\mathcal{L}(h))$ as in (2.9), it also satisfies

$$\mathcal{R}^{\mathcal{L}(h)}W = w_h = \mathcal{R}^{E_h \mathbf{Z}}W$$

²² Of course, this is only half the truth: We have $U^0(E_h \mathbf{Z}) \in \mathcal{D}^{\gamma,\eta}(E_h \mathbf{Z})$ and the RHS *does* depend on \mathbf{Z} . The way that this is meant is that $U^0(E_h \mathbf{Z}) \in \mathcal{D}^{\gamma,\eta}(E_h \bar{\mathbf{Z}})$ for $\mathbf{Z}, \bar{\mathbf{Z}} \in \mathcal{M}$.

for any $\mathbf{Z} \in \mathcal{M}$. Finally, we emphasise that it is not important that \mathbf{h} (in $E_{\mathbf{h}}$) is the minimiser of \mathcal{F} , at least at this stage. However, this property of \mathbf{h} will be used in the sequel and thus makes it the natural choice. \spadesuit

Summary of this section’s findings

For $T \in (0, T_{\infty}^{\mathbf{h}} \wedge T_0)$, Proposition 2.13 justifies to abandon the map

$$I_0 \rightarrow \mathcal{D}_{\mathcal{U}}^{\gamma, \eta, T}(T_{\mathbf{h}} \delta_{\varepsilon} \hat{\mathbf{Z}}), \quad \varepsilon \mapsto \mathcal{S}_{tr}(\mathbf{h}, \delta_{\varepsilon} \hat{\mathbf{Z}})$$

and instead focus on

$$\mathfrak{J}_{E_{\mathbf{h}} \hat{\mathbf{Z}}} : I_0 \rightarrow \mathcal{D}_{\mathcal{U}}^{\gamma, \eta, T}(E_{\mathbf{h}} \hat{\mathbf{Z}}), \quad \varepsilon \mapsto \mathcal{S}_{ex}^{\varepsilon}(\mathbf{h}, \hat{\mathbf{Z}}), \tag{2.30}$$

the solution map to (2.11) (for $\mathbf{Z} := E_{\mathbf{h}} \hat{\mathbf{Z}}$) which we have already encountered in Lemma 2.7, eq. (2.17). Note, however, that we have replaced $\bar{T} \rightsquigarrow T$ and, accordingly, $I \rightsquigarrow I_0$. The target Banach space of this map does not depend on ε anymore, so we can study its Fréchet differentiability in the next section.

2.2. Differentiability of the fixed-point map

Our goal in this section is to establish differentiability of $\mathfrak{J}_{E_{\mathbf{h}} \hat{\mathbf{Z}}}$ via the Implicit Function Theorem in Banach spaces [82, Thm. 4.E.]. Throughout this section, $\mathbf{Z} := E_{\mathbf{h}} \hat{\mathbf{Z}}$ will be fixed; see, however, Remark 2.18. We then introduce the space

$$\mathcal{Y}_T^{\mathbf{Z}} := I_0 \times \mathcal{D}_{\mathcal{U}}^{\gamma, \eta, T}(\mathbf{Z}), \quad \|(\varepsilon, Y)\|_{\mathcal{Y}_T^{\mathbf{Z}}} := (\varepsilon^2 + \|Y\|_{\gamma, \eta; T}^2)^{1/2}$$

and consider the map

$$\Psi_T^{\mathbf{Z}} : \mathcal{Y}_T^{\mathbf{Z}} \rightarrow \mathcal{D}_{\mathcal{U}}^{\gamma, \eta, T}(\mathbf{Z}), \quad (\varepsilon, Y) \mapsto Y - \mathcal{M}_{\mathbf{Z}}(\varepsilon, Y), \tag{2.31}$$

with $\mathcal{M}^{\mathbf{Z}}(\varepsilon, Y) = \mathcal{P}^{\mathbf{Z}}(G(Y)[\varepsilon \mathbf{1} + \mathbf{1}]) + \mathcal{T}Pu_0$ as introduced in (2.13). Whenever convenient and without risk of confusion, we will drop the dependence on T and \mathbf{Z} for all involved quantities.

Proposition A.35 allows to calculate the candidates for the derivatives $D^{(k)}\Psi$.

Lemma 2.20. *Let $k \in \mathbb{N}$ and suppose the function Ψ is C^k F -differentiable in $(\varepsilon, Y) \in \mathcal{Y}$. Then, its k -th F -derivative $D^{(k)}\Psi(\varepsilon, Y) \in \mathcal{L}^{(k)}(\mathcal{D}_{\mathcal{U}}^{\gamma, \eta; T}(\mathbf{Z}))$ ²³ at (ε, Y) is given by*

$$D^{(k)}\Psi(\varepsilon, Y) [(\varepsilon_1, Y_1), \dots, (\varepsilon_k, Y_k)] \tag{2.32}$$

²³ For a normed linear space V and $k \in \mathbb{N}$, we denote by $\mathcal{L}^{(k)}(V)$ the k -linear, bounded maps from V into itself.

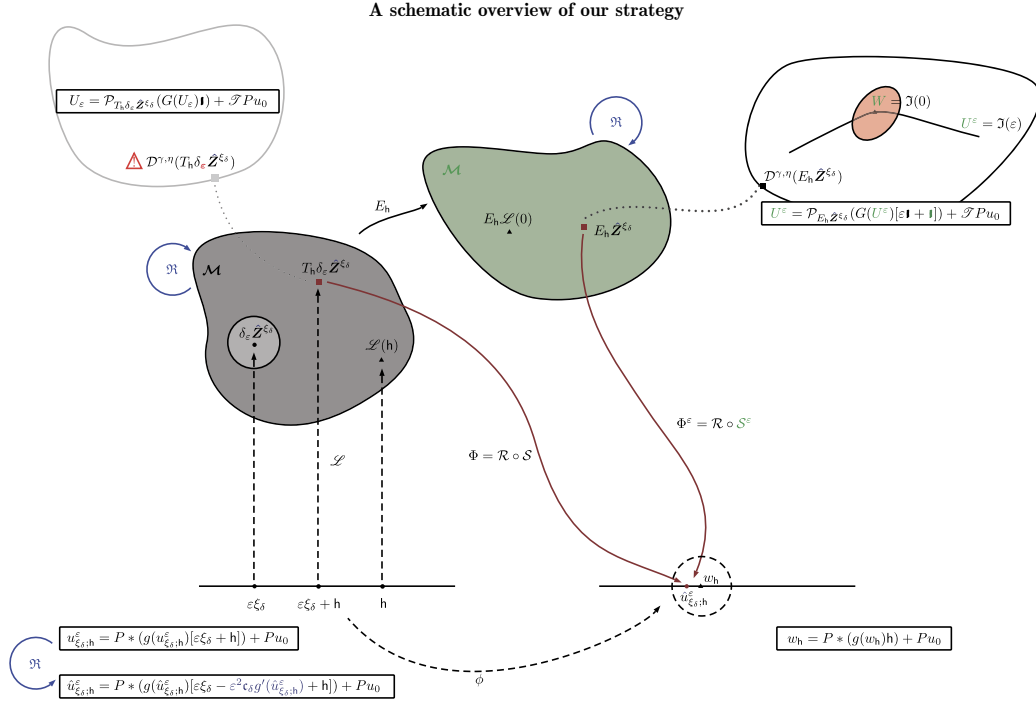


Fig. 2. Fix $\delta > 0$. On the LHS, we see the “noises” h and $\varepsilon \xi_\delta + h$ which are mapped by the solution map ϕ to w_h and $u_{\xi_\delta;h}^\varepsilon$, respectively. Dashed lines such as that for ϕ represent *discontinuous* operations: we thus *lift* h and $\varepsilon \xi_\delta + h$ (and also $\varepsilon \xi_\delta$) to the model space \mathcal{M} . Attached to the model $T_h \delta_\varepsilon \mathbf{Z}^{\kappa_s}$ is a space $\mathcal{D}^{\gamma,\eta}(T_h \delta_\varepsilon \mathbf{Z}^{\kappa_s})$ of modelled distributions: it hosts an abstract FP problem for U_ε which encodes $u_{\xi_\delta;h}^\varepsilon$. The solution map Φ that sends $T_h \delta_\varepsilon \mathbf{Z}^{\kappa_s}$ to U_ε via \mathcal{S} and then to $u_{\xi_\delta;h}^\varepsilon$ via \mathcal{R} is *continuous* now. Unfortunately, \mathbf{Z}^{κ_s} does not converge as $\delta \rightarrow 0$: a shortcoming cured by the *renormalisation group* \mathfrak{R} which acts on \mathcal{M} to change \mathbf{Z}^{κ_s} into $\hat{\mathbf{Z}}^{\kappa_s}$ and, “dually”, $u_{\xi_\delta;h}^\varepsilon$ into $\hat{u}_{\xi_\delta;h}^\varepsilon$. Indeed, $\hat{\mathbf{Z}}^{\kappa_s}$ does have a limit $\hat{\mathbf{Z}}$ as $\delta \rightarrow 0$ and so one can *define* $\hat{u}_\varepsilon := \Phi(T_h \delta_\varepsilon \hat{\mathbf{Z}})$. The lightgray ball around $\delta_\varepsilon \hat{\mathbf{Z}}^{\kappa_s}$ symbolises the localisation procedure of sec. 1 (for technical reasons, it was carried out in \mathcal{M}_- rather than \mathcal{M}). We run into problems once ε starts to vary: Because the Banach space $\mathcal{D}^{\gamma,\eta}(T_h \delta_\varepsilon \mathbf{Z}^{\kappa_s})$ depends on ε itself, we cannot differentiate U_ε w.r.t. that parameter. The remedy is to work on the extended model space \mathcal{M} depicted on the RHS: In the Banach space $\mathcal{D}^{\gamma,\eta}(E_h \hat{\mathbf{Z}}^{\kappa_s})$ attached to $E_h \hat{\mathbf{Z}}^{\kappa_s}$, we set up a FP problem with ε as a parameter. Its solution $U^\varepsilon = \mathcal{S}^\varepsilon(E_h \hat{\mathbf{Z}}^{\kappa_s})$ is reconstructed to $\hat{u}_{\xi_\delta;h}^\varepsilon$ by the operator \mathcal{R} just as U_ε is (Proposition 2.13), as symbolised by the two red arrows. However, the map $\mathfrak{I} : [0, 1] \ni \varepsilon \mapsto U^\varepsilon \in \mathcal{D}^{\gamma,\eta}(E_h \hat{\mathbf{Z}}^{\kappa_s})$ is well-defined and may be investigated for its differentiability properties. The shape of this figure is inspired by L. Zambotti’s slides on *SPDEs and regularity structures* from the Second Haifa Probability School 2017 [81].

$$= Y_1 \mathbf{1}_{k=1} - \mathcal{P} \left(G^{(k)}(Y) \prod_{m=1}^k Y_m [\varepsilon \mathbf{1} + \mathbf{1}] \right) - \sum_{\ell=1}^k \varepsilon_\ell \mathcal{P} \left(G^{(k-1)}(Y) \prod_{\substack{m=1, \\ m \neq \ell}}^k Y_m \mathbf{1} \right). \spadesuit$$

Proof. The claim follows from an inductive argument in k using Proposition A.35 for straightforward direct computations. \square

Note that the premise of the previous proposition is yet to be validated. In any case, however, the expressions on the RHS of (2.32) make perfect sense: We will use this fact to our advantage when verifying the criteria for the Implicit Function Theorem (IFT) to apply.

We start with differentiability of the map Ψ from (2.31). For the corresponding statement in the rough paths setting (without dilation and encoded Cameron-Martin shifts), see Bailleul [12].

Lemma 2.21 (IFT, 1st condition). *Suppose $g \in \mathcal{C}^{\ell+4}$, $\ell \geq 1$, and construct the lift $G^{(m)}$ of $g^{(m)}$, $m = 1, \dots, \ell$, by means of (A.25). Then, there exists a neighbourhood N of $(0, W) \in \mathcal{Y}$ on which the map Ψ is \mathcal{C}^ℓ Fréchet with derivatives given in (2.32). \spadesuit*

Remark 2.22. It has proved futile to use the IFT to prove Malliavin differentiability of the solution map to (singular) stochastic (P)DEs. For gPAM, this is formulated in [23, Prop. 4.7]. In the proof of that proposition, the authors establish the \mathcal{C}^1 F-differentiability of some $\tilde{\Psi}$ (there called F_γ) which coincides with Ψ up to the roles of $\mathbf{1}$ (there denoted by Ξ) and $\mathbf{1}$ (there denoted by H) reversed. Schönbauer [76] has used that idea in full generality, covering any stochastic PDE that can be solved with the theory of regularity structures.

Prior to these recent developments, Nualart and Saussereau [70] have employed the same argument in the context of SDEs. So have Deya and Tindel [30], albeit for stochastic PDEs driven by the fractional BM with Hurst index $H > \frac{1}{2}$. \spadesuit

In the following proof, we will occasionally refer to the proof of Proposition A.35, where similar arguments have been used. We recommend the reader to check that proposition before proceeding.

Proof. Let $M > 0$, denote by $\mathcal{B}_T^{\gamma,\eta}(W, M)$ a closed ball in $\mathcal{D}_U^{\gamma,\eta;T}(\mathcal{Z})$ of radius M , centered at W , and set $B_T^{\gamma,\eta}(W, M) := I_0 \times \mathcal{B}_T^{\gamma,\eta}(W, M) \subseteq \mathcal{Y}$. We fix $\alpha := \text{deg}(\mathbf{1}) = \text{deg}(\mathbf{1}) = -1 - \kappa$.

Suppose we had already established that $\Psi \in \mathcal{C}^k$ at $(\varepsilon, Y) \in B_T^{\gamma,\eta}(W, M)$ for some $k \in \{1, \dots, \ell - 1\}$; its derivatives up to order k are then given by (2.32). We want to do the induction step $k \mapsto k + 1$ and prove that $\Psi \in \mathcal{C}^{k+1}$ at (ε, Y) . As $D^{(k)}\Psi(\varepsilon, Y)$ is linear in ε , it suffices to consider the case $\varepsilon = 0$. We let $(\varepsilon_{k+1}, Y_{k+1}) \in I_0 \times \mathcal{B}_T^{\gamma,\eta}(0, M)$ and study the k -th order remainder (see [82, p. 229, eq. (7)])

$$\begin{aligned} \Delta^{(k)} [(\varepsilon_i, \mathbf{Y}_i)_{i=1}^{k+1}] &\equiv \Delta^{(k)} [(0, \mathbf{Y}); (\varepsilon_i, \mathbf{Y}_i)_{i=1}^{k+1}] \\ &:= D^{(k)} \Psi(\varepsilon_{k+1}, \mathbf{Y} + \mathbf{Y}_{k+1}) [(\varepsilon_i, \mathbf{Y}_i)_{i=1}^k] - D^{(k)} \Psi(0, \mathbf{Y}) [(\varepsilon_i, \mathbf{Y}_i)_{i=1}^k] \\ &\quad - D^{(k+1)} \Psi(0, \mathbf{Y}) [(\varepsilon_i, \mathbf{Y}_i)_{i=1}^{k+1}] \end{aligned} \tag{2.33}$$

where $D^{(k+1)} \Psi(0, \mathbf{Y}) [(\varepsilon_i, \mathbf{Y}_i)_{i=1}^{k+1}]$ is defined by the corresponding quantity in (2.32). At this stage, this notation is only suggestive. However, we will prove that

$$\begin{aligned} \sup_{\substack{\|(\varepsilon_m, \mathbf{Y}_m)\|_{\mathcal{Y}} \leq 1, \\ 1 \leq m \leq k}} \|\Delta^{(k)} [(0, \mathbf{Y}); (\varepsilon_i, \mathbf{Y}_i)_{i=1}^{k+1}]\|_{\gamma, \eta; T} &= o(\|(\varepsilon_{k+1}, \mathbf{Y}_{k+1})\|_{\mathcal{Y}}), \\ (\varepsilon_{k+1}, \mathbf{Y}_{k+1}) &\rightarrow 0, \end{aligned} \tag{2.34}$$

which indeed qualifies $D^{(k+1)} \Psi(0, \mathbf{Y})$ as the $(k + 1)$ -th derivative of Ψ at $(0, \mathbf{Y})$. An elementary calculation gives

$$\Delta^{(k)} [(0, \mathbf{Y}); (\varepsilon_i, \mathbf{Y}_i)_{i=1}^{k+1}] = \Delta_{\mathbf{1}}^{(k)} [\mathbf{Y}; (\varepsilon_i, \mathbf{Y}_i)_{i=1}^{k+1}] + \Delta_{\mathbf{1}}^{(k)} [\mathbf{Y}; (\mathbf{Y}_i)_{i=1}^{k+1}] \tag{2.35}$$

where one summand collects the terms with the noise symbol $\mathbf{1}$, namely

$$\begin{aligned} \Delta_{\mathbf{1}}^{(k)} [\mathbf{Y}; (\varepsilon_i, \mathbf{Y}_i)_{i=1}^{k+1}] &:= \Delta_{\mathbf{1}}^{(k);1} [\mathbf{Y}; (\varepsilon_i, \mathbf{Y}_i)_{i=1}^{k+1}] + \Delta_{\mathbf{1}}^{(k);2} [\mathbf{Y}; (\varepsilon_i, \mathbf{Y}_i)_{i=1}^{k+1}], \\ \Delta_{\mathbf{1}}^{(k);1} [\mathbf{Y}; (\varepsilon_i, \mathbf{Y}_i)_{i=1}^{k+1}] &:= -\varepsilon_{k+1} \mathcal{P} \left([G^{(k)}(\mathbf{Y} + \mathbf{Y}_{k+1}) - G^{(k)}(\mathbf{Y})] \prod_{m=1}^k \mathbf{Y}_m \mathbf{1} \right), \\ \Delta_{\mathbf{1}}^{(k);2} [\mathbf{Y}; (\varepsilon_i, \mathbf{Y}_i)_{i=1}^{k+1}] &:= -\sum_{\ell=1}^k \varepsilon_{\ell} \mathcal{P} \left([G^{(k-1)}(\mathbf{Y} + \mathbf{Y}_{k+1}) - G^{(k-1)}(\mathbf{Y}) - G^{(k)}(\mathbf{Y}) \mathbf{Y}_{k+1}] \prod_{\substack{m=1, \\ m \neq \ell}}^k \mathbf{Y}_m \mathbf{1} \right), \end{aligned} \tag{2.36}$$

and the other one collects those with the Cameron-Martin symbol $\mathbf{1}$, namely

$$\Delta_{\mathbf{1}}^{(k)} [\mathbf{Y}; (\mathbf{Y}_i)_{i=1}^{k+1}] := -\mathcal{P} \left([G^{(k)}(\mathbf{Y} + \mathbf{Y}_{k+1}) - G^{(k)}(\mathbf{Y}) - G^{(k+1)}(\mathbf{Y}) \mathbf{Y}_{k+1}] \prod_{m=1}^k \mathbf{Y}_m \mathbf{1} \right). \tag{2.37}$$

Part 1: Analysis of the Cameron-Martin part $\Delta_{\mathbf{1}}^{(k)}$. At first, we write

$$\Delta_{\mathbf{1}}^{(k)} [\mathbf{Y}; (\mathbf{Y}_i)_{i=1}^{k+1}] = -\mathcal{P} (R^{(k)} [\mathbf{Y}; (\mathbf{Y}_i)_{i=1}^{k+1}] \mathbf{1}) \tag{2.38}$$

with $R^{(k)} [\mathbf{Y}; (\mathbf{Y}_i)_{i=1}^{k+1}]$ implicitly defined from (2.37) and explicitly given in (A.29) in the proof of Proposition A.35, where it is also identified as an element of $\mathcal{D}_{\mathcal{U}}^{\gamma, \eta; T}(\mathbf{Z})$.

As in the proof of [45, Lem. 9.1], the map $z \mapsto \mathbf{1}$ is easily seen to be an element in $\mathcal{D}^{\zeta, \zeta}(\mathbf{Z})$, $\zeta > 0$, with values in a sector of reg. α . From [45, Prop. 6.12], we thus infer

that $R^{(k)} [Y; (Y_i)_{i=1}^{k+1}] \# \in \mathcal{D}^{\gamma+\alpha, \eta+\alpha; T}(\mathbf{Z})$ with values in another sector of reg. α . Since $\eta < \bar{\alpha} := \alpha + 2$ and $\bar{\alpha} > 0$ ²⁴, we have

$$\|\Delta_{\mathbf{I}}^{(k)} [Y; (Y_i)_{i=1}^{k+1}]\|_{\gamma, \eta; T} \lesssim \|\Delta_{\mathbf{I}}^{(k)} [Y; (Y_i)_{i=1}^{k+1}]\|_{\gamma, \bar{\alpha}; T} \lesssim \|\Delta_{\mathbf{I}}^{(k)} [Y; (Y_i)_{i=1}^{k+1}]\|_{\gamma+\bar{\alpha}, \bar{\alpha}; T} \tag{2.39}$$

and now apply [45, Prop.'s 6.16 & 6.12]²⁵ to obtain that

$$\begin{aligned} \|\Delta_{\mathbf{I}}^{(k)} [Y; (Y_i)_{i=1}^{k+1}]\|_{\gamma+\bar{\alpha}, \bar{\alpha}; T} &\lesssim \|R^{(k)} [Y; (Y_i)_{i=1}^{k+1}]\|_{\gamma+\alpha, \eta+\alpha; T} \\ &\lesssim \|R^{(k)} [Y; (Y_i)_{i=1}^{k+1}]\|_{\gamma, \eta; T}. \end{aligned} \tag{2.40}$$

The argument is now identical to that in (A.32): relying on Taylor’s theorem up to order one (i.e. Lemma A.34, eq. (A.26)) we have

$$\begin{aligned} \|R^{(k)} [Y; (Y_i)_{i=1}^{k+1}]\|_{\gamma, \eta; T} &\lesssim (\|Y\|_{\gamma, \eta; T} + \|Y_{k+1}\|_{\gamma, \eta; T}) \|Y_{k+1}\|_{\gamma, \eta; T}^2 \prod_{n=1}^k \|Y_n\|_{\gamma, \eta; T} \\ &\lesssim C(\|W\|_{\gamma, \eta; T}, M) \|Y_{k+1}\|_{\gamma, \eta; T}^2, \end{aligned} \tag{2.41}$$

where we have used that $Y \in \mathcal{B}_{\gamma, \eta}^T(W, M)$, $Y_{k+1} \in \mathcal{B}_{\gamma, \eta}^T(0, M)$, and that $\|Y_m\|_{\gamma, \eta; T} \leq 1$ for $1 \leq m \leq k$ as assumed in (2.34). We then infer that

$$\sup_{\substack{\|Y_m\|_{\gamma, \eta; T} \leq 1, \\ 1 \leq m \leq k}} \|\Delta_{\mathbf{I}}^{(k)} [Y; (Y_i)_{i=1}^{k+1}]\|_{\gamma, \eta; T} = O(\|Y_{k+1}\|_{\gamma, \eta; T}^2), \quad Y_{k+1} \rightarrow 0. \tag{2.42}$$

Part 2: Analysis of the noise part $\Delta_{\mathbf{I}}^{(k)} = \Delta_{\mathbf{I}}^{(k);1} + \Delta_{\mathbf{I}}^{(k);2}$. Modulo minor changes, the arguments are the same as before: We obtain the estimates

$$\sup_{\substack{\|(\varepsilon_m, Y_m)\|_{\mathcal{Y}} \leq 1, \\ 1 \leq m \leq k}} \|\Delta_{\mathbf{I}}^{(k);2} [Y; (\varepsilon_i, Y_i)_{i=1}^{k+1}]\|_{\gamma, \eta; T} = O(\|Y_{k+1}\|_{\gamma, \eta; T}^2), \quad Y_{k+1} \rightarrow 0 \tag{2.43}$$

and

$$\begin{aligned} \sup_{\substack{\|(\varepsilon_m, Y_m)\|_{\mathcal{Y}} \leq 1, \\ 1 \leq m \leq k}} \|\Delta_{\mathbf{I}}^{(k);1} [Y; \varepsilon_{k+1}, (Y_i)_{i=1}^{k+1}]\|_{\gamma, \eta; T} &= O(\varepsilon_{k+1} \|Y_{k+1}\|_{\gamma, \eta; T}), \\ (\varepsilon_{k+1}, Y_{k+1}) &\rightarrow 0, \end{aligned} \tag{2.44}$$

see the extended arXiv version [34] for details. Combining (2.42), (2.43), and (2.44) gives

²⁴ See Remark 2.23 below for a comment on the implications of this constraint for adaptations of our argument to cover the Φ_3^4 equation where $\bar{\alpha} < 0$.

²⁵ Actually, instead of [45, Prop. 6.12], we apply an analogue of [45, Thm. 4.7] (valid for \mathcal{D}^γ spaces *without* blow-ups) in the setting of $\mathcal{D}^{\gamma, \eta}$ spaces *with* blow-ups.

$$\sup_{\substack{\|(\varepsilon_m, Y_m)\|_{\mathcal{Y}} \leq 1, \\ 1 \leq m \leq k}} \|\Delta^{(k)} [(0, Y); (\varepsilon_i, Y_i)_{i=1}^{k+1}]\|_{\gamma, \eta; T} = O(\|(\varepsilon_{k+1}, Y_{k+1})^2\|_{\mathcal{Y}}),$$

$$(\varepsilon_{k+1}, Y_{k+1}) \rightarrow 0,$$

so the claim (2.34) follows. All the estimates are also valid for $k = 0$, which establishes $\Psi \in \mathcal{C}^1$ at $(\varepsilon, Y) \in B_T^{\gamma, \eta}(W, M)$, so we can close the inductive proof. \square

Remark 2.23. At first glance, it might seem that the arguments in (2.39) exclude amendments of our results to cover Φ_3^4 where $\alpha \in (-18/7, -5/2)$ and, as a consequence, $\bar{\alpha} := \alpha + 2 < 0$, cf. [45, Sec. 9.4]. However, this is not the case. Fundamentally, this is due to [45, Lem. 9.7 & Prop. 9.8] (and their proofs) which state that, for $\gamma > |2\alpha + 4|$ and $\eta \in (-2/3, \bar{\alpha})$, the map $U \mapsto U^3$ is strongly locally Lipschitz from $\mathcal{D}_V^{\gamma, \eta}$ to $\mathcal{D}_{\bar{V}}^{\gamma+2\alpha+4, 3\eta}$ for sectors V and \bar{V} of regularity $\bar{\alpha}$ and $3\bar{\alpha}$, respectively. Hence, [45, Prop. 6.16] implies that

$$\begin{aligned} \|\mathcal{P}(U^3)\|_{\gamma, \eta; T} &\lesssim \|\mathcal{P}(U^3)\|_{\gamma, 3\eta+2; T} \lesssim \|\mathcal{P}(U^3)\|_{\gamma+2\alpha+6, 3\eta+2; T} \\ &\lesssim \|U^3\|_{\gamma+2\alpha+4, 3\eta; T} \lesssim \|U\|_{\gamma, \eta; T} \end{aligned}$$

where the first two inequalities hold because $\eta < 3\eta + 2$ and $2\alpha + 6 > 0$. \spadesuit

The following lemma verifies the second condition for the IFT to apply.

Lemma 2.24 (IFT, 2nd condition). *For each $T \in (0, T_\infty^h \wedge T_0)$ and $(\varepsilon, J) \in I \times \mathcal{D}_U^{\gamma, \eta, T}(\mathcal{Z})$, the map*

$$D_2\Psi^{\mathcal{Z}}(\varepsilon, J) = \text{Id} - \mathcal{P}^{\mathcal{Z}}(G'(J) \bullet [\varepsilon \mathbf{1} + \mathbf{1}]) \tag{2.45}$$

is an element of $\mathcal{L}(\mathcal{D}_U^{\gamma, \eta; T}(\mathcal{Z}))$ and bijective. In particular, this is true for the map $\Theta = \Theta_T^{\mathcal{Z}}$ given by

$$\Theta_T^{\mathcal{Z}} := D_2\Psi^{\mathcal{Z}}(0, W) = \text{Id} - \mathcal{P}^{\mathcal{Z}}(G'(W) \bullet \mathbf{1}). \tag{2.46}$$

We will write $\tilde{\Theta} := \Theta^{-1}$ for the inverse of Θ . \spadesuit

Proof. It is clear that $D_2\Psi(\varepsilon, J) \in \mathcal{L}(\mathcal{D}_U^{\gamma, \eta}(\mathcal{Z}))$: linearity is trivial and boundedness follows by exactly the same arguments as (2.39), (2.40), and the paragraph preceding them:

$$\begin{aligned} \|D_2\Psi(\varepsilon, J)(Y)\|_{\gamma, \eta; T} &\lesssim \|Y\|_{\gamma, \eta; T} + \|\mathcal{P}(G'(J)Y [\varepsilon \mathbf{1} + \mathbf{1}])\|_{\gamma+\bar{\alpha}, \bar{\alpha}; T} \\ &\lesssim \|Y\|_{\gamma, \eta; T} + \|G'(W)Y [\varepsilon \mathbf{1} + \mathbf{1}]\|_{\gamma+\alpha, \eta+\alpha; T} \\ &\lesssim C(\|J\|_{\gamma, \eta; T}, g) \|Y\|_{\gamma, \eta} \end{aligned}$$

In the last estimate, we have additionally used [45, Prop. 6.13]. Regarding the bijectivity of $D_2\Psi(\varepsilon, J)$, we need to prove that for each $V \in \mathcal{D}_U^{\gamma, \eta}(\mathbf{Z})$ there exists a $Y \in \mathcal{D}_U^{\gamma, \eta}(\mathbf{Z})$ such that

$$Y = V + \mathcal{P} \left(G'(J)Y [\varepsilon \mathbb{1} + \mathbb{1}] \right). \tag{2.47}$$

This FP problem can again be solved by [45, Thm. 7.8] because $DG_\varepsilon(J) : Y \mapsto G'(J)Y [\varepsilon \mathbb{1} + \mathbb{1}]$

- maps $\mathcal{D}_U^{\gamma, \eta; T}(\mathbf{Z})$ into $\mathcal{D}_U^{\gamma+\alpha, \eta+\alpha; T}(\mathbf{Z})$, $\alpha := \text{deg}(\mathbb{1}) = -1 - \kappa$,
- is strongly locally Lipschitz in the sense of Definition 2.6

The arguments are the same as in the proof of Lemma 2.7, in particular (2.16). Actually, it suffices for $DG_\varepsilon(J)$ to be locally Lipschitz continuous which is true because it is linear and continuous. \square

Remark 2.25. Later on, we will actually derive bounds for the operator norm of Θ for models with norm of order one, see Proposition 2.35 and Remark 2.36. \spadesuit

With these preparations, the following theorem about \mathcal{C}^ℓ F-differentiability of \mathfrak{J}_Z essentially is a consequence of the IFT. In addition, we provide explicit (iterative) FP problems solved by its derivatives $\mathfrak{J}_Z^{(m)}$. For the corresponding statement in the context of Malliavin calculus, see [76, Prop. 4.1].

Theorem 2.26 (Differentiability of \mathfrak{J}). *Let $T \in (0, T_\infty^h \wedge T_0)$, $\mathbf{Z} \in \mathcal{M}$, and set $\mathbf{Z} := E_h \mathbf{Z} \in \mathcal{M}$. Suppose $g \in \mathcal{C}^{\ell+4}$ for some $\ell \geq 1$. Then there exists some $\varepsilon_\star = \varepsilon_\star(T, \mathbf{Z}) > 0$ such that, for $I_\star := [0, \varepsilon_\star)$, we have*

- (i) $\mathfrak{J}_Z \in \mathcal{C}^\ell(I_\star, \mathcal{D}_U^{\gamma, \eta; T}(\mathbf{Z}))$ in the Fréchet sense.
- (ii) For $m = 1, \dots, \ell$ and $\varepsilon \in I_\star$, the derivative $\mathfrak{J}_Z^{(m)}(\varepsilon) \in \mathcal{D}_U^{\gamma, \eta; T}(\mathbf{Z})$ uniquely solves the FP equation

$$\mathfrak{J}_Z^{(m)}(\varepsilon) = \mathcal{M}_m^Z \left(\varepsilon, (\mathfrak{J}_Z^{(k)}(\varepsilon))_{k=0}^m \right), \quad \varepsilon \in I_\star, \tag{2.48}$$

for the FP map \mathcal{M}_m^Z given by

$$\begin{aligned} \mathcal{M}_m^Z \left(\varepsilon, (\mathfrak{J}_Z^{(k)}(\varepsilon))_{k=0}^m \right) &:= \mathcal{P}^Z \left(G^{(m)}(\mathfrak{J}_Z(\varepsilon)) \mathfrak{J}_Z^{(m)}(\varepsilon) [\varepsilon \mathbb{1} + \mathbb{1}] \right) \\ &+ \varepsilon \mathcal{P}^Z \left(B_Z^{(m-1), \mathbb{1}}(\varepsilon) \mathbb{1} \right) + \mathcal{P}^Z \left(B_Z^{(m-1), \mathbb{1}}(\varepsilon) \mathbb{1} \right) \\ &+ m \mathcal{P}^Z \left(A_Z^{(m-1), \mathbb{1}}(\varepsilon) \mathbb{1} \right). \end{aligned} \tag{2.49}$$

Here, we have set²⁶ for $m \geq 1$:

$$A_{\mathbf{Z}}^{(m),\mathbf{1}}(\varepsilon) := m! \sum_{k=1}^m \frac{1}{k!} G^{(k)}(\mathfrak{J}_{\mathbf{Z}}(\varepsilon)) \sum_{\mathbf{i} \in S_k^m} \prod_{\ell=1}^k \frac{1}{i_{\ell}!} \mathfrak{J}_{\mathbf{Z}}^{(i_{\ell})}(\varepsilon), \quad A_{\mathbf{Z}}^{(0),\mathbf{1}}(\varepsilon) := G(\mathfrak{J}_{\mathbf{Z}}(\varepsilon)) \tag{2.50}$$

$$B_{\mathbf{Z}}^{(m-1),\sigma}(\varepsilon) := m! \sum_{k=2}^m \frac{1}{k!} G^{(k)}(\mathfrak{J}_{\mathbf{Z}}(\varepsilon)) \sum_{\mathbf{i} \in S_k^m} \prod_{\ell=1}^k \frac{1}{i_{\ell}!} \mathfrak{J}_{\mathbf{Z}}^{(i_{\ell})}(\varepsilon), \quad \sigma \in \{\mathbf{i}, \mathbf{1}\}. \tag{2.51}$$

where

$$S_k^m := \{\mathbf{i} \in \mathbb{N}_{\geq 1}^k : |\mathbf{i}| = m\}.$$

- (iii) For $m = 1, \dots, \ell$ and $\varepsilon \in I_{\star}$, the map $\mathbf{Z} \mapsto \mathfrak{J}_{\mathbf{Z}}^{(m)}(\varepsilon)$ is strongly locally Lipschitz in the sense of Definition 2.6.
- (iv) It is possible to choose ε_{\star} as²⁷

$$\varepsilon_{\star}(T, \mathbf{Z}) := \sup I_{\infty}^{\mathbf{Z}}(T), \quad I_{\infty}^{\mathbf{Z}}(T) := \{\varepsilon \in I : T_{\infty}(r, E_{\mathbf{h}}\mathbf{Z}) > T \text{ for all } r \in [0, \varepsilon]\}. \tag{2.52}$$

In this case, we have $\varepsilon_{\star}(T, \mathbf{Z}) \geq \varepsilon_0(T, \mathbf{Z})$, the latter defined in (2.28). In particular, $\mathfrak{J}_{\mathbf{Z}} \in \mathcal{C}^{\ell}(I_0, \mathcal{D}_{\mathcal{U}}^{\gamma, \eta; T}(\mathbf{Z}))$ for $I_0 = [0, \varepsilon_0]$ as before. \spadesuit

Remark 2.27. Note that $\mathcal{M}_{\mathbf{Z}}^{\mathbf{Z}}$ in (2.49) is linear in $\mathfrak{J}_{\mathbf{Z}}^{(m)}(\varepsilon)$, so the explosion time of $\mathfrak{J}_{\mathbf{Z}}^{(m)}(\varepsilon)$ coincides with that of $\mathfrak{J}_{\mathbf{Z}}(\varepsilon)$. Also note that $\varepsilon_{\mathbf{1}}$ as well as the second term on the RHS of (2.49) vanish for $\varepsilon = 0$. \spadesuit

Proof. By Lemmas 2.21 and 2.24, we can apply the IFT (see [82, Thm. 4.E]) to Ψ . It asserts that there exists a neighbourhood \mathcal{O} of W and $\varepsilon_{\star} > 0$ and a unique function $Y_{\star} \in \mathcal{C}^{\ell}(I_{\star}, \mathcal{O})$ such that

$$\Psi(\varepsilon, Y_{\star}(\varepsilon)) = 0 \quad \text{for } \varepsilon \in I_{\star} = [0, \varepsilon_{\star}].$$

By uniqueness of solutions to the fixed-point problem (2.11), we find that $Y_{\star}(\varepsilon) = \mathfrak{J}(\varepsilon)$ and hence the claim (i) follows. Regarding (ii), we differentiate m times both sides of the FP equation

²⁶ Note that the terms on the RHS of (2.50) and (2.51) are exactly the same, only the summation in the latter starts with $k = 2$. Note, however that $B_{\mathbf{Z}}^{(m-1),\sigma}$ only depends on $\mathfrak{J}_{\mathbf{Z}}^{(n)}$ for $n \in [m-1]$, whereas $A_{\mathbf{Z}}^{(m),\mathbf{1}}$ does so for $n \in [m]$. The asymmetry in choice of superscript (“ m ” for A , but “ $m - 1$ ” for B) reflects that fact. Also, note that $B_{\mathbf{Z}}^{(m-1),\sigma}(\varepsilon) \equiv 0$ for $m = 1$.

²⁷ Recall that $T_{\infty}(\varepsilon, E_{\mathbf{h}}\mathbf{Z})$ is the existence time of $\mathfrak{J}_{E_{\mathbf{h}}\mathbf{Z}}(\varepsilon)$, cf. Corollary 2.14 and the comments preceding it.

$$\mathfrak{J}_{\mathbf{Z}}(\varepsilon) = \mathcal{P}^{\mathbf{Z}}(G(\mathfrak{J}_{\mathbf{Z}}(\varepsilon))\varepsilon \mathfrak{I}) + \mathcal{P}^{\mathbf{Z}}(G(\mathfrak{J}_{\mathbf{Z}}(\varepsilon)) \mathfrak{I}) + \mathcal{F}Pu_0.$$

For the expression on the RHS, this is valid thanks to the linearity of $\mathcal{P}^{\mathbf{Z}}$, Proposition A.35, and (i) By Leibniz rule, we have

$$\begin{aligned} \partial_{\varepsilon}^m[G(\mathfrak{J}_{\mathbf{Z}}(\varepsilon))\varepsilon \mathfrak{I}] &= \sum_{k=0}^m \binom{m}{k} \partial_{\varepsilon}^{m-k}[G(\mathfrak{J}_{\mathbf{Z}}(\varepsilon))] \partial_{\varepsilon}^k[\varepsilon \mathfrak{I}] \\ &= \partial_{\varepsilon}^m[G(\mathfrak{J}_{\mathbf{Z}}(\varepsilon))]\varepsilon \mathfrak{I} + m\partial_{\varepsilon}^{m-1}[G(\mathfrak{J}_{\mathbf{Z}}(\varepsilon))] \mathfrak{I} \end{aligned}$$

which then leads to

$$\mathfrak{J}_{\mathbf{Z}}^{(m)}(\varepsilon) = \varepsilon \mathcal{P}^{\mathbf{Z}}(\partial_{\varepsilon}^m G(\mathfrak{J}_{\mathbf{Z}}(\varepsilon)) \mathfrak{I}) + \mathcal{P}^{\mathbf{Z}}(\partial_{\varepsilon}^m G(\mathfrak{J}_{\mathbf{Z}}(\varepsilon)) \mathfrak{I}) + m\mathcal{P}^{\mathbf{Z}}(\partial_{\varepsilon}^{m-1} G(\mathfrak{J}_{\mathbf{Z}}(\varepsilon)) \mathfrak{I}). \tag{2.53}$$

We calculate the derivatives by Riordan’s formula (see [34, eq. (2.11)]), which reads

$$\partial_{\varepsilon}^m(G \circ \mathfrak{J}_{\mathbf{Z}})(\varepsilon) = m! \sum_{k=1}^m \frac{1}{k!} G^{(k)}(\mathfrak{J}_{\mathbf{Z}}(\varepsilon)) \sum_{i \in S_k^m} \prod_{\ell=1}^k \frac{1}{i_{\ell}!} \mathfrak{J}_{\mathbf{Z}}^{(i_{\ell})}(\varepsilon). \tag{2.54}$$

when combined with Proposition A.35. Note that the last summand on the RHS of (2.53) does not contribute a term containing $\mathfrak{J}_{\mathbf{Z}}^{(m)}$. For the first two summands, only the term corresponding to $k = 1$ in (2.54) does, more precisely

$$k = 1 \rightsquigarrow S_1^m = \{m\} \rightsquigarrow G^{(m)}(\mathfrak{J}_{\mathbf{Z}}(\varepsilon))\mathfrak{J}_{\mathbf{Z}}^{(m)}(\varepsilon).$$

We single out the corresponding terms in the definition of $\mathcal{M}_m^{\mathbf{Z}}$ in (2.49) and, accordingly, only start the summation in $B_{\mathbf{Z}}^{(m-1),\sigma}$ in (2.51) at $k = 2$. The claim in (2.48) follows.

Regarding (iii), recall that the operations of multiplication and composition with (lifts of sufficiently) regular functions are strongly locally Lipschitz continuous operations (cf. [45, Prop. 6.12] and [49, Prop. 3.11], respectively). As a consequence of [45, Thm. 7.8], so is $\mathbf{Z} \mapsto \mathfrak{J}_{\mathbf{Z}}(\varepsilon)$ for $\varepsilon \in I_{\star}$ fixed, cf. Corollary 2.8 above. Therefore, the map

$$V \mapsto G^{(m)}(\mathfrak{J}_{\mathbf{Z}}(\varepsilon))V [\varepsilon \mathfrak{I} + \mathfrak{I}] + B_{\mathbf{Z}}^{(m-1),\mathfrak{I}}(\varepsilon) \varepsilon \mathfrak{I} + B_{\mathbf{Z}}^{(m-1),\mathfrak{I}}(\varepsilon) \mathfrak{I} + mA_{\mathbf{Z}}^{(m-1),\mathfrak{I}}(\varepsilon) \mathfrak{I}$$

is strongly locally Lipschitz continuous from $\mathcal{D}^{\gamma,\eta,T}(\mathbf{Z})$ to $\mathcal{D}^{\bar{\gamma},\bar{\eta},T}(\mathbf{Z})$ in case $m = 1$. Then, [45, Thm. 7.8] implies strong local Lipschitz continuity of $\mathbf{Z} \mapsto \mathfrak{J}_{\mathbf{Z}}^{(1)}(\varepsilon)$. Iteratively, the same arguments prove the claim for $m \in \{2, \dots, \ell\}$.

The proof of (iv) is almost identical to the corresponding statement in [76]: Suppose ε_{\star} is chosen as in (2.52) and assume there exists $r_{\star} > \varepsilon_{\star}$ such that $T_{\infty}(r, E_{\mathbf{h}}\mathbf{Z}) > T$ for all $r \in [0, r_{\star})$. We can then redo the arguments in Theorem 2.21 with $(0, W) \equiv (0, \mathfrak{J}_{\mathbf{Z}}(0))$ replaced by $(\varepsilon_{\star}, \mathfrak{J}_{\mathbf{Z}}(\varepsilon_{\star}))$ to infer that we can obtain $\theta > 0$ such that the map

$$[0, \theta) \ni \varepsilon \mapsto \mathfrak{J}_{\mathbf{Z}}(\varepsilon_{\star} + \varepsilon) \in \mathcal{D}^{\gamma,\eta,T}(\mathbf{Z})$$

is \mathcal{C}^ℓ in the Fréchet sense. In turn, that implies that $\mathfrak{J}_{\mathbf{Z}} \in \mathcal{C}^\ell([0, \varepsilon_\star + \theta), \mathcal{D}^{\gamma, \eta, T}(\mathbf{Z}))$ which contradicts the maximality of ε_\star . By Corollary 2.15 and eq. (2.28), we immediately have that

$$\varepsilon_0(T, \mathbf{Z}) = \sup I^{\mathbf{Z}}(T) \leq \sup I_\infty^{\mathbf{Z}}(T) = \varepsilon_\star(T, \mathbf{Z}),$$

thus finishing the proof. \square

2.3. Taylor expansion in spaces of modelled distribution

Given Theorem 2.26, Taylor’s formula on Banach spaces [82, Thm. 4.C] provides us with an abstract analogue of Theorem 2 in the space $\mathcal{D}_{\mathcal{U}}^{\gamma, \eta, T}(E_{\mathfrak{h}}\mathbf{Z})$. The following corollary is a direct consequence of the afore-mentioned theorem.

Corollary 2.28. *In the setting of Theorem 2.26, the expansion*

$$U^\varepsilon = W + \sum_{m=1}^{\ell-1} \frac{\varepsilon^m}{m!} U^{(m)} + \mathfrak{R}_\varepsilon^\ell \quad \text{in } \mathcal{D}_{\mathcal{U}}^{\gamma, \eta, T}(\Gamma^{\text{eh}}) \tag{2.55}$$

holds for each $\varepsilon \in I_0$ with

(1) the term W as introduced in Remark 2.19, $U^\varepsilon \equiv U^\varepsilon(\mathbf{Z}) := \mathfrak{J}_{E_{\mathfrak{h}}\mathbf{Z}}(\varepsilon)$, cf. (2.17), and

$$U^{(m)} \equiv U^{(m)}(\mathbf{Z}) := \mathfrak{J}_{E_{\mathfrak{h}}\mathbf{Z}}^{(m)}(0) \quad \text{for } m = 1, \dots, \ell - 1. \tag{2.56}$$

The latter are given iteratively as solutions to the fixed-point problems

$$U^{(m)}(\mathbf{Z}) = \mathcal{P}^{E_{\mathfrak{h}}\mathbf{Z}}(G^{(m)}(W)U^{(m)}(\mathbf{Z}) \mathfrak{!}) + \mathcal{P}^{E_{\mathfrak{h}}\mathbf{Z}}(B_{E_{\mathfrak{h}}\mathbf{Z}}^{(m-1), \mathfrak{!}} \mathfrak{!}) + m\mathcal{P}^{E_{\mathfrak{h}}\mathbf{Z}}(A_{E_{\mathfrak{h}}\mathbf{Z}}^{(m-1), \mathfrak{!}} \mathfrak{!}) \tag{2.57}$$

or, equivalently, by the explicit formula

$$U^{(m)}(\mathbf{Z}) = \tilde{\Theta}^{E_{\mathfrak{h}}\mathbf{Z}}[\mathcal{P}^{E_{\mathfrak{h}}\mathbf{Z}}(B_{E_{\mathfrak{h}}\mathbf{Z}}^{(m-1), \mathfrak{!}} \mathfrak{!}) + m\mathcal{P}^{E_{\mathfrak{h}}\mathbf{Z}}(A_{E_{\mathfrak{h}}\mathbf{Z}}^{(m-1), \mathfrak{!}} \mathfrak{!})]. \tag{2.58}$$

The terms $A_{E_{\mathfrak{h}}\mathbf{Z}}^{(0), \mathfrak{!}} := A_{E_{\mathfrak{h}}\mathbf{Z}}^{(0), \mathfrak{!}}(0) = G(W)$ are given by

$$A_{E_{\mathfrak{h}}\mathbf{Z}}^{(m), \mathfrak{!}} := A_{E_{\mathfrak{h}}\mathbf{Z}}^{(m), \mathfrak{!}}(0) = m! \sum_{k=1}^m \frac{1}{k!} G^{(k)}(W) \sum_{i \in S_n^m} \prod_{n=1}^k \frac{1}{i_n!} U^{(i_n)}(\mathbf{Z}), \tag{2.59}$$

$$B_{E_{\mathfrak{h}}\mathbf{Z}}^{(m-1), \mathfrak{!}} := B_{E_{\mathfrak{h}}\mathbf{Z}}^{(m-1), \mathfrak{!}}(0) = m! \sum_{k=2}^m \frac{1}{k!} G^{(k)}(W) \sum_{i \in S_n^m} \prod_{n=1}^k \frac{1}{i_n!} U^{(i_n)}(\mathbf{Z}). \tag{2.60}$$

given in (2.50) and (2.51), respectively. They only depend on $U^{(k)}(\mathbf{Z})$ for $k = 1, \dots, m - 1$.

(2) the remainder $\mathfrak{R}_\varepsilon^\ell \equiv \mathfrak{R}_\varepsilon^\ell(\mathbf{Z})$ is defined implicitly from (2.55), that is

$$\mathfrak{R}_\varepsilon^\ell := U^\varepsilon - W + \sum_{m=1}^{\ell-1} \frac{\varepsilon^m}{m!} U^{(m)}. \tag{2.61}$$

All the expressions in the expansion (2.55) are continuous in the model \mathbf{Z} . \spadesuit

Remark 2.29. Note that Taylor’s theorem actually gives us an *explicit* formula for the remainder $\mathfrak{R}_\varepsilon^\ell(\mathbf{Z})$, namely

$$\mathfrak{R}_\varepsilon^\ell(\mathbf{Z}) := \int_0^1 \frac{(1-s)^{\ell-1}}{(\ell-1)!} \mathfrak{J}_{E_h \mathbf{Z}}^{(\ell)}(s\varepsilon) ds. \tag{2.62}$$

However, as we will see in subsection 2.3.2, the definition in (2.61) is very well-suited for an inductive argument. \spadesuit

Remark 2.30. The formula for $U^{(m)}$ in (2.57) formally agrees with the corresponding ones by Inahama and Kawabi, specifically [52, eq.’s (4.2) - (4.5)] (in the case that the SDE they consider has no drift b , i.e. $b \equiv 0$). \spadesuit

By itself, Corollary 2.28 is merely a version of Taylor’s theorem. It unfolds its full merit only when complemented with an analysis of the *properties* of the Taylor terms and the remainder. Most importantly in this direction, we will provide estimates for the $U^{(m)}$ ’s in subsection 2.3.1 and for $\mathfrak{R}_\varepsilon^\ell$ in subsection 2.3.2.

Before we turn to the estimates, we study further properties of the terms $U^{(m)}$ which we need in our analysis. To begin with, the next proposition may be understood as m -homogeneity of $U^{(m)}$ w.r.t. dilation of the model.

Proposition 2.31. *With \mathfrak{d}_ε and δ_ε as in section A.2.1, the terms $U^{(m)}(\mathbf{Z}) \in \mathcal{D}^{\gamma, \eta, T}(E_h \mathbf{Z})$ satisfy the identity*

$$\varepsilon^m U^{(m)}(\mathbf{Z}) = \mathfrak{d}_\varepsilon (U^{(m)}(\delta_\varepsilon \mathbf{Z})), \quad \varepsilon \in I_0 = [0, \varepsilon_0(T, \mathbf{Z})]. \tag{2.63}$$

\spadesuit

Before we can prove it, in light of (2.58) we first need to understand how the operator $\tilde{\Theta}$ (see Lemma 2.24) transforms when the underlying model is dilated.

Lemma 2.32. *Let $Z \in \mathcal{M}$. We have*

$$\tilde{\Theta}^{E_h Z} \circ \mathfrak{d}_\varepsilon = \mathfrak{d}_\varepsilon \circ \tilde{\Theta}^{E_h \delta_\varepsilon Z}, \quad \varepsilon \in I_0. \tag{2.64}$$

♠

Proof. The claimed identity is equivalent to

$$\mathfrak{d}_\varepsilon \circ \Theta^{E_h \delta_\varepsilon Z} = \Theta^{E_h Z} \circ \mathfrak{d}_\varepsilon, \quad \varepsilon \in I_0.$$

Let $Y_\varepsilon \in \mathcal{D}_U^{\gamma, \eta; T}(E_h \delta_\varepsilon Z)$. By Lemmas A.21 and A.38(i), we have $\mathfrak{d}_\varepsilon Y_\varepsilon \in \mathcal{D}_U^{\gamma, \eta; T}(E_h Z)$, so we find that

$$\begin{aligned} \Theta^{E_h Z}(\mathfrak{d}_\varepsilon Y_\varepsilon) &= \mathfrak{d}_\varepsilon Y_\varepsilon - \mathcal{P}_{E_h Z}(G'(W)(\mathfrak{d}_\varepsilon Y_\varepsilon) \mathbf{1}) = \mathfrak{d}_\varepsilon Y_\varepsilon - \mathcal{P}_{E_h Z}(\mathfrak{d}_\varepsilon(G'(W_h)Y_\varepsilon) \mathbf{1}) \\ &= \mathfrak{d}_\varepsilon(Y_\varepsilon - \mathcal{P}_{E_h \delta_\varepsilon Z}(G'(W)Y_\varepsilon) \mathbf{1}) = \mathfrak{d}_\varepsilon(\Theta^{E_h \delta_\varepsilon Z}(Y_\varepsilon)). \end{aligned}$$

In the previous equalities, we have used the explicit form of Θ given in Lemma 2.24, the multiplicativity and linearity of \mathfrak{d}_ε , Lemma A.38(iii) (to relate \mathfrak{d}_ε , δ_ε , and \mathcal{P}), and Lemma A.21 (to commute δ_ε and E_h). □

Proof of Proposition 2.31. The proof proceeds by induction, starting with $m = 1$:

$$\begin{aligned} \varepsilon U^{(1)}(Z) &= \tilde{\Theta}^{E_h Z} \mathcal{P}^{E_h Z}(A_{E_h Z}^{(0), \mathbf{1}} \varepsilon \mathbf{1}) = \tilde{\Theta}^{E_h Z} \mathcal{P}^{E_h Z}(G(W) \varepsilon \mathbf{1}) = \tilde{\Theta}^{E_h Z} \mathcal{P}^{E_h Z}(\mathfrak{d}_\varepsilon(G(W)) \mathbf{1}) \\ &= \tilde{\Theta}^{E_h Z} \mathfrak{d}_\varepsilon(\mathcal{P}^{\delta_\varepsilon E_h Z}(G(W)) \mathbf{1}) = \mathfrak{d}_\varepsilon \tilde{\Theta}^{E_h \delta_\varepsilon Z} \mathcal{P}^{E_h \delta_\varepsilon Z}(G(W)) \mathbf{1} = \mathfrak{d}_\varepsilon U^{(1)}(\delta_\varepsilon Z) \end{aligned} \tag{2.65}$$

In the previous equalities, we have used (2.58) for $m = 1$, Lemma 2.32, Lemmas A.38(iii) and A.21 as in the previous proof, and the fact that $\mathfrak{d}_\varepsilon W = W$ since $\langle W, \mathfrak{?} \rangle = 0$, cf. Remark 2.19.

We assume the claim is true for $m \in \{1, \dots, \ell - 2\}$ and do the induction step $m \mapsto m + 1$. First, recall that

$$\varepsilon^{m+1} = \varepsilon^{|\mathbf{i}|} = \prod_{n=1}^k \varepsilon^{i_n} \quad \text{for} \quad \mathbf{i} \in S_k^{m+1} = \{\mathbf{i} \in \mathbb{N}_{\geq 1}^k : |\mathbf{i}| = m + 1\}. \tag{2.66}$$

Using this fact in combination with linearity and multiplicativity of \mathfrak{d}_ε , the induction hypothesis immediately implies that

$$\varepsilon^{m+1} A_{E_h Z}^{(m), \mathbf{1}} = \mathfrak{d}_\varepsilon A_{E_h \delta_\varepsilon Z}^{(m), \mathbf{1}} \varepsilon, \quad \varepsilon^{m+1} B_{E_h Z}^{(m), \mathbf{1}} = \mathfrak{d}_\varepsilon B_{E_h \delta_\varepsilon Z}^{(m), \mathbf{1}}. \tag{2.67}$$

By (2.58) and the same reasoning as in (2.65), we thus have

$$\varepsilon^{m+1} U^{(m+1)}(Z) = \tilde{\Theta}^{E_h Z} [\mathcal{P}^{E_h Z}(\varepsilon^{m+1} B_{E_h Z}^{(m), \mathbf{1}}) \mathbf{1}] + (m + 1) \mathcal{P}^{E_h Z}(\varepsilon^{m+1} A_{E_h Z}^{(m), \mathbf{1}}) \mathbf{1}]$$

$$\begin{aligned}
 &= \tilde{\Theta}^{E_h Z} [\mathcal{P}^{E_h Z} (\mathfrak{d}_\varepsilon B_{E_h \delta_\varepsilon Z}^{(m), \mathbf{1}} \mathbf{!}) + (m + 1) \mathcal{P}^{E_h Z} (\mathfrak{d}_\varepsilon [A_{E_h \delta_\varepsilon Z}^{(m), \mathbf{1}} \mathbf{!}])] \\
 &= (\tilde{\Theta}^{E_h Z} \circ \mathfrak{d}_\varepsilon) [\mathcal{P}^{E_h \delta_\varepsilon Z} (B_{E_h \delta_\varepsilon Z}^{(m), \mathbf{1}} \mathbf{!}) + (m + 1) \mathcal{P}^{E_h \delta_\varepsilon Z} (A_{E_h \delta_\varepsilon Z}^{(m), \mathbf{1}} \mathbf{!})] \\
 &= \mathfrak{d}_\varepsilon (\tilde{\Theta}^{E_h \delta_\varepsilon Z} [\mathcal{P}^{E_h \delta_\varepsilon Z} (B_{E_h \delta_\varepsilon Z}^{(m), \mathbf{1}} \mathbf{!}) + (m + 1) \mathcal{P}^{E_h \delta_\varepsilon Z} (A_{E_h \delta_\varepsilon Z}^{(m), \mathbf{1}} \mathbf{!})]) \\
 &= \mathfrak{d}_\varepsilon U^{(m+1)}(\delta_\varepsilon Z)
 \end{aligned}$$

This completes the proof. \square

The following definition relates the quantities appearing in Corollary 2.28 to those in Theorem 2. Note that it is consistent since the reconstruction operator \mathcal{R} is linear and continuous.

Definition 2.33. As before, let $Z \in \mathcal{M}$. We set $u_h^\varepsilon(Z) := \mathcal{R}(E_h Z; U^\varepsilon)$ and

- $u_h^{(m)}(Z) := \mathcal{R}(E_h Z; U^{(m)})$, $u_{\xi_\delta; h}^{(m)} := u_h^{(m)}(Z^{\xi_\delta})$, $\hat{u}_{\xi_\delta; h}^{(m)} := u_h^{(m)}(\hat{Z}^{\xi_\delta})$, $\hat{u}_h^{(m)} := u_h^{(m)}(\hat{Z})$,
- $R_{h, \varepsilon}^{(\ell)}(Z) := \mathcal{R}(E_h Z; \mathfrak{R}_\varepsilon^\ell)$, $\hat{R}_{h, \varepsilon}^{(\ell)} := R_{h, \varepsilon}^{(\ell)}(\hat{Z})$. \spadesuit

For the sake of curiosity, explicit equations for the terms $\hat{u}_{\xi_\delta; h}^{(m)}$ are derived in appendix 2.5; they are not needed for the proof of Theorem 2.

The following lemma will be needed in subsection 3.1; it states that the term $u_{\xi_\delta; h}^{(1)}$ needs not be renormalised. This is not surprising given that it is a linear function of the noise.

Lemma 2.34. We have $u_{\xi_\delta; h}^{(1)} = \hat{u}_{\xi_\delta; h}^{(1)}$. \spadesuit

Proof. From Theorem 2.26(iii), we know that

$$U^{(1)}(Z) = \mathcal{P}^{E_h Z} (L^{(1)}(Z)), \quad L^{(1)}(Z) := A_{E_h Z}^{(0), \mathbf{1}} \mathbf{!} + B_{E_h Z}^{(0), \mathbf{1}} \mathbf{!} = G'(W)U^{(1)}(Z) \mathbf{!} + G(W) \mathbf{!}$$

As a consequence of Remark 2.19, eq. (2.29), we find $\langle L^{(1)}(Z), \mathfrak{V} \rangle = 0$ for any $Z \in \mathcal{M}$. In turn, since the action of $\Pi^{\xi_\delta; e_h}$ and $\hat{\Pi}^{\xi_\delta; e_h}$ differs only on the symbol \mathfrak{V} , we have

$$\begin{aligned}
 \mathcal{R}^{E_h Z^{\xi_\delta}} (L^{(1)}(Z^{\xi_\delta})) (z) &= \Pi_z^{\xi_\delta; e_h} [L^{(1)}(Z^{\xi_\delta})(z)] (z) = \hat{\Pi}_z^{\xi_\delta; e_h} [L^{(1)}(Z^{\xi_\delta})(z)] (z) \\
 &= \hat{\Pi}_z^{\xi_\delta; e_h} [L^{(1)}(\hat{Z}^{\xi_\delta})(z)] (z) = \mathcal{R}^{E_h \hat{Z}^{\xi_\delta}} (L^{(1)}(\hat{Z}^{\xi_\delta})) (z).
 \end{aligned}$$

With this observation, we then get

$$\begin{aligned}
 \hat{u}_{\xi_\delta; h}^{(1)}(z) &= \mathcal{R}(E_h \hat{Z}^{\xi_\delta}; U^{(1)}(\hat{Z}^{\xi_\delta})) (z) = [P * \mathcal{R}^{E_h \hat{Z}^{\xi_\delta}} (L^{(1)}(\hat{Z}^{\xi_\delta}))] (z) \\
 &= [P * \mathcal{R}^{E_h Z^{\xi_\delta}} (L^{(1)}(Z^{\xi_\delta}))] (z) = \mathcal{R}^{E_h Z^{\xi_\delta}} (U^{(1)}(Z^{\xi_\delta})) (z) = u_{\xi_\delta; h}^{(1)}(z)
 \end{aligned}$$

which is what we wanted to show. \square

Further preparations: Duhamel’s formula. Before we can estimate the terms in (2.55), we need some final preparations. In (2.57) we have seen that $Y \in \{U^{(m)} : m \in [\ell - 1]\}$ satisfies

(1) a linear fixed-point equation of type

$$Y = \mathcal{P}^{\text{eh}} (G'(W)Y \mathfrak{I} + \tilde{V}) \tag{2.68}$$

for some modelled distribution $\tilde{V} \in \mathcal{D}_{\alpha}^{\tilde{\gamma}, \tilde{\eta}}$ for $\tilde{\gamma} := \gamma + \alpha$ and $\tilde{\eta} := \eta + \alpha$. The subscript “ α ” indicates that \tilde{V} takes with values in a sector of regularity $\alpha := \text{deg}(\mathfrak{I})$.

(2) an inhomogeneous linear equation of the form

$$\Theta^{\text{eh}} Y = V \iff Y = \tilde{\Theta}^{\text{eh}} V \tag{2.69}$$

where

$$V := \mathcal{P}^{\text{eh}} \tilde{V} \in \mathcal{D}_{\alpha}^{\gamma, \alpha+2} \hookrightarrow \mathcal{D}_{\alpha}^{\gamma, \eta} \tag{2.70}$$

for $\eta \in (0, \alpha + 2)$ and

$$\Theta^{\text{eh}} := \text{Id} - \mathcal{P}^{\text{eh}} (G'(W) \bullet \mathfrak{I}) \in \mathcal{L}(\mathcal{D}_{\alpha}^{\gamma, \eta}(\Gamma^{\text{eh}})), \quad \tilde{\Theta}^{\text{eh}} := [\Theta^{\text{eh}}]^{-1}, \tag{2.71}$$

introduced in Lemma 2.24 can be interpreted as a “Duhamel operator” and its inverse. In the reasoning above, we have used [45, Prop. 6.16] which asserts that

$$\mathcal{P}^{\text{eh}} : \mathcal{D}_{\alpha}^{\tilde{\gamma}, \tilde{\eta}} \rightarrow \mathcal{D}_{\alpha}^{\tilde{\gamma}+2, \alpha+2} \tag{2.72}$$

is linear and bounded.

In subsection 2.3.2, more precisely Proposition 2.42, we will see that one can also derive an equation of form (2.69) for $Y = \mathfrak{R}_{\varepsilon}^{\ell}$. That makes a strong case for understanding the operator norm of $\tilde{\Theta}$, so that estimating $Y \in \{U^{(m)}, \mathfrak{R}_{\varepsilon}^{\ell} : m \in [\ell - 1]\}$ reduces to estimating the corresponding V ’s in (2.69).

In this spirit, the following proposition is reminiscent of Duhamel’s Formula in linear PDE theory. For its formulation, we define $\tilde{V} := \langle \mathcal{U} \mathfrak{I} \cup \mathcal{U} \mathfrak{I} \rangle = \langle \tau \sigma : \tau \in \mathcal{U}, \sigma \in \{\mathfrak{I}, \mathfrak{I}\} \rangle$, a sector of regularity α .

Proposition 2.35 (Duhamel). *As before, let $\gamma = 1 + 2\kappa$, $\alpha = \text{deg}(\mathfrak{I}) = -1 - \kappa$, and $\eta \in (1/2, 1)$. For each $T \in (0, T_0 \wedge T_{\infty}^{\text{h}})$, $\mathbf{Z} = (\Pi, \Gamma) \in \mathcal{M}$, and $V \in \mathcal{D}^{\gamma, \eta}(\Gamma^{\text{eh}})$, the equation (2.69) admits a unique solution $Y \in \mathcal{D}_{\alpha}^{\gamma, \eta}(\Gamma^{\text{eh}})$ on $(0, T)$. Setting $\lambda = \lambda(\mathbf{Z}) := 1 + \|\mathbf{Z}^{-}\|$ and defining V from $\tilde{V} \in \mathcal{D}_{\alpha}^{\tilde{\gamma}, \tilde{\eta}}(\Gamma^{\text{eh}})$ as in (2.70) we also have the bound*

$$\|\partial_\lambda Y\|_{\gamma,\eta;T;\Gamma^{\text{eh};1/\lambda}} \lesssim \|\partial_\lambda \tilde{V}\|_{\tilde{\gamma},\tilde{\eta};T;\Gamma^{\text{eh};1/\lambda}} \tag{2.73}$$

that holds uniformly over all $Z \in \mathcal{M}$. ♠

The proof of the proposition combines different aspects of the equivalent viewpoints in (1) and (2) above. Since it is highly technical, we only provide a sketch and refer the interested reader to [34, App. E], the extended arXiv version of the article at hand, which contains all the details.

Proof (Sketch). At first, we verify the criteria of [45, Thm. 7.8] to see that the fixed-point problem (2.68) is solvable.

In a second step, we then establish Neumann series bounds on short time intervals. More precisely, for $0 \leq s < t \leq T$ we write $\Theta^{\text{eh}} = \text{Id} - A_s$ for a suitable operator A_s that acts on modelled distributions Y supported on (s, t) and defined w.r.t. the rescaled recentering operator $\Gamma^{\text{eh};1/\lambda}$. By suitably amending [45, Thm. 7.1], the short time bound for the abstract convolution kernel \mathcal{P}^{eh} , one can find t small enough s.t. the operator norm $\|A_s\|_{\text{op};t;\Gamma^{\text{eh};1/\lambda}}$ is bounded by $1/2$ and then the classical Neumann series bound

$$\|[\text{Id} - A_s]^{-1}\|_{\text{op};t;\Gamma^{\text{eh};1/\lambda}} \leq (1 - \|A_s\|_{\text{op};t;\Gamma^{\text{eh};1/\lambda}})^{-1} \leq 2$$

holds uniformly over all $s \in [0, T]$.

These two steps combined show that (2.73) holds with T replaced by some small $s \in [0, T]$. One concludes by restarting the equation after time s , patching together solutions in the spirit of [45, Prop. 7.11], and iterating the previous estimates up to the finite time T is reached. □

Let us make a few remarks about the philosophy behind the previous proposition and existent literature.

Remark 2.36. In the setting of the previous proposition, we could actually prove the slightly stronger statement that

$$\|\partial_\lambda Y\|_{\gamma,\eta;T;\Gamma^{\text{eh};1/\lambda}} \lesssim \|\partial_\lambda V\|_{\gamma,\eta;T;\Gamma^{\text{eh};1/\lambda}},$$

for $V \in \mathcal{D}^{\gamma,\eta,T}(\Gamma^{\text{eh}})$ not necessarily of form (2.70). From a functional analytic perspective, the previous inequality can then equivalently be stated as

$$\|\tilde{\Theta}^{E_h \delta_{1/\lambda}} Z\|_{\text{op};T} \lesssim 1$$

uniformly over each $Z \in \mathcal{M}$, where $\|\cdot\|_{\text{op};T}$ denotes the operator norm in $\mathcal{L}(\mathcal{D}^{\gamma,\eta;T}(\Gamma^{\text{eh};1/\lambda}))$. ♠

Remark 2.37. The purpose of Proposition 2.35 is to prepare for the proof of the estimates in Theorem 2, eq. (0.18): In Definition 2.33, we have seen that the quantities therein are reconstructions of modelled distributions $Y \in \{U^{(m)}, \mathfrak{R}_\varepsilon^\ell : m \in [\ell - 1]\}$. By the trivial identity $Y = \mathfrak{d}_{1/\lambda} \mathfrak{d}_\lambda Y$ and Lemma A.38(ii), we see that

$$\mathcal{R}^{E_h Z} Y = \mathcal{R}^{E_h Z} \mathfrak{d}_{1/\lambda} \mathfrak{d}_\lambda Y = \mathcal{R}^{E_h \delta_{1/\lambda} Z} \mathfrak{d}_\lambda Y = \langle \mathfrak{d}_\lambda Y, \mathbf{1} \rangle$$

where the last identity is true by [45, Prop. 3.28] and the fact that \mathcal{U} is a function-like sector, see [45, Def. 2.5]. Finally, we can estimate

$$\|\mathcal{R}^{E_h Z} Y\|_{\mathcal{X}_T} = \|\langle \mathfrak{d}_\lambda Y, \mathbf{1} \rangle\|_{\mathcal{X}_T} \leq \|\mathfrak{d}_\lambda Y\|_{\gamma, \eta, T; \Gamma^{e_h; 1/\lambda}}. \tag{2.74}$$

By Proposition 2.35, it then only remains to estimate the \tilde{V} corresponding to the specific choice of Y . That will be done in the subsections to follow. \spadesuit

Remark 2.38. The previous proposition is consistent with its rough path counterpart [52, Lem. 2.1], itself a version of Duhamel’s formula. In the framework of regularity structures, Hairer and Mattingly [48] established a version of Duhamel’s formula that applies in our framework but did not consider the estimates provided above. \spadesuit

We now start estimating $Y = U^{(m)}$ in the spirit of Remark 2.37.

2.3.1. Estimates on the Taylor terms

In this subsection, we estimate the Taylor terms $U^{(m)}$ in the expansion (2.55). Recall from (2.58) that

$$U^{(m)}(Z) = \tilde{\Theta}^{E_h Z} [\mathcal{P}^{E_h Z} (\tilde{V}^{(m)}(Z))], \quad \tilde{V}^{(m)}(Z) := mA_{E_h Z}^{(m-1), \mathbf{1}} \mathbf{1} + B_{E_h Z}^{(m-1), \mathbf{1}} \mathbf{1}. \tag{2.75}$$

which is of type (2.69). We obtain the following estimates:

Proposition 2.39. *For each $Z \in \mathcal{M}$, $\lambda = \lambda(Z) := 1 + \|Z^-\|$, and $T \in (0, T_\infty^h \wedge T_0)$, we have the estimate*

$$\|\mathfrak{d}_\lambda U^{(m)}(Z)\|_{\gamma, \eta, T; \Gamma^{e_h; 1/\lambda}} \lesssim (1 + \|Z^-\|)^m \tag{2.76}$$

where the implicit constant differs for each m . \spadesuit

Proof. Since $A_{E_h Z}^{(m-1), \mathbf{1}}$ and $B_{E_h Z}^{(m-1), \mathbf{1}}$ take values in \mathcal{U} , a function-like sector, it is immediately clear that $\tilde{V}^{(m)}$ takes values in $\tilde{\mathcal{V}}$ and thus satisfies the assumptions of Proposition 2.35. We now use the identity

$$\mathfrak{d}_\lambda \tilde{V}^{(m)}(Z) = m \mathfrak{d}_\lambda A_{E_h Z}^{(m-1), \mathbf{1}} \lambda \mathbf{1} + \mathfrak{d}_\lambda B_{E_h Z}^{(m-1), \mathbf{1}} \mathbf{1} \tag{2.77}$$

and run an inductive argument in m . Henceforth, we suppress the subscript $E_h \mathbf{Z}$ of A and B . All the norms $\|\cdot\|$ are taken w.r.t. the model $E_h \delta_{1/\lambda} \mathbf{Z}$.

- $m = 1$: The previous formula reads $\partial_\lambda \tilde{V}^{(1)}(\mathbf{Z}) = \partial_\lambda A^{(0), \dagger} \lambda \# = \partial_\lambda G(W) \lambda \# = G(W) \lambda \#$ since $\langle W, \mathcal{Y} \rangle = 0$, see Remark 2.19. We then obtain the estimate

$$\|\partial_\lambda \tilde{V}^{(1)}(\mathbf{Z})\|_{\bar{\gamma}, \bar{\eta}; T} = \lambda \|G(W) \# \|_{\bar{\gamma}, \bar{\eta}; T} \lesssim \lambda = 1 + \|\mathbf{Z}^-\|$$

for α , γ , and η (and thus $\bar{\gamma}$ and $\bar{\eta}$) as in the formulation of Proposition 2.35. That same proposition verifies the claim.

- $m = 2$: In this case, the formula (2.77) is valid with

$$\partial_\lambda A^{(1), \dagger} = 2G'(W) \partial_\lambda U^{(1)}(\mathbf{Z}), \quad \partial_\lambda B^{(1), \dagger} = G''(W) [\partial_\lambda U^{(1)}(\mathbf{Z})]^2.$$

Analogously to the case $m = 1$, we find

$$\|\partial_\lambda A^{(1), \dagger} \lambda \# \|_{\bar{\gamma}, \bar{\eta}; T} \lesssim \|G'(W) \# \|_{\bar{\gamma}, \bar{\eta}; T} \lambda \|\partial_\lambda U^{(1)}(\mathbf{Z})\|_{\gamma, \eta; T} \lesssim (1 + \|\mathbf{Z}^-\|)^2$$

as well as

$$\|\partial_\lambda B^{(1), \dagger} \# \|_{\bar{\gamma}, \bar{\eta}; T} \lesssim \|G''(W) \# \|_{\bar{\gamma}, \bar{\eta}; T} \|\partial_\lambda U^{(1)}(\mathbf{Z})\|_{\gamma, \eta; T}^2 \lesssim (1 + \|\mathbf{Z}^-\|)^2.$$

These estimates can be combined to

$$\|\partial_\lambda \tilde{V}^{(2)}(\mathbf{Z})\|_{\bar{\gamma}, \bar{\eta}; T} \lesssim (1 + \|\mathbf{Z}^-\|)^2$$

which implies the claimed statement by Proposition 2.35.

Suppose the claim is true for some $m - 1 \in [\ell - 2]$. The argument in case $m = 2$ is a blueprint for the strategy in the induction step

- $m - 1 \mapsto m$: We have

$$\begin{aligned} \|\partial_\lambda B^{(m-1), \dagger} \# \|_{\bar{\gamma}, \bar{\eta}; T} &\lesssim m! \sum_{k=2}^m \frac{1}{k!} \|G^{(k)}(W) \# \|_{\bar{\gamma}, \bar{\eta}; T} \sum_{\mathbf{i} \in S_k^m} \prod_{n=1}^k \frac{1}{i_n!} \|\partial_\lambda U^{(i_n)}(\mathbf{Z})\|_{\gamma, \eta; T} \\ &\lesssim \sum_{k=2}^m \sum_{\mathbf{i} \in S_k^m} (1 + \|\mathbf{Z}^-\|)^{|\mathbf{i}|} = (1 + \|\mathbf{Z}^-\|)^m \sum_{k=2}^m \# [S_k^m] \\ &\lesssim (1 + \|\mathbf{Z}^-\|)^m \end{aligned}$$

where we have used that $\sum_{n=1}^k i_n = m$ for all $\mathbf{i} \in S_k^m$ and the induction hypothesis. Analogously, we find

$$\begin{aligned} & \|\mathfrak{d}_\lambda A^{(m-1),1} \lambda \mathfrak{f}\|_{\tilde{\gamma}, \tilde{\eta}; T} \\ & \lesssim \lambda(m-1)! \sum_{k=1}^{m-1} \frac{1}{k!} \|G^{(k)}(W)\mathfrak{f}\|_{\tilde{\gamma}, \tilde{\eta}; T} \sum_{\mathbf{i} \in S_k^{m-1}} \prod_{n=1}^k \frac{1}{i_n!} \|U^{(i_n)}(\mathbf{Z})\|_{\gamma, \eta; T} \\ & \lesssim \lambda(1 + \|\mathbf{Z}^-\|)^{m-1} = (1 + \|\mathbf{Z}^-\|)^m. \end{aligned}$$

Altogether, this can be combined to give the estimate

$$\|\mathfrak{d}_\lambda \tilde{V}^{(m)}(\mathbf{Z})\|_{\tilde{\gamma}, \tilde{\eta}; T} \lesssim (1 + \|\mathbf{Z}^-\|)^m$$

and again, the claim follows from Proposition 2.35. \square

2.3.2. Estimates on the Taylor remainder

In this subsection, we focus on estimating the Taylor remainder $\mathfrak{R}_\varepsilon^\ell$. Recall from (2.61) that

$$\mathfrak{R}_\varepsilon^{n+1} = \mathfrak{R}_\varepsilon^1 - \sum_{j=1}^n \frac{\varepsilon^j}{j!} U^{(j)} \in \mathcal{D}_U^{\gamma, \eta}(\Gamma^{\text{eh}}) \tag{2.78}$$

where

$$\mathfrak{R}_\varepsilon^1 := U^\varepsilon - W = \mathcal{P}^{\text{eh}}(G(U^\varepsilon)\varepsilon \mathfrak{f} + [G(U^\varepsilon) - G(W)]\mathfrak{f}) \tag{2.79}$$

and, as before, $U^{(m)}$ is given in (2.58). Our aim is to prove the following proposition, the analog to last subsection’s Proposition 2.39.

Proposition 2.40. *Let $n \in [\ell]$ and for $\mathbf{Z} \in \mathcal{M}$, set $\lambda = \lambda(\mathbf{Z}) := 1 + \|\mathbf{Z}^-\|$. Then, for $T \in (0, T_\infty^{\text{h}} \wedge T_0)$ and $\rho \in (0, \rho_0(T))$ with $\rho_0(T) > 0$ as in Lemma B.4, the estimate*

$$\|\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^n(\mathbf{Z})\|_{\gamma, \eta; T; \Gamma^{\text{eh}, 1/\lambda}} \lesssim \varepsilon^n (1 + \|\mathbf{Z}^-\|)^n \tag{2.80}$$

holds uniformly over all $(\varepsilon, \mathbf{Z}) \in I \times \mathcal{M}$ with $\varepsilon \|\mathbf{Z}^-\| < \rho$. \spadesuit

The proof is based on induction. The case $n = 1$ is the content of the following lemma:

Lemma 2.41 (Induction basis). *For $\mathbf{Z} \in \mathcal{M}$, let $\lambda := 1 + \|\mathbf{Z}^-\|$ and assume that $\varepsilon \|\mathbf{Z}^-\| < \rho$. Then, the term $\mathfrak{R}_\varepsilon^1 = U^\varepsilon - W \in \mathcal{D}_U^{\gamma, \eta, T}(\Gamma^{\text{eh}})$ can be estimated by*

$$\|\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^1(\mathbf{Z})\|_{\gamma, \eta, T; \Gamma^{1/\lambda; \text{eh}}} \lesssim_{\rho, T} \varepsilon (1 + \|\mathbf{Z}^-\|). \quad \spadesuit$$

Proof. Recall from (the proof of) Lemma 2.12 that

$$U^\varepsilon = \mathcal{S}_{ex}^\varepsilon(\mathfrak{h}, \mathbf{Z}) = \mathfrak{d}_\varepsilon(U_\varepsilon), \quad U_\varepsilon := \mathcal{S}_{ex}(\mathfrak{h}, \delta_\varepsilon \mathbf{Z}), \tag{2.81}$$

so, bearing in mind that $W = U^0$ (see Remark 2.19), we have

$$\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^1 = \mathfrak{d}_\lambda [\mathfrak{d}_\varepsilon (U_\varepsilon) - \mathfrak{d}_0 (U_0)].$$

Recall that $\mathcal{U} = \{\mathbf{1}, \mathfrak{J}, \mathfrak{I}, X\}$, among which only \mathfrak{J} is *not* invariant under \mathfrak{d}_ε .

Part 1: Let $z \in D_T := (0, T] \times \mathbb{T}^2$. We have

$$\begin{aligned} |\langle \mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^1(z), \mathfrak{J} \rangle| &= |\langle \mathfrak{d}_\lambda [\mathfrak{d}_\varepsilon (U_\varepsilon(z)) - \mathfrak{d}_0 (U_0(z))], \mathfrak{J} \rangle| = |\langle \mathfrak{d}_\lambda \mathfrak{d}_\varepsilon U_\varepsilon(z), \mathfrak{J} \rangle| \\ &\leq \varepsilon \lambda |\langle U_\varepsilon(z) - U_0(z), \mathfrak{J} \rangle| + \varepsilon \lambda |\langle U_0(z), \mathfrak{J} \rangle| \end{aligned}$$

so that we can estimate the first of the two summands defining the norm $\|\cdot\|_{\gamma, \eta, T}$ in Definition A.31, eq. (A.23):

$$\begin{aligned} \|\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^1\|_{\gamma, \eta, T} &= \sup_{\tau \in \mathcal{U}} \sup_{z \in D_T} |t|^{\frac{\deg(\tau) - \eta}{2} \vee 0} |\langle \mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^1(z), \tau \rangle| \\ &\leq \varepsilon \lambda \sup_{z \in D_T} |t|^{\frac{\deg(\mathfrak{J}) - \eta}{2}} (|\langle U_\varepsilon(z) - U_0(z), \mathfrak{J} \rangle| + |\langle U_0(z), \mathfrak{J} \rangle|) \\ &\quad + \sup_{\tau \in \mathcal{U} \setminus \{\mathfrak{J}\}} \sup_{z \in D_T} |t|^{\frac{\deg(\tau) - \eta}{2} \vee 0} |\langle U_\varepsilon(z) - U_0(z), \tau \rangle| \\ &\leq (1 + \varepsilon \lambda) \sup_{\tau \in \mathcal{U}} \sup_{z \in D_T} |t|^{\frac{\deg(\tau) - \eta}{2} \vee 0} |\langle U_\varepsilon(z) - U_0(z), \tau \rangle| \\ &\quad + \varepsilon \lambda \sup_{z \in D_T} |t|^{\frac{\deg(\mathfrak{J}) - \eta}{2} \vee 0} |\langle U_0(z), \mathfrak{J} \rangle| \tag{2.82} \\ &= (1 + \varepsilon \lambda) \|U_\varepsilon - U_0\|_{\gamma, \eta, T} + \varepsilon \lambda \sup_{z \in D_T} |t|^{\frac{\deg(\mathfrak{J}) - \eta}{2} \vee 0} |\langle U_0(z), \mathfrak{J} \rangle| \end{aligned}$$

For now, we leave this term as it is and later analyse it further.

Part 2: We turn to the second term in (A.23), the one containing the Γ 's. At first, by Lemma A.20 in the appendix, we observe that

$$\begin{aligned} [\mathfrak{d}_\lambda \mathfrak{d}_\varepsilon U_\varepsilon; \mathfrak{d}_\lambda \mathfrak{d}_0 U_0](z, \bar{z}) &:= \mathfrak{d}_\lambda \mathfrak{d}_\varepsilon U_\varepsilon(z) - \mathfrak{d}_\lambda \mathfrak{d}_0 U_0(z) - \Gamma_{z\bar{z}}^{\text{eh}; 1/\lambda} \mathfrak{d}_\lambda \mathfrak{d}_\varepsilon U_\varepsilon(\bar{z}) + \Gamma_{z\bar{z}}^{\text{eh}; 1/\lambda} \mathfrak{d}_\lambda \mathfrak{d}_0 U_0(\bar{z}) \\ &= \mathfrak{d}_\lambda \mathfrak{d}_\varepsilon U_\varepsilon(z) - \mathfrak{d}_\lambda \mathfrak{d}_0 U_0(z) - \mathfrak{d}_\lambda \mathfrak{d}_\varepsilon \Gamma_{z\bar{z}}^{\text{eh}; \varepsilon} U_\varepsilon(\bar{z}) + \mathfrak{d}_\lambda \mathfrak{d}_0 \Gamma_{z\bar{z}}^{\text{eh}; 0} U_0(\bar{z}) \end{aligned} \tag{2.83}$$

holds for any $z, \bar{z} \in D_T$. Now recall that

$$\Gamma_{z\bar{z}}^{\text{eh}; \varepsilon} \mathfrak{J} = \mathfrak{J} + \gamma_{z\bar{z}}^{\text{eh}; \varepsilon}(\mathcal{J}\mathfrak{I})\mathbf{1} \quad \text{with} \quad \gamma_{z\bar{z}}^{\text{eh}; \varepsilon}(\mathcal{J}\mathfrak{I}) = \varepsilon \langle \Pi \mathbf{1}, K(\cdot - \bar{z}) - K(\cdot - z) \rangle$$

so that, in particular, $\gamma_{z\bar{z}}^{\text{eh}; 0}(\mathcal{J}\mathfrak{I}) = 0$ and thus $\Gamma_{z\bar{z}}^{\text{eh}; 0} \mathfrak{J} = \mathfrak{J}$. Thus, we find that

$$\begin{aligned} & \mathfrak{d}_\lambda \mathfrak{d}_\varepsilon \langle U_\varepsilon(z), \mathfrak{J} \rangle \mathfrak{J} - \mathfrak{d}_\lambda \mathfrak{d}_\varepsilon \Gamma_{z\bar{z}}^{\text{eh};\varepsilon} \langle U_\varepsilon(\bar{z}), \mathfrak{J} \rangle \mathfrak{J} \\ &= \varepsilon \lambda [\langle U_\varepsilon(z), \mathfrak{J} \rangle - \langle U_\varepsilon(\bar{z}), \mathfrak{J} \rangle] \mathfrak{J} - \langle U_\varepsilon(\bar{z}), \mathfrak{J} \rangle \gamma_{z\bar{z}}^{\text{eh};\varepsilon} (\mathcal{J}\mathbf{1}) \mathbf{1} \\ &= \varepsilon \lambda \langle U_\varepsilon(z) - \Gamma_{z\bar{z}}^{\text{eh};\varepsilon} U_\varepsilon(\bar{z}), \mathfrak{J} \rangle \mathfrak{J} - \langle \Gamma_{z\bar{z}}^{\text{eh};\varepsilon} \langle U_\varepsilon(\bar{z}), \mathfrak{J} \rangle \mathfrak{J}, \mathbf{1} \rangle \mathbf{1}. \end{aligned}$$

As a consequence, we find that

$$\langle [\mathfrak{d}_\lambda \mathfrak{d}_\varepsilon U_\varepsilon; \mathfrak{d}_\lambda \mathfrak{d}_0 U_0](z, \bar{z}), \tau \rangle = \langle U_\varepsilon(z) - U_0(z) - \Gamma_{z\bar{z}}^{\text{eh};\varepsilon} U_\varepsilon(\bar{z}) + \Gamma_{z\bar{z}}^{\text{eh};0} U_0(\bar{z}), \tau \rangle, \tag{2.84}$$

for $\tau \in \mathcal{U} \setminus \{\mathfrak{J}\}$. For $\tau = \mathfrak{J}$, the situation is different:

$$\begin{aligned} & \langle [\mathfrak{d}_\lambda \mathfrak{d}_\varepsilon U_\varepsilon; \mathfrak{d}_\lambda \mathfrak{d}_0 U_0](z, \bar{z}), \mathfrak{J} \rangle = \varepsilon \lambda \langle U_\varepsilon(z) - \Gamma_{z\bar{z}}^{\text{eh};\varepsilon} U_\varepsilon(\bar{z}), \mathfrak{J} \rangle \\ &= \varepsilon \lambda \langle U_\varepsilon(z) - U_0(z) - \Gamma_{z\bar{z}}^{\text{eh};\varepsilon} U_\varepsilon(\bar{z}) + \Gamma_{z\bar{z}}^{\text{eh};0} U_0(\bar{z}), \mathfrak{J} \rangle + \varepsilon \lambda \langle U_0(z) - \Gamma_{z\bar{z}}^{\text{eh};0} U_0(\bar{z}), \mathfrak{J} \rangle \tag{2.85} \\ &= \varepsilon \lambda \langle U_\varepsilon(z) - U_0(z) - \Gamma_{z\bar{z}}^{\text{eh};\varepsilon} U_\varepsilon(\bar{z}) + \Gamma_{z\bar{z}}^{\text{eh};0} U_0(\bar{z}), \mathfrak{J} \rangle + \varepsilon \lambda \langle U_0(z) - U_0(\bar{z}), \mathfrak{J} \rangle \end{aligned}$$

We can now combine (2.83), (2.84), and (2.85) to obtain the following estimate:

$$\begin{aligned} & \sup_{\substack{z, \bar{z} \in D_T, \tau \in \mathcal{U} \\ |z - \bar{z}| \leq 1}} (|t| \wedge |\bar{t}|)^{\frac{\text{deg}(\tau) - \eta}{2} \vee 0} \frac{|\langle [\mathfrak{d}_\lambda \mathfrak{d}_\varepsilon U_\varepsilon; \mathfrak{d}_\lambda \mathfrak{d}_0 U_0](z, \bar{z}), \tau \rangle|}{|z - \bar{z}|^{\gamma - \text{deg}(\tau)}} \\ &\leq (1 + \varepsilon \lambda) \sup_{\substack{z, \bar{z} \in D_T, \tau \in \mathcal{U} \\ |z - \bar{z}| \leq 1}} (|t| \wedge |\bar{t}|)^{\frac{\text{deg}(\tau) - \eta}{2} \vee 0} \frac{|\langle U_\varepsilon(z) - U_0(z) - \Gamma_{z\bar{z}}^{\text{eh};\varepsilon} U_\varepsilon(\bar{z}) + \Gamma_{z\bar{z}}^{\text{eh};0} U_0(\bar{z}), \tau \rangle|}{|z - \bar{z}|^{\gamma - \text{deg}(\tau)}} \\ &+ \varepsilon \lambda \sup_{\substack{z, \bar{z} \in D_T, \\ |z - \bar{z}| \leq 1}} (|t| \wedge |\bar{t}|)^{\frac{\text{deg}(\mathfrak{J}) - \eta}{2} \vee 0} \frac{|\langle U_0(z) - U_0(\bar{z}), \mathfrak{J} \rangle|}{|z - \bar{z}|^{\gamma - \text{deg}(\mathfrak{J})}} \tag{2.86} \end{aligned}$$

Part 3: Observe that U_0 solves the fixed-point equation

$$U_0 = \mathcal{P}^{E_h \delta_0} \mathcal{Z} G(U_0)[\mathfrak{I} + \mathfrak{I}] + \mathcal{T} P u_0 = \mathcal{I} (G(U_0)[\mathfrak{I} + \mathfrak{I}]) + \bar{\mathcal{T}} \text{ in } \mathcal{D}_{\mathcal{U}}^{\gamma, \eta}(\Gamma^{\text{eh};0}),$$

so a quick comparison of coefficients shows that $\langle U_0, \mathfrak{J} \rangle = \langle G(U_0), \mathbf{1} \rangle = g(\langle U_0, \mathbf{1} \rangle)$. We remind the reader of the identity

$$\langle U_0, \mathbf{1} \rangle = \mathcal{R}^{E_h \delta_0} \mathcal{Z} U_0 = \mathcal{R}^{E_h} \mathcal{Z} \mathfrak{d}_0 U_0 = \mathcal{R}^{E_h} \mathcal{Z} W = \langle W, \mathbf{1} \rangle,$$

so that $\langle U_0, \mathfrak{J} \rangle = \langle G(W), \mathbf{1} \rangle$. Next, we emphasise that for $z = (t, x), \bar{z} = (\bar{t}, \bar{x}) \in D_T$ with $|z - \bar{z}| \leq 1$ we have

$$|z - \bar{z}|^\gamma \leq |z - \bar{z}|^{\gamma - \text{deg}(\mathfrak{J})}, \quad (|t| \vee |\bar{t}|)^{\frac{\text{deg}(\mathfrak{J}) - \eta}{2} \vee 0} \leq T^{\frac{\text{deg}(\mathfrak{J}) - \eta}{2} \vee 0}.$$

Since $\text{deg}(\mathbf{1}) = 0$ and $\Gamma_{z\bar{z}}^{\text{eh};0} \mathbf{1} = \mathbf{1}$, we then infer that

$$\begin{aligned}
 & \varepsilon \lambda \left[\sup_{z \in D_T} |t|^{\frac{\deg(\gamma) - \eta}{2} \vee 0} |\langle U_0(z), \gamma \rangle| + \sup_{\substack{z, \bar{z} \in D_T, \\ |z - \bar{z}| \leq 1}} (|t| \wedge |\bar{t}|)^{\frac{\deg(\gamma) - \eta}{2} \vee 0} \frac{|\langle U_0(z) - U_0(\bar{z}), \gamma \rangle|}{|z - \bar{z}|^{\gamma - \deg(\gamma)}} \right] \\
 & \lesssim_T \varepsilon \lambda \left[\sup_{z \in D_T} |\langle G(W)(z), \mathbf{1} \rangle| + \sup_{\substack{z, \bar{z} \in D_T, \\ |z - \bar{z}| \leq 1}} \frac{|\langle G(W)(z) - \Gamma_{z\bar{z}}^{\text{eh}; 0} G(W)(\bar{z}), \mathbf{1} \rangle|}{|z - \bar{z}|^\gamma} \right] \\
 & \leq \varepsilon \lambda \|G(W)\|_{\gamma, \eta, T}.
 \end{aligned} \tag{2.87}$$

Part 4: The estimates in (2.82), (2.86), and (2.87) then lead to the inequality

$$\begin{aligned}
 \|\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^1(\mathbf{Z})\|_{\gamma, \eta, T; \Gamma^{1/\lambda; \text{eh}}} & \leq (1 + \varepsilon \lambda) \|U_\varepsilon; U_0\|_{\gamma, \eta, T} + \varepsilon \lambda \|G(W)\|_{\gamma, \eta, T} \\
 & = (1 + \varepsilon \lambda) \|\mathcal{S}_{\text{ex}}(\mathbf{h}, \delta_\varepsilon \mathbf{Z}); \mathcal{S}_{\text{ex}}(\mathbf{h}, \delta_0 \mathbf{Z})\|_{\gamma, \eta, T} + \varepsilon \lambda \|G(W)\|_{\gamma, \eta, T}.
 \end{aligned} \tag{2.88}$$

where the equality is true by definition of U_ε in (2.81). By joint local Lipschitz continuity of \mathcal{S}_{ex} as established in [23, Prop. 3.25],²⁸ we can further estimate

$$\begin{aligned}
 \|\mathcal{S}_{\text{ex}}(\mathbf{h}, \delta_\varepsilon \mathbf{Z}); \mathcal{S}_{\text{ex}}(\mathbf{h}, \delta_0 \mathbf{Z})\|_{\gamma, \eta, T} & \lesssim \|\delta_\varepsilon \mathbf{Z} - \delta_0 \mathbf{Z}\| \lesssim \|\delta_\varepsilon \mathbf{Z}^- - \mathbf{0}\| \\
 & = \varepsilon \|\Pi^-\|_{\mathfrak{I}} \vee \varepsilon^2 \|\Pi^-\|_{\mathfrak{J}}^2 \leq \varepsilon(1 + \rho) \|\mathbf{Z}^-\|.
 \end{aligned}$$

By assumption, we have $\varepsilon \lambda \leq 1 + \rho$ since $\varepsilon \leq 1$, so the previous estimate may be combined with (2.88) to give

$$\|\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^1(\mathbf{Z})\|_{\gamma, \eta, T; \Gamma^{1/\lambda; \text{eh}}} \lesssim \varepsilon \|\mathbf{Z}^-\| + \varepsilon \lambda \lesssim \varepsilon(1 + \|\mathbf{Z}^-\|)$$

where the implicit constant depends on ρ , T , and the norm of $G(W)$, a deterministic quantity. \square

We proved the case $n = 1$ by leveraging the continuity of the involved solution maps. The general strategy, however, is to invoke Duhamel’s formula, Proposition 2.35, again. As alluded to earlier, our first step towards this goal is to derive an equation of type (2.69) for $\mathfrak{R}_\varepsilon^{n+1}$.

Proposition 2.42. *For $n \geq 1$ we have*

$$\Theta^{\text{eh}} \mathfrak{R}_\varepsilon^{n+1} = \mathcal{P}^{\text{eh}} \left([\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}] \varepsilon \mathfrak{I} + [\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}] \mathfrak{I} \right), \tag{2.89}$$

where

$$[\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}] := G(U^\varepsilon) - G(W) - \sum_{j=1}^{n-1} \sum_{i \in \cup_{l=1}^{n-1} S_j^l} \frac{1}{j!} G^{(j)}(W) \prod_{m=1}^j \frac{\varepsilon^{i_m}}{i_m!} U^{(i_m)}, \tag{2.90}$$

²⁸ In [23, Prop. 3.25], the map \mathcal{S}_{ex} is denoted by $\mathcal{S}_{\text{ex}}^H$.

$$[\mathfrak{R}_\varepsilon^{n+1}; \mathbf{i}] := G(U^\varepsilon) - G(W) - G'(W)(U^\varepsilon - W) - \sum_{j=2}^n \sum_{\mathbf{i} \in \cup_{l=1}^n S_l^j} \frac{1}{j!} G^{(j)}(W) \prod_{m=1}^j \frac{\varepsilon^{i_m}}{i_m!} U^{(i_m)}, \tag{2.91}$$

and $S_k^n := \{\mathbf{i} \in \mathbb{N}_{\geq 1}^k : |\mathbf{i}| = n\}$. \spadesuit

Proof. We start with the case $n = 1$. Recall from Corollary 2.28 that $A^{(0), \mathbf{1}} = G(W)$, $B^{(0), \mathbf{1}} = 0$, thus $U^{(1)} = \mathcal{P}^{\text{eh}}(G'(W)U^{(1)} \mathbf{1}) + \mathcal{P}^{\text{eh}}(G(W) \mathbf{1})$. Combined with (2.79) we find

$$\mathfrak{R}_\varepsilon^2 = \mathfrak{R}_\varepsilon^1 - \varepsilon U^{(1)} = \mathcal{P}^{\text{eh}}([G(U^\varepsilon) - G(W)]\varepsilon \mathbf{1}) + \mathcal{P}^{\text{eh}}(G(U^\varepsilon) - G(W) - G'(W)\varepsilon U^{(1)} \mathbf{1})$$

and then use the identity $\varepsilon U^{(1)} = \mathfrak{R}_\varepsilon^1 - \mathfrak{R}_\varepsilon^2 = U^\varepsilon - W - \mathfrak{R}_\varepsilon^2$ in the last term of the second summand. This leads to the claimed identity

$$\begin{aligned} \mathfrak{R}_\varepsilon^2 &= \mathcal{P}^{\text{eh}}([G(U^\varepsilon) - G(W)]\varepsilon \mathbf{1}) + \mathcal{P}^{\text{eh}}([G(U^\varepsilon) - G(W) - G'(W)[U^\varepsilon - W]] \mathbf{1}) \\ &\quad + \mathcal{P}^{\text{eh}}(G'(W)\mathfrak{R}_\varepsilon^2 \mathbf{1}) \end{aligned} \tag{2.92}$$

For $n \geq 2$, we use the fact that

$$\mathfrak{R}_\varepsilon^{n+1} = \mathfrak{R}_\varepsilon^2 - \sum_{j=2}^n \frac{\varepsilon^j}{j!} U^{(j)}$$

combined with (2.57) and (2.92) to obtain the identity

$$\begin{aligned} &\mathfrak{R}_\varepsilon^{n+1} - \mathcal{P}^{\text{eh}}([G(U^\varepsilon) - G(W)]\varepsilon \mathbf{1}) - \mathcal{P}^{\text{eh}}([G(U^\varepsilon) - G(W) - G'(W)[U^\varepsilon - W]] \mathbf{1}) \\ &= \mathfrak{R}_\varepsilon^2 - \mathcal{P}^{\text{eh}}([G(U^\varepsilon) - G(W)]\varepsilon \mathbf{1}) - \mathcal{P}^{\text{eh}}([G(U^\varepsilon) - G(W) - G'(W)[U^\varepsilon - W]] \mathbf{1}) \\ &\quad - \sum_{j=2}^n \frac{\varepsilon^j}{j!} U^{(j)} \\ &= \mathcal{P}^{\text{eh}}(G'(W)\mathfrak{R}_\varepsilon^2 \mathbf{1}) \\ &\quad - \sum_{j=2}^n \frac{\varepsilon^j}{j!} \left[\mathcal{P}^{\text{eh}}(G'(W)U^{(j)} \mathbf{1}) + \mathcal{P}^{\text{eh}}(A^{(j-1), \mathbf{1}} \mathbf{1}) + \mathcal{P}^{\text{eh}}(B^{(j-1), \mathbf{1}} \mathbf{1}) \right] \\ &= \mathcal{P}^{\text{eh}}(G'(W)\mathfrak{R}_\varepsilon^{n+1} \mathbf{1}) - \sum_{j=2}^n \frac{\varepsilon^j}{j!} \left[\mathcal{P}^{\text{eh}}(A^{(j-1), \mathbf{1}} \mathbf{1}) + \mathcal{P}^{\text{eh}}(B^{(j-1), \mathbf{1}} \mathbf{1}) \right]. \end{aligned} \tag{2.93}$$

Next, observe that $S_k^j = \emptyset$ for $k > j$ (*). By definition of $B^{(j-1), \mathbf{1}}$ in (2.60), we have

$$\sum_{j=2}^n \frac{\varepsilon^j}{j!} B^{(j-1), \mathbf{1}} = \sum_{j=2}^n \varepsilon^j \sum_{k=2}^j \frac{1}{k!} G^{(k)}(W) \sum_{\mathbf{i} \in S_k^j} \prod_{m=1}^k \frac{1}{i_m!} U^{(i_m)}$$

$$\begin{aligned} &\stackrel{(*)}{=} \sum_{k=2}^n \frac{1}{k!} G^{(k)}(W) \sum_{j=1}^n \varepsilon^j \sum_{\mathbf{i} \in S_k^j} \prod_{m=1}^k \frac{1}{i_m!} U^{(i_m)} \\ &= \sum_{k=2}^n \frac{1}{k!} G^{(k)}(W) \sum_{\mathbf{i} \in \cup_{j=1}^n S_k^j} \prod_{m=1}^k \frac{\varepsilon^{i_m}}{i_m!} U^{(i_m)} \end{aligned}$$

where the last equality uses that $\varepsilon^j = \varepsilon^{|\mathbf{i}|} = \prod_{m=1}^k \varepsilon^{i_m}$ for $\mathbf{i} \in S_k^j$. Analogously, by definition of $A^{(j), \mathbf{1}}$ in (2.59) we have

$$\sum_{j=2}^n \frac{\varepsilon^j}{j!} j A^{(j-1), \mathbf{1}} = \varepsilon \sum_{j=1}^{n-1} \frac{\varepsilon^j}{j!} A^{(j), \mathbf{1}} = \varepsilon \sum_{k=1}^{n-1} \frac{1}{k!} G^{(k)}(W) \sum_{\mathbf{i} \in \cup_{j=1}^{n-1} S_k^j} \prod_{m=1}^k \frac{\varepsilon^{i_m}}{i_m!} U^{(i_m)}.$$

Once we update (2.93) by the previous two identities, the claim follows. \square

Remark 2.43. Formally, the expression for $\mathfrak{R}_\varepsilon^{n+1}$ from Proposition 2.42 agrees with the one by Inahama and Kawabi, see the penultimate equation on page 303 in [52]. \spadesuit

For $Y := \mathfrak{R}_\varepsilon^{n+1}$, we learn from the previous proposition that the corresponding \tilde{V} and V in (2.70) resp. (2.69) are given by

$$\tilde{V}^{[n+1]} := [\mathfrak{R}_\varepsilon^{n+1}; \mathbf{1}] \varepsilon \mathbf{1} + [\mathfrak{R}_\varepsilon^{n+1}; \mathbf{1}] \mathbf{1}, \quad V^{[n+1]} := \mathcal{P}^{\text{eh}}(\tilde{V}). \tag{2.94}$$

In the next two lemmas, we derive more explicit expressions for $[\mathfrak{R}_\varepsilon^{n+1}; \sigma]$, $\sigma \in \{ \mathbf{1}, \mathbf{1} \}$.

Lemma 2.44. *The identity*

$$\begin{aligned} [\mathfrak{R}_\varepsilon^{n+1}; \mathbf{1}] &= \sum_{j=1}^{n-1} \frac{1}{j!} G^{(j)}(W) \left(\sum_{\substack{\mathbf{i} \in [n-1]^j \\ |\mathbf{i}| \geq n}} \prod_{m=1}^j \frac{\varepsilon^{i_m}}{i_m!} U^{(i_m)} + \sum_{m=1}^j \binom{j}{m} (\mathfrak{S}_\varepsilon^{n-1})^{j-m} (\mathfrak{R}_\varepsilon^n)^m \right) \\ &\quad + \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} G^{(n)}(W + s\mathfrak{R}_\varepsilon^1) [\mathfrak{R}_\varepsilon^1]^n ds \end{aligned} \tag{2.95}$$

holds with $\mathfrak{S}_\varepsilon^{n-1} := \sum_{i=1}^{n-1} \frac{\varepsilon^i}{i!} U^{(i)}$. \spadesuit

Proof. By definition, $\mathfrak{R}_\varepsilon^1 = U^\varepsilon - W$. We expand $G(U^\varepsilon)$ at $W \equiv U^0$ to write

$$\begin{aligned} &G(U^\varepsilon) - G(W) \\ &= \sum_{j=1}^{n-1} \frac{1}{j!} D^{(j)} G(W) [\mathfrak{R}_\varepsilon^1, \dots, \mathfrak{R}_\varepsilon^1] + \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} D^{(n)} G(W + s\mathfrak{R}_\varepsilon^1) [\mathfrak{R}_\varepsilon^1, \dots, \mathfrak{R}_\varepsilon^1] ds \end{aligned}$$

$$= \sum_{j=1}^{n-1} \frac{1}{j!} G^{(j)}(W) [\mathfrak{R}_\varepsilon^1]^j + \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} G^{(n)}(W + s\mathfrak{R}_\varepsilon^1) [\mathfrak{R}_\varepsilon^1]^n ds \tag{2.96}$$

where we used Proposition A.35 in the last equality. At the same time, by the identity $\mathfrak{R}_\varepsilon^1 = \mathfrak{S}_\varepsilon^{n-1} + \mathfrak{R}_\varepsilon^n$ and the binomial theorem, the first term in the expansion (2.96) reads

$$\sum_{j=1}^{n-1} \frac{1}{j!} G^{(j)}(W) [\mathfrak{R}_\varepsilon^1]^j = \sum_{j=1}^{n-1} \frac{1}{j!} G^{(j)}(W) \left[(\mathfrak{S}_\varepsilon^{n-1})^j + \sum_{m=1}^j \binom{j}{m} (\mathfrak{S}_\varepsilon^{n-1})^{j-m} (\mathfrak{R}_\varepsilon^n)^m \right]. \tag{2.97}$$

For $[\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}]$ as in (2.90), we have thus far established that

$$\begin{aligned} & [\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}] \\ &= \sum_{j=1}^{n-1} \frac{1}{j!} G^{(j)}(W) \left[(\mathfrak{S}_\varepsilon^{n-1})^j + \sum_{m=1}^j \binom{j}{m} (\mathfrak{S}_\varepsilon^{n-1})^{j-m} (\mathfrak{R}_\varepsilon^n)^m - \sum_{i \in \cup_{l=1}^{n-1} S_j^l} \prod_{m=1}^j \frac{\varepsilon^{i_m}}{i_m!} U^{(i_m)} \right] \\ &+ \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} G^{(n)}(W + s\mathfrak{R}_\varepsilon^1) [\mathfrak{R}_\varepsilon^1]^n ds. \end{aligned} \tag{2.98}$$

We further rewrite²⁹

$$(\mathfrak{S}_\varepsilon^{n-1})^j = \left(\sum_{i=1}^{n-1} \frac{\varepsilon^i}{i!} U^{(i)} \right)^j = \prod_{r=1}^j \left(\sum_{\ell_r=1}^{n-1} \frac{\varepsilon^{\ell_r}}{\ell_r!} U^{(\ell_r)} \right) = \sum_{\ell \in [n-1]^j} \prod_{m=1}^j \frac{\varepsilon^{\ell_m}}{\ell_m!} U^{(\ell_m)} \tag{2.99}$$

and observe that

$$\bigcup_{l=1}^{n-1} S_j^l = \bigcup_{l=1}^{n-1} \{i \in \mathbb{N}_{\geq 1}^j : |i| = l\} = \{i \in \mathbb{N}_{\geq 1}^j : |i| \in [n-1]\} = \{i \in [n-1]^j : |i| \in [n-1]\}.$$

Combining this identity with the formula (2.99) for $(\mathfrak{S}_\varepsilon^{n-1})^j$, we see that

$$\begin{aligned} (\mathfrak{S}_\varepsilon^{n-1})^j - \sum_{i \in \cup_{l=1}^{n-1} S_j^l} \prod_{m=1}^j \frac{\varepsilon^{i_m}}{i_m!} U^{(i_m)} &= \sum_{i \in [n-1]^j} \prod_{m=1}^j \frac{\varepsilon^{i_m}}{i_m!} U^{(i_m)} - \sum_{\substack{i \in [n-1]^j \\ |i| \in [n-1]}} \prod_{m=1}^j \frac{\varepsilon^{i_m}}{i_m!} U^{(i_m)} \\ &= \sum_{\substack{i \in [n-1]^j \\ |i| \geq n}} \prod_{m=1}^j \frac{\varepsilon^{i_m}}{i_m!} U^{(i_m)} \end{aligned}$$

²⁹ Recall that $[n] := \{1, \dots, n\}$ for $n \in \mathbb{N}$.

Plugging this identity into (2.98) finishes the proof. \square

A similar identity holds for $[\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}]$ and can be proved analogously.

Lemma 2.45. For $\mathfrak{S}_\varepsilon^{n-1}$ as in Lemma 2.44, the following identity holds:

$$\begin{aligned}
 [\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}] &= \sum_{j=2}^n \frac{1}{j!} G^{(j)}(W) \left(\sum_{\substack{\mathbf{i} \in [n-1]^j \\ |\mathbf{i}| \geq n+1}} \prod_{m=1}^j \frac{\varepsilon^{i_m}}{i_m!} U^{(i_m)} + \sum_{m=1}^j \binom{j}{m} (\mathfrak{S}_\varepsilon^{n-1})^{j-m} (\mathfrak{R}_\varepsilon^n)^m \right) \\
 &\quad + \int_0^1 \frac{(1-s)^n}{n!} G^{(n+1)}(W + s\mathfrak{R}_\varepsilon^1) [\mathfrak{R}_\varepsilon^1]^{n+1} ds
 \end{aligned} \tag{2.100}$$

Finally, we have gathered all the tools to prove Proposition 2.40.

Proof of Proposition 2.40. The statement for $n = 1$ is the content of Lemma 2.41, so we assume the claim is true for $n \in [\ell - 1]$ and consider the induction step $n \mapsto n + 1$. Using Proposition 2.42 in conjunction with Lemmas 2.32 and A.38, we obtain the identity

$$\begin{aligned}
 \mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^{n+1}(\mathbf{Z}) &= (\mathfrak{d}_\lambda \circ \tilde{\Theta}^{E_h \mathbf{Z}}) [\mathcal{P}^{E_h \mathbf{Z}}([\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}] \varepsilon \mathfrak{I} + [\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}] \mathfrak{I})] \\
 &= \tilde{\Theta}^{E_h \delta_{1/\lambda} \mathbf{Z}} [\mathcal{P}^{E_h \delta_{1/\lambda} \mathbf{Z}}(\mathfrak{d}_\lambda [\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}] \varepsilon \mathfrak{I} \lambda + \mathfrak{d}_\lambda [\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}] \mathfrak{I})],
 \end{aligned}$$

so by Duhamel’s principle presented in Proposition 2.35, we have the estimate

$$\|\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^{n+1}(\mathbf{Z})\|_{\gamma, \eta; T; \Gamma^{\text{eh}; 1/\lambda}} \lesssim \varepsilon \lambda \|\mathfrak{d}_\lambda [\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}]\|_{\gamma, \eta; T; \Gamma^{\text{eh}; 1/\lambda}} + \|\mathfrak{d}_\lambda [\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}]\|_{\gamma, \eta; T; \Gamma^{\text{eh}; 1/\lambda}}. \tag{2.101}$$

We have to estimate the terms $\|\mathfrak{d}_\lambda [\mathfrak{R}_\varepsilon^{n+1}; \sigma]\|_{\gamma, \eta; \Gamma^{1/\lambda, \text{eh}}}$ on the right hand side and start with $\sigma = \mathfrak{I}$. In this case, the factor $\varepsilon \lambda = \varepsilon(1 + \|\mathbf{Z}^-\|)$ already contributes one power towards the exponent in the claim. By Lemma 2.44, terms of three types are to be estimated:³⁰

- (1) Those only containing Taylor terms $\mathfrak{d}_\lambda U^{(i)}$ with $\mathbf{i} \in [n-1]$, for which Proposition 2.39 implies the estimate

$$\sum_{\substack{\mathbf{i} \in [n-1]^j \\ |\mathbf{i}| \geq n}} \prod_{m=1}^j \frac{\varepsilon^{i_m}}{i_m!} \|\mathfrak{d}_\lambda U^{(i_m)}(\mathbf{Z})\|_{\gamma, \eta; T; \Gamma^{\text{eh}; 1/\lambda}} \lesssim \sum_{\substack{\mathbf{i} \in [n-1]^j \\ |\mathbf{i}| \geq n}} \prod_{m=1}^j \frac{\varepsilon^{i_m}}{i_m!} (1 + \|\mathbf{Z}^-\|)^{i_m}$$

³⁰ Recall that the operator \mathfrak{d}_λ is linear, multiplicative, and commutes with the application of lifts G of functions g , see eq. (A.17) and Lemma A.38(i).

$$\lesssim \sum_{\substack{\mathbf{i} \in [n-1]^j \\ |\mathbf{i}| \geq n}} [\varepsilon(1 + \|\mathbf{Z}^-\|)]^{|\mathbf{i}|} \lesssim_\rho \varepsilon^n (1 + \|\mathbf{Z}^-\|)^n,$$

where we have used $\varepsilon\|\mathbf{Z}^-\| < \rho$ to estimate terms with powers bigger than n .

(2) Those that contain at least one factor of $\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^n$, for which we have

$$\sum_{m=1}^j \binom{j}{m} \|\mathfrak{d}_\lambda \mathfrak{G}_\varepsilon^{n-1}\|_{\gamma, \eta; T; \Gamma^{\text{eh}; 1/\lambda}}^{j-m} \|\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^n\|_{\gamma, \eta; T; \Gamma^{\text{eh}; 1/\lambda}}^m \lesssim \varepsilon^n (1 + \|\mathbf{Z}^-\|)^n \tag{2.102}$$

Here, we have used the induction hypothesis, Proposition 2.39 to estimate the terms $\mathfrak{d}_\lambda U^{(k)}$ (hidden in $\mathfrak{d}_\lambda \mathfrak{G}_\varepsilon^{n-1}$), and the fact that $\varepsilon\|\mathbf{Z}^-\| < \rho$ to estimate powers higher than n by a ρ -dependent constant.

(3) The Bochner integral in (2.95). In order to control this term, we observe that

$$\|W + s\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^1\|_{\gamma, \eta; T; \Gamma^{\text{eh}; 1/\lambda}} \lesssim \|W\|_{\gamma, \eta; T; \Gamma^{\text{eh}; 1/\lambda}} + (1 + \rho)$$

holds uniformly over $s \in [0, 1]$. In this estimate, we have used the induction basis (Lemma 2.41) and the assumption that $\varepsilon\|\mathbf{Z}^-\| < \rho$. Hence, we can estimate

$$\begin{aligned} & \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} \|G^{(n)}(W + s\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^1)\|_{\gamma, \eta; T; \Gamma^{\text{eh}; 1/\lambda}} \, ds \|\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^1\|_{\gamma, \eta; T; \Gamma^{\text{eh}; 1/\lambda}}^n \\ & \lesssim \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} \|W + s\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^1\|_{\gamma, \eta; T; \Gamma^{\text{eh}; 1/\lambda}} \, ds \|\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^1\|_{\gamma, \eta; T; \Gamma^{\text{eh}; 1/\lambda}}^n \lesssim \varepsilon^n (1 + \|\mathbf{Z}^-\|)^n \end{aligned}$$

again by the induction basis. Note that the first estimate, depends on the norms of $\|g^{(r)}\|_\infty$, $r \in \{n, n + 1\}$, as well as $\|\Gamma^{\text{eh}; 1/\lambda}\|$; the latter, however, satisfies a deterministic bound that depends on $\|h\|_{\mathcal{H}}$, see Lemma A.24 in the appendix.

By definition of λ , these estimates can be combined to

$$\varepsilon \lambda \|\mathfrak{d}_\lambda [\mathfrak{R}_\varepsilon^{n+1}; \sigma]\|_{\gamma, \eta; T; \Gamma^{1/\lambda; \text{eh}}} \lesssim_\rho \varepsilon^{n+1} (1 + \|\mathbf{Z}^-\|)^{n+1},$$

in case $\sigma = \mathfrak{I}$. For $\sigma = \mathfrak{I}$, we argue analogously using Lemma 2.45 instead of Lemma 2.44. However, note that no additional factor $\varepsilon \lambda$ appears in $\|\mathfrak{d}_\lambda [\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}]\|_{\gamma, \eta; T; \Gamma^{\text{eh}; 1/\lambda}}$ in (2.101), so all estimates have to be of order $n + 1$ rather than n . In fact, they are: For terms of types (1) and (3) appearing in $[\mathfrak{R}_\varepsilon^{n+1}; \mathfrak{I}]$, this can be seen immediately from their definition. For terms of type (2) note that the first sum in (2.100) starts with $j = 2$, so $(\mathfrak{G}_\varepsilon^{n-1})^{j-m}$ always comes with an exponent $j - m \geq 1$, contributing one additional power to the exponent of $\varepsilon(1 + \|\mathbf{Z}^-\|)$ in (2.102). \square

2.4. Proof of Theorem 2

The whole section is occupied by the proof.

Proof. Note that

$$\mathcal{U} = \langle \mathbf{1}, \mathfrak{J}, \mathfrak{J}, X \rangle \subseteq \bar{\mathcal{T}} \oplus \mathcal{T}_{\geq \alpha+2}, \quad 1 - \kappa = \alpha + 2 < \gamma = 1 + 2\kappa$$

and recall that $\mathcal{X}_T := \mathcal{C}([0, T], \mathcal{C}^\eta(\mathbb{T}^2))$ for fixed $\eta \in (1/2, 1)$. As a consequence of [45, Prop. 3.28], it thus follows that

$$V \in \mathcal{D}_{\mathcal{U}}^{\gamma, \eta, T}(\mathbf{Z}) \implies \mathcal{R}^{\mathbf{Z}} V \in \mathcal{C}((0, T], \mathcal{C}^{\alpha+2}(\mathbb{T}^2)) \cap \mathcal{C}([0, T], \mathcal{C}^\eta(\mathbb{T}^2)) \subseteq \mathcal{X}_T.$$

More generally, the reconstruction operators

$$\mathcal{R} : \mathcal{M} \times \mathcal{D}_{\mathcal{U}}^{\gamma, \eta; T} \rightarrow \mathcal{X}_T, \quad \mathcal{R}^{E_h \bullet} : \mathcal{M} \times \mathcal{D}_{\mathcal{U}}^{\gamma, \eta; T}(E_h \bullet) \rightarrow \mathcal{X}_T.$$

are jointly locally Lipschitz continuous by the reconstruction theorem [45, Thm. 3.10] and Proposition A.13. Since $\mathcal{R}^{E_h \mathbf{Z}}$ is linear and bounded for any fixed $\mathbf{Z} \in \mathcal{M}$ (and thus Fréchet \mathcal{C}^∞), it immediately follows from Theorem 2.26 that

$$u_h^\bullet(\mathbf{Z}) = \mathcal{R}^{E_h \mathbf{Z}} \circ \mathfrak{J}_{E_h \mathbf{Z}} \in \mathcal{C}^{(\ell)}(I_0(T, \mathbf{Z}), \mathcal{X}_T).$$

• **Property (i): Continuous dependence on the model and estimates.**

By Definition 2.33, continuity of $u_h^{(m)}(\mathbf{Z})$, $m \in [\ell - 1]$, and $R_{h, \varepsilon}^{(\ell)}(\mathbf{Z})$ in \mathbf{Z} is a direct consequence of joint continuity of $\mathcal{R}^{E_h \bullet}$ and Corollary 2.28.

In view of Remark 2.37, the estimates in (0.18) are straightforward consequences of Proposition 2.39 and Proposition 2.40, respectively. We have

$$\|u_h^{(m)}(\mathbf{Z})\|_{\mathcal{X}_T} = \|\mathcal{R}^{E_h \mathbf{Z}} U^{(m)}(\mathbf{Z})\|_{\mathcal{X}_T} \leq \|\mathfrak{d}_\lambda U^{(m)}(\mathbf{Z})\|_{\gamma, \eta; T; \Gamma^{e_h; 1/\lambda}} \lesssim (1 + \|\mathbf{Z}^-\|)^m$$

and

$$\|R_{h, \varepsilon}^{(\ell)}(\mathbf{Z})\|_{\mathcal{X}_T} = \|\mathcal{R}^{E_h \mathbf{Z}} \mathfrak{R}_\varepsilon^\ell(\mathbf{Z})\|_{\mathcal{X}_T} \leq \|\mathfrak{d}_\lambda \mathfrak{R}_\varepsilon^\ell(\mathbf{Z})\|_{\gamma, \eta; T; \Gamma^{e_h; 1/\lambda}} \lesssim \varepsilon^\ell (1 + \|\mathbf{Z}^-\|)^\ell.$$

• **Property (ii): Homogeneity w.r.t. model dilation.**

Recall that we want to prove the equality $\varepsilon^m u_h^{(m)}(\mathbf{Z}) = u_h^{(m)}(\delta_\varepsilon \mathbf{Z})$ for $\varepsilon \in I_0(T, \mathbf{Z})$. By now, this is an easy consequence of Proposition 2.31: We have

$$\begin{aligned} \varepsilon^m u_h^{(m)}(\mathbf{Z}) &= \mathcal{R}^{E_h \mathbf{Z}} (\varepsilon^m U^{(m)}(\mathbf{Z})) = \mathcal{R}^{E_h \mathbf{Z}} (\mathfrak{d}_\varepsilon U^{(m)}(\delta_\varepsilon \mathbf{Z})) \\ &= \mathcal{R}^{E_h \delta_\varepsilon \mathbf{Z}} (U^{(m)}(\delta_\varepsilon \mathbf{Z})) = u_h^{(m)}(\delta_\varepsilon \mathbf{Z}) \end{aligned}$$

where we have additionally used Lemma A.38(ii) and Lemma A.21). \square

2.5. Stochastic PDEs for the Taylor terms

In this section, we derive stochastic PDEs that are satisfied by the Taylor terms $\hat{u}_{\xi_\delta;h}^{(m)}$ in the expansion (0.16) in Theorem 2 (with the notational convention introduced in Definition 2.33). Even though these equations are not used elsewhere in this article, we feel that they are still interesting to the reader. For a corresponding statement in the framework of rough paths theory, see Inahama and Kawabi [52, Def. 4.1]. In what follows, we will use the following convention:

Notation 2.46. For $U^{(m)}(\mathcal{Z})$, $A_{E_h \mathcal{Z}}^{(m), \mathbf{1}}$, and $B_{E_h \mathcal{Z}}^{(m-1), \mathbf{1}}$ as in Corollary 2.28, we set

$$\hat{U}_{\xi_\delta, h}^{(m)} := U^{(m)}(\hat{\mathcal{Z}}^{\xi_\delta}), \quad \hat{A}_{\xi_\delta, h}^{(m), \mathbf{1}} := A_{E_h \hat{\mathcal{Z}}^{\xi_\delta}}^{(m), \mathbf{1}}, \quad \hat{B}_{\xi_\delta, h}^{(m-), \mathbf{1}} := B_{E_h \hat{\mathcal{Z}}^{\xi_\delta}}^{(m-), \mathbf{1}}. \quad \spadesuit$$

With this convention, the fixed-point equation (2.57) for $\hat{U}_{\xi_\delta, h}^{(m)}$ reads

$$\hat{U}_{\xi_\delta, h}^{(m)} = \mathcal{P}^{E_h \hat{\mathcal{Z}}^{\xi_\delta}} (G'(W) \hat{U}_{\xi_\delta, h}^{(m)} \mathbf{1}) + m \mathcal{P}^{E_h \hat{\mathcal{Z}}^{\xi_\delta}} (\hat{A}_{\xi_\delta, h}^{(m-), \mathbf{1}} \mathbf{1}) + \mathcal{P}^{E_h \hat{\mathcal{Z}}^{\xi_\delta}} (\hat{B}_{\xi_\delta, h}^{(m-), \mathbf{1}} \mathbf{1}) \quad (2.103)$$

The announced equations are:

Proposition 2.47 (Explicit equations for $\hat{u}_{\xi_\delta;h}^{(m)}$). *The term $\hat{u}_{\xi_\delta;h}^{(m)}$ satisfies the linear stochastic PDE given by*

$$(\partial_t - \Delta) \hat{u}_{\xi_\delta;h}^{(m)} = \hat{a}_{\xi_\delta;h}^{(m-), \mathbf{1}} \xi_\delta - \hat{a}_{\xi_\delta;h}^{(m-), \mathbf{c}} \mathbf{c}_\delta + [\hat{b}_{\xi_\delta;h}^{(m-), \mathbf{1}} + \hat{u}_{\xi_\delta;h}^{(m)} g'(w_h)] \mathbf{h}, \quad \hat{u}_{\xi_\delta;h}^{(m)}(0, \cdot) = 0. \quad (2.104)$$

The coefficient functions of ξ_δ and \mathbf{h} are defined by³¹

$$\begin{aligned} \hat{a}_{\xi_\delta;h}^{(m-), \mathbf{1}} &:= m \mathcal{R}^{e_h} (\hat{A}_{\xi_\delta;h}^{(m-), \mathbf{1}}) = m \langle \hat{A}_{\xi_\delta;h}^{(m-), \mathbf{1}}, \mathbf{1} \rangle, \\ \hat{b}_{\xi_\delta;h}^{(m-), \mathbf{1}} &:= \mathcal{R}^{e_h} (\hat{B}_{\xi_\delta;h}^{(m-), \mathbf{1}}) = \langle \hat{B}_{\xi_\delta;h}^{(m-), \mathbf{1}}, \mathbf{1} \rangle. \end{aligned} \quad (2.105)$$

They calculate to the same expressions as the corresponding terms in (2.59), multiplied by m , and (2.60), respectively, only with the replacements $G \rightsquigarrow g$, $W \rightsquigarrow w_h$, and $U^{(k)} \rightsquigarrow \hat{u}_{\xi_\delta;h}^{(k)}$. For the coefficient of \mathbf{c}_δ , we have

$$\hat{a}_{\xi_\delta;h}^{(m-), \mathbf{c}} := m \langle \hat{A}_{\xi_\delta;h}^{(m-), \mathbf{1}}, \mathcal{J} \rangle = m! \sum_{k=1}^{m-1} \frac{1}{k!} g^{(k)}(w_h) \sum_{i \in S_k^{m-1}} \sum_{r=1}^k \frac{1}{i_r!} \hat{a}_{\xi_\delta;h}^{(i_r-), \mathbf{1}} \prod_{\substack{n=1, \\ n \neq r}}^k \frac{1}{i_n!} \hat{u}_{\xi_\delta;h}^{(i_n)}.$$



³¹ The second equalities with $\langle \dots, \mathbf{1} \rangle$ are due to an analogue of [45, Prop. 3.28] to the spaces $\mathcal{D}_{\mathcal{U}}^{\gamma, \eta}(\hat{\mathcal{Z}}^{\xi_\delta; e_h})$.

Remark 2.48. As a sanity check, one can verify that the equation for $\hat{u}_{\xi_\delta; \mathbf{h}}^{(m)}$ in Proposition 2.47 can also be obtained by naively applying $\partial_\varepsilon^m|_{\varepsilon=0}$ to

$$(\partial_t - \Delta)\hat{u}_{\xi_\delta; \mathbf{h}}^\varepsilon = g(\hat{u}_{\xi_\delta; \mathbf{h}}^\varepsilon) (\varepsilon\xi_\delta + \mathbf{h} - \varepsilon^2\mathbf{c}_\delta g'(\hat{u}_{\xi_\delta; \mathbf{h}}^\varepsilon)), \quad \hat{u}_{\xi_\delta; \mathbf{h}}^\varepsilon(0, \cdot) = u_0.$$

In particular, we find

$$\begin{aligned} (m = 1) \quad \hat{a}_{\xi_\delta; \mathbf{h}}^{(0), \mathbf{1}} &= g(w_{\mathbf{h}}), \quad \hat{a}_{\xi_\delta; \mathbf{h}}^{(0), \mathbf{c}} = 0 = \hat{b}_{\xi_\delta; \mathbf{h}}^{(0), \mathbf{1}} \\ (m = 2) \quad \hat{a}_{\xi_\delta; \mathbf{h}}^{(1), \mathbf{1}} &= 2g'(w_{\mathbf{h}})\hat{u}_{\xi_\delta; \mathbf{h}}^{(1)}, \quad \hat{a}_{\xi_\delta; \mathbf{h}}^{(1), \mathbf{c}} = 2g'(w_{\mathbf{h}})g(w_{\mathbf{h}}), \quad \hat{b}_{\xi_\delta; \mathbf{h}}^{(1), \mathbf{1}} = g''(w_{\mathbf{h}}) [\hat{u}_{\xi_\delta; \mathbf{h}}^{(1)}]^2 \end{aligned}$$

such that

$$\begin{aligned} (m = 1) \quad (\partial_t - \Delta)\hat{u}_{\xi_\delta; \mathbf{h}}^{(1)} &= g(w_{\mathbf{h}})\xi_\delta + \hat{u}_{\xi_\delta; \mathbf{h}}^{(1)}g'(w_{\mathbf{h}})\mathbf{h} \\ (m = 2) \quad (\partial_t - \Delta)\hat{u}_{\xi_\delta; \mathbf{h}}^{(2)} &= 2g'(w_{\mathbf{h}})\hat{u}_{\xi_\delta; \mathbf{h}}^{(1)}\xi_\delta - 2g'(w_{\mathbf{h}})g(w_{\mathbf{h}})\mathbf{c}_\delta \\ &\quad + [g''(w_{\mathbf{h}}) [\hat{u}_{\xi_\delta; \mathbf{h}}^{(1)}]^2 + \hat{u}_{\xi_\delta; \mathbf{h}}^{(2)}g'(w_{\mathbf{h}})] \mathbf{h} \end{aligned} \tag{2.106}$$

with $\hat{u}_{\xi_\delta; \mathbf{h}}^{(i)}(0, \cdot) = 0$ for $i = 1, 2$. \spadesuit

The equations for the Taylor terms become useful when one attempts to calculate (or numerically approximate) the coefficients a_k in (0.13) explicitly, see the formula (3.16) in Proposition 3.6 below. A proof of Proposition 2.47 can be found in [34, Sec. 2.5].

3. Local analysis in the vicinity of the minimiser

Recall that the application of the Cameron-Martin theorem in Proposition 1.2 has led to the expression

$$J_\rho(\varepsilon) = \exp\left(-\frac{\mathcal{F}(\mathbf{h})}{\varepsilon^2}\right) \mathbb{E} \left[\exp\left(-\frac{\tilde{F}_{\Phi}^-(\mathbf{h}, \varepsilon)}{\varepsilon^2}\right); \varepsilon \|\hat{\mathbf{Z}}^-\| < \rho \right], \tag{3.1}$$

with $\tilde{F}_{\Phi}^-(\mathbf{h}, \varepsilon)$ given in (1.11). By plugging in the expansion obtained in Corollary 2.1, we obtain the identity

$$\tilde{F}_{\Phi}^-(\mathbf{h}, \varepsilon) = \varepsilon [DF|_{w_{\mathbf{h}}}(\hat{u}_{\mathbf{h}}^{(1)}) + \xi(\mathbf{h})] + \frac{\varepsilon^2}{2} \hat{Q}_{\mathbf{h}} + \sum_{m=3}^{N+2} \frac{\varepsilon^m}{m!} \hat{F}_{\mathbf{h}}^{(m)} + \hat{R}_{\mathbf{h}}^{F; \varepsilon; N+3} \tag{3.2}$$

for each $\varepsilon \in I_0 := [0, \varepsilon_0)$ with $\varepsilon_0 = \varepsilon_0(T, \hat{\mathbf{Z}})$ as in (2.28).

3.1. *Vanishing of the linear term by first-order optimality*

In this subsection, we want to use first-order optimality implied by assumption (H2) to prove the proposition that follows. Recall our convention from Definition 2.33 that $\hat{u}_h^{(1)}(\omega) = u_h^{(1)}(\hat{Z}(\omega))$ for $\omega \in \Omega$.

Proposition 3.1. *We have*

$$DF|_{w_h}(\hat{u}_h^{(1)}(\omega)) + \xi(\omega)(h) = 0. \tag{3.3}$$

for \mathbb{P} -a.e. $\omega \in \Omega$. Hence,

$$J_\rho(\varepsilon) = \exp\left(-\frac{\mathcal{F}(h)}{\varepsilon^2}\right) \mathbb{E}\left[\exp\left(-\frac{1}{2}\hat{Q}_h - \hat{S}_h^{F;\varepsilon;N} - \frac{1}{\varepsilon^2}\hat{R}_h^{F;\varepsilon;N+3}\right); \varepsilon\|\hat{Z}^-\| < \rho\right] \tag{3.4}$$

where we have set

$$\hat{S}_h^{F;\varepsilon;N} := \sum_{m=3}^{N+2} \frac{\varepsilon^{m-2}}{m!} \hat{F}_h^{(m)}. \tag{3.5}$$

♠

Proof. By assumption (H2), the Cameron-Martin function h is the unique minimiser of $\mathcal{F} = (F_\Phi \circ \mathcal{L}) + \mathcal{I}$, so $D\mathcal{F}|_h \equiv 0 \in \mathcal{H}'$. We calculate the Fréchet derivatives of the two summands of \mathcal{F} . Since ξ_δ is smooth a.s., we have $\xi_\delta(\omega) \in \mathcal{H}$ for \mathbb{P} -a.e. $\omega \in \Omega$.

(1) For Schilder’s rate function \mathcal{I} , we obtain

$$D\mathcal{I}|_h(\xi_\delta(\omega)) = \frac{1}{2} \frac{d}{d\lambda} \Big|_{\lambda=0} \langle h + \lambda\xi_\delta(\omega), h + \lambda\xi_\delta(\omega) \rangle_{\mathcal{H}} = \langle \xi_\delta(\omega), h \rangle_{\mathcal{H}}. \tag{3.6}$$

(2) Recall that $(\Phi \circ \mathcal{L})(h) = w_h$ and, by Proposition 2.13 (with $Z := Z^{\xi_\delta}(\omega)$),

$$\begin{aligned} (\Phi \circ \mathcal{L})(\varepsilon\xi_\delta(\omega) + h) &= \mathcal{R}(\mathcal{S}(T_h \delta_\varepsilon Z^{\xi_\delta}(\omega))) = \mathcal{R}(\mathcal{S}_{tr}(u_0, h, \delta_\varepsilon Z^{\xi_\delta}(\omega))) \\ &= \mathcal{R}(\mathcal{S}_{ex}^\varepsilon(u_0, h, Z^{\xi_\delta}(\omega))) = \mathcal{R}(\mathcal{J}_{E_h Z^{\xi_\delta}(\omega)}(\varepsilon)) = u_h^{(\varepsilon)}(Z^{\xi_\delta}(\omega)), \end{aligned}$$

which in turn implies that

$$D(\Phi \circ \mathcal{L})|_h(\xi_\delta(\omega)) = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} u_h^{(\varepsilon)}(Z^{\xi_\delta}(\omega)) = u_h^{(1)}(Z^{\xi_\delta}(\omega)) = u_h^{(1)}(\hat{Z}^{\xi_\delta}(\omega)).$$

where $\hat{Z}^{\xi_\delta}(\omega)$ denotes the BPHZ model associated to $\xi_\delta(\omega)$ and the last equality is due to Lemma 2.34. We combine these observations to obtain

$$\begin{aligned}
 D(F \circ \Phi \circ \mathcal{L})|_{\mathbf{h}}(\xi_\delta(\omega)) &= DF|_{(\Phi \circ \mathcal{L})(\mathbf{h})} \left(D(\Phi \circ \mathcal{L})|_{\mathbf{h}}(\xi_\delta(\omega)) \right) \\
 &= DF|_{w_{\mathbf{h}}}(u_{\mathbf{h}}^{(1)}(\hat{\mathbf{Z}}^{\xi_\delta}(\omega))).
 \end{aligned}
 \tag{3.7}$$

Altogether, (3.6) and (3.7) lead to the equality

$$DF|_{w_{\mathbf{h}}}(u_{\mathbf{h}}^{(1)}(\hat{\mathbf{Z}}^{\xi_\delta}(\omega))) + \langle \xi_\delta(\omega), \mathbf{h} \rangle_{\mathcal{H}} = D\mathcal{F}|_{\mathbf{h}}(\xi_\delta(\omega)) = 0.
 \tag{3.8}$$

We recall two more results:

(1) From [45, Rmk. 10.6] and the comments thereafter, we know that

$$\lim_{\delta \rightarrow 0} \langle \xi_\delta, \mathbf{h} \rangle_{\mathcal{H}} = \xi(\mathbf{h}) \quad \text{in } L^2(\mathbb{P}).
 \tag{3.9}$$

(2) From [45, Thm. 10.19] (or, more generally, [24, Thm. 2.33]), we know that $\hat{\mathbf{Z}}^{\xi_\delta}$ converges to $\hat{\mathbf{Z}}$ in probability in \mathcal{M} as $\delta \rightarrow 0$.

Therefore, we may pass to a common subsequence to obtain \mathbb{P} -a.s. limits in the preceding two statements. As $DF|_{w_{\mathbf{h}}}$ is continuous by assumption (H3) and $u_{\mathbf{h}}^{(1)}$ is continuous in the model by Theorem 2(i), taking $\delta \rightarrow 0$ in (3.8) establishes the claim in (3.3). \square

By the previous proposition, we now need to analyse the behaviour as $\varepsilon \rightarrow 0$ of

$$\mathcal{E}_\rho(\varepsilon) := \mathbb{E} \left[\exp \left(-\frac{1}{2} \hat{Q}_{\mathbf{h}} - \hat{S}_{\mathbf{h}}^{F;\varepsilon;N} - \frac{1}{\varepsilon^2} \hat{R}_{\mathbf{h}}^{F;\varepsilon;N+3} \right); \varepsilon \|\hat{\mathbf{Z}}^-\| < \rho \right].
 \tag{3.10}$$

3.2. Exponential integrability of the quadratic term

In a first step, we study the exponential integrability of the quadratic term $-1/2 \hat{Q}_{\mathbf{h}}$ in (3.10), as it will give us the leading order term a_0 in the expansion (0.13), cf. section 3.3.

Proposition 3.2. *There exists some $\beta > 0$ such that $\exp\left(-\frac{1}{2}\hat{Q}_{\mathbf{h}}\right) \in L^p(\mathbb{P})$ with $p := 1 + \beta$. \spadesuit*

The proof of the preceding proposition crucially relies on the non-degeneracy assumption (H4) on the minimiser \mathbf{h} . It will be combined with the following elementary lemma.

Lemma 3.3. *Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable and $\lambda \in \mathbb{R}$. Suppose there exists some $C > \lambda$ and $k, K > 0$ such that for all $r \in (k, \infty)$ we have the estimate $\mathbb{P}(X \geq r) \leq K \exp(-Cr)$. Then, there exists some $\beta > 0$ such that $\exp(\lambda X) \in L^{1+\beta}(\mathbb{P})$. \spadesuit*

Recall that $\hat{Q}_h = Q_h(\hat{Z})$. In Corollary 2.3(ii), we have established that $\varepsilon^2 Q_h(\hat{Z}) = Q_h(\delta_\varepsilon \hat{Z})$. Since Q_h is continuous in the model (by Corollary 2.3(i)) and the family $(\delta_\varepsilon \hat{Z} : \varepsilon > 0)$ satisfies a LDP on model space \mathcal{M} with good rate function (RF)

$$\mathcal{I}_{\mathcal{M}}(\mathbf{Z}) = \inf\{\mathcal{I}(h) : h \in \mathcal{H} \text{ with } \mathcal{L}(h) = \mathbf{Z}\}, \tag{3.11}$$

cf. Theorem C.1, we can apply the *contraction principle* to infer that the family $(-\frac{1}{2}Q_h(\delta_\varepsilon \hat{Z}) : \varepsilon > 0)$ satisfies a LDP on \mathbb{R} with good RF

$$\begin{aligned} \Lambda(y) &:= \inf\{\mathcal{I}_{\mathcal{M}}(\mathbf{Z}) : \mathbf{Z} \in \mathcal{M} \text{ with } -1/2 Q_h(\mathbf{Z}) = y\} \\ &= \inf\{\mathcal{I}(h) : h \in \mathcal{H} \text{ with } -1/2 Q_h(\mathcal{L}(h)) = y\}. \end{aligned}$$

Recall that \mathcal{I} denotes Schilder’s RF, that is: $\mathcal{I}(h) = \frac{1}{2}\|h\|_{\mathcal{H}}^2$ for $h \in \mathcal{H}$.

Proof of Proposition 3.2. We want to apply Lemma 3.3 with $X := -\frac{1}{2}Q_h(\hat{Z})$ and $\lambda = 1$. We set

$$C_* := \inf\{\mathcal{I}_{\mathcal{M}}(\mathbf{Z}) : \mathbf{Z} \in \mathcal{M}, -1/2 Q_h(\mathbf{Z}) \geq 1\} \tag{3.12}$$

and immediately find³²

$$C_* = \inf\{1/2 \|h\|_{\mathcal{H}}^2 : h \in \mathcal{H}_*\}, \quad \mathcal{H}_* := \{h \in \mathcal{H} : -1/2 Q_h(\mathcal{L}(h)) \geq 1\}. \tag{3.13}$$

The LDP upper bound for $(-\frac{1}{2}Q_h(\delta_\varepsilon \hat{Z}) : \varepsilon > 0)$ now implies that for each $C \in (0, C_*)$ there exists an $\varepsilon_0 = \varepsilon_0(C) > 0$ such that

$$\mathbb{P}\left(-\frac{1}{2}Q_h(\hat{Z}) \geq \varepsilon^{-2}\right) = \mathbb{P}\left(-\frac{1}{2}Q_h(\delta_\varepsilon \hat{Z}) \geq 1\right) \leq \exp(-\varepsilon^{-2}C),$$

holds for all $\varepsilon \leq \varepsilon_0$. Hence, all we need to show is $C_* > \lambda = 1$, or equivalently

$$C_* - 1 > 0. \tag{3.14}$$

If $\mathcal{H}_* = \emptyset$, then $C_* = +\infty$ and there is nothing to prove, so we assume $\mathcal{H}_* \neq \emptyset$. By a compactness argument, outsourced to Lemma 3.4 below, we find $h_* \in \mathcal{H}_*$ with $h_* \neq 0$ and $C_* = \frac{1}{2}\|h_*\|_{\mathcal{H}}^2$. Now take $v, w \in \mathcal{H}$ and observe the following:

³² We remind the reader that assumption (H4) reads $D^2\mathcal{F}|_h > 0$ for $\mathcal{F} = F \circ \Phi \circ \mathcal{L} + \mathcal{I}$. As we will see shortly, it implies $Q_h \circ \mathcal{L} > -\text{Id}$ (and *not* $Q_h \circ \mathcal{L} > 0$), so the condition $-1/2 Q_h(\mathcal{L}(h)) \geq 1$ for $h \in \mathcal{H}$ in (3.13) is indeed meaningful. In particular, it is consistent with earlier work by Ben Arous [11].

(1) From (3.6), we know that $D\mathcal{J}|_h(v) = \langle h, v \rangle_{\mathcal{H}}$. For the second derivative, we find

$$D^2\mathcal{J}|_h[v, v] = \frac{d}{dt} \Big|_{t=0} D\mathcal{J}|_{h+tv}(w) = \frac{d}{dt} \Big|_{t=0} \langle h + tv, w \rangle_{\mathcal{H}} = \langle v, w \rangle_{\mathcal{H}}.$$

In particular, $C_* = \frac{1}{2}\|h_*\|_{\mathcal{H}}^2 = \frac{1}{2}D^2\mathcal{J}|_h[h_*, h_*]$.

(2) The definitions in Corollary 2.1 immediately imply

$$\begin{aligned} \frac{1}{2}Q_h(\mathcal{L}(h_*)) &= \frac{1}{2}\partial_\varepsilon^2 \Big|_{\varepsilon=0} F(u_h^{(\varepsilon)}(\mathcal{L}(h_*))) = \frac{1}{2}\partial_\varepsilon^2 \Big|_{\varepsilon=0} F \circ \Phi(T_h\delta_\varepsilon\mathcal{L}(h_*)) \\ &= \frac{1}{2}\partial_\varepsilon^2 \Big|_{\varepsilon=0} F \circ \Phi \circ \mathcal{L}(h + \varepsilon h_*) = \frac{1}{2}D^2(F \circ \Phi \circ \mathcal{L})|_h[h_*, h_*]. \end{aligned} \tag{3.15}$$

where we additionally use Remark A.15 and Proposition A.18 in the third equality.

Altogether, we obtain (3.14) from

$$-1 + C_* \geq \frac{1}{2}D^2((F \circ \Phi \circ \mathcal{L}) + \mathcal{J})|_h[h_*, h_*] = \frac{1}{2}D^2\mathcal{F}|_h[h_*, h_*] > 0,$$

where we have used that $h_* \in \mathcal{H}_*$ in the first estimate and assumption (H4) in the last. \square

Lemma 3.4. *In the setting of the previous proof, assume $\mathcal{H}_* \neq \emptyset$. Then, $C_* < \infty$ and there exists $h_* \in \mathcal{H}_*$ with $h_* \neq 0$ and $C_* = \frac{1}{2}\|h_*\|_{\mathcal{H}}^2$. \spadesuit*

Proof. Notice that $C_* \leq \mathcal{J}_{\mathcal{M}}(\mathcal{L}(h)) < \infty$ for any $h \in \mathcal{H}_* \neq \emptyset$ and recall that $\mathcal{J}_{\mathcal{M}}$ is a good RF, so that the set $L_k := \{\mathbf{Z} \in \mathcal{M} : \mathcal{J}_{\mathcal{M}}(\mathbf{Z}) \leq k\}$ is compact in \mathcal{M} for each $k \in [0, \infty)$. Also, for any such k the set

$$A_k := L_k \cap \{\mathbf{Z} \in \mathcal{M} : -1/2 Q_h(\mathbf{Z}) \geq 1\} \subseteq \mathcal{M}$$

is compact again because $(-1/2 Q_h)^{-1}([1, \infty))$ is closed in \mathcal{M} by continuity of Q_h . Now choose k_* sufficiently large such that $A_{k_*} \neq \emptyset$. Then,

$$C_* \equiv \inf\{\mathcal{J}_{\mathcal{M}}(\mathbf{Z}) : \mathbf{Z} \in \mathcal{M}, -1/2 Q_h(\mathbf{Z}) \geq 1\} = \inf\{\mathcal{J}_{\mathcal{M}}(\mathbf{Z}) : \mathbf{Z} \in A_{k_*}\}$$

and since $\mathcal{J}_{\mathcal{M}}$ is l.s.c. as a RF, it attains its minimum on compact sets, so there exists a $\mathbf{Z}_* \in A_{k_*}$ with $C_* = \mathcal{J}_{\mathcal{M}}(\mathbf{Z}_*)$. By definition of $\mathcal{J}_{\mathcal{M}}$, we may thus find $h_* \in \mathcal{H}_*$ with $C_* = \frac{1}{2}\|h_*\|_{\mathcal{H}}^2$.

Suppose $h_* = 0$. By (3.15) and the fact that $h_* \in \mathcal{H}_*$, the latter defined in (3.13), we find that

$$-1 \geq \frac{1}{2}Q_h(\mathcal{L}(0)) = \frac{1}{2}D^2(F \circ \Phi \circ \mathcal{L})|_h[0, 0] = 0,$$

because $D^2(F \circ \Phi \circ \mathcal{L})|_{\mathcal{H}}$ is a bilinear form on \mathcal{H} . Therefore, we clearly have $h_* \neq 0$. \square

Remark 3.5. Ben Arous has also employed a large deviations argument to prove [11, Lem. 1.51], the equivalent of Proposition 3.2 in his setting. He uses [11, Lem. 1.11] to close that argument and the proof of the latter, in turn, is based on [11, Lem. 1.9, 1.23, and 1.26]. These lemmas crucially rely on the correspondence between trace-class operators, Carleman-Fredholm determinants, and exponential integrability for elements in the second Wiener chaos that we alluded to in the introduction.

This is the point where our line of reasoning diverts: We use the continuity of the Itô-Lyons solution map (resp. its counterpart in regularity structures), one of the main results of rough paths theory only developed ten years after Ben Arous’s work. More precisely, this idea entered our argument via continuity of $Q_{\mathfrak{h}}$ that we used to identify the set $(-1/2Q_{\mathfrak{h}})^{-1}([1, \infty]) \subseteq \mathcal{M}$ as closed.

The connection to Ben Arous’s line of reasoning is elaborated upon in the recent follow-up article [64] by the second author, see also Remark 0.6. \spadesuit

3.3. The coefficients in the asymptotic expansion

It remains to calculate the coefficients $(a_k)_{k=0}^N \subseteq [0, \infty)$ in the expansion (0.13). The arguments for that are completely analogous to those of Inahama and Kawabi [52, sec. 6]: Thus, we will only state the formulas for the a_k ’s and provide a sketch of the proof. A complete argument with all the details can be found in the extended arXiv version [34] of the present article.

For $N \geq 1$, $m \in [N]$, and $k \in [m]$, let $\mathcal{G}(k, m)$ be the set of all maps $\pi : \{1, \dots, k\} \rightarrow \{3, \dots, m + 2\}$. For $\pi \in \mathcal{G}(k, m)$ and $i \in \{1, \dots, k\}$, we also set

$$\pi_i := \pi(i), \quad \ell(\pi) := \sum_{i=1}^k (\pi_i - 2), \quad |\pi| := \sum_{i=1}^k \pi_i$$

and then define $\mathcal{G}_=(k, m) := \{\pi \in \mathcal{G}(k, m) : \ell(\pi) = m\}$.

Proposition 3.6 (The coefficients a_m). *We have*

$$a_m \equiv a_m(\mathfrak{h}, \hat{\mathbf{Z}}, F) := \mathbb{E} \left[\exp \left(-\frac{1}{2} \hat{Q}_{\mathfrak{h}} \right) W_m \right] < \infty \tag{3.16}$$

where $W_0 := 1$ and

$$W_m := \sum_{k=1}^N \frac{(-1)^k}{k!} \sum_{\pi \in \mathcal{G}_=(k, m)} \prod_{i=1}^k \frac{\hat{F}_{\mathfrak{h}}^{(\pi_i)}}{\pi_i!}, \quad m \in [N]. \quad \spadesuit$$

Remark 3.7. Recall that the terms $\hat{F}_h^{(i)}$ have been introduced in Corollary 2.1 above. Further, note that the terms W_m can be calculated *explicitly* by Lemma 2.4 and Proposition 2.47. \spadesuit

Proof (Sketch). We decompose $\mathcal{E}_\rho(\varepsilon)$ from (3.10) as $\mathcal{E}_\rho(\varepsilon) = \sum_{m=0}^N a_m \varepsilon^m + \sum_{j=1}^4 I_\rho^{(j)}(\varepsilon)$ in such a way that we can show that $I_\rho^{(j)}(\varepsilon) = o(\varepsilon^N)$ as $\varepsilon \rightarrow 0$ for each $j = 1, \dots, 4$.

Essentially the latter is a consequence of the localisation in Proposition 1.2. More precisely, since $\varepsilon \in I_0$ is such that $\varepsilon \|\hat{Z}^-\| < \rho$, the estimates

$$\begin{aligned} \varepsilon^m |\hat{F}_h^{(m+2)}| \mathbf{1}_{\varepsilon \|\hat{Z}^-\| < \rho} &\lesssim \varepsilon^m (1 + \|\hat{Z}^-\|)^{m+2} \mathbf{1}_{\varepsilon \|\hat{Z}^-\| < \rho} \lesssim (\varepsilon + \rho)^m (1 + \|\hat{Z}^-\|)^2 \\ \frac{1}{\varepsilon^2} |\hat{R}_h^{F;\varepsilon;N+3}| \mathbf{1}_{\varepsilon \|\hat{Z}^-\| < \rho} &\lesssim_\rho \varepsilon^{N+1} (1 + \|\hat{Z}^-\|)^{N+3} \mathbf{1}_{\varepsilon \|\hat{Z}^-\| < \rho} \lesssim (\varepsilon + \rho)^{N+1} (1 + \|\hat{Z}^-\|)^2 \end{aligned} \tag{3.17}$$

follow from Corollary 2.3. Combined with the Fernique-type Theorem D.1 and Proposition 3.2, they can be used to prove the claimed asymptotic identities.

It remains to prove that the a_m 's are finite. By Proposition 3.2, we know that $\exp(-1/2\hat{Q}_h) \in L^p(\mathbb{P})$ for some $p > 1$. We denote the conjugate Hölder exponent to p by q to infer that

$$\left\| \exp\left(-\frac{1}{2}\hat{Q}_h\right) W_m \right\|_{L^1(\mathbb{P})} \leq \left\| \exp\left(-\frac{1}{2}\hat{Q}_h\right) \right\|_{L^p(\mathbb{P})} \|W_m\|_{L^q(\mathbb{P})} < \infty,$$

Here we have used that $W_m \in L^q(\mathbb{P})$ because it belongs to a finite inhomogeneous Wiener chaos and thus has moments of all orders [61, Thm. 3.50]. \square

Appendix A. Regularity structures – background knowledge

In this section, we provide the basis on regularity structures needed in this article. We try to keep the presentation self-contained but the sheer scope of the material forces us to refer the reader to the literature where appropriate. For a short introduction on regularity structures, we refer the reader to Hairer [46] as well as Chandra and Weber [27].

A.1. The extended regularity structure

The specific regularity structure for $(\text{gPAM}_\varepsilon)$ has been constructed frequently, see Hairer [45] and Hairer and Pardoux [49]

However, as exemplified by eq. (0.8), we want to consider noises ξ shifted by Cameron-Martin functions $h \in \mathcal{H}$ during our analysis. Hence, we need to extend \mathcal{T} by introducing another noise symbol $H \equiv \mathfrak{!}$ that represents h . This procedure has already been carried out in the context of Malliavin calculus, for $(\text{gPAM}_\varepsilon)$ by Cannizzaro, Friz, and Gassiat [23], and more recently by Schönbauer [76] in a much more general framework. We adopt the setting of the former and consider the extended regularity structure

$\mathcal{F} = (\mathcal{A}, \mathcal{T}, \mathcal{G})$: the extended model space $\mathcal{T} = \mathcal{T}_{\mathcal{W}} \oplus \mathcal{T}_{\mathcal{U}}$ is given by $\mathcal{T}_{\mathcal{W}} := \langle \mathcal{W} \rangle$ and $\mathcal{T}_{\mathcal{U}} := \langle \mathcal{U} \rangle$, where

$$\begin{aligned} \mathcal{W} &:= \{\Xi, H, \mathcal{I}(\Xi)\Xi, \mathcal{I}(H)\Xi, \mathcal{I}(\Xi)H, \mathcal{I}(H)H, X_i\Xi, X_iH : i = 1, 2\} \\ &= \{\mathfrak{!}, \mathfrak{!}, \mathfrak{!}, \mathfrak{!}, \mathfrak{!}, \mathfrak{!}, X_i\mathfrak{!}, X_i\mathfrak{!} : i = 1, 2\} \\ \mathcal{U} &:= \{\mathbf{1}, \mathcal{I}(\Xi), \mathcal{I}(H), X_i : i = 1, 2\} = \{\mathfrak{!}, \mathfrak{!}, \mathfrak{!}, X_i : i = 1, 2\} \end{aligned}$$

Sometimes, we will also refer to the extended list of symbols as $\mathcal{F} := \mathcal{U} \cup \mathcal{W}$, so that $\mathcal{T} = \langle \mathcal{F} \rangle$. Further, we introduce $\mathcal{A} := \{\text{deg}(\tau) : \tau \in \mathcal{F}\}$, where for $i = 1, 2$ the symbols have homogeneities

τ	$\mathfrak{!}, \mathfrak{!}$	$\mathfrak{!}, \mathfrak{!}, \mathfrak{!}, \mathfrak{!}$	$X_i\mathfrak{!}, X_i\mathfrak{!}$	$\mathbf{1}$	$\mathfrak{!}, \mathfrak{!}$	X_i
$\text{deg}(\tau)$	$-1 - \kappa$	-2κ	$-\kappa$	0	$1 - \kappa$	1

for fixed $\kappa > 0$ with $\kappa \ll 1$. The sets $\mathcal{U}, \mathcal{W}, \mathcal{F}, \mathcal{T}$, and \mathcal{A} are defined analogously, leaving out all the symbols with an instance of $\mathfrak{!}$.

For the sake of brevity, we focus on the construction of \mathcal{G} as in [23, Sec. 3.2]. This is in analogy with [45, Sec. 8.1] where the structure group \mathcal{G} is constructed. The latter also serves as a reference for further details on our construction; for a far-reaching generalisation, see Bruned, Hairer, and Zambotti [19].

We introduce the vector space \mathcal{T}_+ built from finite linear combinations of the basis vectors

$$X^k \prod_j \mathcal{J}_{\ell_j} \tau_j, \quad j, k \in \mathbb{N}^2, \tau_j \in \mathcal{T}, \text{deg}(\tau_j) + 2 - |j| > 0,$$

on which we act by a co-module Δ , i.e. a linear map $\Delta : \mathcal{T} \rightarrow \mathcal{T} \otimes \mathcal{T}_+$ given by

$$\Delta\tau := \tau \otimes \mathbf{1} \text{ for } \tau \in \{\mathbf{1}, \mathfrak{!}, \mathfrak{!}\}, \quad \Delta X_i := X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i,$$

as well as, for $\tau, \sigma \in \mathcal{T}$ and $B_\tau := \{m \in \mathbb{N}^2 : \text{deg}(\tau) + 2 - |m| > 0\}$, recursively by

$$\Delta(\tau\sigma) := (\Delta\tau)(\Delta\sigma), \quad \Delta(\mathcal{I}\tau) := (\mathcal{I} \otimes \text{Id})\Delta\tau + \sum_{k+\ell \in B_\tau} \frac{1}{k!\ell!} X^k \otimes X^\ell \mathcal{J}_{k+\ell}(\tau)$$

In the previous expression, note that $X^k \in \mathcal{T}$ and $X^\ell \in \mathcal{T}_+$: By abuse of notation, we do not distinguish between these symbols on the left and right hand side of the tensor product.

One then defines \mathcal{G}_+ as the set of characters of \mathcal{T}_+ , that is: linear functionals $f : \mathcal{T}_+ \rightarrow \mathbb{R}$ such that $f(\tau\sigma) = f(\tau)f(\sigma)$. The structure group \mathcal{G} is then given by all $\Gamma_f, f \in \mathcal{G}_+$, where

$$\Gamma_f \tau := (\text{Id} \otimes f) \Delta \tau, \quad \tau \in \mathcal{T}. \tag{A.1}$$

A matrix representation for the structure groups \mathcal{G} and \mathcal{G} is given in [23, eq. (3.3)] and [23, eq. (3.4)], respectively.

Remark A.1. For convenience, the constructions that follow are presented for the regularity structure \mathcal{T} . Mutatis mutandis, they also hold for \mathcal{F} . \spadesuit

A.2. Admissible models

In practice, the cornerstone of Hairer’s theory are the so-called *models*: a family of analytical objects tailored to the equation at hand to mimic the role of classical Taylor polynomials.

In particular, *admissible* models relate last paragraph’s abstract integration operator \mathcal{I} to a concrete integral kernel K . We follow Hairer [45, Sec. 10.4] in choosing

$$K(x) = -\frac{1}{2\pi} \log \|x\| \tag{A.2}$$

for $x \in \mathbb{R}^2$ in a sufficiently small neighbourhood of 0. Outside of that neighbourhood, we choose K smooth, compactly supported, and such that $\int_{\mathbb{R}^2} x^k K(x) dx = 0$ for $|k| \leq r$ and some fixed degree r that is sufficiently large. The definitions that follow are in adapted to this choice (and thus our equation at hand) but they work in much greater generality.

Definition A.2 (*Smooth admissible model*). Let $\Pi : \mathbb{R}^2 \rightarrow \mathcal{L}(\mathcal{T}, \mathcal{C}^\infty(\mathbb{R}^2))$ be a smooth map that satisfies the following four conditions:

- (II1) For each $\mathfrak{K} \subseteq \mathbb{R}^2$ compact, the bound $|(\Pi_y \tau)(\bar{y})| \lesssim_{\mathfrak{K}} |\bar{y} - y|^{\deg(\tau)}$ holds uniformly over $y, \bar{y} \in \mathfrak{K}$ and $\tau \in \mathcal{F}$.
- (II2) For each $y, \bar{y} \in \mathbb{R}^2$, we have $\Pi_y \mathbf{1}(\bar{y}) \equiv 1$.
- (II3) For any $k \in \mathbb{N}^2$, $y, \bar{y} \in \mathbb{R}^2$ and $\tau \in \mathcal{F}$ with $X^k \tau \in \mathcal{T}$ we have

$$(\Pi_y [X^k \tau])(\bar{y}) = (\bar{y} - y)^k (\Pi_y \tau)(\bar{y}). \tag{A.3}$$

- (II4) For any $\tau \in \mathcal{F}$ with $\mathcal{I}\tau \in \mathcal{T}$ and all $y, \bar{y} \in \mathbb{R}^2$ we have

$$(\Pi_y \mathcal{I}\tau)(\bar{y}) = (K * \Pi_y \tau)(\bar{y}) + \sum_{|\ell| < \deg(\tau) + 2} \frac{(\bar{y} - y)^\ell}{\ell!} f_y(\mathcal{J}_\ell \tau). \tag{A.4}$$

where the map $f : \mathbb{R}^2 \rightarrow \mathcal{G}_+$, $y \mapsto f_y$, is uniquely specified from Π by imposing multiplicativity and

- (II5) $f_y(X^\ell) = (-y)^\ell$, $f_y(\mathcal{J}_\ell \tau) = -(D^\ell K * \Pi_y \tau)(y)$ for $|\ell| < \deg(\tau) + 2$.

We then define $F : \mathbb{R}^2 \rightarrow \mathcal{G}$ from f by $F_y := \Gamma_{f_y}$ according to (A.1) and impose the algebraic relation

$$(II6) \quad \Pi_y \circ F_y^{-1} = \Pi_{\bar{y}} \circ F_{\bar{y}}^{-1} \text{ for every } y, \bar{y} \in \mathbb{R}^2.$$

The pair $\mathbf{Z} := (\Pi, F)$ is then called a *smooth admissible model* for \mathcal{T} , the collection of which we denote by \mathcal{M}_∞ . \spadesuit

Remark A.3. A priori, one would have to account for time-dependence of (gPAM $_\varepsilon$) in the models. However, given that SWN ξ does not depend on time, one would immediately find that

$$\Pi_z \tau(\bar{z}), \quad z = (t, x), \quad \bar{z} = (\bar{t}, \bar{x}) \in \mathbb{R}^3$$

is independent both of t and \bar{t} , cf. [45, sec. 10.4]; this fact is reflected in our definitions.

In particular, it has informed our choice of the integral kernel K in (A.2): While initially, we need to take a suitable “cut-off” version of the heat kernel P that complies with [45, Lem. 5.5], we can actually integrate out the temporal variable and then arrive at the Green’s function K of the Laplacian. \spadesuit

Remark A.4. Notice that the map Π uniquely specifies f (and a fortiori F through (A.1) and Γ by $\Gamma_{y\bar{y}} = F_y^{-1} \circ F_{\bar{y}}$) via (II5). Hence, we will interchangeably write and speak of an admissible model

$$\Pi \longleftrightarrow \mathbf{Z} = (\Pi, f) \longleftrightarrow \mathbf{Z} = (\Pi, F) \longleftrightarrow \mathbf{Z} = (\Pi, \Gamma). \quad \spadesuit$$

In addition, we consider a set of test functions

$$\mathcal{B} \equiv \mathcal{B}_2 := \{\eta \in C^2(\mathbb{R}^2) : \|\eta\|_{C^2} \leq 1, \text{ supp } \eta \subseteq B(0, 1)\}, \tag{A.5}$$

whose rescaled versions are given by $\eta_y^\lambda(\bar{y}) = \lambda^{-2}\eta(\lambda^{-1}(\bar{y} - y))$ for $\lambda \in (0, 1]$ and $y, \bar{y} \in \mathbb{R}^2$. Since we are studying (gPAM $_\varepsilon$) with periodic boundary conditions, it is then natural to enforce periodicity:

Definition A.5 (Periodicity). Let $\mathbf{Z} = (\Pi, F) \in \mathcal{M}_\infty$. We denote by $\{\mathbf{e}_1, \mathbf{e}_2\}$ the standard basis of \mathbb{R}^2 and introduce the *translation maps*

$$T_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad T_i(y) = y + \mathbf{e}_i \quad i = 1, 2. \tag{A.6}$$

These maps T_i act on test functions ϕ by $T_i\phi := \phi \circ T_i^{-1}$. The model $\mathbf{Z} = (\Pi, F)$ is said to be *periodic* if

$$(\Pi_{T_i y} \tau)(T_i \phi) = (\Pi_y \tau)(\phi), \quad F_{T_i y} = F_y \tag{A.7}$$

holds for every $y \in \mathbb{R}^2$, every test function ϕ , every $\tau \in \mathcal{T}$ and every $i = 1, 2$. The set of periodic admissible smooth models is denoted by $\mathcal{M}_{\infty,p}$. \spadesuit

Definition A.6 (*Admissible model*). Let $\mathbf{Z} = (\Pi, F)$, $\bar{\mathbf{Z}} = (\bar{\Pi}, \bar{F}) \in \mathcal{M}_{\infty,p}$, and $\mathfrak{K} \subseteq \mathbb{R}^2$ a compact domain. We set

$$\|\Pi - \bar{\Pi}\|_{\mathfrak{K}} := \sup_{\lambda, \phi, \tau, z} \lambda^{-\deg(\tau)} |(\Pi_z \tau - \bar{\Pi}_z \tau)(\phi_z^\lambda)| \tag{A.8}$$

where the suprema are taken over $\lambda \in (0, 1]$, $\phi \in \mathcal{B}$, $\tau \in \mathcal{F}$, and $z \in \mathfrak{K}$. The set $(\mathcal{M}_{\infty, \|\cdot\|})$ then forms a pseudometric space, from which we define the (separable) space

$$\mathcal{M} := \text{cl}_{\|\cdot\|} \mathcal{M}_{\infty,p}. \tag{A.9}$$

The elements of \mathcal{M} are called (periodic) *admissible models* for \mathcal{T} . Another metric on \mathcal{M} , in the literature (e.g. [31, Sec. A.3]) by abuse of terminology referred to as the *homogeneous model norm* $\|\cdot\|$, is given by³³

$$\|\Pi - \bar{\Pi}\|_{\mathfrak{K}} := \sup_{\lambda, \phi, \tau, z} \left(\lambda^{-\deg(\tau)} |(\Pi_z \tau - \bar{\Pi}_z \tau)(\phi_z^\lambda)| \right)^{\frac{1}{|\tau|}} \tag{A.10}$$

where the supremum runs over the same set as in (A.8). In addition, $[\tau] := \#\{\mathfrak{I} \text{ in } \tau\} \vee 1$ denotes the number of noises \mathfrak{I} in the symbol τ , e.g. $[\mathfrak{V}] = 2$. Analogously to \mathcal{M} , we define the space \mathcal{M} of models for \mathcal{T} . \spadesuit

Remark A.7. Periodicity implies that we can always work on the spatial domain $\mathfrak{K} = \mathbb{T}^2$, so we will omit the subscript \mathfrak{K} in $\|\cdot\|$ and $\|\cdot\|$. \spadesuit

• Minimal admissible models.

We want to characterise a model given as minimal information as possible. For that purpose, we fix

- $\mathcal{W}_- := \{\mathfrak{I}, \mathfrak{V}\} \subseteq \mathcal{W}$ and $\mathcal{T}_- := \langle \mathcal{W}_- \rangle \subseteq \mathcal{T}$,
- $\mathcal{W}_- := \{\mathfrak{I}, \mathfrak{I}, \mathfrak{V}, \mathfrak{V}, \mathfrak{V}, \mathfrak{V}\} \subseteq \mathcal{W}$ and $\mathcal{T}_- := \langle \mathcal{W}_- \rangle \subseteq \mathcal{T}$,

i.e. the symbols of negative homogeneity without factors of X_i (in \mathcal{F} or \mathcal{F} , respectively) and the linear spaces they span.

Definition A.8 (*Minimal admissible model*). Let $\Pi^- : \mathbb{R}^2 \rightarrow \mathcal{L}(\mathcal{T}_-, \mathcal{C}^\infty(\mathbb{R}^2))$ be a smooth periodic map such that $\|\Pi^-\| < \infty$ for

³³ The reader might wonder why the metric $\|\cdot\|$ does not involve Γ . This fact has extensively been elaborated on in the literature: We refer to [23, Rmk. 3.5] for further details and references.

$$\|\Pi^-\| := \max_{\tau \in \mathcal{W}_-} \|\Pi^-\|_\tau, \quad \|\Pi^-\| := \max_{\tau \in \mathcal{W}_-} \|\Pi^-\|_\tau \tag{A.11}$$

with

$$\|\Pi^-\|_\tau := \sup_{\lambda, \phi, z} \lambda^{-\deg(\tau)} |\langle \Pi_z^- \tau, \phi_z^\lambda \rangle|, \quad \|\Pi^-\|_\tau := \sup_{\lambda, \phi, z} \left(\lambda^{-\deg(\tau)} |\langle \Pi_z^- \tau, \phi_z^\lambda \rangle| \right)^{\frac{1}{|\tau|}}, \tag{A.12}$$

where

- $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2 and
- the suprema are taken over $\lambda \in (0, 1]$, $\phi \in \mathcal{B}$, $z \in D$ as in (A.8).

In addition, we impose the constraints (II3) to (II6) from Definition A.2. The closure of the set of all such maps Π^- under $\|\cdot\|$ is denoted by \mathcal{M}_- . Its elements are called *minimal admissible models*. \spadesuit

The consistency of the preceding definition is checked in [23, Sec. 3.3] and [50, Sec. 2.4]. We emphasise that changing $\mathcal{W}_- \rightsquigarrow \mathcal{W}_-$ leads to the space \mathcal{M}_- of minimal admissible models w.r.t. \mathcal{F} .

The following lemma is straightforward but helpful.

Lemma A.9. *Convergence w.r.t. $\|\cdot\|$ and $\|\cdot\|$ in \mathcal{M}_- are equivalent.* \spadesuit

Proof. This follows immediately from the continuity of the maps $x \mapsto x^k$ and $x \mapsto x^{1/k}$, $k \in \mathbb{N}$, at the origin. \square

The introduction of minimal admissible model is justified by the following theorem, see [23, Thm. 3.9] in case of (gPAM $_\varepsilon$); the general case is considered in [50, Thm. 2.10].

Theorem A.10 (Extension theorem). *For every $\Pi^- \in \mathcal{M}_-$, there exists a unique admissible model $Z = (\Pi, F) \in \mathcal{M}$ such that $\Pi_z \tau = \Pi_z^- \tau$ for all $\tau \in \mathcal{W}_-$ and $z \in \mathbb{R}^2$. In addition, the extension map*

$$\mathcal{E} : \mathcal{M}_- \rightarrow \mathcal{M}, \quad \Pi^- \mapsto (\Pi, F) \tag{A.13}$$

is locally Lipschitz continuous on $(\mathcal{M}_-, \|\cdot\|)$ ³⁴ and bijective.

Notation A.11. Given a model $Z = (\Pi, f) \in \mathcal{M}$, by the previous theorem there always exists a $\Pi^- \in \mathcal{M}_-$ (simply given by restriction of Π to \mathcal{T}_-) such that $Z = \mathcal{E}(\Pi^-)$. We will also write $Z^- := \Pi^-$ and sometimes also write Z^- for a generic element in \mathcal{M}_- .

³⁴ Note that \mathcal{E} is *not* locally Lipschitz continuous on $(\mathcal{M}_-, \|\cdot\|)$ simply because the square root is not Lipschitz. However, \mathcal{E} is still continuous w.r.t. $\|\cdot\|$ by Lemma A.9.

With a superscript “-”-sign on an operation we signify that it acts on \mathcal{M}_- (rather than \mathcal{M}), that is: it is composed with \mathcal{E} . In this article, this convention for example concerns the solution map $\Phi^- := \Phi \circ \mathcal{E}$, the functional $F_{\Phi}^- := F \circ \Phi^-$, the explosion time $T_{\infty}^- := T_{\infty} \circ \mathcal{E}$ to be introduced in Appendix B below, etc. \spadesuit

Finally, we present how to lift smooth functions ζ to \mathcal{M}_- and then to \mathcal{M} . The example the reader should have in mind is $\zeta = \xi_{\delta}$, a realisation of smoothed SWN ξ .

Definition A.12 (Canonical lift). Let $\zeta \in C^{\infty}(\mathbb{T}^2)$. We define $\mathcal{L}_-(\zeta) := \Pi^{\zeta} \in \mathcal{M}_-$ by

$$\Pi_y^{\zeta} \mathfrak{I}(\bar{y}) := \zeta, \quad \Pi_y^{\zeta} \mathfrak{J}(\bar{y}) := \Pi_y^{\zeta} \mathfrak{J}(\bar{y}) \Pi_y^{\zeta} \mathfrak{I}(\bar{y}) \tag{A.14}$$

where $\Pi_y^{\zeta} \mathfrak{J}(\bar{y})$ is defined by (II4) and (II5) above. In addition, we impose (II2) and (II3) and then define $\mathcal{L}(\zeta) = (\Pi^{\zeta}, F^{\zeta}) := \mathcal{E}(\mathcal{L}_-(\zeta))$. \spadesuit

The fact that $\mathcal{L}(\zeta) \in \mathcal{M}_{\infty}$ is commonplace in the literature by now; we refer to Hairer [45, Prop. 8.27] for details.

A.2.1. Analytical operations for admissible models

In this section, we introduce analytical operations on \mathcal{M} . Let us emphasise that all these operations have obvious counterparts on \mathcal{M}_- : actually, given Theorem A.10, we could even *define* them on \mathcal{M}_- and simply “push them forward” onto \mathcal{M} via \mathcal{E} .

We start with a recap of *extension*³⁵ and *translation* studied by Cannizzaro, Friz, and Gassiat [23].

• Extension and translation.

On an algebraic level, we have introduced the symbol \mathfrak{I} to encode shifts of the driving noise ξ into Cameron-Martin direction $h \in \mathcal{H}$, thus extending the regularity structure \mathcal{T} to \mathcal{T} . The following proposition provides an analytical counterpart on the level of models, see [23, Prop. 3.10].

Proposition A.13 (Extension). Let $Z = (\Pi, f) \in \mathcal{M}$ and $h \in \mathcal{H}$. Then, there exists a unique admissible model $Z^{e_h} = (\Pi^{e_h}, f^{e_h}) \in \mathcal{M}$ such that for all $z \in \mathbb{R}^3$,

- (1) $\Pi^{e_h} |_{\mathcal{T}} = \Pi, f^{e_h} |_{\mathcal{T}_+} = f, \Pi_z^{e_h} \mathfrak{I} \equiv h$, and
- (2) $\Pi_z^{e_h} \tau = (\Pi_z^{e_h} \tau_1)(\Pi_z^{e_h} \tau_2)$ for all $\tau = \tau_1 \tau_2 \in \mathcal{W} \setminus \mathcal{W}$ with $\tau_1 \in \mathcal{U}$ and $\tau_2 \in \{\mathfrak{I}, \mathfrak{I}\}$.
- (3) the conditions (II3) to (II6) hold.

³⁵ This is not to be confused with the extension operator \mathcal{E} from Theorem A.10. Here, we will focus on extending models for \mathcal{T} to models for \mathcal{T} , that is: extend $Z \in \mathcal{M}$ to \mathcal{M} .

In addition, the extension map E given by

$$E : \mathcal{H} \times \mathcal{M} \rightarrow \mathcal{M}, (h, \mathbf{Z}) \mapsto E_h \mathbf{Z} := \mathbf{Z}^{e_h}, \tag{A.15}$$

is jointly locally Lipschitz continuous. \spadesuit

Eventually, we want to build a *translated* model $T_h \mathbf{Z} \in \mathcal{M}$ for the original regularity structure \mathcal{S} , so we introduce abstract linear translation operations $\mathbf{t}_1 : \mathcal{T} \rightarrow \mathcal{T}^1$ and $\mathbf{t}_1^+ : \mathcal{T}_+ \rightarrow \mathcal{T}_+^1$ by

$$\begin{aligned} \mathbf{t}_1 \mathbb{1} &:= \mathbb{1} + \mathbb{1}, & \mathbf{t}_1 X^k &:= X^k, & \mathbf{t}_1(\tau\sigma) &:= (\mathbf{t}_1\tau)(\mathbf{t}_1\sigma), & \mathbf{t}_1(\mathcal{I}\tau) &:= \mathcal{I}(\mathbf{t}_1\tau), \\ \mathbf{t}_1^+ X^k &:= X^k, & \mathbf{t}_1^+(\tau\sigma) &:= (\mathbf{t}_1^+\tau)(\mathbf{t}_1^+\sigma), & \mathbf{t}_1^+(\mathcal{J}_\ell\tau) &:= \mathcal{J}_\ell(\mathbf{t}_1\tau) \end{aligned}$$

where the very last relations hold for all $\tau \in \mathcal{T}$ such that $\mathcal{I}\tau \in \mathcal{T}$ and $\deg(\tau) + 2 - |\ell| > 0$. We then have the following result, see [23, Prop. 3.12]:

Proposition A.14 (Translation). *Let $\mathbf{Z} = (\Pi, f) \in \mathcal{M}$ and $h \in \mathcal{H}$. Then, $\mathbf{Z}^h = (\Pi^h, f^h)$ given by $\Pi^h := \Pi^{e_h} \circ \mathbf{t}_1$ and $f^h := f^{e_h} \circ \mathbf{t}_1^+$ is an element of \mathcal{M} , i.e. an admissible model for \mathcal{S} . In addition, the translation map T given by*

$$T : \mathcal{H} \times \mathcal{M} \rightarrow \mathcal{M}, (h, \mathbf{Z}) \mapsto T_h \mathbf{Z} := \mathbf{Z}^h, \tag{A.16}$$

is jointly locally Lipschitz continuous. \spadesuit

Remark A.15. From the previous proposition, we also see that $\mathcal{L}(h)$ is well-defined for $h \in \mathcal{H}$. Indeed, because $T_h \mathbf{0}$ for $\mathbf{0} := \mathcal{L}(0)$ is a model, we know that the analytical and algebraic relations for constructing $\mathcal{L}(h)$ by Definition A.12 are true. In fact, one simply has $\mathcal{L}(h) = T_h \mathbf{0}$, where the RHS may even serve as a *definition* of the LHS. In the most general setting, this relation also holds, see the first equation of p. 28 in the work of Schönbauer [76].

In addition, it is easy to see that for $h, k \in \mathcal{H}$, we have $T_h \mathcal{L}(k) = \mathcal{L}(k + h)$. \spadesuit

The following lemma will be crucial in Appendix D below when we investigate Gaussian concentration on $(\mathcal{M}_-, \|\cdot\|)$.

Lemma A.16. *Let $h \in \mathcal{H}$ and $\mathbf{Z}^- \in \mathcal{M}_-$. Then, we have the inequality*

$$\|T_h \mathbf{Z}^-\| \lesssim \|\mathbf{Z}^-\| + \|h\|_{\mathcal{H}}. \spadesuit$$

Remark A.17. Note that the inequality in the previous lemma is uniform in $h \in \mathcal{H}$. This is to be compared to Proposition A.14, where joint Lipschitz continuity w.r.t. $\|\cdot\|$ holds locally. \spadesuit

• **Dilation.**

The operator T_h from Proposition A.14 accounts for translations $\xi \rightsquigarrow \xi + h$ in Cameron-Martin directions $h \in \mathcal{H}$. Reviewing eq. (gPAM $_\varepsilon$), we also need to analytically encode the operation $\xi \rightsquigarrow \varepsilon\xi$ which goes by the name *dilation* by $\varepsilon \in I$. In analogy with translation, we first introduce some abstract linear dilation operations $\mathfrak{d}_\varepsilon : \mathcal{T} \rightarrow \mathcal{T}$ and $\mathfrak{d}_\varepsilon^+ : \mathcal{T}_+ \rightarrow \mathcal{T}_+$ by

$$\begin{aligned} \mathfrak{d}_\varepsilon \mathfrak{I} &:= \varepsilon \mathfrak{I}, & \mathfrak{d}_\varepsilon \mathfrak{J} &:= \mathfrak{J}, & \mathfrak{d}_\varepsilon X^k &:= X^k, & \mathfrak{d}_\varepsilon(\tau\sigma) &:= (\mathfrak{d}_\varepsilon\tau)(\mathfrak{d}_\varepsilon\sigma), & \mathfrak{d}_\varepsilon(\mathcal{I}\tau) &:= \mathcal{I}(\mathfrak{d}_\varepsilon\tau), \\ & & & & \mathfrak{d}_\varepsilon^+ X^k &:= X^k, & \mathfrak{d}_\varepsilon^+(\tau\sigma) &:= (\mathfrak{d}_\varepsilon^+\tau)(\mathfrak{d}_\varepsilon^+\sigma), & \mathfrak{d}_\varepsilon^+(\mathcal{J}_\ell\tau) &:= \mathcal{J}_\ell(\mathfrak{d}_\varepsilon\tau) \end{aligned} \tag{A.17}$$

where the very last relations hold for all $\tau \in \mathcal{T}$ such that $\mathcal{I}\tau \in \mathcal{T}$ and $\deg(\tau) + 2 - |\ell| > 0$.

Proposition A.18 (Dilation). *Let $\mathbf{Z} = (\Pi, f) \in \mathcal{M}$ be an admissible model. Then, $\delta_\varepsilon \mathbf{Z} = (\Pi^\varepsilon, f^\varepsilon)$ given by $\Pi^\varepsilon := \Pi \circ \mathfrak{d}_\varepsilon$ and $f^\varepsilon := f \circ \mathfrak{d}_\varepsilon^+$ is an element of \mathcal{M} . In addition, the dilation map δ given by*

$$\delta : \mathbb{R} \times \mathcal{M} \rightarrow \mathcal{M}, \quad (\varepsilon, \mathbf{Z}) \mapsto \delta_\varepsilon \mathbf{Z}, \tag{A.18}$$

is jointly continuous and $\Pi^\varepsilon \tau = \varepsilon^{[\tau]} \Pi \tau$ for any $\tau \in \mathcal{T}$. For $h \in \mathcal{H}$ and the canonical lift \mathcal{L} introduced in Definition A.12 (cf. also Remark A.15), we have $\delta_\varepsilon \mathcal{L}(h) = \mathcal{L}(\varepsilon h)$. ♠

Remark A.19. Observe that the dilation operator δ also maps \mathcal{M} resp. \mathcal{M}_- onto themselves. ♠

In order to prove that δ is well-defined and thus the preceding proposition, we need the following lemma which investigates how the structure group transforms under dilations.

Lemma A.20. *In the setting of Proposition A.18, consider the operators $\Gamma := \Gamma_f \in \mathcal{G}$ and $\Gamma^\varepsilon := \Gamma_{f^\varepsilon} \in \mathcal{G}$ built via (A.1). Then, $\mathfrak{d}_\varepsilon \circ \Gamma^\varepsilon = \Gamma \circ \mathfrak{d}_\varepsilon$. ♠*

As is easily seen, the operations of extension and dilation commute.

Lemma A.21. *Let $h \in \mathcal{H}$ and $\varepsilon > 0$. Then, $E_h \circ \delta_\varepsilon = \delta_\varepsilon \circ E_h$. ♠*

On the other hand, the operations of translation and dilation do *not* commute, but instead satisfy a relationship that is easy to state.

Lemma A.22. *For each $h \in \mathcal{H}$ and $\varepsilon > 0$, we have $T_h \circ \delta_\varepsilon = \delta_\varepsilon \circ T_{h/\varepsilon}$. ♠*

Proof. The assertion follows by $\mathfrak{d}_\varepsilon \circ \mathfrak{t}_1 = \mathfrak{t}_{1/\varepsilon} \circ \mathfrak{d}_\varepsilon$ and $\mathfrak{d}_\varepsilon^+ \circ \mathfrak{t}_1^+ = \mathfrak{t}_{1/\varepsilon}^+ \circ \mathfrak{d}_\varepsilon^+$, both of which are straightforward consequences of the definitions of \mathfrak{d}_ε , $\mathfrak{d}_\varepsilon^+$, \mathfrak{t}_1 , and \mathfrak{t}_1^+ . □

The following lemma allows us to normalise models to have norm of order one.

Lemma A.23. Let $h \in \mathcal{H}$, $\mathbf{Z} \in \mathcal{M}$, and $\lambda = \lambda(\mathbf{Z}) := 1 + \|\mathbf{Z}^-\|$. Then, the estimate

$$\|E_h \delta_{1/\lambda} \mathbf{Z}\| \lesssim 1 \tag{A.19}$$

holds with an implicit constant that depends on $\|h\|_{\mathcal{H}}$. ♠

As a corollary, this allows us to control the extended and appropriately normalised operators $\Gamma \in \mathcal{G}$.

Lemma A.24. Let $h \in \mathcal{H}$, $\mathbf{Z} = (\Pi, \Gamma) \in \mathcal{M}$, and set $\lambda := 1 + \|\mathbf{Z}^-\|$. We have the bound

$$\|\Gamma^{e_h, 1/\lambda}\| := \sup_{\tau \in \mathcal{T}} \sup_{\beta < \deg(\tau)} \sup_{z \neq \bar{z}} \frac{\|\Gamma_{z\bar{z}}^{e_h, 1/\lambda} \tau\|_{\beta}}{|z - \bar{z}|^{\deg(\tau) - \beta}} \lesssim \|h\| \quad \spadesuit$$

Proof. In the case of gPAM, this can be proved directly, see [23, Rmk. 3.5] and its proof in their appendix B. However, as pointed out in [50, Rmk. 2.4] and [49, Rmk. 3.5], it is also a direct consequence of [45, Thm. 5.14] in the general case. □

The merit of the following lemma lies in the fact that it explains why the homogeneous norm $\|\cdot\|$ from eq.’s (A.11) and (A.12) is called *homogeneous* in the first place. It also explains the very reason for working with the *minimal* rather than the full model: The latter includes the polynomials and therefore is *not homogeneous* (w.r.t. dilation) under the homogeneous norm.

Lemma A.25. For $\varepsilon \geq 0$ and $\mathbf{Z}^- \in \mathcal{M}_-$, we have $\|\delta_{\varepsilon} \mathbf{Z}^-\| = \varepsilon \|\mathbf{Z}^-\|$. ♠

Proof. The claim follows directly from the definition of $\|\cdot\|$ and the fact that $\Pi^{\varepsilon} \tau = \varepsilon^{[\tau]} \Pi \tau$, cf. Proposition A.18. □

A.2.2. Renormalisation

Renormalisation in regularity structures is encoded on the level of models. In recent years, it has been set up in great generality in the series [19,24,14] of seminal papers; for the relatively simple gPAM, however, it can be done “by hand”, as carried out in Hairer’s foundational article [45].

We just review the results we need; for details, see [45, Sec.’s 8.3 and 9.1]. We fix $c_{\delta} \in \mathbb{R}$ and suitably amend Definition A.12 of the canonical lift $\mathcal{L}(\xi_{\delta})$: For $y, \bar{y} \in \mathbb{T}^2$, we define

$$\hat{\Pi}_y^{\xi_{\delta}} \mathbf{1}(\bar{y}) := \Pi_y^{\xi_{\delta}} \mathbf{1}(\bar{y}), \quad \hat{\Pi}_y^{\xi_{\delta}} \mathfrak{I}(\bar{y}) := \Pi_y^{\xi_{\delta}} \mathfrak{I}(\bar{y}), \quad \hat{\Pi}_y^{\xi_{\delta}} \mathfrak{J}(\bar{y}) := \hat{\Pi}_y^{\xi_{\delta}} \mathfrak{J}(\bar{y}) \hat{\Pi}_y^{\xi_{\delta}} \mathbf{1}(\bar{y}) - c_{\delta}, \tag{A.20}$$

impose (II2) to (II5) above, set $\hat{\mathbf{Z}}^{\xi_{\delta}, -} := \hat{\Pi}^{\xi_{\delta}} \in \mathcal{M}_-$ and then $\hat{\mathbf{Z}}^{\xi_{\delta}} := \mathcal{E}(\hat{\mathbf{Z}}^{\xi_{\delta}, -}) \in \mathcal{M}$. We then have the following convergence result [45, Thm. 10.19]:

Proposition A.26. Let $\mathfrak{c}_\delta := \langle K_\delta, \rho_\delta \rangle$. Then, there exists a (random) model $\hat{\mathbf{Z}} \in \mathcal{M}$ independent of the specific choice of mollifier ρ such that

$$\lim_{\delta \rightarrow 0} \hat{\mathbf{Z}}^{\xi_\delta} = \hat{\mathbf{Z}} \quad \text{in prob. in } \mathcal{M}.$$

We call $\hat{\mathbf{Z}}$ the BPHZ model. \spadesuit

Remark A.27. For $\hat{\mathbf{Z}}$ to be the BPHZ model, one defines $\mathfrak{c}_\delta := \mathbb{E} [\mathbf{\Pi}^{\xi_\delta} \mathfrak{J}(0)]$. That is consistent with our choice, as the following short calculation using the covariance structure of SWN ξ shows:

$$\begin{aligned} \mathbb{E} [\mathbf{\Pi}^{\xi_\delta} \mathfrak{J}(0)] &= \mathbb{E} [\mathbf{\Pi}^{\xi_\delta} \mathfrak{J}(0) \mathbf{\Pi}^{\xi_\delta} \mathfrak{I}(0)] = \mathbb{E} [(K * \xi_\delta)(0) \xi_\delta(0)] = \mathbb{E} [\langle \xi, K_\delta(-\cdot) \rangle \langle \xi, \rho_\delta(-\cdot) \rangle] \\ &= \mathbb{E} [\langle \xi, K_\delta(-\cdot) \rangle \langle \xi, \rho_\delta(-\cdot) \rangle] = \int K_\delta(-x) \rho_\delta(-x) \, dx \tag{A.21} \\ &= \int K_\delta(x) \rho_\delta(x) \, dx = \langle K_\delta, \rho_\delta \rangle. \end{aligned}$$

Furthermore, the statement of the previous proposition is also true on minimal model space: We denote the limiting model by $\hat{\mathbf{Z}}^- \in \mathcal{M}_-$ and call it the *minimal BPHZ model*. \spadesuit

We recall another result from Cannizzaro, Friz, and Gassiat, namely [23, Lem. 3.20], in slightly updated terminology.

Lemma A.28. Let $\hat{\mathbf{Z}} \in \mathcal{M}$ be the BPHZ model. There exists a null set N such that, for every $h \in \mathcal{H}$ and $\omega \in N^c$, we have

$$T_h [\hat{\mathbf{Z}}(\omega)] = \hat{\mathbf{Z}}(\omega + h). \tag{A.22}$$

\spadesuit

The following lemma is a consequence of [23, Lem. 3.20] and Lemma A.22:

Lemma A.29. There exists a nullset N such that, for all $\omega \in N^c$, $h \in \mathcal{H}$, and $\varepsilon > 0$, we have

$$(T_h \circ \delta_\varepsilon) [\hat{\mathbf{Z}}(\omega)] = \delta_\varepsilon [\hat{\mathbf{Z}}(\omega + h/\varepsilon)]. \quad \spadesuit$$

Proof. Let N be the null set from Lemma A.28. Now consider eq. (A.22), replace $h \rightsquigarrow h/\varepsilon$, and apply δ_ε on both sides. We then conclude by Lemma A.22. \square

Trivially, both of the previous lemmas also hold for $\hat{\mathbf{Z}}^-$.

A.3. Modelled distributions

Spaces of modelled distributions play a key role in Hairer’s theory: In case of the polynomial regularity structure, they are simply Hölder continuous functions and for the rough paths regularity structure, they coincide with controlled paths, see [33, sec. 13.3].

Definition A.30. For $s \in [0, \infty)$, we define the *hyperplane at time s* by

$$P_s := \{z = (t, x) \in \mathbb{R}^3 : t = s\}.$$

We also set $P := P_0$. \spadesuit

Definition A.31 (*Modelled distribution*). Let $\mathbf{Z} = (\Pi, \Gamma) \in \mathcal{M}$ be a model for the regularity structure \mathcal{T} . Fix $\gamma > 0$, $\eta \in \mathbb{R}$, and $s \in [0, \infty)$.

(i) We say that

$$U : (0, \infty) \times \mathbb{R}^2 \rightarrow \mathcal{T}_{<\gamma} := \bigoplus_{\beta < \gamma} \mathcal{T}_\beta$$

belongs to $\mathcal{D}_s^{\gamma, \eta}(\Gamma)$ if, for every compact set $D \subseteq (0, \infty) \times \mathbb{R}^2$, the quantity

$$\begin{aligned} \|U\|_{\gamma, \eta; \Gamma; D; P_s} &= \sup_{z \in D \setminus P_s} \sup_{\beta < \gamma} |t - s|^{\frac{\beta - \eta}{2} \vee 0} \|U(z)\|_\beta \\ &+ \sup_{\substack{z, \bar{z} \in D \setminus P_s, \\ |z - \bar{z}| \leq 1}} \sup_{\beta < \gamma} (|t - s| \wedge |\bar{t} - s|)^{\frac{\beta - \eta}{2} \vee 0} \frac{\|U(z) - \Gamma_{z\bar{z}}U(\bar{z})\|_\beta}{|z - \bar{z}|^{\gamma - \beta}} \end{aligned} \tag{A.23}$$

is finite, where $z = (t, x)$ and $\bar{z} = (\bar{t}, \bar{x})$. We denote the first summand on the RHS of (A.23) by $\|U\|_{\gamma, \eta; D; P_s}$.

(ii) Let $\bar{\mathbf{Z}} = (\bar{\Pi}, \bar{\Gamma}) \in \mathcal{M}$ be another model. For $U \in \mathcal{D}_s^{\gamma, \eta}(\Gamma)$ and $\bar{U} \in \mathcal{D}_s^{\gamma, \eta}(\bar{\Gamma})$, we write $\|U; \bar{U}\|_{\gamma, \eta; D; P_s}$ for the quantity in (A.23) where we change $U(z) \rightsquigarrow U(z) - \bar{U}(z)$ in the first summand and

$$U(z) - \Gamma_{z\bar{z}}U(\bar{z}) \rightsquigarrow U(z) - \bar{U}(z) - \Gamma_{z\bar{z}}U(\bar{z}) + \bar{\Gamma}_{z\bar{z}}\bar{U}(\bar{z})$$

in the second. \spadesuit

Remark A.32. In practice, when the model \mathbf{Z} is clear from the context, we refrain from indicating Γ in the norm in (A.23). In case $s = 0$, we also omit the subscript P_s . In addition, recall that we are only considering *periodic* models, cf. Definition A.5, complying with our assumption of periodic boundary conditions. Hence, the domain D can always be taken as $D = I \times \mathbb{T}^2$ for some interval $I \subseteq \mathbb{R}$. We will use the following notation:

$$\|\cdot\|_{\gamma,\eta,(s,T]} \quad \text{when } D := (s, T] \times \mathbb{T}^2, \quad \|\cdot\|_{\gamma,\eta,T} \quad \text{when } D = D_T := (0, T] \times \mathbb{T}^2.$$

Finally, observe that the notation $\|U; \bar{U}\|$ in (ii) above indicates that it is *not* a function of $U - \bar{U}$. In fact, U and \bar{U} do not even live in the same space, a consequence of the “fibred” structure of $\mathcal{M} \times \mathcal{D}^{\gamma,\eta}$, cf. Fig. 1 on p. 26. For any fixed domain D , the space $(\mathcal{D}_{P_s}^{\gamma,\eta}(\Gamma), \|\cdot\|_{\gamma,\eta;D;P_s})$ is a bona fide Banach space. \spadesuit

• **Fréchet differentiability and Taylor’s formula for modelled distributions**

We can always write a modelled distribution $Y \in \mathcal{D}_{\mathcal{U}}^{\gamma,\eta}(\mathcal{Z})$ that takes values in the sector $\mathcal{U} = \langle \mathbf{1}, \mathfrak{J}, \mathfrak{J}, X \rangle$ as

$$Y(z) = \phi_{\mathbf{1}}(z)\mathbf{1} + \phi_{\mathfrak{J}}(z)\mathfrak{J} + \phi_{\mathfrak{J}}(z)\mathfrak{J} + \langle \phi_X(z), X \rangle \tag{A.24}$$

for some coefficient functions $\phi_{\mathbf{1}}, \phi_{\mathfrak{J}}, \phi_{\mathfrak{J}}$, and ϕ_X . Further layed out in [49, sec. 3.4], the function g then induces a function G that acts on such modelled distributions by

$$G(Y)(z) := g(\phi_{\mathbf{1}}(z))\mathbf{1} + g'(\phi_{\mathbf{1}}(z))\phi_{\mathfrak{J}}(z)\mathfrak{J} + g'(\phi_{\mathbf{1}}(z))\phi_{\mathfrak{J}}(z)\mathfrak{J} + g'(\phi_{\mathbf{1}}(z))\langle \phi_X(z), X \rangle. \tag{A.25}$$

Remark A.33 (*Projection onto $\mathcal{T}_{<\gamma}$*). In following Hairer [45, sec. 4.2], the function G should actually be denoted G_{γ} to keep track of the (here implicit) projection onto $\mathcal{T}_{<\gamma}$. In the whole article, we will keep $\gamma = 1 + 2\kappa$ for some $0 < \kappa \ll 1$ fixed and thus suppress the subindex. In the same spirit, we will not explicitly indicate said projection when multiplying two elements in $\mathcal{D}_{\mathcal{U}}^{\gamma,\eta}(\mathcal{Z})$ and when expanding Pu_0 into its Taylor jet

$$\mathcal{T}Pu_0 := \sum_{|k|<\gamma} \frac{X^k}{k!} (D^k(Pu_0)) \in \mathcal{D}_{\mathcal{U}}^{\gamma,\eta}$$

of order γ . \spadesuit

In a next step, we will quantify the Fréchet differentiability that G inherits from g and, as a natural byproduct, obtain Taylor’s formula for G .

Lemma A.34. Fix $\mathcal{Z} \in \mathcal{M}$ and consider two modelled distributions $Y, \tilde{Y} \in \mathcal{D}_{\mathcal{U}}^{\gamma,\eta}(\mathcal{Z})$ with coefficient functions ϕ_{τ} and $\tilde{\phi}_{\tau}$, $\tau \in \{\mathbf{1}, \mathfrak{J}, \mathfrak{J}, X\}$, respectively. Given a function $f \in \mathcal{C}^4(\mathbb{R}; \mathbb{R})$ with F defined as in (A.25), we then have the identity

$$F(Y + \tilde{Y}) - F(Y) - F'(Y)\tilde{Y} = \int_0^1 (1-s)F''(Y + s\tilde{Y})\tilde{Y}^2 ds. \tag{A.26}$$

For $f \in \mathcal{C}^3(\mathbb{R}, \mathbb{R})$, we have

$$F(Y + \tilde{Y}) - F(Y) = \int_0^1 F'(Y + s\tilde{Y})\tilde{Y} ds. \tag{A.27}$$

♠

This lemma is true in greater generality than stated here, see [76, Prop. B.2] and its proof. It is instrumental in the proof of the following proposition.

Proposition A.35. Fix $Z \in \mathcal{M}$, let $\mathcal{D} := \mathcal{D}_{\mathcal{U}}^{\gamma, \eta}(Z)$, and suppose $g \in \mathcal{C}^{\ell+4}$, $\ell \geq 1$. Then, $G \in \mathcal{C}^{\ell}(\mathcal{D}, \mathcal{D})$ (in the Fréchet sense) and

$$D^{(k)}G(Y) [(Y_m)_{m=1}^k] := D^{(k)}G(Y)[Y_1, \dots, Y_k] = G^{(k)}(Y) \prod_{m=1}^k Y_m, \quad k \leq \ell, \tag{A.28}$$

where $G^{(k)}$ is built from $g^{(k)}$ by means of (A.25). ♠

Remark A.36 (Regularity assumptions on g). Let us comment on the order of differentiability imposed on g . First of all, recall that $\gamma = 1 + 2\kappa$ and that \mathcal{U} is a function-like sector, the lowest non-zero homogeneity of which is

$$\chi := \deg(\mathfrak{Y}) = \deg(\mathfrak{Z}) = 1 - \kappa.$$

As detailed in [45, Prop. 6.13], for $G^{(\ell)}$ to be well-defined we actually only need $g^{(\ell)} \in \mathcal{C}^n$ with $n \in \mathbb{N}$ and $n \geq \gamma/\chi \vee 1$, i.e. $n \geq 2$ and $g \in \mathcal{C}^{\ell+2}$. However, in our proof we will make use of Lemma A.34 with $F' := G^{(\ell)}$, so that $G^{(\ell+1)}$ appears on the RHS of (A.26) and, by the same token, necessitates $g \in \mathcal{C}^{(\ell+3)}$. However, we will also want to use (strong) local Lipschitz continuity of $G^{(\ell+1)}$, so [45, Prop. 6.13] and [49, Prop. 3.11] imply the need for an additional “+1” in regularity, resulting in $g \in \mathcal{C}^{(\ell+4)}$.

In the setting of Theorem 1, we have $\ell = N+3$ and so the above reasoning requires $g \in \mathcal{C}^{(N+7)}$. ♠

The following proof borrows from that of [23, Prop. 4.7].

Proof. Suppose we had already established that $G \in \mathcal{C}^k$ for some $k \in \{1, \dots, \ell - 1\}$ with $D^{(m)}G$, $m = 1, \dots, k$, given by (A.28). We want to do the induction step $k \mapsto k + 1$ and prove that $G \in \mathcal{C}^{k+1}$, so let $Y \in \mathcal{D}$ and consider

$$R^{(k)} [Y; (Y_i)_{i=1}^{k+1}] := D^{(k)}G(Y+Y_{k+1}) [(Y_i)_{i=1}^k] - D^{(k)}G(Y) [(Y_i)_{i=1}^k] - G^{(k+1)}(Y) \prod_{m=1}^{k+1} Y_m. \tag{A.29}$$

We want to prove that

$$\sup_{\substack{\|Y_m\|_{\gamma,\eta} \leq 1, \\ 1 \leq m \leq k}} \|R^{(k)} [Y; (Y_i)_{i=1}^{k+1}] \|_{\gamma,\eta} = o(\|Y_{k+1}\|_{\gamma,\eta}), \quad Y_{k+1} \rightarrow 0. \tag{A.30}$$

Setting $P_k := \prod_{n=1}^k Y_n$, an element of \mathcal{D} by [45, Prop. 6.12], we easily see that

$$R^{(k)} [Y; (Y_i)_{i=1}^{k+1}] = (G^{(k)}(Y + Y_{k+1}) - G^{(k)}(Y) - G^{(k+1)}(Y)Y_{k+1}) P_k. \tag{A.31}$$

Since $k \leq \ell - 1$, we have $g^{(k)} \in \mathcal{C}^m$ with $m \geq 5$, so we may invoke Lemma A.34, more specifically (A.26), to obtain

$$R^{(k)} [Y; (Y_i)_{i=1}^{k+1}] = \int_0^1 (1-s) G^{(k+2)}(Y + sY_{k+1}) Y_{k+1}^2 ds P_k =: I_{k+2}(Y, Y_{k+1}) P_k$$

From [45, Prop. 6.13], we know that $G^{(k+2)}(Y + sY_{k+1}) \in \mathcal{D}$ for each $s \in [0, 1]$; [45, Prop. 6.12] then implies that $G^{(k+2)}(Y + sY_{k+1}) [Y_{k+1}]^2$ is also in \mathcal{D} . Hence, so is the Bochner integral $I_{k+2}(Y, Y_{k+1})$ and, by the same token as before, the product $I_{k+2}(Y, Y_{k+1}) P_k$. The quoted propositions then give

$$\begin{aligned} \|R^{(k)} [Y; (Y_i)_{i=1}^{k+1}] \|_{\gamma,\eta} &\lesssim \sup_{s \in [0,1]} \|G^{(k+2)}(Y + sY_{k+1})\|_{\gamma,\eta} \|Y_{k+1}\|_{\gamma,\eta}^2 \prod_{n=1}^k \|Y_n\|_{\gamma,\eta} \\ &\lesssim (\|Y\|_{\gamma,\eta} + \|Y_{k+1}\|_{\gamma,\eta}) \|Y_{k+1}\|_{\gamma,\eta}^2 \prod_{n=1}^k \|Y_n\|_{\gamma,\eta}, \end{aligned} \tag{A.32}$$

using, in particular, that $G^{(k+2)}$ is locally Lipschitz continuous. The previous equation implies that

$$\sup_{\substack{\|Y_m\|_{\gamma,\eta} \leq 1, \\ 1 \leq m \leq k}} \|R^{(k)} [Y; (Y_i)_{i=1}^{k+1}] \|_{\gamma,\eta} = O(\|Y_{k+1}\|_{\gamma,\eta}^2), \quad Y_{k+1} \rightarrow 0, \tag{A.33}$$

and so (A.30) follows. Since the previous argument also works for $k = 0$, where

$$R^{(0)} [Y; Y_1] = G(Y + Y_1) - G(Y) - G'(Y)Y_1,$$

the induction argument can be closed. \square

As a corollary of the last proposition, we may lift Taylor’s theorem from g to G .

Corollary A.37. *In the setting of Proposition A.35, the identity*

$$G(Y + \tilde{Y}) + G(Y) + \sum_{m=1}^{\ell-1} \frac{1}{m!} G^{(m)}(Y) \tilde{Y}^m + R_\ell(Y, \tilde{Y}), \quad Y, \tilde{Y} \in \mathcal{D},$$

holds true with

$$R_\ell(Y, \tilde{Y}) = \int_0^1 \frac{(1-s)^{\ell-1}}{(\ell-1)!} G^{(\ell)}(Y + s\tilde{Y}) \tilde{Y}^\ell ds. \quad \spadesuit$$

• **Dilation of modelled distributions**

The following lemma clarifies the relation between modelled distributions based on an admissible model and its dilated counterpart. It is the analogue of [23, Lem. 3.22] and proved similarly.

Lemma A.38. *Let $Z \in \mathcal{M}$ be an admissible model for a regularity structure \mathcal{T} , $\varepsilon \in I = [0, 1]$, and $\delta_\varepsilon Z = (\Pi^\varepsilon, \Gamma^\varepsilon)$. For each $\gamma > 0$, $\eta \in [0, \gamma]$, and $U^\varepsilon \in \mathcal{D}^{\gamma, \eta}(\Gamma^\varepsilon)$, we have:*

- (i) $\mathfrak{d}_\varepsilon U^\varepsilon \in \mathcal{D}^{\gamma, \eta}(\Gamma)$, i.e. $\mathfrak{d}_\varepsilon (\mathcal{D}^{\gamma, \eta}(\Gamma^\varepsilon)) \subseteq \mathcal{D}^{\gamma, \eta}(\Gamma)$. In addition, \mathfrak{d}_ε commutes with the operations of composition with smooth functions and product between modelled distributions.
- (ii) $\mathcal{R}^{\delta_\varepsilon Z} U^\varepsilon = \mathcal{R}^Z \mathfrak{d}_\varepsilon U^\varepsilon$.
- (iii) $\mathfrak{d}_\varepsilon (\mathcal{P}^{\delta_\varepsilon Z} U^\varepsilon) = \mathcal{P}^Z \mathfrak{d}_\varepsilon U^\varepsilon$. \spadesuit

Remark A.39. We emphasise that the preceding lemma is valid for *any* regularity structure, in particular for those introduced in section A.1 above. \spadesuit

Appendix B. Deterministic gPAM and explosion times

In this section, we summarise facts about explosion times of the (stochastic) PDEs under investigation.

We start with gPAM driven by a Cameron-Martin function. The following proposition, in essence, is a version of [22, Prop. A.1], with some appropriate changes to cover our case of interest. Recall that \mathcal{X}_T is the space of functions $u \in \mathcal{C}([0, T]; \mathcal{C}^\eta(\mathbb{T}^2))$ with $u(0, \cdot) = u_0$ and $\eta \in (1/2, 1)$.

Proposition B.1. *Let $T > 0$. Given $g \in \mathcal{C}_b^1(\mathbb{R})$, $u_0 \in \mathcal{C}^\eta(\mathbb{T}^2)$ for $\eta \in (0, 1)$, and $h \in \mathcal{H} \equiv L^2(\mathbb{T}^2)$, there exists a unique global solution $w_h \in \mathcal{X}_T$ to the equation*

$$(\partial_t - \Delta)w_h = g(w_h)h, \quad w_h(0, \cdot) = u_0. \quad \spadesuit$$

• **Explosion times.**

Next, recall that the explosion time T_∞ of the solution to eq. (2.9) – that is: the smallest time for which the equation

$$U = \mathcal{P}^Z (G(U) \mathbf{1}) + \mathcal{T} P u_0 \quad \text{in} \quad \mathcal{D}_U^{\gamma, \eta}(Z), \tag{B.1}$$

does *not* have a solution (cf. [45, Coro. 9.3]) – is lower semicontinuous (abbr. “l.s.c.”) as a function of $\mathbf{Z} \in \mathcal{M}$ (u_0 is fixed), see [23, Prop. 3.23]. Since the extension operator $\mathcal{E} : \mathcal{M}_- \rightarrow \mathcal{M}$ is continuous (cf. Theorem A.10), the map

$$T_\infty^- := T_\infty(u_0, \cdot) \circ \mathcal{E} : \mathcal{M}_- \rightarrow [0, +\infty]$$

is l.s.c. both on $(\mathcal{M}_-, \|\cdot\|)$ and $(\mathcal{M}_-, \|\cdot\|)$. We introduce

- the explosion time T_0 of the *deterministic* PDE when the noise is set to zero ($\varepsilon = 0$). More precisely, we set

$$T_0 := T_\infty^-(\mathbf{0}) \tag{B.2}$$

where $\mathbf{0} \in \mathcal{M}_-$ is the lift of the function that is constant 0.

- For $\varepsilon \in [0, 1]$, let $T^\varepsilon := T_\infty^-(\delta_\varepsilon \hat{\mathbf{Z}}^-)$. In other words, T^ε is the explosion time of \hat{u}^ε from Theorem 0.1 in the introduction.
- For $h \in \mathcal{H}$, let $T_\infty^h := T_\infty^-(\mathcal{L}_-(h))$ denote the *deterministic* explosion time of the PDE driven by the non-random noise h .

Remark B.2. In the specific case at hand, T_0 is the explosion time of the homogeneous heat equation with initial condition $u_0 \in \mathcal{C}^\eta(\mathbb{T}^2)$. Hence, $T_0 = +\infty$ by linearity of the equation. Furthermore, for gPAM we have $T_\infty^h = +\infty$ for any $h \in \mathcal{H}$ as Proposition B.1 above shows. ♠

The following two lemmas are general statements for stochastic PDEs amenable to analysis via regularity structures and not specific to gPAM.

Lemma B.3. *Let $T < T_0$. Then $\mathbb{P}(T^\varepsilon > T) \rightarrow 1$ as $\varepsilon \rightarrow 0$.* ♠

Loosely speaking, the next lemma states that if one controls the noise (in the form of the model $\delta_\varepsilon \hat{\mathbf{Z}}^-$), then one can ensure that the stochastic PDE driven by $h + \varepsilon \xi$ lives as long as that driven by h . Its proof is informed by [23, Cor. 3.27].

Lemma B.4. *Fix $h \in \mathcal{H}$ and let $T < T_\infty^h$. Then, there exists $\rho = \rho(T) > 0$ such that, for all $\varepsilon \in [0, 1]$ and $\mathbf{Z}^- \in \mathcal{M}_-$ with $\varepsilon \|\mathbf{Z}^-\| < \rho$, we have $T_\infty^-(T_h \delta_\varepsilon \mathbf{Z}^-) > T$.* ♠

Appendix C. Large deviations

In this section, we will give a precise formulation of Theorem 0.2 that characterises the large deviation behaviour of the solutions $(\hat{u}^{(\varepsilon)} : \varepsilon \in I)$ from Theorem 0.1.

Basically, this amounts to proving applicability of [50, thm. 3.5] in the setting of the abstract Wiener space (B, \mathcal{H}, μ) introduced in the prequel to Proposition 1.2. Therefore we only state the three LDP’s that we will need. For their proofs, the interested reader is

referred to the extended arXiv version [34, App. C] of the article at hand. It also contains a complete specification of the setup for (gPAM $_{\varepsilon}$) and a proof that the afore-mentioned result of Hairer and Weber indeed applies in this case.

The first LDP concerns the space of minimal models \mathcal{M}_- and is, in fact, the one we are working with. Recall that $\mathcal{I}(h) = \frac{1}{2}\|h\|_{\mathcal{H}}^2$ for $\mathcal{H} = L^2(\mathbb{T}^2)$ denotes Schilder’s RF.

Theorem C.1. *The family $(\delta_{\varepsilon}\hat{\mathcal{Z}}^- : \varepsilon \in I)$ of dilated, minimal BPHZ models satisfies a large deviation principle on \mathcal{M}_- (both equipped with $\|\cdot\|$ and $\|\cdot\|$) under μ with rate ε^2 and good rate function*

$$\mathcal{I}_{\mathcal{M}_-} : \mathcal{M}_- \rightarrow [0, +\infty], \quad \mathcal{I}_{\mathcal{M}_-}(\mathbf{Z}^-) = \inf\{\mathcal{I}(h) : h \in \mathcal{H}, \mathcal{L}_-(h) = \mathbf{Z}^-\}. \quad (\text{C.1})$$

Its proof is virtually the same as the one of [50, Thm. 4.3]. This LDP can then be “pushed forward” via the generalised contraction principle [50, Lem. 3.3], leading to the following corollary.

Corollary C.1. *The family $(\delta_{\varepsilon}\hat{\mathcal{Z}} : \varepsilon \in I)$ of dilated BPHZ models satisfies a large deviation principle on \mathcal{M} under μ with rate ε^2 and good rate function*

$$\mathcal{I}_{\mathcal{M}} : \mathcal{M} \rightarrow [0, +\infty], \quad \mathcal{I}_{\mathcal{M}}(\mathbf{Z}) = \inf\{\mathcal{I}(h) : h \in \mathcal{H}, \mathcal{L}(h) = \mathbf{Z}\}. \quad (\text{C.2})$$

Finally, we can give a precise statement of Theorem 0.2 announced in the introduction. Its proof entirely mirrors that of [50, Thm. 4.4]. Recall that $\mathcal{X}_T := \{u \in \mathcal{C}([0, T], \mathcal{C}^{\eta}(\mathbb{T}^2)) : u(0, \cdot) = u_0\}$, where $u_0 \in \mathcal{C}^{\eta}(\mathbb{T}^2)$ is the fixed initial condition, and that “ u^{\clubsuit} ” denotes a “graveyard path” which we postulate to have distance 1 from every element $u \in \mathcal{X}_T$.

Theorem C.2. *Let $T > 0$, $\eta \in (1/2, 1)$, and $\bar{\mathcal{X}}_T := \mathcal{X}_T \cup \{u^{\clubsuit}\}$. Then, the family $(\hat{u}^{\varepsilon} : \varepsilon \in I)$ in Theorem 0.1 satisfies a LDP in $\bar{\mathcal{X}}_T$ with RF*

$$\mathcal{I}(u) = \inf\{\mathcal{I}(h) : h \in \mathcal{H}, \Phi(\mathcal{L}(h)) = u\}, \quad \mathcal{I}(u^{\clubsuit}) := \infty$$

Here, we have set $\hat{u}^{\varepsilon} := u^{\clubsuit}$ in case $T > T^{\varepsilon}$, the latter introduced in Appendix B above. ♠

Appendix D. Fernique’s theorem on model space

Let (B, \mathcal{H}, μ) be the abstract Wiener space from Appendix C which is also the setting for the generalised Fernique theorem proved by Friz and Oberhauser [35].

By $\hat{\mathcal{Z}} \in \mathcal{M}$, we denote the BPHZ model and let $\hat{\mathcal{Z}}^- \in \mathcal{M}_-$ be its minimal part, that is $\hat{\mathcal{Z}} = \mathcal{L}(\hat{\mathcal{Z}}^-)$. The following theorem establishes Gaussian concentration.

Theorem D.1 (Fernique). *There exists an $\chi > 0$ such that*

$$\mathbb{E}_\mu [\exp(\chi \|\hat{\mathbf{Z}}^-\|^2)] \equiv \int_B \exp(\chi \|\hat{\mathbf{Z}}^-(\omega)\|^2) \mu(d\omega) < \infty. \quad (\text{D.1})$$

♠

Proof. Let $h \in \mathcal{H}$ and $N \subseteq B$ be the nullset from Lemma A.28. For each $\omega \in N^c$, we have

$$\|\hat{\mathbf{Z}}^-(\omega)\| = \|T_h [T_{-h}(\hat{\mathbf{Z}}^-(\omega))]\| \lesssim \|T_{-h}(\hat{\mathbf{Z}}^-(\omega))\| + \|h\|_{\mathcal{H}} = \|\hat{\mathbf{Z}}^-(\omega - h)\| + \|h\|_{\mathcal{H}},$$

where the inequality is due to Lemma A.16 and the equalities due to Lemma A.28, which trivially also holds for the minimal admissible BPHZ model $\hat{\mathbf{Z}}_-$. The claim now follows from the generalised Fernique theorem [35, Thm. 2]. \square

References

- [1] Sergio Albeverio, Luca Di Persio, Elisa Mastrogioacomo, Small noise asymptotic expansions for stochastic PDE's, I. The case of a dissipative polynomially bounded nonlinearity, *Tohoku Math. J.* 63 (4) (2011) 877–898.
- [2] Shigeki Aida, Semiclassical limit of the lowest eigenvalue of a Schrödinger operator on a Wiener space, *J. Funct. Anal.* 203 (2) (2003) 401–424.
- [3] Shigeki Aida, Semi-classical limit of the bottom of spectrum of a Schrödinger operator on a path space over a compact Riemannian manifold, *J. Funct. Anal.* 251 (1) (2007) 59–121.
- [4] Shigeki Aida, Semi-classical limit of the lowest eigenvalue of a Schrödinger operator on a wiener space: I. Unbounded one particle hamiltonians, in: Dai Xianzhe, Léandre Rémi, Xiaonan Ma, Zhang Weiping (Eds.), *From Probability to Geometry (I) - Volume in Honor of the 60th Birthday of Jean-Michel Bismut*, in: *Astérisque*, vol. 327, Société Mathématique de France, 2009, pp. 1–16.
- [5] Shigeki Aida, Semi-classical limit of the lowest eigenvalue of a Schrödinger operator on a Wiener space: II. $P(\phi)_2$ -model on a finite volume, *J. Funct. Anal.* 256 (10) (2009) 3342–3367.
- [6] Shigeki Aida, Tunneling for spatially cut-off $P(\phi)_2$ -Hamiltonians, *J. Funct. Anal.* 263 (9) (2012) 2689–2753.
- [7] Sergio Albeverio, Elisa Mastrogioacomo, Boubaker Smii, Small noise asymptotic expansions for stochastic PDE's driven by dissipative nonlinearity and Lévy noise, *Stoch. Process. Appl.* 123 (6) (2013) 2084–2109.
- [8] Robert Azencott, Formule de Taylor stochastique et développement asymptotique d'intégrales de Feynman, *Sémin. Probab. Strasbourg S16* (1982) 237–285.
- [9] Robert Azencott, Petites perturbations aleatoires des systemes dynamiques: developpements asymptotiques, *Bull. Sci. Math.* 109 (3) (1985) 253–308.
- [10] Gérard Ben Arous, Développement asymptotique du noyau de la chaleur hypoelliptique hors du cut-locus, *Ann. Sci. Éc. Norm. Supér.* (4) 21 (3) (1988) 307–331.
- [11] Gérard Ben Arous, Methods de Laplace et de la phase stationnaire sur l'espace de Wiener, *Stochastics* 25 (3) (1988) 125–153.
- [12] Ismaël Bailleul, Regularity of the Itô-Lyons map, *Confluentes Math.* 7 (1) (2015) 3–11.
- [13] Nikolay Barashkov, A variational approach to Gibbs measures on function spaces, *Doctoral thesis*, Rheinische Friedrich-Wilhelms-Universität Bonn, 2022.
- [14] Yvain Bruned, Ajay Chandra, Ilya Chevyrev, Martin Hairer, Renormalising SPDEs in regularity structures, *J. Eur. Math. Soc.* 23 (3) (2020) 869–947.
- [15] Christian Bayer, Peter K. Friz, Paul Gassiat, Jörg Martin, Benjamin Stemper, A regularity structure for rough volatility, *Math. Finance* (2019) 1–51.
- [16] N. Barashkov, M. Gubinelli, A variational method for Φ_3^4 , *Duke Math. J.* 169 (17) (2020) 3339–3415.
- [17] Fabrice Baudoin, Martin Hairer, A version of Hörmander's theorem for the fractional Brownian motion, *Probab. Theory Relat. Fields* 139 (3–4) (2007) 373–395.

- [18] I. Bailleul, M. Hoshino, A tourist's guide to regularity structures and singular stochastic PDEs, preprint, arXiv:2006.03524v2, 2021.
- [19] Y. Bruned, M. Hairer, L. Zambotti, Algebraic renormalisation of regularity structures, *Invent. Math.* 215 (3) (Mar 2019) 1039–1156.
- [20] Fabrice Baudoin, Cheng Ouyang, Small-time kernel expansion for solutions of stochastic differential equations driven by fractional Brownian motions, *Stoch. Process. Appl.* 121 (4) (2011) 759–792.
- [21] Thomas Cass, Peter Friz, Densities for rough differential equations under Hörmander's condition, *Ann. Math. (2)* 171 (3) (2010) 2115–2141.
- [22] K. Chouk, P.K. Friz, Support theorem for a singular SPDE: The case of gPAM, *Ann. Inst. Henri Poincaré Probab. Stat.* 54 (1) (2018) 202–219.
- [23] G. Cannizzaro, P.K. Friz, P. Gassiat, Malliavin calculus for regularity structures: the case of gPAM, *J. Funct. Anal.* 272 (1) (2017) 363–419.
- [24] Ajay Chandra, Martin Hairer, An analytic BPHZ theorem for regularity structures, preprint, arXiv:1612.08138, December 2016.
- [25] Thomas Cass, Martin Hairer, Christian Litterer, Samy Tindel, Smoothness of the density for solutions to Gaussian rough differential equations, *Ann. Probab.* 43 (1) (2015) 188–239.
- [26] Thomas Cass, Christian Litterer, Terry Lyons, Integrability and tail estimates for Gaussian rough differential equations, *Ann. Probab.* 41 (4) (2013) 3026–3050.
- [27] Ajay Chandra, Hendrik Weber, Stochastic PDEs, regularity structures, and interacting particle systems, *Ann. Fac. Sci. Univ. Toulouse Sci. Math.* 26 (4) (2017) 847–909.
- [28] Giuseppe Da Prato, Jerzy Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, 1992.
- [29] Jean-Dominique Deuschel, Daniel W. Stroock, *Large Deviations*, Pure and Applied Mathematics, vol. 137, Academic Press, Inc., 1989.
- [30] Aurélien Deya, Samy Tindel, Malliavin calculus for fractional heat equation, in: *Malliavin Calculus and Stochastic Analysis: a Festschrift in Honor of David Nualart*, in: Springer Proceedings in Mathematics & Statistics, Springer, 2013, 583 p., Dedicated to David Nualart on occasion of his 60th birthday.
- [31] P.K. Friz, P. Gassiat, P. Pigato, Precise asymptotics: robust stochastic volatility models, *Ann. Appl. Probab.* 31 (2) (2021) 896–940.
- [32] Richard P. Feynman, Albert R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, 1965.
- [33] Peter K. Friz, Martin Hairer, *A Course on Rough Paths*, With an introduction to regularity structures, 2 edition, Universitext, Springer, 2020.
- [34] Peter K. Friz, Tom Klose, Precise Laplace asymptotics for singular stochastic PDEs: the case of 2D gPAM, extended arXiv version of this article, arXiv:2103.00028v2, 2022.
- [35] Peter Friz, Harald Oberhauser, A generalized Fernique theorem and applications, *Proc. Am. Math. Soc.* 138 (10) (October 2010) 3679–3688.
- [36] Peter Friz, Nicolas Victoir, A variation embedding theorem and applications, *J. Funct. Anal.* 239 (2006) 631.
- [37] P. Friz, N. Victoir, Differential equations driven by Gaussian signals, *Ann. Inst. Henri Poincaré B, Probab. Stat.* 46 (2) (2010) 369–413.
- [38] Peter K. Friz, Nicolas B. Victoir, *Multidimensional Stochastic Processes as Rough Paths. Theory and Applications*, Cambridge University Press, 2010.
- [39] Massimiliano Gubinelli, Peter Imkeller, Nicolas Perkowski, Paracontrolled distributions and singular PDEs, *Forum Math. Pi* 3 (6) (2015) 1–75.
- [40] James Glimm, Arthur Jaffe, *Quantum Physics. A Functional Integral Point of View*, Springer, 1981.
- [41] Paul Gassiat, Cyril Labbé, Existence of densities for the dynamic Φ_3^4 model, *Ann. Inst. Henri Poincaré Probab. Stat.* 56 (1) (2020) 326–373.
- [42] Leonard Gross, Abstract Wiener spaces, in: *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, Volume 2: Contributions to Probability Theory, Part 1, University of California Press, Berkeley, Calif., 1967, pp. 31–42.
- [43] M. Ghani Varzaneh, S. Riedel, M. Scheutzwow, A dynamical theory for singular stochastic delay differential equations I: linear equations and a multiplicative ergodic theorem on fields of Banach spaces, *SIAM J. Appl. Dyn. Syst.* 21 (1) (2022) 542–587.
- [44] Martin Hairer, Solving the KPZ equation, *Ann. Math.* 178 (2013) 599–664.
- [45] Martin Hairer, A theory of regularity structures, *Invent. Math.* 198 (2014) 269–504.
- [46] Martin Hairer, Introduction to regularity structures, *Braz. J. Probab. Stat.* 29 (2) (2015) 175–210.
- [47] M. Hairer, K. Matetski, Discretisations of rough stochastic PDEs, *Ann. Probab.* 46 (3) (2018) 1651–1709.

- [48] M. Hairer, J. Mattingly, The strong Feller property for singular stochastic PDEs, *Ann. Inst. Henri Poincaré Probab. Stat.* 54 (3) (2018) 1314–1340.
- [49] Martin Hairer, Etienne Pardoux, A Wong-Zakai theorem for stochastic PDEs, *J. Math. Soc. Jpn.* 67 (4) (2015) 1551–1604.
- [50] Martin Hairer, Hendrik Weber, Large deviations for white-noise driven, nonlinear stochastic PDEs in two and three dimensions, *Ann. Fac. Sci. Toulouse* 24 (1) (2015) 55–92.
- [51] Yuzuru Inahama, Hiroshi Kawabi, Large deviations for heat kernel measures on loop spaces via rough paths, *J. Lond. Math. Soc.* 73 (3) (2006) 797–816.
- [52] Yuzuru Inahama, Hiroshi Kawabi, Asymptotic expansions for the Laplace approximations for Itô functionals of Brownian rough paths, *J. Funct. Anal.* 243 (1) (2007) 270–322.
- [53] Yuzuru Inahama, Hiroshi Kawabi, On asymptotics of Banach space-valued Itô functionals of Brownian rough paths, in: *Stochastic Analysis and Applications. The Abel Symposium 2005 - Proceedings of the 2nd Abel Symposium, Held in Honor of Kiyosi Ito, 2007*, pp. 415–434.
- [54] Yuzuru Inahama, Nobuaki Naganuma, Asymptotic expansion of the density for hypoelliptic rough differential equation, 2019, pp. 1–33.
- [55] Yuzuru Inahama, Laplace’s method for the laws of heat processes on loop spaces, *J. Funct. Anal.* 232 (2006) 148–194.
- [56] Yuzuru Inahama, A stochastic Taylor-like expansion in the rough path theory, *J. Theor. Probab.* 23 (2010) 04.
- [57] Yuzuru Inahama, Laplace approximation for rough differential equation driven by fractional Brownian motion, *Ann. Probab.* 41 (1) (2013) 170–205.
- [58] Yuzuru Inahama, Malliavin differentiability of solutions of rough differential equations, *J. Funct. Anal.* 267 (5) (2014) 1566–1584.
- [59] Yuzuru Inahama, Short time kernel asymptotics for rough differential equation driven by fractional Brownian motion, *Electron. J. Probab.* 21 (2016).
- [60] Yuzuru Inahama, Short time kernel asymptotics for Young SDE by means of Watanabe distribution theory, *J. Math. Soc. Jpn.* 68 (2) (2016) 535–577.
- [61] Svante Janson, *Gaussian Hilbert Spaces*, Cambridge University Press, 1997.
- [62] Warren P. Johnson, The curious history of Faà di Bruno’s formula, *Am. Math. Mon.* 109 (3) (2002) 217–234.
- [63] J. Jost, *Partial Differential Equations*, third edition, Graduate Texts in Mathematics, vol. 214, Springer, 2013.
- [64] Tom Klose, The constant coefficient in precise Laplace asymptotics for gPAM, preprint, arXiv: 2202.03358, 2022.
- [65] Shigeo Kusuoka, Yasufumi Osajima, A remark on the asymptotic expansion of density function of Wiener functionals, *J. Funct. Anal.* 255 (9) (2008) 2545–2562.
- [66] Wolfgang König, *The Parabolic Anderson Model. Random Walk in Random Potential*, Pathways in Mathematics, Birkhäuser, 2016.
- [67] Shigeo Kusuoka, Daniel W. Stroock, Precise asymptotics of certain Wiener functionals, *J. Funct. Anal.* 99 (1) (1991) 1–74.
- [68] Pierre Simon Laplace, Memoir on the Probability of the Causes of Events, *Stat. Sci.* 1 (3) (1986) 364–378.
- [69] Terry J. Lyons, Differential equations driven by rough signals, *Rev. Mat. Iberoam.* 14 (2) (1998) 215–310.
- [70] David Nualart, Bruno Saussereau, Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion, *Stoch. Process. Appl.* 119 (2) (2009) 391–409.
- [71] Yasufumi Osajima, General asymptotics of Wiener functionals and application to implied volatilities, in: *Large Deviations and Asymptotic Methods in Finance*, Springer, 2015, pp. 137–173.
- [72] V.I. Piterbarg, V.R. Fatalov, The Laplace method for probability measures in Banach spaces, *Russ. Math. Surv.* 50 (6) (1995) 1151–1239.
- [73] Carles Rovira, Samy Tindel, Sharp Laplace asymptotics for a parabolic SPDE, *Stoch. Stoch. Rep.* 69 (1–2) (2000) 11–30.
- [74] C. Rovira, S. Tindel, Sharp Laplace asymptotics for a hyperbolic SPDE, in: *Laurent Decreasefond, Bernt K. Oksendal, Ali Süleyman Üstünel (Eds.), Stochastic Analysis and Related Topics VII*, Birkhäuser Boston, Boston, MA, 2001, pp. 225–244.
- [75] M. Schilder, Some asymptotic formulas for Wiener integrals, *Trans. Am. Math. Soc.* 125 (1966) 63–85.
- [76] Philipp Schönbauer, Malliavin calculus and density for singular stochastic partial differential equations, preprint, arXiv:1809.03570, Sep 2018.

- [77] Barry Simon, *Trace Ideals and Their Applications*, second edition, *Mathematical Surveys and Monographs*, vol. 120, American Mathematical Society, 2005.
- [78] S. Takano, S. Watanabe, Asymptotic expansion formulas of the Schilder type for a class of conditional Wiener functional integrations, in: *Asymptotic Problems in Probability Theory: Wiener Functionals and Asymptotics*, Sanda/Kyoto, 1990, in: *Pitman Res. Notes Math. Ser.*, vol. 284, Longman Sci. Tech, Harlow, 1993, pp. 194–241.
- [79] S.R.S. Varadhan, Asymptotic probabilities and differential equations, *Commun. Pure Appl. Math.* 19 (3) (1966) 261–286.
- [80] Shinzo Watanabe, Analysis of Wiener functionals (Malliavin calculus) and its applications to heat kernels, *Ann. Probab.* 15 (1) (1987) 1–39.
- [81] Lorenzo Zambotti, SPDEs and regularity structures, Available on [Lorenzo Zambotti's webpage](#), Dec 2017. (Accessed 17 November 2020).
- [82] Eberhard Zeidler, *Applied Functional Analysis. Main Principles and Their Applications*, *Applied Mathematical Sciences*, vol. 109, Springer, 1995.