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Stability of Weak Solutions to Parabolic Problems with Nonstandard Growth and Cross–Diffusion

André H. Erhardt

Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, 0316 Oslo, Norway; andreerh@math.uio.no

Abstract: We study the stability of a unique weak solution to certain parabolic systems with nonstandard growth condition, which are additionally dependent on a cross-diffusion term. More precisely, we show that two unique weak solutions of the considered system with different initial values are controlled by their initial values.

Keywords: stability; nonlinear parabolic problem; nonstandard growth; cross-diffusion

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1. Introduction

In [1], we have recently established the existence of a unique weak solution to the following parabolic problem involving nonstandard $p(x,t)$-growth conditions and a cross-diffusion term $\delta\Delta u$ with $\delta \geq 0$:

$$\begin{cases} \partial_t u - \operatorname{div}(a(x,t,\nabla u)) = \operatorname{div}(|F|^{p(x,t)-2}F), & \text{in } \Omega_T, \\ \partial_t v - \operatorname{div}(a(x,t,\nabla v)) = \delta\Delta u, & \text{in } \Omega_T, \\ u = v = 0, & \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0,\ v(\cdot,0) = v_0, & \text{on } \Omega \times \{0\}, \end{cases} \tag{1}$$

where the vector-field $a(x,t,\cdot)$ fulfils nonstandard growth and monotonicity properties, which we will specify later. Moreover, $\Omega \subset \mathbb{R}^n$, $n \geq 2$ denotes an open, bounded Lipschitz domain and $\Omega_T := \Omega \times (0,T)$ represents the space-time cylinder over Ω of height $T > 0$. Furthermore, the partial derivatives of a function u are represented with respect to time t by u_t or $\partial_t u$ and with respect to the space variable x by ∇u.

Since in [1] the stability of the solution remained open, we will now catch up this and we will establish the desired stability result. More precisely, we will prove that two (unique) weak solutions with different initial values are controlled by their initial values.

The investigation of problems like the one in [1] is motivated by several aspects and issues. First of all, many applications, e.g., in physics or biology, motivate the study of parabolic problems. In particular, evolutionary equations are used to model biological or physical processes, see [2,3], including climate modelling and climatology [4].

The second interesting aspect here is the variable exponent $p(x,t)$ and the nonstandard growth setting, which arises by modelling and investigating certain classes of stationary and non-stationary non-Newtonian fluids such as electro-rheological fluids or fluids with viscosity depending on the temperature [5–9]. In addition, one uses such diffusion models in the context of the restoration in image processing [10–12] and applications include also models for flows in porous media [13] or parabolic obstacle problems [14]. Moreover, in past years parabolic nonstandard growth problems gained increased interest in mathematics, see e.g., [15–20].

A further important aspect is the effect of a cross-diffusion term, which arises for instance by the modelling of interaction between species [21]. As already mentioned in [1,22], this may lead to unexpected behaviour, e.g., in our case the cross-diffusion term $\delta\Delta u$, $\delta \geq 0$ requires that the growth exponent $p(x,t) \geq 2$. Only if $\delta = 0$ we may assume that $\frac{2n}{n+2} < p(x,t)$, $n \geq 2$. Furthermore, the cross-diffusion term $\delta\Delta u$ for $\delta > 0$ will complicate the derivation of the desired stability estimate and requires certain additional assumption, which we will discuss later in detail.

2. General Assumptions, Settings and Notation

We suppose that the vector-fields $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ are Carathéodory functions, i.e., $a(z,w)$ is measurable in the first argument for every $w \in \mathbb{R}^n$ and continuous in the second one for a.e. $z = (x,t) \in \Omega_T \subset \mathbb{R}^{n+1}$. In addition, $a(z,w)$ satisfy the following nonstandard growth and monotonicity conditions with variable exponents $p : \Omega_T \to [2,\infty)$, $\mu \in [0,1]$ and $0 < \nu \leq 1 \leq L$:

$$|a(z,w)| \leq \; L(1+|w|)^{p(z)-1}, \tag{2}$$

$$(a(z,w)-a(z,w_0)) \cdot (w-w_0) \geq \nu(\mu^2+|\nabla w|^2+|\nabla w_0|^2)^{\frac{p(z)-2}{2}}|\nabla(w-w_0)|^2, \tag{3}$$

for all $z \in \Omega_T$ and $w,\ w_0 \in \mathbb{R}^n$, cf. [23], where the exponent function $p : \Omega_T \to [2,\infty)$ fulfils the following conditions: there exist two constants γ_1 and γ_2, such that

$$2 \leq \gamma_1 \leq p(z) \leq \gamma_2 < \infty \quad \text{and} \quad |p(z_1)-p(z_2)| \leq \omega(d_{\mathcal{P}}(z_1,z_2)) \tag{4}$$

hold true for any choice of $z_1,\ z_2 \in \Omega_T$, where $\omega : [0,\infty) \to [0,1]$ denotes a modulus of continuity, which is assumed to be a concave, non-decreasing function with $\lim_{\rho\downarrow 0}\omega(\rho) = 0 = \omega(0)$. Furthermore, the parabolic distance is given by $d_{\mathcal{P}}(z_1,z_2) := \max\{|x_1-x_2|, \sqrt{|t_1-t_2|}\}$ for $z_1 = (x_1,t_1),\ z_2 = (x_2,t_2) \in \mathbb{R}^{n+1}$ and we suppose the following weak logarithmic continuity condition

$$\limsup_{\rho\downarrow 0}\, \omega(\rho)\log\left(\frac{1}{\rho}\right) < +\infty. \tag{5}$$

The spaces $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$ denote the usual Lebesgue and Sobolev spaces, while the nonstandard $p(z)$–Lebesgue space $L^{p(z)}(\Omega_T,\mathbb{R}^k)$ is defined as the set of those measurable functions $v : \Omega_T \to \mathbb{R}^k$ for $k \in \mathbb{N}$, which satisfy $|v|^{p(z)} \in L^1(\Omega_T,\mathbb{R}^k)$, i.e.,

$$L^{p(z)}(\Omega_T,\mathbb{R}^k) := \left\{ v : \Omega_T \to \mathbb{R}^k \text{ is measurable in } \Omega_T : \int_{\Omega_T} |v|^{p(z)}\mathrm{d}z < +\infty \right\}.$$

The set $L^{p(z)}(\Omega_T,\mathbb{R}^k)$ equipped with the Luxemburg norm

$$\|v\|_{L^{p(z)}(\Omega_T)} := \inf\left\{ \delta > 0 : \int_{\Omega_T}\left|\frac{v}{\delta}\right|^{p(z)}\mathrm{d}z \leq 1 \right\}$$

becomes a Banach space. Now, we are able to specify the regularity assumption on the inhomogeneity, i.e., we suppose that $F \in L^{p(z)}(\Omega_T,\mathbb{R}^n)$. Note as an abbreviation for the exponent $p(z)$ we will also write $p(\cdot)$.

Furthermore, we denote by $W^{p(\cdot)}_g(\Omega_T)$ the Banach space

$$W^{p(\cdot)}_g(\Omega_T) := \left\{ u \in [g + L^1(0,T;W^{1,1}_0(\Omega))] \cap L^{p(\cdot)}(\Omega_T) \mid \nabla u \in L^{p(\cdot)}(\Omega_T,\mathbb{R}^n) \right\}$$

equipped with the norm $\|u\|_{W^{p(\cdot)}(\Omega_T)} := \|u\|_{L^{p(\cdot)}(\Omega_T)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega_T)}$. If $g = 0$ we write $W^{p(\cdot)}_0(\Omega_T)$ instead of $W^{p(\cdot)}_g(\Omega_T)$. In addition, we denote by $W^{p(\cdot)}(\Omega_T)'$ the dual of the space $W^{p(\cdot)}_0(\Omega_T)$.

Finally, we can state the definition of a weak solution to system (1):

Definition 1. *We call $u, v \in C^0([0,T]; L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T)$ a (weak) solution to the Dirichlet problem (1), if and only if*

$$\begin{cases} \int_{\Omega_T} [u \cdot \varphi_t - a(z, \nabla u) \cdot \nabla \varphi] \mathrm{d}z = \int_{\Omega_T} |F|^{p(x,t)-2} F \cdot \nabla \varphi \mathrm{d}z, \\ \int_{\Omega_T} [v \cdot \zeta_t - a(z, \nabla v) \cdot \nabla \zeta] \mathrm{d}z = \int_{\Omega_T} \delta \nabla u \cdot \nabla \zeta \mathrm{d}z, \end{cases} \tag{6}$$

whenever $\varphi, \zeta \in C_0^\infty(\Omega_T)$, $\delta \geq 0$, the boundary condition $u = v = 0$ on $\partial\Omega \times \{0\}$ and initial conditions $u(\cdot, 0) = u_0 \in L^2(\Omega)$, $v(\cdot, 0) = v_0 \in L^2(\Omega)$ a.e. on Ω are satisfied.

Remark 1. *We will also use the notation $(u, v) \in (C^0([0,T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T))^2$ instead of $u, v \in C^0([0,T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T)$ and similarly, $(u_0, v_0) \in (L^2(\Omega))^2$ instead of $u_0, v_0 \in L^2(\Omega)$.*

3. Main Result

In this section, we will state and prove our main result, which reads as follows:

Theorem 1. *Let $\delta \geq 0$, $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain and $p : \Omega_T \to [\gamma_1, \gamma_2]$ satisfies (4) and (5). Furthermore, assume that $F \in L^{p(z)}(\Omega_T, \mathbb{R}^n)$ and suppose that the vector-field $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function satisfying the growth condition (2) and the monotonicity condition (3). Then, for two unique weak solutions $(u, v), (u^*, v^*) \in (C^0([0,T]; L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T))^2$ with $(\partial_t u, \partial_t v), (\partial_t u^*, \partial_t v^*) \in (W^{p(\cdot)}(\Omega_T)')^2$ and different initial values $(u_0, v_0), (u_0^*, v_0^*) \in (L^2(\Omega))^2$ [i.e., $(u_0, v_0) \neq (u_0^*, v_0^*)$] to system (1) the stability estimates*

$$\|u(\cdot, t) - u^*(\cdot, t)\|_{L^2(\Omega)}^2 \leq \|u_0 - u_0^*\|_{L^2(\Omega)}^2 \tag{7}$$

and

$$\|v(\cdot, t) - v^*(\cdot, t)\|_{L^2(\Omega)}^2 \leq \|v_0 - v_0^*\|_{L^2(\Omega)}^2 + \frac{\delta^2}{\gamma_1 \nu^2} \|u_0 - u_0^*\|_{L^2(\Omega)}^2 + \frac{\gamma_2 - 2}{\gamma_1 - 1} 4\delta |\Omega_T| \tag{8}$$

holds true for a.e. $t \in [0, T)$, i.e., the solutions are controlled by their initial values.

Remark 2. *Our first remark related to our result is that we have to assume a different monotonicity condition as in [1] to be able to prove the stability estimate. Nevertheless, the existence result from [1] still holds true under this assumption, cf. [23,24]. Additionally, we want to remark that regarding the stability estimates (7) and (8), we see that in the case $\delta = 0$, i.e., problem (1) without cross-diffusion, we would derive the 'standard' stability estimates:*

$$\begin{aligned} \|u(x,t) - u^*(x,t)\|_{L^2(\Omega)} \leq \|u_0 - u_0^*\|_{L^2(\Omega)} \quad \text{and} \\ \|v(x,t) - v^*(x,t)\|_{L^2(\Omega)} \leq \|v_0 - v_0^*\|_{L^2(\Omega)} \end{aligned} \tag{9}$$

for a.e. $t \in [0, T)$. Furthermore, in the case $p(\cdot) \equiv 2$ or $\mu \equiv 1$ the stability estimate (8) will be reduce to

$$\|v(x,t) - v^*(x,t)\|_{L^2(\Omega)}^2 \leq \|v_0 - v_0^*\|_{L^2(\Omega)}^2 + \frac{\delta^2}{2\nu^2} \|u_0(x) - u_0^*(x)\|_{L^2(\Omega)}^2 \tag{10}$$

for a.e. $t \in [0, T)$, cf. proof of Theorem 1. Finally, we want to refer to ([24] [Remark 2 & Remark 3]) regarding L^1-estimates for the solutions, i.e., using Hölder's inequality to derive

$$\|u(x,t) - u^*(x,t)\|_{L^1(\Omega)} \leq |\Omega|^{\frac{1}{2}} \|u(x,t) - u^*(x,t)\|_{L^2(\Omega)} \leq |\Omega|^{\frac{1}{2}} \|u_0 - u_0^*\|_{L^2(\Omega)}$$

for a.e. $t \in [0, T)$ (similar holds true for $v - v^*$), and the possible extension of problem (1).

Proof. Our aim is to prove the stability of weak solutions to the Dirichlet problem (1). To this end, we consider the unique weak solution $(u,v) \in (C^0([0,T];L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T))^2$ with $(\partial_t u, \partial_t v) \in (W^{p(\cdot)}(\Omega_T)')^2$ to system (1) and the unique weak solution $(u^*, v^*) \in (C^0([0,T];L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T))^2$ with $(\partial_t u^*, \partial_t v^*) \in (W^{p(\cdot)}(\Omega_T)')^2$ to the following Dirichlet problem:

$$\begin{cases} \partial_t u^* - \operatorname{div}(a(x,t,\nabla u^*)) = \operatorname{div}(|F|^{p(x,t)-2}F), & \text{in } \Omega_T, \\ \partial_t v^* - \operatorname{div}(a(x,t,\nabla v^*)) = \delta \Delta u^*, & \text{in } \Omega_T, \\ u^* = v^* = 0 & \text{on } \partial\Omega \times (0,T), \\ u^*(\cdot,0) = u_0^*,\ v^*(\cdot,0) = v_0^* & \text{on } \Omega \times \{0\}, \end{cases} \tag{11}$$

where the initial values $(u_0, v_0), (u_0^*, v_0^*) \in (L^2(\Omega))^2$ are not equal. Next, we choose $\varphi = u - u^* \in W_0^{p(\cdot)}(\Omega_T)$ and $\zeta = v - v^* \in W_0^{p(\cdot)}(\Omega_T)$, since $W^{p(\cdot)}(\Omega_T)'$ is the dual of $W_0^{p(\cdot)}(\Omega_T)$, as admissible test functions and subtract the weak formulations of system (11) from the one of (1), which yields

$$\begin{cases} \int_{\Omega_T} [(u-u^*)\cdot \varphi_t - (a(z,\nabla u) - a(z,\nabla u^*))\cdot \nabla\varphi]\mathrm{d}z = 0, \\ \int_{\Omega_T} [(v-v^*)\cdot \zeta_t - (a(z,\nabla v) - a(z,\nabla v^*))\cdot \nabla\zeta]\mathrm{d}z = \int_{\Omega_T} \delta\nabla(u-u^*)\cdot\nabla\zeta\mathrm{d}z, \end{cases}$$

cf. (6). Then, using integration by parts and inserting $\varphi = u - u^*$ and $\zeta = v - v^*$, we get

$$\begin{cases} \int_{\Omega_T} (u-u^*)_t(u-u^*) + (a(z,\nabla u) - a(z,\nabla u^*))\nabla(u-u^*)\mathrm{d}z = 0, \\ \int_{\Omega_T} (v-v^*)_t(v-v^*) + (a(z,\nabla v) - a(z,\nabla v^*))\nabla(v-v^*) + \delta\nabla(u-u^*)\nabla(v-v^*)\mathrm{d}z = 0. \end{cases} \tag{12}$$

From the first equation of (12), we can conclude by applying the monotonicity condition (3) that

$$0 \geq \int_{\Omega_T} (u-u^*)_t(u-u^*)\mathrm{d}z + \nu \int_{\Omega_T} 2^{\frac{2-p(\cdot)}{2}} |\nabla(u-u^*)|^{p(\cdot)}\mathrm{d}z, \tag{13}$$

where we also used that

$$\begin{aligned} |\nabla(u-u^*)|^{p(\cdot)} &= (|\nabla(u-u^*)|^2)^{\frac{p(\cdot)-2}{2}} |\nabla(u-u^*)|^2 \\ &\leq 2^{\frac{p(\cdot)-2}{2}} (|\nabla u|^2 + |\nabla u^*|^2)^{\frac{p(\cdot)-2}{2}} |\nabla(u-u^*)|^2, \end{aligned}$$

if $p(\cdot) \geq 2$. Therefore, if follows from (13) that

$$\|u_0(x) - u_0^*(x)\|^2_{L^2(\Omega)} \geq \|u(x,t) - u^*(x,t)\|^2_{L^2(\Omega)} + 2\nu \int_{\Omega_T} 2^{\frac{2-p(\cdot)}{2}} |\nabla(u-u^*)|^{p(\cdot)}\mathrm{d}z$$

and finally, the stability estimate (7). Moreover, we have

$$\int_{\Omega_T} 2^{\frac{2-p(\cdot)}{2}} |\nabla(u-u^*)|^{p(\cdot)}\mathrm{d}z \leq \frac{1}{2\nu} \|u_0(x) - u_0^*(x)\|^2_{L^2(\Omega)}. \tag{14}$$

Notice that in case $\mu = 1$ the monotonicity condition (3) implies that

$$(a(z,\nabla u) - a(z,\nabla u^*))\nabla(u-u^*) \geq \nu|\nabla(u-u^*)|^2,$$

since $(\mu^2+|\nabla u|^2+|\nabla u^*|^2)^{\frac{p(\cdot)-2}{2}}\geq 1$ and hence, we can conclude that

$$\int_{\Omega_T}|\nabla(u-u^*)|^2\mathrm{d}z\leq\frac{1}{2\nu}\|u_0(x)-u_0^*(x)\|^2_{L^2(\Omega)}. \tag{15}$$

Our next aim is to treat the last term of the second equation in (12). Here, we will see the reason to choose a slightly different monotonicity condition compared to [1]. To this end, we first consider the case that $p(\cdot)\equiv 2$ or $(\mu^2+|\nabla u|^2+|\nabla u^*|^2)^{\frac{p(\cdot)-2}{2}}\geq 1$ to conclude by means of Hölder's inequality and (15) that

$$\begin{aligned}
-\delta\int_{\Omega_T}\nabla(u-u^*)\nabla(v-v^*)\mathrm{d}z &\leq \delta\left|\int_{\Omega_T}\nabla(u-u^*)\nabla(v-v^*)\mathrm{d}z\right|\\
&\leq \delta\left(\int_{\Omega_T}|\nabla(u-u^*)|^2\mathrm{d}z\right)^{\frac12}\left(\int_{\Omega_T}|\nabla(v-v^*)|^2\mathrm{d}z\right)^{\frac12}\\
&\leq \frac{\delta}{\sqrt{2\nu}}\|u_0(x)-u_0^*(x)\|_{L^2(\Omega)}\left(\int_{\Omega_T}|\nabla(v-v^*)|^2\mathrm{d}z\right)^{\frac12}\\
&= \frac{\delta}{\sqrt{2}\nu}\|u_0(x)-u_0^*(x)\|_{L^2(\Omega)}\left(\nu\int_{\Omega_T}|\nabla(v-v^*)|^2\mathrm{d}z\right)^{\frac12}.
\end{aligned}$$

Then, by Cauchy's inequality we gain (10). Finally, for $p(\cdot)\geq\gamma_1>2$ we can conclude by using Young's inequality twice (first with $\frac{1}{p(\cdot)}+\frac{p(\cdot)-1}{p(\cdot)}=1$ and then with $\frac{1}{p(\cdot)-1}+\frac{p(\cdot)-2}{p(\cdot)-1}=1$) and (14) the following:

$$\begin{aligned}
\delta\int_{\Omega_T}|\nabla(u-u^*)||\nabla(v-v^*)|\mathrm{d}z &= \int_{\Omega_T}\left(\frac{\delta^2}{\nu}2^{\frac{2-p(\cdot)}{2}}\right)^{\frac{1}{p(\cdot)}}|\nabla(u-u^*)|\times\\
&\qquad\left(\delta^{p(\cdot)-2}\frac{\nu}{2^{\frac{2-p(\cdot)}{2}}}\right)^{\frac{1}{p(\cdot)}}|\nabla(v-v^*)|\mathrm{d}z\\
&\leq \frac{1}{\gamma_1}\frac{\delta^2}{\nu}\int_{\Omega_T}2^{\frac{2-p(\cdot)}{2}}|\nabla(u-u^*)|^{p(\cdot)}\mathrm{d}z\\
&+\int_{\Omega_T}\frac{p(\cdot)-1}{p(\cdot)}\left(\delta^{p(\cdot)-2}\frac{\nu}{2^{\frac{2-p(\cdot)}{2}}}\right)^{\frac{1}{p(\cdot)-1}}|\nabla(v-v^*)|^{p'(\cdot)}\mathrm{d}z\\
&\leq \frac{1}{\gamma_1}\frac{\delta^2}{2\nu^2}\|u_0(x)-u_0^*(x)\|^2_{L^2(\Omega)}\\
&+\int_{\Omega_T}\left(\delta^{p(\cdot)-2}\frac{\nu}{2^{\frac{2-p(\cdot)}{2}}}\right)^{\frac{1}{p(\cdot)-1}}|\nabla(v-v^*)|^{p'(\cdot)}\mathrm{d}z\\
&= \frac{1}{\gamma_1}\frac{\delta^2}{2\nu^2}\|u_0(x)-u_0^*(x)\|^2_{L^2(\Omega)}\\
&+\int_{\Omega_T}(2\delta)^{\frac{p(\cdot)-2}{p(\cdot)-1}}\left(\nu 2^{\frac{2-p(\cdot)}{2}}|\nabla(v-v^*)|^{p(\cdot)}\right)^{\frac{1}{p(\cdot)-1}}\mathrm{d}z\\
&\leq \frac{1}{\gamma_1}\frac{\delta^2}{2\nu^2}\|u_0(x)-u_0^*(x)\|^2_{L^2(\Omega)}+\frac{\gamma_2-2}{\gamma_1-1}2\delta|\Omega_T|\\
&+\frac{\nu}{\gamma_1-1}\int_{\Omega_T}2^{\frac{2-p(\cdot)}{2}}|\nabla(v-v^*)|^{p(\cdot)}\mathrm{d}z.
\end{aligned}$$

Then, we get from the second equation in (12) the following estimate:

$$\|v(x,t)-v^*(x,t)\|^2_{L^2(\Omega)}+2\int_{\Omega_T}(a(z,\nabla v)-a(z,\nabla v^*))\nabla(v-v^*)\mathrm{d}z\le\|v_0-v_0^*\|^2_{L^2(\Omega)}$$
$$+\frac{\delta^2}{\gamma_1\nu^2}\|u_0(x)-u_0^*(x)\|^2_{L^2(\Omega)}+\frac{\gamma_2-2}{\gamma_1-1}4\delta|\Omega_T|+\frac{2\nu}{\gamma_1-1}\int_{\Omega_T}2^{\frac{2-p(\cdot)}{2}}|\nabla(v-v^*)|^{p(\cdot)}\mathrm{d}z,$$

provided $p(\cdot)\ge\gamma_1>2$. Finally, applying the monotonicity condition (3) as for concluding estimate (13) and reabsorbing the last term on the right-hand side on its left-hand side, we derive at (8). □

4. Conclusions and Discussion

Summarising, we were able to show that the unique weak solution to system (1) satisfies certain stability estimates, i.e., we proved that two weak solutions to system (1) with different initial values are controlled by their initial values. Remarkable is that the solution of the first equation of system (1) is controlled only by its corresponding initial values, while the solution of the second equation of system (1) is controlled by its corresponding initial values and the ones from the first equation due to the cross-diffusion term. Moreover, the stability estimate (8) is also dependent on several system parameters. Notable is also that the left-hand side of the stability estimate (8) may increase or decrease independently on the initial values of the solution but dependent on the system parameters δ, ν, γ_1 and γ_2, and the measure of Ω_T, which affects the stability. Notice that for small δ, e.g., $0\le\delta\ll1$, the last two terms in (8) are less or not dominating and in the limit case $\delta\to0$ they disappear, which is to expect. Finally, we want to discuss some special cases.

(i) If one is a priori able to guarantee that $\delta\int_{\Omega_T}\nabla(u-u^*)\nabla(v-v^*)\mathrm{d}z\le0$, then one can immediately conclude that the stability estimate (9) holds true and one can use the structure assumption on the vector-field $a(z,\cdot)$ from [1].

(ii) Furthermore, if one can assume that the vector-field $a(z,\cdot)$ satisfies the monotonicity condition $(a(z,w)-a(z,w_0))\cdot(w-w_0)\ge|\nabla(w-w_0)|^2$ for all $z\in\Omega_T$ and $w,\ w_0\in\mathbb{R}^n$, then again, the stability estimates (7) and (10) are satisfied.

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